Noncommutative geometry and twisted conformal symmetry

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The twist-deformed conformal algebra is constructed as a Hopf algebra with twisted coproduct. This allows for the definition of conformal symmetry in a noncommutative background geometry. The twisted coproduct is reviewed for the Poincaré algebra and the construction is then extended to the full conformal algebra. The case of Moyal-type noncommutativity of the coordinates is considered. It is demonstrated that conformal invariance need not be viewed as incompatible with noncommutative geometry; the noncommutativity of the coordinates appears as a consequence of the twisting, as has been shown in the literature in the case of the twisted Poincaré algebra.

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I. INTRODUCTION

Noncommutative geometry is often introduced by postulating that the coordinates of space (or spacetime) do not commute, so that

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x). \tag{1}$$

In order to formulate Physics in this context [1,2], fields must be defined on such a background, and of course they inherit algebraic properties from those of the background itself. The algebra of fields is used to construct an action and to express physical principles. One way of performing calculations is to avoid dealing with the algebra of functions on noncommutative coordinates as defined above, and instead formulate a theory in terms of the algebra of functions on a commutative space, but with a deformed 'star product,' defined such that these two algebras are isomorphic. The application of such ideas and their appearance in string theory can be found in the well-known paper by Seiberg and Witten [3] and references therein.

In practice, dealing with an arbitrary noncommutativity function $\theta(x)$ is very difficult. Rather than attempt this, authors have mainly considered constant, linear, and quadratic noncommutativity, referred to, respectively, as the 'canonical structure,' 'Lie algebra,' and 'quantum space' types of noncommutative space, as explained in [4]. In the present paper, we consider the canonical structure case and will eventually show the property

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2}$$

where θ is a constant matrix, not depending on x. At first glance, this relation seems to be more than just a statement of noncommutativity; it appears to break Poincaré[1] invariance, and to introduce 'preferred directions' into the formalism at the outset. Given that such preferred directions disappear in the $\theta = 0$ case, it is not clear that the simple limit $\theta \to 0$ reproduces the Lorentz symmetry which we are used to seeing in nature. Furthermore, although the commutation relation (2) has been used as the basis of many analyses of physics in the language of quantum theory on noncommutative space, it does not make sense to talk of vectors or spin, which depend on representations of the Lorentz group, when it is not clear that there is a Lorentz group present.

These problems were pointed out and addressed systematically in the interesting paper [5] and in the lecture [6].[2] It is true that the noncommutativity relation (2) is not a Poincaré-invariant statement, but it is twist-deformed Poincaré invariant. 'Relativistically invariant' in the usual sense means that a theory is Poincaré-invariant, and this may be consistently modified to mean invariance under twisted Poincaré transformations in the case of a noncommutative background. This is demonstrated in [5,7] and we shall briefly review the construction in order that the present work be somewhat self-contained. We note that the construction may also be generalized to the supersymmetric case, as investigated in [8].

The commutator (2) does not look Poincaré-invariant, but can be understood to be relativistically invariant in the twisted Poincaré sense. It also does not look conformally invariant, and the purpose of the present paper is to show that it is consistent to twist deform the conformal algebra along the same lines as the Poincaré algebra and therefore consider conformal symmetry in a noncommutative background. The relation (2) *is* twist-conformal invariant.

In Section II we review the construction of the Hopf algebra and coproduct used by the authors of [5] to define the twisted Poincaré algebra. In Section III we calculate the twisted coproduct for the generators of the full conformal algebra, including for completeness the Poincaré subalgebra. Finally, in Section IV we give some explicit examples of the twisted conformal transformations. As in the treatment in [5], we show that the noncommutativity of Eq. (2)

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[1] It might be pointed out that Eq. (2) only obviously breaks Lorentz invariance, as it is clearly invariant under $x \to x + a$. This is only (naively) true if the coordinate shift a commutes with the coordinates, which is a nontrivial assumption in itself.

[2] We will follow more closely the notations of [5].

is a simple consequence of the deformed algebra, in this case the twisted conformal algebra. We conclude with some discussion in Section V.

II. COPRODUCT AND TWIST DEFORMATION

Here we briefly explain the formalism so that we may apply it to the conformal algebra; for a detailed treatment of Hopf algebras, the reader may consult the comprehensive reference [9].

To deform the universal enveloping algebra $\mathcal{U}(A)$ of a Lie algebra A, one first constructs a representation of $\mathcal{U}(A)$ in the tensor product $\mathcal{U}(A) \otimes \mathcal{U}(A)$ by defining a coproduct

$$\Delta : \mathcal{U}(A) \rightarrow \mathcal{U}(A) \otimes \mathcal{U}(A). \tag{3}$$

Starting with the primitive coproduct Δ_0, defined by

$$\Delta_0(X) \equiv X \otimes 1 + 1 \otimes X, \tag{4}$$

a twist element $\mathcal{F} \in \mathcal{U}(A) \otimes \mathcal{U}(A)$ may be chosen, and a twisted coproduct Δ_t defined as

$$\Delta_t(X) \equiv \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}. \tag{5}$$

The twist element must be an invertible element and satisfy the twist equation $\mathcal{F}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}$ for the original coproduct $\Delta = \Delta_0$ [9]. Consequently, the Hopf algebra retains its properties and remains a Hopf algebra under the twist. The twisted and untwisted coproducts Δ_t and Δ_0 of the generators satisfy the same Lie algebra; that is, the coproduct map is an obvious homomorphism thereof.

In [5] the algebra A was taken to be the Poincaré algebra $\mathcal{P}$, with momentum generators P_μ and Lorentz generators $M_{\mu\nu}$. In that case, the twist element was taken to be a so-called Abelian (since the P_μ form an Abelian subalgebra) twist,

$$\mathcal{F} = e^{(i/2)\theta^{\mu\nu}P_\mu \otimes P_\nu}, \tag{6}$$

constructed to reproduce the noncommutativity relation (2), as we will see below. Our main result will be that this same twist element can be used to construct the twist-deformed conformal algebra under which the noncommutativity (2) is invariant. With this in mind, let us recall the full conformal algebra for $d > 2$, including the dilatation generator D, and the special conformal transformation (SCT) K_μ,

$$\begin{aligned}
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\
[K_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \\
[P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} \\
&\quad - \eta_{\nu\sigma}M_{\mu\rho}).
\end{aligned} \tag{7}$$

The conformal algebra contains the Poincaré algebra and the Abelian algebra of the P_μ as subalgebras. The representation of the conformal algebra on an algebra $\mathcal{A}$ of functions on spacetime is given by [10]

$$\begin{aligned}
\hat{P}_\mu &= -i\partial_\mu, & \hat{M}_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\
\hat{D} &= -ix \cdot \partial, & \hat{K}_\mu &= 2x_\mu\hat{D} - x^2\hat{P}_\mu,
\end{aligned} \tag{8}$$

where $\hat{G}$ denotes the representation of G on the algebra of functions $\mathcal{A}$.

In the primitive case, using the untwisted coproduct Δ_0, the multiplication in $\mathcal{A}$ is given by a map $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ defined simply as $m(f \otimes g) \equiv fg$. In the twisted case, this multiplication map must be modified by composition with the representation of the inverse twist element, to give the twisted map m_t which is the "star product" in $\mathcal{A}$,

$$f \star g \equiv m_t(f \otimes g) = m(\hat{\mathcal{F}}^{-1}(f \otimes g)). \tag{9}$$

The algebra $\mathcal{A}$ endowed with the new twisted multiplication m_t is then consistent with the twisted conformal algebra, exhibited explicitly in the following section. By 'consistent' is meant that statements (i.e. physics) expressed in terms of the elements and operations of the twisted algebra of functions are covariant under the twist-deformed conformal algebra. This includes the relation (2), as we show in Section IV.

III. TWISTED CONFORMAL ALGEBRA

Using the twisted coproduct Δ_t given in Eq. (5) with the twist element of Eq. (6), we may calculate the twisted coproducts of the generators. As mentioned in [7], the momentum generators do not get twisted due to the commutativity of P_μ;

$$\Delta_t(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu = \Delta_0(P_\mu). \tag{10}$$

For the Lorentz generators, $\Delta_t(M_{\mu\nu})$ has been calculated in [6,7], and is given by

$$\Delta_t(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\rho\sigma}((\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu)). \tag{11}$$

For the dilatation generator D we find

$$\Delta_t(D) = D \otimes 1 + 1 \otimes D + \theta^{\rho\sigma}P_\rho \otimes P_\sigma, \tag{12}$$

while for the SCT generator K_μ a short calculation gives

$$\Delta_t(K_\mu) = K_\mu \otimes 1 + 1 \otimes K_\mu + \theta^{\rho\sigma}((\eta_{\rho\mu}D + M_{\rho\mu}) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu}D + M_{\sigma\mu})) + \frac{1}{4}\theta^{\rho\sigma}\theta^{\lambda\pi}((\eta_{\lambda\mu}P_\rho - \eta_{\lambda\rho}P_\mu + \eta_{\rho\mu}P_\lambda) \otimes P_\pi P_\sigma + P_\lambda P_\rho \otimes (\eta_{\pi\mu}P_\sigma - \eta_{\pi\sigma}P_\mu + \eta_{\sigma\mu}P_\pi)). \tag{13}$$

As a consistency check, it may be shown explicitly that these objects satisfy the conformal algebra (7), as by construction they should.

IV. NONCOMMUTATIVITY AND TRANSFORMATIONS

The noncommutativity relation (2) is a consequence of the twisted algebra, as shown in [5]. Evaluating the commutator of x_μ with x_ν in $\mathcal{A}$, we have

$$[x_\mu, x_\nu]_{\mathcal{A}} \equiv m_t(x_\mu \otimes x_\nu) - m_t(x_\nu \otimes x_\mu) = \theta_{\mu\nu}. \tag{14}$$

In particular, $\theta_{\mu\nu}$ is an invariant under twisted Poincaré transformations. Significantly, $\theta_{\mu\nu}$ is also invariant under twisted conformal transformations, as we will show at the end of this section.

Now, let us choose some simple example functions and demonstrate explicitly that they transform correctly under the twisted conformal symmetry. Following [5] consider the second-rank tensor $f_{\rho\sigma} = x_\rho x_\sigma$. In that paper, the action of the Lorentz algebra on this object is calculated to show how the twisted covariance works in practice on a Lorentz tensor. A similar test can be performed using the twisted conformal algebra. The actions of the original (untwisted) dilatation generator D and SCT generator K_μ on $f_{\rho\sigma}$ are given by

$$Df_{\rho\sigma} = -2if_{\rho\sigma}, \tag{15}$$

showing that $f_{\rho\sigma}$ has conformal dimension two, and

$$K_\mu f_{\rho\sigma} = -4ix_\mu x_\rho x_\sigma + ix^2(\eta_{\mu\rho}x_\sigma + \eta_{\mu\sigma}x_\rho). \tag{16}$$

In the twisted case, $f_{\rho\sigma}$ is replaced by the twisted object $f^t_{\rho\sigma} = m_t(x_\rho \otimes x_\sigma)$, and the conformal generators are now applied through the twisted coproduct, so that for a generator G,

$$G^t f^t_{\rho\sigma} = m_t(\Delta_t(\hat{G})(x_\rho \otimes x_\sigma)). \tag{17}$$

Using the twisted coproduct from Eq. (12), we first calculate

$$\Delta_t(\hat{D})x_\rho \otimes x_\sigma = -2ix_\rho \otimes x_\sigma - \theta_{\rho\sigma}1 \otimes 1 \tag{18}$$

so that, using an expansion for $\mathcal{F}^{-1}$ (which terminates in this case),

$$\hat{\mathcal{F}}^{-1}\Delta_t(\hat{D})x_\rho \otimes x_\sigma = -2ix_\rho \otimes x_\sigma. \tag{19}$$

We can now read off

$$D^t f^t_{\rho\sigma} = -2if^t_{\rho\sigma}, \tag{20}$$

analogous to the untwisted case, Eq. (15), showing that the conformal dimension of the twisted tensor f^t is two, under the twisted conformal algebra. Now, considering the action of the twisted coproduct of K_μ from Eq. (13) on f^t, we have

$$\Delta_t(\hat{K}_\mu)x_\rho \otimes x_\sigma = -i(2x_\mu x_\rho - x^2\eta_{\mu\rho}) \otimes x_\sigma - ix_\rho \otimes (2x_\mu x_\sigma - x^2\eta_{\mu\sigma}) + \theta^{\lambda\tau}((x_\lambda\eta_{\mu\rho} - x_\rho\eta_{\lambda\mu} - x_\mu\eta_{\lambda\rho}) \otimes \eta_{\tau\sigma} + \eta_{\lambda\rho} \otimes (x_\tau\eta_{\mu\sigma} - x_\sigma\eta_{\tau\mu} - x_\mu\eta_{\tau\sigma})). \tag{21}$$

Acting on this with $\mathcal{F}^{-1}$ we find

$$\hat{\mathcal{F}}^{-1}\Delta_t(\hat{K}_\mu)x_\rho \otimes x_\sigma = -i(2x_\mu x_\rho - x^2\eta_{\mu\rho}) \otimes x_\sigma - ix_\rho \otimes (2x_\mu x_\sigma - x^2\eta_{\mu\sigma}), \tag{22}$$

and performing the multiplication m we obtain

$$K^t_\mu f^t_{\rho\sigma} = -4ix_\mu x_\rho x_\sigma + ix^2(x_\sigma\eta_{\mu\rho} + x_\rho\eta_{\mu\sigma}). \tag{23}$$

This equation is in agreement with the untwisted case (16), reflecting that the full conformal algebra is represented.

Now, returning to the question of invariance of the noncommutativity matrix $\theta_{\rho\sigma}$ under the twisted conformal group, we may write $i\theta_{\rho\sigma}$ as the commutator $[x_\rho, x_\sigma]$ in the twisted algebra as we did at the beginning of this section. This is nothing but the antisymmetric part of the example tensor $f^t_{\rho\sigma}$ which we have considered above. Noticing that the right-hand sides of Eqs. (20) and (23) are symmetric in ρ and σ, we immediately have

$$D^t\theta_{\rho\sigma} = 0 \quad \text{and} \quad K^t_\mu\theta_{\rho\sigma} = 0, \tag{24}$$

showing the expected result that $\theta_{\rho\sigma}$ is invariant under twisted conformal symmetry.

V. FINAL REMARKS

We have constructed the twist-deformed conformal algebra as a Hopf algebra with a twisted coproduct. The twist element $\mathcal{F}$ has been chosen so that the resulting algebra of functions reproduces the noncommutativity of coordinates often considered to define noncommutative geometry. The conclusion is that a theory formulated in the twisted algebra of functions, by using the star-product in place of the traditional product in the Lagrangian, will be invariant under the twist-deformed conformal algebra.

It would be interesting to investigate the full conformal group and examine global and local transformations. This could be done along the lines of the twisted diffeomorphism invariance constructed in [11]. Using this structure in field theory it should be possible to construct a well-defined notion of conformal field theory in a noncommutative background. A natural extension would be to investigate the two-dimensional case, where the conformal algebra is infinite-dimensional.

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[1] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[2] R. J. Szabo, Phys. Rep. **378**, 207 (2003).
[3] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[4] J. Madore, S. Schraml, P. Schupp, and J. Wess, Eur. Phys. J. C **16**, 161 (2000).
[5] M. Chaichian, P. Presnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005).
[6] J. Wess, hep-th/0408080.
[7] M. Chaichian, P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004).
[8] Y. Kobayashi and S. Sasaki, hep-th/0410164.
[9] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, England, 1994).
[10] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
[11] M. Dimitrijevic and J. Wess, hep-th/0411224.