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Five dimensional 2-branes from special Lagrangian wrapped M5-branes

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We present an ansatz for black 2-branes in five dimensional $\mathcal{N} = 2$ supergravity theory that carry magnetic charge with respect to general hypermultiplet scalars. We find explicit solutions in certain special cases and examine the constraints on the general case. These branes may be thought of as arising from M-branes by wrapping 11 dimensional supergravity over special Lagrangian calibrated cycles of a Calabi-Yau 3-fold.

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I. INTRODUCTION

In this paper, we study black 2-branes in the context of ungauged $D = 5$ $\mathcal{N} = 2$ supergravity theory that arise from dimensionally reducing $D = 11$ supergravity on a Calabi-Yau 3-fold $\mathcal{M}$ [1]. These branes carry magnetic charge with respect to the hypermultiplet scalars and are dual to the electrically charged instantons discussed in [2,3]. From a higher dimensional viewpoint they correspond to M-branes wrapping special Lagrangian (SLAG) cycles of $\mathcal{M}$.

In recent years, interest in nonperturbative solutions to $\mathcal{N} = 2$ supersymmetric theories has increased due to their relevance to the conjectured equivalence between string theory on anti-de Sitter space and certain superconformal gauge theories living on the boundary of the space (the anti-de Sitter/conformal field theory duality) [4]. In $\mathcal{N} = 2$ supergravity (SUGRA), however, an abundance of studies of solitonic solutions coupled to the vector multiplets exist (for example [5,6] and other sources cited below), while very little work on the hypermultiplets sector has been produced (for example [3]). It then becomes of particular importance to fill in this gap, and this paper presents a step in this direction.

Another motivation for this work is to extend recent analysis of supergravity solutions for branes wrapping Kähler calibrated cycles to the SLAG case: $D = 11$ Bogomol'nyi-Prasad-Sommerfield (BPS) spacetimes, known as Fayyazuddin-Smith (FS) spacetimes [7], that describe M2- and M5-branes wrapping Kähler calibrated cycles of Calabi-Yau manifolds have been discussed in [7–15]. FS spacetimes [7] are specified entirely in terms of a Hermitian metric $g_{m\bar{n}}$ on the Calabi-Yau space $\mathcal{M}$, which varies as a function of the noncompact coordinates transverse to the wrapped brane. Far from the brane in the transverse space, the Hermitian metric $g_{m\bar{n}}$ asymptotes to a fixed Ricci-flat Kähler metric describing the Calabi-Yau vacuum. Moving closer to the brane, variation of $g_{m\bar{n}}$ describes how the wrapped brane distorts the shape of the Calabi-Yau space. Near the brane, the metric $g_{m\bar{n}}$ at a fixed transverse position is not even approximately Ricci-flat (see, for example, [12]).

Supersymmetry requires that the Hermitian metric $g_{m\bar{n}}$ satisfy certain additional properties that depend on the precise case under consideration. For example, for M2-branes wrapping 2-cycles of Calabi-Yau N-folds with $N = 2, 3, 4, 5$, it was shown in [13] that the Kähler form $J = ig_{m\bar{n}}dz^m \wedge dz^{\bar{n}}$ at a fixed transverse position satisfies the condition

$$dJ^{N-1} = 0. \tag{1}$$

The metric also satisfies a nonlinear equation of motion, the precise form of which again depends on the dimensions of the Calabi-Yau and on the dimensions of the brane and the wrapped cycles. Explicit solutions to the FS equations of motion are difficult to find.[1] On the other hand, one can gain intuition by studying approximate solutions that hold far away from the wrapped brane and correspond to black brane solutions of the dimensionally reduced theory. If we focus on Calabi-Yau 3-folds, for example, then M2-branes wrapping 2-cycles are black holes of the corresponding $D = 5$ $\mathcal{N} = 2$ supergravity theory, and M5-branes wrapping 4-cycles are black strings [21]. Such black hole and black string spacetimes have been studied [5,6,22–24] in the context of the attractor phenomenon of $D = 4, 5$ $\mathcal{N} = 2$ supergravity discovered in [25–28].

Recently, an ansatz for $D = 11$ FS spacetimes corresponding to M5-branes wrapping SLAG cycles of a Calabi-Yau 3-fold has been constructed [29] using the G-structures approach [30–34]. The Calabi-Yau metric in this case also satisfies a nonlinear equation that is difficult to solve exactly. The black 2-branes introduced here are the dimensionally reduced counterparts of these $D = 11$ spacetimes.

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[1]Exact solutions have been found only in near horizon limits [9,16–20].

In future work we plan to analyze the relation between these results.

The paper is structured as follows: Section II begins with a review of $D = 5$ $\mathcal{N} = 2$ SUGRA theory coupled to the universal hypermultiplet. We then proceed to discuss our solutions (Sec. II B) representing $D = 5$ black 2-branes coupled to the universal hypermultiplet. There are two such special cases, corresponding to the dimensional reduction of M2- and M5-branes over a six dimensional torus. Section III considers the theory with the full hypermultiplets spectrum. For this, we review the special geometry that arises in the space of moduli of the complex structures of the Calabi-Yau space, following the notation of [3]. We construct the form of the general ansatz in Sec. IV, representing a two brane coupled to the full set of hypermultiplets, and find the differential equations constraining the solution. These depend on the explicit form of the underlying compact Calabi-Yau space. Finally, we conclude and motivate further investigation.

II. THE UNIVERSAL HYPERMULTIPLET SOLUTIONS

Dimensionally reducing $D = 11$ supergravity on a Calabi-Yau 3-fold $\mathcal{M}$ yields $D = 5$ $\mathcal{N} = 2$ supergravity coupled to $(h_{1,1} - 1)$ vector multiplets and $(h_{2,1} + 1)$ hypermultiplets [1]; the h's being the Hodge numbers of $\mathcal{M}$. M-branes wrapping Kähler calibrated cycles of $\mathcal{M}$ deform the Kähler structure of $\mathcal{M}$ and reduce to configurations in which the vector multiplets are excited. SLAG wrapped M-branes, on the other hand, our focus in this paper, deform the complex structure of $\mathcal{M}$ and reduce to configurations carrying charge under the hypermultiplet scalars. The two sectors of the theory decouple and we only keep the hypermultiplets in our presentation. The universal hypermultiplet is present even if $h_{2,1}$ vanishes, and in this section we present $D = 5$ 2-branes for which only the scalar fields of the universal hypermultiplet are excited. We will see that these solutions are easily obtained by compactifying known M-brane solutions of $D = 11$ supergravity on a 6-torus.

We will use the following notation: The volume form is related to the totally antisymmetric Levi-Civita symbol by $\varepsilon_{\mu_1\mu_2\cdots\mu_D} = e\bar{\varepsilon}_{\mu_1\mu_2\cdots\mu_D}$, where $e = \sqrt{\det(g_{\mu\nu})}$ and $\bar{\varepsilon}_{012\cdots} = +1$.

A. $D = 5$ $\mathcal{N} = 2$ supergravity coupled to the universal hypermultiplet

We briefly sketch the derivation of the $D = 5$ Lagrangian following the discussion in [2]. The bosonic fields of $D = 11$ supergravity theory are the metric and the 3-form gauge potential A. The action is given by

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{48}F^2\right) - \frac{1}{12\kappa_{11}^2}\int A \wedge F \wedge F. \quad (2)$$

The $D = 5$ couplings of gravity to the universal hypermultiplet follow from $D = 11$ fields of the form

$$ds^2 = e^{-\sigma/3}ds^2_{\mathrm{CY}} + e^{2\sigma/3}g_{\mu\nu}dx^\mu dx^\nu, \quad \mu, \nu = 0, \ldots, 4$$
$$A = \frac{1}{3!}A_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho + \frac{1}{\sqrt{2}\|\Omega\|}\chi\Omega + \frac{1}{\sqrt{2}\|\Omega\|}\bar{\chi}\bar{\Omega}, \quad (3)$$

where ds^2_{CY} is a fixed Ricci-flat metric on the Calabi-Yau space $\mathcal{M}$ and Ω is the holomorphic 3-form on $\mathcal{M}$ with norm $\|\Omega\|$. The $D = 5$ fields of the universal hypermultiplet are the real overall volume modulus σ of the Calabi-Yau space, the $D = 5$ 3-form gauge potential $A_{\mu\nu\rho}$ and the complex scalar χ. The $D = 5$ action is given by ($\kappa_5^2 = \kappa_{11}^2/V_{\mathrm{CY}}$):

$$S_5 = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\sigma)^2 - \frac{1}{48}e^{-2\sigma}F^2 - e^\sigma|\nabla\chi|^2\right] - \frac{1}{4\kappa_5^2}\int F \wedge (\chi d\bar{\chi} - \bar{\chi}d\chi). \quad (4)$$

The equations of motion of σ, $F_{\mu\nu\rho\sigma}$ and $(\chi, \bar{\chi})$ are, respectively,

$$\begin{aligned}
&\nabla^2\sigma - \frac{1}{2}e^\sigma(\partial_\mu\chi)(\partial^\mu\bar{\chi}) + \frac{1}{24}e^{-2\sigma}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} = 0\\
&\nabla^\mu\left(e^{-2\sigma}F_{\mu\nu\rho\sigma} + \bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}\frac{i}{2}[\chi(\partial^\alpha\bar{\chi}) - \bar{\chi}(\partial^\alpha\chi)]\right) = 0\\
&\nabla^\mu\left[e^\sigma(\partial_\mu\chi) + \frac{i}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\nu\rho\sigma\alpha}\chi\right] = -\frac{i}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}(\partial^\alpha\chi)\\
&\nabla^\mu\left[e^\sigma(\partial_\mu\bar{\chi}) - \frac{i}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\nu\rho\sigma\alpha}\bar{\chi}\right] = +\frac{i}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}(\partial^\alpha\bar{\chi}).
\end{aligned} \quad (5)$$

The full action is invariant under the set of supersymmetry (SUSY) transformations:

$$\begin{aligned}\delta\psi^1_\mu &= (\partial_\mu\epsilon_1) + \frac{1}{4}\omega_\mu^{\hat\mu\hat\nu}\Gamma_{\hat\mu\hat\nu}\epsilon_1 + i\frac{e^{-\sigma}}{96}\varepsilon_{\mu\nu\rho\sigma\lambda}F^{\nu\rho\sigma\lambda}\epsilon_1 - \frac{e^{\sigma/2}}{\sqrt{2}}(\partial_\mu\chi)\epsilon_2\\ \delta\psi^2_\mu &= (\partial_\mu\epsilon_2) + \frac{1}{4}\omega_\mu^{\hat\mu\hat\nu}\Gamma_{\hat\mu\hat\nu}\epsilon_2 - i\frac{e^{-\sigma}}{96}\varepsilon_{\mu\nu\rho\sigma\lambda}F^{\nu\rho\sigma\lambda}\epsilon_2 + \frac{e^{\sigma/2}}{\sqrt{2}}(\partial_\mu\bar\chi)\epsilon_1\\ \delta\xi_1 &= \frac{1}{2}(\partial_\mu\sigma)\Gamma^\mu\epsilon_1 - \frac{i}{48}e^{-\sigma}\varepsilon_{\mu\nu\rho\sigma\lambda}F^{\mu\nu\rho\sigma}\Gamma^\lambda\epsilon_1 + \frac{e^{\sigma/2}}{\sqrt{2}}(\partial_\mu\chi)\Gamma^\mu\epsilon_2\\ \delta\xi_2 &= \frac{1}{2}(\partial_\mu\sigma)\Gamma^\mu\epsilon_2 + \frac{i}{48}e^{-\sigma}\varepsilon_{\mu\nu\rho\sigma\lambda}F^{\mu\nu\rho\sigma}\Gamma^\lambda\epsilon_2 - \frac{e^{\sigma/2}}{\sqrt{2}}(\partial_\mu\bar\chi)\Gamma^\mu\epsilon_1,\end{aligned} \tag{6}$$

where ψ and ξ are the gravitini and hyperini fermions, respectively, the ϵ's are the $\mathcal{N}=2$ SUSY spinors, ω is the spin connection and the hatted indices are frame indices in a flat tangent space.

B. The solutions

We find explicit solutions with only the universal hypermultiplet fields excited. The first one may be thought of as a direct reduction of the M2-brane (i.e. without wrapping) to $D=5$. The $D=11$ M2-brane solution is given by

$$\begin{aligned}ds^2_{11} &= f^{-2/3}(-dx_0^2 + dx_1^2 + dx_2^2) + f^{1/3}(dx_3^2 + \cdots \\ &\quad + dx_{10}^2)\\ A_{012} &= f^{-1},\end{aligned} \tag{7}$$

where for the direct reduction to $D=5$ we take $f=f(x_3,x_4)$ with $(\partial_3^2+\partial_4^2)f=0$. If we let $x^5,\ldots,x^{10}$ be coordinates on a flat 6-torus, then putting this into the form of the ansatz for the universal hypermultiplet we find

$$\begin{aligned}ds^2_5 &= -dx_0^2 + dx_1^2 + dx_2^2 + f(dx_3^2 + dx_4^2)\\ e^\sigma &= f^{-1}, \qquad A_{012} = f^{-1}, \qquad \chi = \bar\chi = 0.\end{aligned} \tag{8}$$

This satisfies both the $D=5$ equations of motion (5) and the SUSY equations (6) in a straightforward way, as can be easily checked.

Another more interesting solution we find is one which may be thought of as the wrapping of a M5-brane on a particular SLAG cycle of the 6-torus. Exciting only the universal hypermultiplet requires a configuration of 4 sets of intersecting M5-branes as follows:

$$\begin{array}{lcccccccccc} M5: & 1 & 2 & & & 5 & 6 & 7 & & & \\ \overline{M5}: & 1 & 2 & & & 5 & & & & 9 & 10\\ M5: & 1 & 2 & & & & 6 & & 8 & & 10\\ \overline{M5}: & 1 & 2 & & & & & 7 & 8 & 9 & . \end{array} \tag{9}$$

The $D=11$ metric and 6-form gauge potential can then be constructed as follows:

$$\begin{aligned}ds^2_{11} &= h^{-4/3}(-dx_0^2 + dx_1^2 + dx_2^2) + h^{8/3}(dx_3^2 + dx_4^2)\\ &\quad + h^{2/3}(dx_5^2 + \ldots + dx_{10}^2)\\ A &= \tfrac{1}{2}h^{-1}dx^0\wedge dx^1\wedge dx^2\wedge(\Omega+\bar\Omega)\end{aligned} \tag{10}$$

where $\Omega = dz^1\wedge dz^2\wedge dz^3$ is the holomorphic 3-form associated with the complex coordinates $z^1 = x^5 + ix^8$, $z^2 = x^6 + ix^9$ and $z^3 = x^7 + ix^{10}$ on the 6-torus. This satisfies the $D=11$ equations of motion and SUSY variation equations provided that the supersymmetry parameters satisfy $\epsilon = h^{-1/3}\epsilon_0$, where ϵ_0 is a constant spinor.

The corresponding $D=5$ fields are then found to be

$$\begin{aligned}ds^2_5 &= -dx_0^2 + dx_1^2 + dx_2^2 + h^4(dx_3^2 + dx_4^2)\\ e^\sigma &= h^{-2}, \qquad A_{\mu\nu\rho} = 0, \qquad (\partial_\mu\chi) = i\sqrt{2}\bar\varepsilon_\mu{}^\nu(\partial_\nu h),\\ &\qquad (\partial_\mu\bar\chi) = -i\sqrt{2}\bar\varepsilon_\mu{}^\nu(\partial_\nu h),\end{aligned} \tag{11}$$

where the equations of motion are satisfied provided that the function h is harmonic in the transverse space. The SUSY equations are also satisfied provided the constant spinors are simply related by $\epsilon_1 = \pm i\epsilon_2$.

III. SPECIAL GEOMETRY OF THE COMPLEX STRUCTURE MODULI SPACE

The $D=5$ $\mathcal{N}=2$ supergravity Lagrangian including the full set of $(h_{2,1}+1)$ hypermultiplets can be written in terms of geometric quantities on the moduli space of the complex structures on the Calabi-Yau manifold $\mathcal{M}$. These structures are discussed in detail in [35] and we will give a brief review here. Start by taking a basis of the homology 3-cycles (A^I, B_J) with $I,J = 0,1,\ldots,h_{2,1}$ and a dual cohomology basis of 3-forms (α_I, β^J) such that

$$\begin{aligned}\int_{A^J}\alpha_I &= \int_{\mathcal{M}}\alpha_I\wedge\beta^J = \delta_I^J,\\ \int_{B_I}\beta^J &= \int_{\mathcal{M}}\beta^J\wedge\alpha_I = -\delta_I^J.\end{aligned} \tag{12}$$

Define the periods of the holomorphic 3-form Ω on $\mathcal{M}$ by

$$Z^I = \int_{A^I}\Omega, \qquad F_I = \int_{B_I}\Omega. \tag{13}$$

The periods Z^I can be regarded as coordinates on the complex structure moduli space. Since Ω can be multiplied by an arbitrary complex number without changing the complex structure, the Z^I are projective coordinates. The remaining periods F_I can then be regarded as functions

$F_I(Z)$. One can further show that F_I is the gradient of a function $F(Z)$, known as the prepotential, that is homogeneous of degree two in the coordinates, i.e. $F_I = \partial_I F(Z)$ with $F(\lambda Z) = \lambda^2 F(Z)$. The quantity $F_{IJ}(Z) = \partial_I \partial_J F(Z)$ will also play an important role. Nonprojective coordinates can then be given by taking e.g. $z^i = Z^i/Z^0$ with $i = 1, \ldots, h_{2,1}$. The Kähler potential of the complex structure moduli space is $\mathcal{K} = -\log(i\int_{\mathcal{M}} \Omega \wedge \bar{\Omega})$. Given the expansion of Ω in terms of the periods

$$\Omega = Z^I \alpha_I - F_I \beta^I, \tag{14}$$

the Kähler potential is determined in terms of the prepotential $F(Z)$ according to

$$\mathcal{K} = -\ln[i(Z^I \bar{F}_I - \bar{Z}^I F_I)]. \tag{15}$$

An important ingredient in carrying out the dimensional reduction is the expression for the Hodge-Kähler duals of the cohomology basis (α_I, β^J) expressed in terms of the basis itself. This is given by

$$\begin{aligned} {}^*\alpha_I &= (\gamma_{IJ} + \theta_{IK}\gamma^{KL}\theta_{LJ})\beta^J - \theta_{IK}\gamma^{KJ}\alpha_J \\ {}^*\beta^I &= \gamma^{IK}\theta_{KJ}\beta^J - \gamma^{IJ}\alpha_J. \end{aligned} \tag{16}$$

Here θ_{IJ} and γ_{IJ} are real matrices defined by

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} - 2i\frac{N_{IK}Z^K N_{JL}Z^L}{Z^P N_{PQ} Z^Q} = \theta_{IJ} - i\gamma_{IJ} \tag{17}$$

where $N_{IJ} = \mathrm{Im}(F_{IJ})$, $\gamma^{IJ}\gamma_{JK} = \delta^I_K$ and $\mathcal{N}_{IJ}$ is known as the period matrix.

IV. THE GENERAL CASE

The derivation of the Lagrangian for the bosonic fields of the $D = 5$ theory is sketched in [3], where Euclidean instantons carrying electric charge with respect to the hypermultiplet scalars are constructed. We follow the notation of [3], with the exception that we will be working in Lorentzian signature. The reduction of the $D = 11$ action is done over the metric (3), and the $D = 11$ 3-form is now expanded in terms of the cohomology basis as follows:

$$A = \frac{1}{3!}A_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho + \sqrt{2}(\zeta^I \alpha_I + \tilde{\zeta}_I \beta^I), \qquad \mu, \nu = 0, \ldots, 4. \tag{18}$$

The resulting $D = 5$ action for the bosonic fields is

$$\begin{aligned} S_5 = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\Big[& R - \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) \\ & - G_{i\bar{j}}(\partial_\mu z^i)(\partial^\mu z^{\bar{j}}) - \frac{1}{48}e^{-2\sigma}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \\ & - \frac{1}{24}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}K^\alpha(\zeta, \tilde{\zeta}) + e^\sigma L^\mu_\mu(\zeta, \tilde{\zeta})\Big], \end{aligned} \tag{19}$$

where we have defined:

$$\begin{aligned} K_\alpha(\zeta, \tilde{\zeta}) &= [\zeta^I(\partial_\alpha \tilde{\zeta}_I) - \tilde{\zeta}_I(\partial_\alpha \zeta^I)] \\ L_{\mu\nu}(\zeta, \tilde{\zeta}) &= -(\gamma + \gamma^{-1}\theta^2)(\partial_\mu \zeta)(\partial_\nu \zeta) - \gamma^{-1}(\partial_\mu \tilde{\zeta})(\partial_\nu \tilde{\zeta}) \\ &\quad - 2\gamma^{-1}\theta(\partial_\mu \zeta)(\partial_\nu \tilde{\zeta}). \end{aligned} \tag{20}$$

The scalar fields z^i, $z^{\bar{i}}$ with $i = 1, \ldots, h_{2,1}$ are complex coordinates on the complex structure moduli space with metric $G_{i\bar{j}}$. The scalar fields ζ^I, $\tilde{\zeta}_I$ with $I = 0, 1, \ldots, h_{2,1}$ arise from the dimensional reduction of the $D = 11$ 3-form gauge potential. The scalar field σ is the overall volume scalar of the Calabi-Yau $\mathcal{M}$ and $F_{\mu\nu\rho\sigma}$ is the $D = 5$ 4-form field strength. Each hypermultiplet has 4 scalar fields. The scalar fields $(z^i, z^{\bar{i}}, \zeta^i, \tilde{\zeta}_i)$ make up $h_{2,1}$ of the hypermultiplets. The additional universal hypermultiplet is comprised of the fields $(a, \sigma, \zeta^0, \tilde{\zeta}_0)$, where the axion a is the scalar dual of the 3-form gauge potential $A_{\mu\nu\rho}$.

The equations of motion for σ, $F_{\mu\nu\rho\lambda}$, $(z, \bar{z})$ and $(\zeta, \tilde{\zeta})$ are

$$\begin{aligned} \nabla^2\sigma + e^\sigma L^\mu_\mu + \frac{1}{24}e^{-2\sigma}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} &= 0 \\ \nabla^\mu(e^{-2\sigma}F_{\mu\rho\sigma\lambda} + \bar{\varepsilon}_{\mu\rho\sigma\lambda\nu}K^\nu) &= 0 \\ \nabla^2 z^i + \Gamma^i_{jk}(\partial_\alpha z^j)(\partial^\alpha z^k) + e^\sigma(\partial^i L^\mu_\mu) &= 0 \\ \nabla^2 z^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}(\partial_\alpha z^{\bar{j}})(\partial^\alpha z^{\bar{k}}) + e^\sigma(\partial^{\bar{i}} L^\mu_\mu) &= 0 \\ \nabla^\mu\Big[e^\sigma(\gamma + \gamma^{-1}\theta^2)(\partial_\mu\zeta) + \gamma^{-1}\theta e^\sigma(\partial_\mu\tilde{\zeta}) - \frac{1}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}\tilde{\zeta}\Big] &= \frac{1}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}(\partial^\alpha\tilde{\zeta}) \\ \nabla^\mu\Big[e^\sigma\gamma^{-1}\theta(\partial_\mu\zeta) + e^\sigma\gamma^{-1}(\partial_\mu\tilde{\zeta}) + \frac{e^\sigma}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}\zeta\Big] &= -\frac{1}{48}\bar{\varepsilon}_{\mu\nu\rho\sigma\alpha}F^{\mu\nu\rho\sigma}(\partial^\alpha\zeta). \end{aligned} \tag{21}$$

Further study of the structure of the theory (see [3] and the references within) reveals that the hypermultiplets define a $(h_{2,1} + 1)$ dimensional quaternionic space. This structure, in five dimensions, is dual to the special Kähler geometry of the $D = 4$ vector multiplets sector via the so-called c-map (e.g. [36]). This duality justifies the use of the special geometry defined in Sec. III as opposed to the explicit quaternionic form.

Furthermore, one finds that the theory is invariant under the symplectic group $Sp(2h_{2,1}, \mathbb{R})$, i.e. (19) actually defines a family of Lagrangians that differ from each other only by a rotation in symplectic space that has no effect on the physics. In fact, if we define

$$V = \begin{pmatrix} L^I \\ M_J \end{pmatrix} \equiv e^{\mathcal{K}/2} \begin{pmatrix} Z^I \\ F_J \end{pmatrix} \tag{22}$$

satisfying

$$\nabla_{\bar{i}} V = [\partial_{\bar{i}} - \tfrac{1}{2}(\partial_{\bar{i}}\mathcal{K})]V = 0, \tag{23}$$

then V is a basis vector in symplectic space that satisfies the inner product

$$i\langle V|\bar{V}\rangle = i(\bar{L}^I M_I - L^I \bar{M}_I) = 1. \tag{24}$$

An orthogonal vector may be defined by

$$U_i \equiv \begin{pmatrix} f_i^I \\ h_{J|i} \end{pmatrix} = \nabla_i V, \tag{25}$$

such that

$$\langle V|U_i\rangle = \langle V|U_{\bar{i}}\rangle = 0. \tag{26}$$

Based on this, the following useful identities may be derived:

$$\begin{gathered} \mathcal{N}_{IJ} L^J = M_I, \qquad \mathcal{N}_{IJ} f_i^J = h_{I|i} \\ \gamma^{IJ} = 2(G^{i\bar{j}} f_i^I f_{\bar{j}}^J + L^I \bar{L}^J) \qquad (\nabla_{\bar{j}} f_i^I) = G_{i\bar{j}} L^I, \\ (\nabla_{\bar{j}} h_{iI}) = G_{i\bar{j}} M_I \qquad \gamma_{IJ} L^I \bar{L}^J = \tfrac{1}{2} \qquad G_{i\bar{j}} = 2 f_i^I \gamma_{IJ} f_{\bar{j}}^J. \end{gathered} \tag{27}$$

The gravitini variation equations are

$$\begin{aligned} \delta\psi_\mu^A &= (\partial_\mu \epsilon^A) + \tfrac{1}{4}\omega_\mu{}^{\hat{\mu}\hat{\nu}}\Gamma_{\hat{\mu}\hat{\nu}}\epsilon^A + [\mathcal{Q}_\mu]^A{}_B \epsilon^B \\ [\mathcal{Q}_\mu] &= \begin{bmatrix} \frac{1}{4}\left(v_\mu - \bar{v}_\mu - \frac{\bar{Z}^I N_{IJ}(\partial_\mu Z^J) - Z^I N_{IJ}(\partial_\mu \bar{Z}^J)}{\bar{Z}^I N_{IJ} Z^J}\right) & -\bar{u}_\mu \\ u_\mu & -\frac{1}{4}\left(v_\mu - \bar{v}_\mu - \frac{\bar{Z}^I N_{IJ}(\partial_\mu Z^J) - Z^I N_{IJ}(\partial_\mu \bar{Z}^J)}{\bar{Z}^I N_{IJ} Z^J}\right) \end{bmatrix} \end{aligned} \tag{28}$$

where the indices A and B run over $(1, 2)$, and

$$\begin{aligned} u_\mu &= -ie^{\sigma/2}[M_I(\partial_\mu \zeta^I) + L^I(\partial_\mu \tilde{\zeta}_I)] \\ \bar{u}_\mu &= +ie^{\sigma/2}[\bar{M}_I(\partial_\mu \zeta^I) + \bar{L}^I(\partial_\mu \tilde{\zeta}_I)] \\ v_\mu &= \frac{1}{2}(\partial_\mu \sigma) + \frac{i}{2}e^\sigma[(\partial_\mu a) - K_\mu] \\ \bar{v}_\mu &= \frac{1}{2}(\partial_\mu \sigma) - \frac{i}{2}e^\sigma[(\partial_\mu a) - K_\mu]. \end{aligned} \tag{29}$$

The hyperini equations are

$$\begin{aligned} \delta\xi_1^I &= e_\mu^{1I}\Gamma^\mu \epsilon_1 - \bar{e}_\mu^{2I}\Gamma^\mu \epsilon_2 \\ \delta\xi_2^I &= e_\mu^{2I}\Gamma^\mu \epsilon_1 + \bar{e}_\mu^{1I}\Gamma^\mu \epsilon_2, \end{aligned} \tag{30}$$

written in terms of the beins:

$$e_\mu^{1I} = \begin{pmatrix} u_\mu \\ E^{\hat{i}}{}_\mu \end{pmatrix} \qquad e_\mu^{2I} = \begin{pmatrix} v_\mu \\ e^{\hat{i}}{}_\mu \end{pmatrix} \tag{31}$$

$$\begin{aligned} E^{\hat{i}}{}_\mu &= -ie^{\sigma/2}e^{\hat{i}j}[h_{jI}(\partial_\mu \zeta^I) + f_j^I(\partial_\mu \tilde{\zeta}_I)] \\ \bar{E}^{\hat{i}}{}_\mu &= +ie^{\sigma/2}e^{\hat{i}\bar{j}}[h_{\bar{j}I}(\partial_\mu \zeta^I) + f_{\bar{j}}^I(\partial_\mu \tilde{\zeta}_I)], \end{aligned} \tag{32}$$

and the beins of the special Kähler metric:

$$\begin{gathered} e^{\hat{i}}{}_\mu = e^{\hat{i}}{}_j(\partial_\mu z^j), \qquad \bar{e}^{\hat{i}}{}_\mu = e^{\hat{i}}{}_{\bar{j}}(\partial_\mu z^{\bar{j}}) \\ G_{i\bar{j}} = e^{\hat{k}}{}_i e^{\hat{l}}{}_{\bar{j}} \delta_{\hat{k}\hat{l}}. \end{gathered} \tag{33}$$

A. Constructing the ansatz

Solitonic solutions coupled to the $\mathcal{N} = 2$ hypermultiplets are quite rare in the literature. On the other hand, there is an abundance of solutions coupled to the vector multiplets. $D = 4, 5$ black holes coupled to vector multiplets, for example, have been extensively studied. In what follows, we are only interested in the case $A_{\mu\nu\rho} = 0$ representing the wrapping of a M5-brane.

For the scalar fields, we expect that their most general form will be an expansion in terms of more than one harmonic function. In the previous simpler cases, a single function, harmonic in the transverse space, sufficed. We now need to generalize this by introducing a number $2(h_{2,1} + 1)$ electric charges q_I and a similar number of magnetic charges $\tilde{q}^I$, defined by harmonic functions corresponding to each homology cycle on the Calabi-Yau submanifold as follows:

$$\begin{gathered} H_I = h_I + q_I \ln r, \qquad \tilde{H}^I = \tilde{h}^I + \tilde{q}^I \ln r, \\ I = 0, \dots, h_{2,1}, \end{gathered} \tag{34}$$

where h and $\tilde{h}$ are constants and r is the radial coordinate in the two dimensional space transverse to the brane. We find that the BPS 2-brane metric is given by

$$\begin{gathered} ds^2 = \eta_{ab}dx^a dx^b + e^{-2\sigma}\delta_{\mu\nu}dx^\mu dx^\nu, \\ a, b = 0, 1, 2 \qquad \mu, \nu = 3, 4, \end{gathered} \tag{35}$$

satisfying both the SUSY and field equations, yielding:

$$\begin{aligned}(\partial_\mu \sigma) &= -2e^{\sigma/2}[L^I(\partial_\mu H_I) - M_I(\partial_\mu \tilde{H}^I)]\\ (\partial_\mu z^i) &= -e^{\sigma/2}G^{i\bar{j}}[f_{\bar{j}}^I(\partial_\nu H_I) - h_{\bar{j}I}(\partial_\nu \tilde{H}^I)]\\ (\partial_\mu z^{\bar{i}}) &= -e^{\sigma/2}G^{\bar{i}j}[f_j^I(\partial_\mu H_I) - h_{jI}(\partial_\mu \tilde{H}^I)]\\ (\partial_\mu \zeta^I) &= \pm\bar{\varepsilon}_\mu{}^\nu(\partial_\nu \tilde{H}^I)\\ (\partial_\mu \tilde{\zeta}_I) &= \pm\bar{\varepsilon}_\mu{}^\nu(\partial_\nu H_I).\end{aligned} \tag{36}$$

The Bianchi identity on the $(\zeta, \tilde{\zeta})$ fields gives the harmonic condition on $(H, \tilde{H})$, and the SUSY spinors satisfy $\epsilon_1 = \pm\epsilon_2$. One also finds that the condition

$$H_I(\partial_\mu \tilde{H}^I) - \tilde{H}^I(\partial_\mu H_I) = h\tilde{q} - \tilde{h}q = 0 \tag{37}$$

is satisfied, guaranteeing the vanishing of the M5-brane charge.

We note that there is a relationship between the charges q and $\tilde{q}$ and the central charge Z of the theory as follows [37]:

$$Z = (L^I q_I - M_I\tilde{q}^I) \qquad \bar{Z} = (\bar{L}^I q_I - \bar{M}_I\tilde{q}^I). \tag{38}$$

Based on this, Eqs. (36) become:

$$\begin{aligned}\frac{d\sigma}{dr} = -2e^{\sigma/2}\frac{Z}{r} \qquad \frac{dz^i}{dr} = -e^{-\sigma/2}\frac{\nabla^i\bar{Z}}{r}\\ \frac{dz^{\bar{i}}}{dr} = -e^{-\sigma/2}\frac{\nabla^{\bar{i}}Z}{r}.\end{aligned} \tag{39}$$

The solution may be further specified in terms of the moduli by adopting Sabra's ansatz [6]:

$$H_I = i(F_I - \bar{F}_I) \qquad \tilde{H}^I = i(Z^I - \bar{Z}^I) \qquad \sigma = -\mathcal{K}. \tag{40}$$

The details of how this does indeed satisfy (39) are very similar to those in Sabra's paper and will not be reproduced here. Finally, one can easily check that these equations immediately yield the universal hypermultiplet solution (11) by setting ($h_{2,1} = 0$) and integrating (39).

V. CONCLUSION

We have discussed the coupling of BPS black 2-branes to the hypermultiplets of five dimensional $\mathcal{N} = 2$ supergravity theory and found explicit solutions in the special case of the universal hypermultiplet. In analyzing the general case, we have used the fact that the quaternionic structure of the theory can be mapped to the special geometry of the four dimensional vector multiplets sector, allowing us to use the more familiar symplectically invariant form of the action for the five dimensional hypermultiplets, following the work of [3].

Those 2-branes can be seen as various types of M-branes wrapped over supersymmetric cycles of Calabi-Yau manifolds. We have particularly focused on the case of wrapped M5-branes. Further analysis is left for future work.

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