

Massless and massive three-dimensional super Yang-Mills theory and mini-twistor string theoryDah-Wei Chiou,^{*} Ori J. Ganor,[†] Yoon Pyo Hong,[‡] Bom Soo Kim,[§] and Indrajit Mitra^{||}*Department of Physics, University of California, Berkeley, California 94720, USA
and Theoretical Physics Group, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA
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We propose various ways of adding mass terms to three-dimensional twistor string theory. We begin with a review of mini-twistor space—the reduction of $D = 4$ twistor space to $D = 3$. We adapt the two proposals for twistor string theory, Witten’s and Berkovits’s, to $D = 3$ super Yang-Mills theory. In Berkovits’s model, we identify the enhanced R symmetry. We then construct B-model topological string theories that, we propose, correspond to $D = 3$ Yang-Mills theory with massive spinors and massive and massless scalars in the adjoint representation of the gauge group. We also analyze the counterparts of these constructions in Berkovits’s model. Some of our constructions can be lifted to $D = 4$, where infinitesimal mass terms correspond to vacuum expectation values of certain superconformal gravity fields.

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I. INTRODUCTION

Over the last 20 years following the work of Parke and Taylor [1] (among others, see also [2]) it has become clear that the scattering amplitudes of Yang-Mills theory in four dimensions are much simpler than one would guess. This simplicity was known to persist not only for tree-level results, but also at the one-loop level (for a nice review, see [3]). Witten has recently shown [4] that these amplitudes are most succinctly expressed not in terms of the polarization and momenta of the incoming and outgoing photons, but rather in terms of Penrose’s twistor variables [5] (which of course encode information about the polarization and momenta of the particles). By applying the so-called twistor transform to the amplitudes expressed in terms of spinor variables, Witten showed that the results collapse to simple algebraic curves in twistor space. The use of twistors to study the classical Yang-Mills theory had been discovered much earlier, see e.g. [6].

Twistor theory uncovers holomorphic structure underlying massless free field equations of motion. The “twistor transform” converts harmonic functions on a manifold \mathcal{M} to meromorphic functions on its “twistor space” $\mathbb{T}\mathcal{M}$. The twistor transform can be used to convert scattering amplitudes of n massless gluons in perturbative Yang-Mills theory to a meromorphic function (i.e., a section of a certain line bundle) of n points on $\mathbb{T}(\mathbb{R}^4)$. It was conjectured in [4] that the l -loop contribution to the transformed amplitude is nonvanishing only if the n points on twistor space that correspond to the n gluons lie on a holomorphic curve whose degree and genus are determined by the helicity of the particles and by l . This led Witten to con-

jecture that there exists a dual string theory and that it is a topological B model [4]. For related follow-up work, see for instance [7]. The target space of this string theory is the twistor space of \mathbb{R}^4 , and Witten showed how the D instantons of this string theory can compute the scattering amplitudes of the perturbative gauge theory. (The issue of whether one should consider connected or disconnected instantons in order to reproduce the gauge theory amplitudes was rather nicely resolved in [8].) This surprising duality is a “weak-weak” duality in the sense that a weakly coupled string theory is dual to a weakly coupled gauge theory, unlike the “strong-weak” duality [9–11] where a strongly coupled string theory was dual to a weakly coupled gauge theory.

We wish to extend these results to three dimensions and to understand the string dual of weakly coupled super Yang-Mills theory in $D = 3$. The target space of this string theory is the twistor space of \mathbb{R}^3 . This turns out to be the space of oriented lines in \mathbb{R}^3 (we mostly follow [12] and the nice review article [13], see also [14]). It is called “mini-twistor” space [12], and is related to the twistor space of $\mathbb{R}^{3,1}$ by dimensional reduction. This well-known elegant relation [12] will guide us in developing an algorithm to obtain gauge theory scattering amplitudes in $D = 3$ from the corresponding ones in $D = 4$. We shall see that the $D = 3$ amplitudes are still supported on holomorphic curves. The twistor equations in $D = 6$ have been studied in [15].

Recall that the twistor space of Minkowski space $\mathbb{R}^{3,1}$ is $\mathbb{T}(\mathbb{R}^{3,1}) \simeq \mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1$ (i.e., $\mathbb{C}\mathbb{P}^3$ with a rational curve excised¹); the twistor space of \mathbb{R}^3 is $T\mathbb{C}\mathbb{P}^1$ (the tangent space of $\mathbb{C}\mathbb{P}^1$) [12,13].² The mini-twistor space does not possess

^{*}Email address: dwchiou@socrates.berkeley.edu[†]Email address: origa@socrates.berkeley.edu[‡]Email address: yph@socrates.berkeley.edu[§]Email address: bskim@socrates.berkeley.edu^{||}Email address: imitra@socrates.berkeley.edu¹We use the operator \setminus to denote the set-theory “minus” and it is not to be confused with division.²This space has also been discussed in footnote 13 of [4], and in [16] it was derived by dimensional reduction with constraints, from 2-Time physics.

the full $D = 3$ superconformal symmetry $SO(3, 2)$, but only its Poincaré subgroup $SO(3) \ltimes \mathbb{R}^3$. It can be obtained by dimensional reduction as follows [12]. The 3(complex)-dimensional $D = 4$ twistor space $\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1$ can be written as a fiber bundle with the 2(complex)-dimensional $D = 3$ mini-twistor space $T\mathbb{C}\mathbb{P}^1$ as the base, and the fiber is \mathbb{C} . The structure group is the additive translation group $\sim \mathbb{C}$ (as opposed to the multiplicative group \mathbb{C}^*). However, the fibration is not canonical; it depends on a choice of direction in the physical space \mathbb{R}^4 . This is the direction of the dimensional reduction. For a given choice of this direction (which we will refer to as the “4th direction”) there is a natural projection from the $D = 4$ twistor space $\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1$ onto the $D = 3$ mini-twistor space $T\mathbb{C}\mathbb{P}^1$. We will use this fibration to calculate mini-twistor amplitudes of $D = 3$ Yang-Mills theory by taking the $D = 4$ amplitudes and integrating them over the \mathbb{C} fibers of the above fibration. In the world-sheet theory of the $D = 4$ twistor string theory—the B model with target space $\mathbb{C}\mathbb{P}^{3|4}$ —we realize dimensional reduction by gauging one of the four translation symmetries. It is an element of the B-model symmetry group $PSL(4 | 4)$. The resulting string theory is the B model with target space $T\mathbb{C}\mathbb{P}^1$ and four additional local fermionic coordinates that transform as sections of the pullback of the $\mathcal{O}(1)$ line bundle over $\mathbb{C}\mathbb{P}^1$.

One aspect that is not so obvious in this construction is the enhanced R symmetry. The R-symmetry group of $D = 3$ super Yang-Mills with $N = 8$ supersymmetry (SUSY) is $Spin(7)$, but we will find that only an $SU(4)$ subgroup is manifest in the B-model string theory. There is, however, another version of twistor string theory due to Berkovits [17]. This is an open string theory. There is a prescription due to Berkovits and Witten [18] which allows one to go from one picture to the other, and we will use this somewhat extensively. We will implement dimensional reduction from $D = 4$ to $D = 3$ in Berkovits’s model as well, and we will derive the $D = 3$ version of this string theory. In this string theory we will be able to construct the full R-symmetry current.

We will also describe a twistor string theory dual of a certain massive $D = 3$ super Yang-Mills theory, and this is one of our main new results. The target space of this string theory is also $T\mathbb{C}\mathbb{P}^1$ with four additional local fermionic coordinates, but the way in which they are fibered over $T\mathbb{C}\mathbb{P}^1$ is modified from the massless case. It corresponds to a Yang-Mills theory with massive scalars and fermions. We will systematically study what adding small mass terms to the fermions means for the string theory. We will study this question both in the context of the B model and Berkovits’s open string theory. In the dimensionally reduced $D = 3$ gauge theory, mass terms have two different origins. They either come from mass terms in the original $D = 4$ theory, or they come by coupling the R-symmetry current to a constant background gauge field (i.e., an R-symmetry twist).

In the B model, the effect of a mass term for the space-time fermions can be achieved by deforming the (super-)complex structure of the target space. Such a deformation corresponds to a closed string vertex operator. We will identify the vertex operators which give rise to the mass terms. Then, using the prescription of Berkovits and Witten, we convert these operators to open string operators which deform the boundary of the world sheet. In making this transformation, we encounter a surprise: The operators that one gets using this dictionary do not lie in an irreducible representation of the R-symmetry group $Spin(7)$, as they should. Some of the operators have to be modified to fit into the required irreducible representation.

We also propose that an infinitesimal mass term in $D = 4$ can be achieved by a certain marginal deformation of the world-sheet theory. In physical space, this corresponds to a small vacuum expectation value (VEV) for a B-model closed string field, which according to [4,18] is part of a conformal supergravity multiplet. We identify this field.

The paper is organized as follows. In Sec. II we review the construction of mini-twistor space and its geometrical interpretation as the space of oriented lines, following [12,13]. We review the relation between harmonic functions on \mathbb{R}^3 and meromorphic forms on mini-twistor space, and we apply it to the scalar propagator. In Sec. III we review the connection [12] between the $D = 3$ and $D = 4$ twistor spaces, and we discuss the supersymmetric extensions. We derive the tree-level amplitudes of $D = 3$ super Yang-Mills theory by dimensional reduction, and we find that they have support on holomorphic curves, like their $D = 4$ counterparts. We comment on a possible physical interpretation of this result. In Sec. IV we augment the theory with mass terms, and we relate the infinitesimal mass terms to world-sheet operators in the B model and in Berkovits’s model. We conclude in Sec. V. (For convenience we list in Tables IV and V the various symbols that are used throughout the text.)

II. THE (MINI-)TWISTOR SPACE OF \mathbb{R}^3

The twistor space of \mathbb{R}^3 is $T\mathbb{C}\mathbb{P}^1$ —the tangent bundle of $\mathbb{C}\mathbb{P}^1$, and harmonic functions on \mathbb{R}^3 can be converted into meromorphic functions on $T\mathbb{C}\mathbb{P}^1$. The space $T\mathbb{C}\mathbb{P}^1$ is called mini-twistor space [12,19]. We will now explain these statements in detail and apply them to convert the propagator of massless fields on \mathbb{R}^3 to a meromorphic function of two points in mini-twistor space. Our initial discussion is based mostly on the nice review paper by Baird [13].

Our discussion in this section is limited to scalar fields. This is not too much of a restriction, since in $D = 3$ massless gauge fields can be converted by duality to massless scalars, as we will review in Sec. III A. Massless spinors in $D = 3$ also have just one helicity state, and solutions to the massless Dirac equation can be readily

converted to mini-twistor space. We refer the reader to [13] for further details.

A. Harmonic functions on \mathbb{R}^3

Pick coordinates x_1, x_2, x_3 on \mathbb{R}^3 . For any fixed $0 \leq \theta \leq 2\pi$, the linear expression $x_1 + ix_2 \sin\theta + ix_3 \cos\theta$ is a harmonic function, and so is any analytic function of this expression. We can construct a more complicated harmonic function on \mathbb{R}^3 by taking linear combinations of analytic functions of $x_1 + ix_3 \cos\theta + ix_2 \sin\theta$ for various values of θ . Whittaker's formula states that a complex-valued harmonic function ϕ on \mathbb{R}^3 can be given by an integral

$$\phi(x_1, x_2, x_3) = \int_0^{2\pi} d\theta f(x_1 + ix_2 \sin\theta + ix_3 \cos\theta, \theta), \tag{2.1}$$

where $f(z, \theta)$ is analytic in the first variable. To prove this formula, note that the right-hand side of (2.1) is obviously harmonic. In order to write a Whittaker formula for an arbitrary harmonic function ϕ , pick polar coordinates (r, u, v) such that

$$x_1 = r \cos u, \quad x_2 = r \sin u \sin v, \quad x_3 = r \sin u \cos v.$$

Then, for $l \geq 0$, and $|m| \leq l$, the spherical harmonics can be written as

$$\begin{aligned} r^l Y_{lm}(u, v) &= \frac{\sqrt{(2l+1)(l-m)!(l+m)!}}{4\pi^{3/2} i^{3m} l!} \\ &\times \int_0^{2\pi} d\theta e^{im\theta} (x_1 + ix_2 \sin\theta + ix_3 \cos\theta)^l, \end{aligned} \tag{2.2}$$

and later we will also need the identity

$$\begin{aligned} \frac{1}{r^{l+1}} Y_{l,m}(u, v) &= \pm \frac{i^{3m} l!}{4\pi^{3/2}} \sqrt{\frac{2l+1}{(l-m)!(l+m)!}} \\ &\times \int_0^{2\pi} \frac{d\theta e^{im\theta}}{(x_1 + ix_2 \sin\theta + ix_3 \cos\theta)^{l+1}}, \end{aligned} \tag{2.3}$$

where the sign on the right-hand side of (2.3) is the same as that of x_1 . The formula (2.2) is a standard integral representation of spherical harmonics, while (2.3) can be derived by starting from [20]

$$\begin{aligned} P_l^m(\cos u) &= \frac{(-1)^m}{\pi} \frac{l!}{(l-m)!} e^{m\pi i/2} \\ &\times \int_0^\pi \frac{\cos m\theta d\theta}{(\cos u + i \sin u \cos\theta)^{l+1}} \quad (\cos u > 0). \end{aligned}$$

Any harmonic function that is regular at the origin can be expanded as a linear combination of spherical harmonics

$$\phi(x_1, x_2, x_3) = \sum_{l=0}^\infty \sum_{m=-l}^l C_{lm} r^l Y_{lm}(u, v).$$

This allows us to express ϕ as a Whittaker integral (2.1). A possible choice for the analytic function to be used on the right-hand side of (2.1) is

$$f(\zeta, \theta) = \sum_{l=0}^\infty \sum_{m=-l}^l C_{lm} \frac{\sqrt{(2l+1)(l-m)!(l+m)!}}{4\pi^{3/2} i^{3m} l!} e^{im\theta} \zeta^l.$$

B. Identification of mini-twistor space with $T\mathbb{CP}^1$

As explained in [12,13,21], the twistor space of \mathbb{R}^3 can be identified with the space of oriented lines in \mathbb{R}^3 . This space is isomorphic to the 2-complex dimensional space $T\mathbb{CP}^1$. We will now review how this works.

First we rewrite Whittaker's formula (2.1) by introducing complex coordinates

$$w = 2e^{i\theta}(x_1 + ix_2 \sin\theta + ix_3 \cos\theta), \quad z = e^{i\theta}.$$

Given the analytic function f on the right-hand side of (2.1), it is convenient to define a related analytic function φ by

$$\varphi(e^{i\theta}, w) := e^{-i\theta} f\left(\frac{1}{2}e^{-i\theta} w, \theta\right).$$

We assume that we can extend φ to an analytic function $\varphi(z, w)$ defined in a neighborhood of the circle $|z| = 1$. Formula (2.1) can now be rewritten as

$$\begin{aligned} \phi(\vec{x}) &= \frac{1}{2\pi i} \oint \varphi(z, -[x_2 - ix_3] + 2zx_1 \\ &+ z^2[x_2 + ix_3]) dz. \end{aligned} \tag{2.4}$$

We take z and w as local coordinates on mini-twistor space, which will be identified with $T\mathbb{CP}^1$ soon. Under favorable conditions, $\varphi(z, w)$ can be analytically continued to a meromorphic function for all $z \in \mathbb{C}$ and $w \in \mathbb{C}$. For simplicity of the discussion we will assume that this is the case. (Although, in the more general case we can assume that φ can be analytically continued to a neighborhood around $|z| = 1$. We defer the discussion of this case till the end of this subsection, since we will need to use sheaf cohomology.) We would actually like to view z as a coordinate on a \mathbb{CP}^1 by identifying $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$ (say, by stereographic projection), so that $z = \infty$ will be an allowed value. Then, for every $\vec{x} \in \mathbb{R}^3$, the equation

$$\begin{aligned} w &= -(x_2 - ix_3) + 2x_1 z + (x_2 + ix_3)z^2 \\ &\text{(incidence relation),} \end{aligned} \tag{2.5}$$

which is analytic in z , defines an algebraic curve in (z, w) space. We get the integrand of (2.4) from $\varphi(z, w)$ by setting (2.5). The right-hand side of (2.5) has a double pole at $z = \infty$. We define

$$z' = \frac{1}{z}, \quad w' = -\frac{w}{z^2} \tag{2.6}$$

to be regular local coordinates on mini-twistor space near $z = \infty$, instead of (z, w) . Equation (2.5) then becomes

$$w' = -(x_2 + ix_3) - 2x_1 z' + (x_2 - ix_3)z'^2,$$

which has no pole at $z = \infty$. The two coordinate systems (z, w) [$z \neq \infty$] and (z', w') [$z' \neq \infty$], with the transition rules (2.6), describe the holomorphic line bundle $\mathcal{O}(2)$ over $\mathbb{C}\mathbb{P}^1$, w being the local coordinate on the fiber and z being the coordinate on the base. $\mathcal{O}(2)$ can be identified with the tangent bundle $T\mathbb{C}\mathbb{P}^1$, since from (2.6) it is obvious that w transforms like a vector on $\mathbb{C}\mathbb{P}^1$. Thus we identify the mini-twistor space $\mathbb{T}(\mathbb{R}^3)$ with $T\mathbb{C}\mathbb{P}^1$.

The relation (2.5) describes a holomorphic section of the line bundle $T\mathbb{C}\mathbb{P}^1$, and it varies with the point $\vec{x} \in \mathbb{R}^3$. It is called the *incidence relation* [13]. In Sec. II C, following [13], we will give the mini-twistor space and the incidence relation a more geometric interpretation.

From (2.4) we see that it is natural to think of φ as a holomorphic 1-form $\varphi(z, w)dz$. This 1-form is defined in the neighborhood of $|z| = 1$. A 1-form on $\mathbb{C}\mathbb{P}^1$, by definition, takes values in the holomorphic sheaf $\Omega^1 \simeq \mathcal{O}(-2)$ over $\mathbb{C}\mathbb{P}^1$. We can think of φdz as taking values in the pullback $\tilde{\Omega}^1$ of Ω^1 to the tangent bundle $T\mathbb{C}\mathbb{P}^1$. Also, the integral (2.4) is unchanged if we replace φ with $\varphi(z, w)dz + g_0(z, w)dz - g_\infty(z, w)dz$, for any pair of local holomorphic 1-forms g_0 and g_∞ (taking values in the sheaf $\tilde{\Omega}^1$, by definition) such that g_0 has no poles for all $|z| \leq 1$ and g_∞ has no poles for all $|z| \geq 1$ (including $z = \infty$). This defines an equivalence class of 1-forms $\varphi \sim \varphi + g_0 - g_\infty$, which defines the *sheaf cohomology* $H^1(T\mathbb{C}\mathbb{P}^1, \tilde{\Omega}^1)$ [22]. This notion will be useful in Sec. III D when we derive mini-twistor tree-level amplitudes of $D = 3$ super Yang-Mills theory by dimensional reduction of $D = 4$ amplitudes.

C. Geometric picture of mini-twistor space

As explained in [12,13], the mini-twistor space $T\mathbb{C}\mathbb{P}^1$, whose construction was reviewed above, can be identified with the space of oriented lines in \mathbb{R}^3 , and the incidence relation (2.5) is the condition that the line that corresponds to the mini-twistor (z, w) passes through $\vec{x} \in \mathbb{R}^3$. We will now review how this works.

To describe an oriented line ℓ in \mathbb{R}^3 , we need its direction, which is a unit vector \vec{n} , and its displacement vector \vec{A} , which is the vector from the origin of \mathbb{R}^3 to a point on ℓ

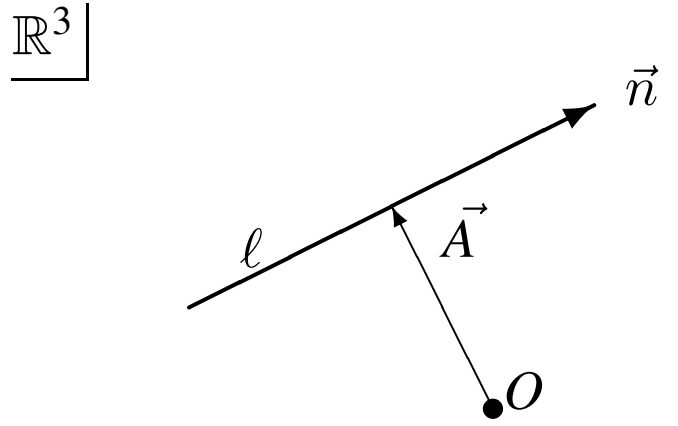


FIG. 1. Mini-twistor space can be identified with the space of oriented lines $\ell \in \mathbb{R}^3$. The direction of ℓ is \vec{n} , and the displacement vector is \vec{A} .

closest to the origin (see Fig. 1). Therefore, \vec{A} is perpendicular to \vec{n} . The space of unit vectors \vec{n} in \mathbb{R}^3 is the sphere S^2 , which we identify with $\mathbb{C}\mathbb{P}^1$ by the stereographic projection:

$$\|\vec{n}\| = 1 \Rightarrow z = \frac{n^2 - in^3}{1 + n^1} \in \mathbb{C}\mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}. \quad (2.7)$$

Given \vec{n} , the space of vectors \vec{A} that satisfy $\vec{n} \cdot \vec{A} = 0$ is the tangent plane to \vec{n} on S^2 . With the identification $S^2 \simeq \mathbb{C}\mathbb{P}^1$ we find that the space of (\vec{n}, \vec{A}) is $T\mathbb{C}\mathbb{P}^1$. The holomorphic coordinate w on the fiber of $T\mathbb{C}\mathbb{P}^1$ can be defined as

$$w = -A^k \frac{\partial z}{\partial n^k} = \frac{-(1 + n^1)(A^2 - iA^3) + (n^2 - in^3)A^1}{(1 + n^1)^2},$$

where A^i and n^i ($i = 1, 2, 3$) are the components of \vec{A} and \vec{n} respectively. Given z , we can recover \vec{n} by

$$\vec{n} = \left(\frac{1 - |z|^2}{1 + |z|^2}, \frac{z + \bar{z}}{1 + |z|^2}, \frac{i(z - \bar{z})}{1 + |z|^2} \right), \quad (2.8)$$

and, noting that $\vec{n} \cdot \vec{A} = 0$, we can recover \vec{A} from (z, w) :

$$\vec{A} = \left(2 \frac{w\bar{z} + \bar{w}z}{(1 + |z|^2)^2}, -\frac{w(1 - \bar{z}^2) + \bar{w}(1 - z^2)}{(1 + |z|^2)^2}, -i \frac{w(1 + \bar{z}^2) - \bar{w}(1 + z^2)}{(1 + |z|^2)^2} \right). \quad (2.9)$$

One can check [13] that the oriented line given by $\vec{x} = (x_1, x_2, x_3) = \vec{A} + c\vec{n}$ is the solution set to the incidence relation (2.5).

D. The Poincaré group

It will be useful for us to express the generators of the symmetry group of $T\mathbb{C}\mathbb{P}^1$ in mini-twistor variables. This will allow us to easily check symmetry properties of vari-

ous expressions that are given in terms of z and w . The Poincaré group in three dimensions has generators \vec{P} (translations) and \vec{J} (rotations). It acts on \vec{n} and \vec{A} as follows:

$$\begin{aligned} [P_i, n_j] &= 0, & [P_i, A_j] &= i\delta_{ij} - in_i n_j, \\ [J_i, n_j] &= i\epsilon_{ijk} n_k, & [J_i, A_j] &= i\epsilon_{ijk} A_k. \end{aligned}$$

From this and (2.8) and (2.9) it is easy to find the expressions in terms of z and w . We get

$$\begin{aligned} P_1 &= iz \frac{\partial}{\partial w} + i\bar{z} \frac{\partial}{\partial \bar{w}}, \\ P_+ &:= P_2 + iP_3 = -i \frac{\partial}{\partial w} + i\bar{z}^2 \frac{\partial}{\partial \bar{w}}, \\ P_- &:= P_2 - iP_3 = iz^2 \frac{\partial}{\partial w} - i \frac{\partial}{\partial \bar{w}}, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} J_1 &= -z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} - w \frac{\partial}{\partial w} + \bar{w} \frac{\partial}{\partial \bar{w}}, \\ J_+ &:= J_2 + iJ_3 = \frac{\partial}{\partial z} + \bar{z}^2 \frac{\partial}{\partial \bar{z}} + 2\bar{z}\bar{w} \frac{\partial}{\partial \bar{w}}, \\ J_- &:= J_2 - iJ_3 = -z^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} - 2zw \frac{\partial}{\partial w}. \end{aligned} \quad (2.11)$$

E. Extension to superspace

We can easily extend the discussion to accommodate supersymmetry. For $D = 3$, $N = 2$ supersymmetry, the generators are Q_{\pm} and their Hermitian conjugates are $Q_{\pm}^{\dagger} \equiv \bar{Q}_{\pm}$. The SUSY algebra is

$$\begin{aligned} [P_i, Q_{\pm}] &= 0, & [P_i, \bar{Q}_{\pm}] &= 0, \\ [J_1, Q_{\pm}] &= \pm \frac{1}{2} Q_{\pm}, & [J_1, \bar{Q}_{\pm}] &= \pm \frac{1}{2} \bar{Q}_{\pm}, \\ [J_{\pm}, Q_{\mp}] &= Q_{\pm}, & [J_{\pm}, \bar{Q}_{\mp}] &= -\bar{Q}_{\pm}, \\ [J_{\pm}, Q_{\pm}] &= 0, & [J_{\pm}, \bar{Q}_{\pm}] &= 0, \end{aligned}$$

and

$$\{Q_{\pm}, Q_{\pm}\} = \{Q_{\pm}, Q_{\mp}\} = \{\bar{Q}_{\pm}, \bar{Q}_{\mp}\} = \{\bar{Q}_{\pm}, \bar{Q}_{\pm}\} = 0,$$

and

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = P_{\pm}, \quad \{Q_{\pm}, \bar{Q}_{\mp}\} = \pm P_1.$$

We add a superspace coordinate θ and its complex conjugate $\bar{\theta}$. We can now express Q_{\pm} and \bar{Q}_{\pm} in terms of (z, w, θ) and their conjugates $(\bar{z}, \bar{w}, \bar{\theta})$:

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial \theta} + i\bar{\theta} \bar{z} \frac{\partial}{\partial \bar{w}}, & \bar{Q}_+ &= \bar{z} \frac{\partial}{\partial \bar{\theta}} - i\theta \frac{\partial}{\partial w}, \\ Q_- &= z \frac{\partial}{\partial \theta} - i\bar{\theta} \frac{\partial}{\partial \bar{w}}, & \bar{Q}_- &= \frac{\partial}{\partial \bar{\theta}} + i\theta z \frac{\partial}{\partial w}. \end{aligned} \quad (2.12)$$

The angular momentum operators change slightly with the fermionic contributions:

$$\begin{aligned} J_1 &= -z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} - w \frac{\partial}{\partial w} + \bar{w} \frac{\partial}{\partial \bar{w}} - \frac{1}{2} \theta \frac{\partial}{\partial \theta} + \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}}, \\ J_+ &\equiv J_2 + iJ_3 = \frac{\partial}{\partial z} + \bar{z}^2 \frac{\partial}{\partial \bar{z}} + 2\bar{z}\bar{w} \frac{\partial}{\partial \bar{w}} + \bar{z}\bar{\theta} \frac{\partial}{\partial \bar{\theta}}, \\ J_- &\equiv J_2 - iJ_3 = -z^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} - 2zw \frac{\partial}{\partial w} - z\theta \frac{\partial}{\partial \theta}. \end{aligned} \quad (2.13)$$

Note that \bar{z} and \bar{w} have a J_1 eigenvalue of $+1$, $\bar{\theta}$ and Q_+ , \bar{Q}_+ have a J_1 eigenvalue $+\frac{1}{2}$, θ and Q_- , \bar{Q}_- have a J_1 eigenvalue $-\frac{1}{2}$, and z, w have a J_1 eigenvalue -1 . For $N = 8$ supersymmetry, we take four copies of the fermionic coordinates $\theta^A, \bar{\theta}^A$ ($A = 1, \dots, 4$). We will denote this super mini-twistor space by \mathbb{T}_3 . It is $T\mathbb{C}\mathbb{P}^1$ with the four fermionic coordinates θ^A fibered over it.

F. The scalar propagator

For a fixed point $\vec{x}' \in \mathbb{R}^3$ the Green's function for Laplace's equation on \mathbb{R}^3 ,

$$G(\vec{x}, \vec{x}') = \frac{1}{\|\vec{x} - \vec{x}'\|},$$

is harmonic away from $\vec{x} = \vec{x}'$, and therefore it should be possible to express it as in Whittaker's formula (2.4).³ In fact, it is not hard to check that

$$\frac{1}{\|\vec{x} - \vec{x}'\|} \operatorname{sgn}(x'_1 - x_1) = \frac{1}{2\pi i} \oint \varphi(z, -[x_2 - ix_3] + 2zx_1 + z^2[x_2 + ix_3]; \vec{x}') dz, \quad (2.14)$$

where

$$\varphi(z, w; \vec{x}') = \frac{2}{(x'_2 + ix'_3)z^2 + 2x'_1z - (x'_2 - ix'_3) - w}. \quad (2.15)$$

Note that φ has a simple pole whenever the mini-twistor (z, w) and the point \vec{x}' satisfy the incidence relation (2.5). Thus, the integral (2.4) diverges when $\vec{x} = \vec{x}'$, as it should. The extra sign $\operatorname{sgn}(x'_1 - x_1)$ on the left-hand side is required if we take the contour of integration in (2.4) to be the unit circle $|z| = 1$.

The left-hand side of (2.14) is translationally invariant. Likewise, the mini-twistor transform $\varphi(z, w; \vec{x}')$ is also translationally invariant in the sense that it satisfies

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial w} - \frac{1}{2} \frac{\partial}{\partial x'_2} - \frac{i}{2} \frac{\partial}{\partial x'_3} \right) \varphi(z, w; \vec{x}'), \\ 0 &= \left(z \frac{\partial}{\partial w} + \frac{1}{2} \frac{\partial}{\partial x'_1} \right) \varphi(z, w; \vec{x}'), \\ 0 &= \left(-z^2 \frac{\partial}{\partial w} - \frac{1}{2} \frac{\partial}{\partial x'_2} + \frac{i}{2} \frac{\partial}{\partial x'_3} \right) \varphi(z, w; \vec{x}'). \end{aligned} \quad (2.16)$$

Here we have used (2.10) to write the translation generators in terms of z, w . The translational invariance of $\varphi(z, w; \vec{x}')$ is not a completely trivial statement, because the integral on the right-hand side of (2.14) would have been translationally invariant even if, say, the left-hand sides of (2.16) were not zero but were holomorphic functions of z, w .

³Twistor methods are actually often used to compute the propagators in nontrivial geometries, see e.g. [23].

Can we go one step further and mini-twistor transform $\varphi(z, w; \vec{x})$ with respect to \vec{x}' to get a meromorphic function of two mini-twistors (z, w) and (z', w') ? Using the familiar expansion

$$\frac{1}{\|\vec{x} - \vec{x}'\|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \times \frac{\|\vec{x}\|^l}{\|\vec{x}'\|^{l+1}} Y_{lm}\left(\frac{\vec{x}}{\|\vec{x}\|}\right) Y_{lm}^*\left(\frac{\vec{x}'}{\|\vec{x}'\|}\right) \quad (2.17)$$

(for $\|\vec{x}\| < \|\vec{x}'\|$)

and the mini-twistor transforms (2.2) and (2.3), we arrive at the mini-twistor transform of the scalar propagator

$$\tilde{G}(t, t') \equiv \tilde{G}(z, w; z', w') = \frac{2(w'z + wz')}{(w - w')(wz'^2 - w'z^2)}, \quad (2.18)$$

where $t \equiv (z, w)$ and $t' \equiv (z', w')$ are shorthand for our twistor variables.

It can be explicitly checked that if $\|\vec{x}\| < \|\vec{x}'\|$, then

$$\frac{1}{\|\vec{x} - \vec{x}'\|} = \text{sgn}(x'_1) \oint_{|z|=1} \frac{dz}{2\pi i} \oint_{|z'|=1} \frac{dz'}{2\pi i} \times \tilde{G}(z, -[x_2 - ix_3] + 2x_1z + [x_2 + ix_3]z^2; z', -[x'_2 - ix'_3] + 2x'_1z' + [x'_2 + ix'_3]z'^2) \quad (2.19)$$

holds, unless all of the inequalities

$$|x'_1| \leq \|\vec{x}\| \leq \|\vec{x}'\| \quad \text{and} \quad 4x_1^2 \leq x_2^2 + x_3^2 \quad \text{and} \quad 4x_1'^2 \leq x_2'^2 + x_3'^2 \quad (2.20)$$

are satisfied. If condition (2.20) holds, there are poles along the integration path which need a special treatment. Identity (2.19) does not necessarily hold for values of \vec{x} and \vec{x}' that do not satisfy (2.20). This is because when we derive (2.18) from (2.2), (2.3), and (2.17), we have to change the order of integration and summation, and for an infinite series that does not necessarily converge.

If, on the other hand, $\|\vec{x}\| > \|\vec{x}'\|$, then (2.19) still holds, except that $\text{sgn}(x'_1)$ needs to be replaced by $\text{sgn}(x_1)$. The analog of the condition (2.20) is now

$$|x_1| \leq \|\vec{x}'\| \leq \|\vec{x}\| \quad \text{and} \quad 4x_1^2 \leq x_2^2 + x_3^2 \quad \text{and} \quad 4x_1'^2 \leq x_2'^2 + x_3'^2. \quad (2.21)$$

The mini-twistor transform $G(t, t')$ is not uniquely defined, because we can, for example, add an arbitrary meromorphic function that has no poles in the region $|z| \leq 1$, and we can also add an arbitrary meromorphic function with no poles in the region $|z| \geq 1$ (including $z = \infty$). We can also add functions with similar properties for z' . Note also that the propagator (2.18) is not invariant under translations. The total translation generators can be read off from (2.10). When acting on holomorphic functions, they reduce to

$$P_+ = -i \frac{\partial}{\partial w} - i \frac{\partial}{\partial w'}, \quad P_1 = iz \frac{\partial}{\partial w} + iz' \frac{\partial}{\partial w'},$$

$$P_- = iz^2 \frac{\partial}{\partial w} + iz'^2 \frac{\partial}{\partial w'}.$$

It can be checked that $P_{\pm} \tilde{G}$ and $P_1 \tilde{G}$ do not vanish.

It is interesting that the off-shell propagator can be recast in terms of mini-twistors, in some region of parameter space. This seems to be the $D = 3$ analog of the off-shell twistor propagator of [24]. It is possible that these formulas could be used to convert Feynman diagram rules, which are usually expressed in terms of momenta or coordinates, to diagrams in terms of mini-twistor variables. We were unsuccessful in putting such rules to practical use. This is partly because (2.19) only holds under the assumption (2.20), and partly because individual Feynman diagrams of gauge theories are not gauge invariant.

G. Minkowski space $\mathbb{R}^{2,1}$

We can also define mini-twistor space for Minkowski signature. Pick a Majorana representation of the Clifford algebra, say,

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.22)$$

$$\Gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For a 3-momentum $\vec{k} = (k^0, k^1, k^2)$, set

$$k_{\mu} \Gamma^{\mu} = \begin{pmatrix} -k_1 & k_2 + k_0 \\ k_2 - k_0 & k_1 \end{pmatrix} = \begin{pmatrix} -k^1 & k^2 - k^0 \\ k^2 + k^0 & k^1 \end{pmatrix}$$

$$\Rightarrow k_{\alpha}^{\beta} = k_{\mu} (\Gamma^{\mu})_{\alpha}^{\beta}.$$

For a lightlike \vec{k} we get

$$0 = \det(k_{\mu} \Gamma^{\mu}) \Rightarrow k_{\alpha}^{\beta} = \lambda_{\alpha} \tilde{\lambda}^{\beta},$$

and if \vec{k} is real, λ and $\tilde{\lambda}$ are also real.

We get mini-twistor space by Fourier transforming with respect to $\tilde{\lambda}$:

$$e^{i\vec{k} \cdot \vec{x}} \rightarrow \frac{1}{(2\pi)^2} \int d^2 \tilde{\lambda} e^{ix^{\alpha} \lambda_{\alpha} \tilde{\lambda}^{\beta}} e^{i\tilde{\lambda}^{\beta} \mu_{\beta}}$$

$$= \delta^{(2)}(x^{\alpha} \lambda_{\alpha} + \mu_{\beta}),$$

where

$$x_{\beta}^{\alpha} = \begin{pmatrix} -x^1 & x^2 + x^0 \\ x^2 - x^0 & x^1 \end{pmatrix}.$$

The condition

$$\mu_{\beta} + x^{\alpha} \lambda_{\alpha} = 0$$

is related to the incidence relation (2.5) as follows. Expanding the components we get

$$\begin{aligned} -\mu_1 &= x^1_1 \lambda_1 + x^2_1 \lambda_2 = -x^1 \lambda_1 + (x^2 - x^0) \lambda_2, \\ -\mu_2 &= x^1_2 \lambda_1 + x^2_2 \lambda_2 = (x^2 + x^0) \lambda_1 + x^1 \lambda_2. \end{aligned} \quad (2.23)$$

Now set

$$z = \frac{\lambda_1}{\lambda_2}, \quad w = \frac{\mu_1 \lambda_2 - \mu_2 \lambda_1}{(\lambda_2)^2}. \quad (2.24)$$

Then we find from (2.23) that

$$w = 2zx^1 - (x^2 - x^0) + (x^2 + x^0)z^2. \quad (2.25)$$

Renaming $x^0 \rightarrow ix^3$, the Minkowski incidence relation (2.25) becomes the Euclidean version (2.5). For Minkowski space, z and w are real. z takes values in $\mathbb{R}P^1 \simeq S^1$ and w takes values in its tangent space, so together they parametrize the tangent bundle $TS^1 \simeq \mathbb{R} \times S^1$.

III. DIMENSIONAL REDUCTION

We can calculate amplitudes of $D = 3$ super Yang-Mills theory by dimensional reduction of $D = 4$ amplitudes. For this purpose, we need to understand the connection [12] between the twistor space $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ of \mathbb{R}^4 and the mini-twistor space $T\mathbb{C}P^1$ of \mathbb{R}^3 . We will review this connection in Sec. III B, after a brief review of dimensional reduction for super Yang-Mills theory. For a recent comprehensive review of various aspects of $D = 3$ Yang-Mills theory see [25].

A. Field theory

Euclidean $D = 4$ Yang-Mills theory with $N = 4$ supersymmetry has an $SU(4)$ R-symmetry group and $SU(2)_L \times SU(2)_R$ is the (double cover of) rotation group. We will now review the $SU(2)_L \times SU(2)_R \times SU(4)$ representations of the fields, and at the same time introduce our notation. The field content is given by the following: a bosonic gauge field A_i ($i = 1, \dots, 4$) in $(\mathbf{2}, \mathbf{2}, \mathbf{1})$, bosonic scalars Φ^I ($I = 1, \dots, 6$) in $(\mathbf{1}, \mathbf{1}, \mathbf{6})$, fermionic left spinors ψ_α^A ($A = 1, \dots, 4$ and $\alpha = 1, 2$) in $(\mathbf{2}, \mathbf{1}, \mathbf{4})$, and fermionic right spinors $\psi_{\dot{\alpha}A}$ ($\dot{\alpha} = \dot{1}, \dot{2}$) in $(\mathbf{1}, \mathbf{2}, \mathbf{4})$.

Dimensional reduction to $D = 3$ proceeds by taking all the fields to be independent of x_4 and defining $\Phi^7 := A_4$. The (double cover of the) rotation group in (Euclidean) $D = 3$ is $SU(2)$ and the distinction between dotted and undotted spinors disappears. The $D = 3$ R-symmetry group is $\text{Spin}(7)$. We will now review the $SU(2) \times \text{Spin}(7)$ representations of the $D = 3$ fields. The fermions ψ_α^A and $\psi_{\dot{\alpha}A}$ combine to form fields in the $(\mathbf{2}, \mathbf{8})$, the gauge field components A_1, \dots, A_3 form a gauge field in the $(\mathbf{3}, \mathbf{1})$ (that is dual to a scalar in $D = 3$), and the scalars Φ^I ($I = 1, \dots, 7$) are in the $(\mathbf{1}, \mathbf{7})$ representation.

We will denote the $D = 3$ fermions by χ_α^a with $a = 1, \dots, 8$ being the index of the spinor representation of $so(7)$ and $\alpha = 1, 2$ the $D = 3$ space-time spinor index. The $D = 3$ Lagrangian is given by

$$\begin{aligned} g_3^2 \mathcal{L} &= \text{tr} \left(\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \sum_{i=1}^7 D_i \Phi^I D^i \Phi^I - \frac{1}{4} \sum_{I,J} [\Phi^I, \Phi^J]^2 \right. \\ &\quad \left. + \sum_{a=1}^8 \chi_\alpha^a \sigma^{i\alpha\beta} \partial_i \chi_\beta^a + \sum_{a,b,I} \epsilon^{\alpha\beta} \Gamma_{ab}^I \Phi^I \chi_\alpha^a \chi_\beta^b \right), \end{aligned} \quad (3.1)$$

where g_3 is the $D = 3$ coupling constant, $\epsilon^{\alpha\beta}$ is the standard antisymmetric lowering and raising matrix for 2-component spinors, $\sigma^{i\alpha\beta}$ are Pauli matrices, and Γ_{ab}^I are $so(7)$ Dirac matrices.

Helicity in $D = 3$ is defined as follows. We Wick rotate to Minkowski metric $\mathbb{R}^{2,1}$ and let k^μ ($\mu = 0, \dots, 2$) be a 3-momentum of a massless particle. Choose a reference frame and let $\vec{k} \equiv k\vec{n} \in \mathbb{R}^2$ be the spatial component of k^μ , with \vec{n} a unit vector. Then, in the temporal gauge $A_0 = 0$, a (\pm) helicity photon has a wave function \vec{A} (the two-component spatial part of A_i) satisfying $\vec{n} \times \vec{A} = \pm i\Phi^7$. This can be described more conveniently as follows. In $D = 3$ a photon is equivalent to a scalar Φ^8 , by duality. The field strength is then given by

$$F_{ij} = \epsilon_{ijl} \partial_l \Phi^8.$$

A (\pm) helicity photon then satisfies $\Phi^7 = \pm i\Phi^8$. Note that the condition for a particular helicity breaks the $\text{Spin}(7)$ R symmetry to $\text{Spin}(6) \simeq SU(4)$. We will see later in Sec. III C that, indeed, our super-mini-twistor space only has a manifest $SU(4)$ symmetry and not $\text{Spin}(7)$.

B. Dimensional reduction of $D = 4$ twistor space

What is the relation between the twistor space $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ of \mathbb{C}^4 (regarded here as complexified $\mathbb{R}^{2,2}$) and the mini-twistor space $T\mathbb{C}P^1$ of \mathbb{R}^3 ? The answer is that $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ is a fibration over $T\mathbb{C}P^1$ [12]. To see this, consider an arbitrary lightlike 4-momentum k^μ in \mathbb{C}^4 . It can be written as

$$k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}.$$

In accordance with the Clifford algebra in (2.22), we choose the Majorana-Weyl spinor representation of $\mathbb{R}^{2,2}$ as

$$\begin{aligned} \sigma_{\alpha\dot{\alpha}}^0 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \sigma_{\alpha\dot{\alpha}}^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_{\alpha\dot{\alpha}}^2 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & \sigma_{\alpha\dot{\alpha}}^3 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (3.2)$$

where we are working in signature $(+ - - +)$. Now suppose that we set $k^3 = 0$, so that k^μ will lie in \mathbb{C}^3 (complexified $\mathbb{R}^{2,1}$). This implies the constraint

$$k^{\alpha\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^3 = 0 \Rightarrow \lambda^1 \tilde{\lambda}^1 + \lambda^2 \tilde{\lambda}^2 = 0. \quad (3.3)$$

In general, we will refer to the direction on which we dimensionally reduce as the 4th *direction*. We can now see how a choice of direction for dimensional reduction naturally defines a fibration structure of $\mathbb{C}P^3 \setminus \mathbb{C}P^1$. For

example, the choice of 4th direction in (3.3) defines the fibration by the condition that two twistors (λ, μ) and (λ', μ') in $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ belong to the same fiber if

$$\lambda'_\alpha = \lambda_\alpha, \quad \mu'_1 = \mu_1 - t\lambda^1, \quad \mu'_2 = \mu_2 - t\lambda^2, \quad (3.4)$$

for some $t \in \mathbb{C}$. The equivalence relation (3.4) arises naturally from (3.3) if we recall that the twistor transform of $\mathbb{R}^{2,2}$ is the Fourier transform from $\tilde{\lambda}$ to μ . In general, had we chosen another direction n^μ on which to dimensionally reduce [instead of a unit vector $(0, 0, 0, 1)$], we would have gotten the condition

$$\lambda'_\alpha = \lambda_\alpha, \quad \mu'_{\dot{\alpha}} = \mu_{\dot{\alpha}} - tn_\mu \sigma^\mu_{\alpha\dot{\alpha}} \lambda^\alpha.$$

To see that (3.4) gives $T\mathbb{CP}^1$ as the base of the fibration, consider the two patches of $\mathbb{CP}^3 \setminus \mathbb{CP}^1$, defined by the conditions $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, respectively. If $\lambda_2 \neq 0$ we can set $z = \lambda_1/\lambda_2$, and rescale by λ_2 to get

$$(\lambda_1, \lambda_2, \mu^1, \mu^2) \rightarrow (z, 1, \mu^1/\lambda_2, \mu^2/\lambda_2).$$

After raising and lowering indices, the equivalence relation (3.4) can be written as

$$\lambda'_\alpha = \lambda_\alpha, \quad \mu'^1 = \mu^1 + t\lambda_1, \quad \mu'^2 = \mu^2 + t\lambda_2. \quad (3.5)$$

Therefore, it has a unique representative given by $\mu^2 = 0$. We get to that representative by picking $t = -\mu^2/\lambda_2$ in (3.5), and using μ' instead of μ . Thus, the twistor

$$\left(z, 1, \frac{\mu^1}{\lambda_2} + tz, \frac{\mu^2}{\lambda_2} + t \right) = \left(z, 1, \frac{\mu^1 \lambda_2 - \lambda_1 \mu^2}{(\lambda_2)^2}, 0 \right)$$

represents the equivalence class (3.4). We set

$$w := \frac{\mu^1 \lambda_2 - \lambda_1 \mu^2}{(\lambda_2)^2}. \quad (3.6)$$

Now consider the other patch $\lambda_1 \neq 0$. By similar arguments

$$\left(1, \frac{1}{z}, 0, \frac{\mu^2 \lambda_1 - \lambda_2 \mu^1}{(\lambda_1)^2} \right)$$

represents (3.4). In this patch we choose the coordinates

$$z' = \frac{\lambda_2}{\lambda_1} = \frac{1}{z}, \quad w' = \frac{\mu^2 \lambda_1 - \lambda_2 \mu^1}{(\lambda_1)^2} = -\frac{w}{z^2}.$$

A given space-time point $x \in \mathbb{R}^{2,2}$ corresponds to a holomorphic section on $\mathbb{T}(\mathbb{R}^{2,2})$ through the incidence relation [4]:

$$\mu_{\dot{\alpha}} + x_{\alpha\dot{\alpha}} \lambda^\alpha = 0.$$

In particular, if $x \in \mathbb{R}^{2,1}$ ($x^0 = 0$), the incidence relation gives

$$\mu^1 = x^{11} \lambda_1 + x^{21} \lambda_2 = x^1 \lambda_1 - (x^2 - x^0) \lambda_2,$$

$$\mu^2 = x^{12} \lambda_1 + x^{22} \lambda_2 = -(x^2 + x^0) \lambda_1 - x^1 \lambda_2,$$

where we used the conventions (3.2). This together with (3.6) leads to the same three-dimensional incidence relation as (2.25), except that z and w are now complex numbers. Again, (2.25) becomes (2.5) by taking $x^0 \rightarrow ix^3$. Therefore, (z, w) and (z', w') parametrize the mini-twistor space $T\mathbb{CP}^1$ of \mathbb{R}^3 exactly as described in Sec. II B.

Thus, the twistor space $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ of \mathbb{C}^4 is a fibration over the mini-twistor space $T\mathbb{CP}^1$ of \mathbb{R}^3 . The fiber is $F \simeq \mathbb{C}$ and the structure group is the group \mathbb{C} of translations of \mathbb{C} . To see this, note that on the patch of $T\mathbb{CP}^1$ where z is a good coordinate, $u := \mu^2/\lambda_2$ is a good coordinate on F , and on the other patch where z' is a good coordinate, $u' := \mu^1/\lambda_1$ is a good coordinate on F . On the intersection of the two patches, where both z and z' are good coordinates,

$$u' = \frac{\mu^1}{\lambda_1} = u + \frac{w}{z}.$$

We will see in Sec. III D that tree-level $D = 3$ amplitudes in mini-twistor space $T\mathbb{CP}^1$ can be obtained from tree-level $D = 4$ amplitudes in twistor space $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ by integration over the fiber F .

To understand the fibration structure more geometrically, note that $\mu = 0$ defines a rational curve (which is homeomorphic to \mathbb{CP}^1) in $\mathbb{CP}^3 \setminus \mathbb{CP}^1$. The normal bundle to the curve $\mu = 0$ is isomorphic to the direct sum $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of line bundles. [Embedded in $\mathbb{CP}^3 \setminus \mathbb{CP}^1$, the normal bundle to $\mu = 0$ can be parametrized as $(\lambda_1, \lambda_2, d\mu^1, d\mu^2) \sim (z, 1, \xi_1, \xi_2)$ for $z = \lambda_1/\lambda_2 \neq \infty$ and $(\lambda_1, \lambda_2, d\mu^1, d\mu^2) \sim (1, z', \xi'_1, \xi'_2) \equiv (1, 1/z, \xi_1/z, \xi_2/z)$ for $z = 1/z' \neq 0$. Thus, the normal bundle is $\mathcal{O}(1) \oplus \mathcal{O}(1)$. See also [26–29] for a related discussion and further details.] This holomorphic vector bundle has nowhere-vanishing holomorphic sections. For example, we define a section \mathfrak{s} of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as follows. Set $\mathfrak{s}(z, 1) = (z, 1, z, 1)$ for $z \neq \infty$ and $\mathfrak{s}(1, z') = (1, z', 1, z')$ for $z' = 1/z \neq \infty$. Obviously, the section values agree on the intersection of the two patches because $(z, 1, z, 1) \sim (\lambda_1, \lambda_2, \lambda_1, \lambda_2) \sim (1, z', 1, z')$, and \mathfrak{s} is nowhere zero.

Based on the section \mathfrak{s} , we can define the sub-bundle of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as $c\mathfrak{s}$ ($c \in \mathbb{C}$), which is a trivial line bundle over $\mu = 0$, since c is a good global coordinate for the line fibers. Modding out $\mathcal{O}(1) \oplus \mathcal{O}(1)$ by this trivial line bundle, we get a quotient space. This means that we impose the equivalence relation on $\mathcal{O}(1) \oplus \mathcal{O}(1)$ by $c\mathfrak{s}$:

$$\begin{aligned} (\xi_1, \xi_2) + c(z, 1) &\sim (\xi_1, \xi_2), \quad z \neq \infty; \\ (\xi'_1, \xi'_2) + c'(1, z') &\sim (\xi'_1, \xi'_2), \quad z' \neq \infty. \end{aligned} \quad (3.7)$$

We get the representatives $(\eta, 0) = (\xi_1 - \xi_2 z, 0)$ and $(0, \eta') = (0, \xi'_2 - \xi'_1 z')$ by choosing $c = -\xi_2$ and $c' = -\xi'_1$. It can be easily shown that $\eta' = -\eta/z^2$ and therefore the quotient space is a line bundle $\mathcal{O}(2)$.

The resulting $\mathcal{O}(2)$ line bundle can be identified with the mini-twistor space $T\mathbb{C}\mathbb{P}^1$. The $\mu = 0$ rational curve is then identified with the base of $T\mathbb{C}\mathbb{P}^1$. We denote the projection by

$$\pi: \mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1 \rightarrow T\mathbb{C}\mathbb{P}^1. \quad (3.8)$$

These observations can be readily modified to space-time $\mathbb{R}^{2,2}$ with signature $(++--)$, for which the twistor space is $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$. All the equations above still apply, but z , w and λ , μ should be real. In particular, if we choose a timelike 4th direction ($0'$) for dimensional reduction we get $D = 3$ Minkowski space $\mathbb{R}^{2,1}$. The choice of 4th direction defines a fibration structure on twistor space $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$, with the projection

$$\pi': \mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1 \rightarrow T\mathbb{R}\mathbb{P}^1. \quad (3.9)$$

The base of this fibration is the mini-twistor space of $D = 3$ Minkowski space $T\mathbb{R}\mathbb{P}^1$. The incidence relation is (2.25), as discussed in Sec. II G.

C. Dimensional reduction of $D = 4$ supertwistor space

The dimensional reduction of supertwistor space $\mathbb{C}\mathbb{P}^{3|4}$ proceeds in a similar fashion. We take homogeneous variables

$$(\lambda_1, \lambda_2, \mu^1, \mu^2, \theta^1, \theta^2, \theta^3, \theta^4)$$

on $\mathbb{C}\mathbb{P}^{3|4}$ and begin with the patch $\lambda_2 \neq 0$. At the end of Sec. III B we identified the three-dimensional mini-twistor space $T\mathbb{C}\mathbb{P}^1$ with a quotient of the normal bundle to the $\mu = 0$ rational curve, where we modded out by a trivial sub-bundle. We can repeat the same procedure for $\mathbb{C}\mathbb{P}^{3|4}$. Pick the rational curve given by $\mu = 0$ and $\theta = 0$. The normal superspace is a sum of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ corresponding to the μ directions, and four copies of anticommuting $\mathcal{O}(1)$ spaces, corresponding to the θ directions. As before, a choice of 4th direction on which to dimensionally reduce defines, as in (3.4), a trivial sub-bundle of the commuting $\mathcal{O}(1) \oplus \mathcal{O}(1)$ vector bundle. Modding out by this subspace leaves

$$\mathcal{O}(2|0) \oplus \mathcal{O}(0|1)^4,$$

where the first factor is commuting and the last four are anticommuting. This is the $D = 3$ super-mini-twistor space. It can be covered with two patches U_1 and U_2 . On the first patch U_1 the local coordinates are $(z, w, \theta_1, \dots, \theta_4)$ (where z, w are commuting), and on the second patch U_2 the local coordinates are $(z', w', \theta'_1, \dots, \theta'_4)$, with transition functions

$$z' = \frac{1}{z}, \quad w' = -\frac{w}{z^2}, \quad \theta'_i = \frac{1}{z} \theta_i \quad (i = 1, \dots, 4), \quad (3.10)$$

generalizing (2.6). Note that this super-mini-twistor space is a noncompact Calabi-Yau supermanifold [30]. Defining

x to be the generator of the cohomology group $H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z})$, we find that the first Chern class of $\mathbb{C}\mathbb{P}^1$ is $2x$, the $\mathcal{O}(2|0)$ factor contributes another $2x$, and the four $\mathcal{O}(0|1)$ anticommuting factors contribute $-x$ each, to a total of 0.

D. Tree-level amplitudes

For simplicity, we will start with $D = 4$ with signature $\mathbb{R}^{2,2}$ and dimensionally reduce by picking a timelike 4th direction to obtain $D = 3$ Minkowski space $\mathbb{R}^{2,1}$. We have seen in Sec. III B that the twistor space $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$ is a fibration over $T\mathbb{R}\mathbb{P}^1$. We will now argue that the tree-level $D = 3$ amplitudes are obtained by integrating the $D = 4$ tree-level amplitudes over the fiber.

In [4], the twistor space $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$ is obtained as the parameter space of the Fourier transforms of functions of $k_{\alpha\dot{\alpha}}^\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ with respect to $\tilde{\lambda}$. Alternatively, if we have a function $\tilde{F}(\lambda, \mu)$ on the twistor space, we can get back a function $F(\lambda, \tilde{\lambda})$ by Fourier transforming with respect to μ .

To dimensionally reduce a given amplitude of the $D = 4$ theory in its twistor space $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$, we need to enforce the condition (3.3). Let us integrate $\tilde{F}(\lambda, \mu)$ over the fiber of the fibration (3.9). We get

$$\begin{aligned} & \frac{1}{2\pi} \int dt \tilde{F}(\lambda_1, \lambda_2, \mu_i + t\lambda^1, \mu_j + t\lambda^2) \\ &= \frac{1}{2\pi} \int dt d^2 \tilde{\lambda} e^{i\tilde{\lambda}^1(\mu_1 + t\lambda^1) + i\tilde{\lambda}^2(\mu_2 + t\lambda^2)} F(\lambda_1, \lambda_2, \tilde{\lambda}^1, \tilde{\lambda}^2) \\ &= \int d^2 \tilde{\lambda} e^{i\tilde{\lambda}^\alpha \mu_\alpha} \delta(\lambda^1 \tilde{\lambda}^1 + \lambda^2 \tilde{\lambda}^2) F(\lambda_1, \lambda_2, \tilde{\lambda}^1, \tilde{\lambda}^2). \end{aligned} \quad (3.11)$$

Thus, integrating over the fiber is equivalent to inserting a delta function $\delta(\lambda^1 \tilde{\lambda}^1 + \lambda^2 \tilde{\lambda}^2) = \delta(k^3)$. To transform an n -particle $D = 4$ tree-level amplitude $A_4(\lambda^{(1)}, \mu^{(1)}; \dots; \lambda^{(n)}, \mu^{(n)})$ on $\mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}\mathbb{P}^1$ to $T\mathbb{R}\mathbb{P}^1$, all we need then is to integrate A_4 over the fibers of n twistors:

$$\begin{aligned} A_3(\lambda^{(1)}, \mu^{(1)}; \dots; \lambda^{(n)}, \mu^{(n)}) &= \int \prod_{j=1}^n dt_j A_4(\lambda^{(1)}, \\ & \mu^{(1)} + t_1 \lambda^{(1)}; \dots; \lambda^{(n)}, \\ & \mu^{(n)} + t_n \lambda^{(n)}). \end{aligned} \quad (3.12)$$

In momentum space, each dt_j integration inserts a $\delta(k_j^3)$, as in (3.11). Because of total 4-momentum conservation, this is one $\delta(k_j^3)$ too many,

$$\delta\left(\sum_{j=1}^n k_j^3\right) \prod_{j=1}^n \delta(k_j^3) = \delta(0) \prod_{j=1}^n \delta(k_j^3).$$

The infinite factor $\delta(0)$ can be regularized if we take a compact 4th dimension. The singular $\delta(0)$ can then be interpreted as the spatial size ($\sim 2\pi R$) in the 4th direction, which is absorbed in the $D = 3$ coupling constant $g_3 = g_4/2\pi R$. (Here g_4 is the $D = 4$ coupling constant.) The

resulting amplitude A_3 is invariant under the equivalence relation (3.4) for each particle separately. It can therefore be written as a function of only $t_j := \pi'(\lambda^{(j)}, \mu^{(j)})$ (for $j = 1, \dots, n$) [where the projection π' was defined in (3.9)].

We are now ready to show that tree-level amplitudes have support on algebraic curves, as in $D = 4$ [4]. Let $A_3(t_1, \dots, t_n)$ be a tree-level n -point $D = 3$ amplitude as a function of n mini-twistors in $T\mathbb{R}P^1$. As argued above, this amplitude can be obtained from a $D = 4$ amplitude $A_4(\tilde{t}_1, \dots, \tilde{t}_n)$ where \tilde{t}_i is a twistor in $\mathbb{R}P^3 \setminus \mathbb{R}P^1$ that projects to t_i in the fibration above. That is, $A_3(t_1, \dots, t_n)$ is obtained from $A_4(\tilde{t}_1, \dots, \tilde{t}_n)$ by integrating with respect to $\tilde{t}_1, \dots, \tilde{t}_n$ over the fibers above t_1, \dots, t_n . In the notation of (3.9) we have $\pi'(\tilde{t}_i) = t_i$. According to [4], A_4 is nonzero only if its arguments $\tilde{t}_1, \dots, \tilde{t}_n$ lie on an algebraic curve of degree $d = q - 1$, where q is the number of negative-helicity gluons. Since this curve is given by algebraic equations in the homogeneous coordinates of $\mathbb{R}P^3$, it can be analytically continued to $\mathbb{C}P^3$. Let this analytically continued curve be $\tilde{\Sigma} \subset \mathbb{C}P^3$. According to [4], this curve must be of genus 0, otherwise the amplitude vanishes. We denote its projection by $\Sigma = \pi(\tilde{\Sigma}) \subset T\mathbb{C}P^1$, where we used π from (3.8). The genus of a projection of a sphere cannot be >0 , so Σ is also of genus 0. In the local coordinates (z, w) on $T\mathbb{C}P^1$, Σ can be expressed as a polynomial equation

$$0 = \sum_{r,s} C_{r,s} z^r w^s, \quad (3.13)$$

with some coefficients $C_{r,s} \in \mathbb{C}$. In order to reduce Witten's conjectures [4] to $D = 3$, we need to define the degree of Σ . We can do that by identifying $T\mathbb{C}P^1$ with an open subset of weighted projective space $W\mathbb{C}P^{1,1,2}$ as follows. Let ξ_1, ξ_2, ξ_3 be projective coordinates on $W\mathbb{C}P^{1,1,2}$ defined with the equivalence relation

$$(\xi_1, \xi_2, \xi_3) \sim (\zeta \xi_1, \zeta \xi_2, \zeta^2 \xi_3), \quad 0 \neq \zeta \in \mathbb{C}. \quad (3.14)$$

If we take the singular point $\xi_1 = \xi_2 = 0$ out of $W\mathbb{C}P^{1,1,2}$, we get the mini-twistor space $T\mathbb{C}P^1$ as follows:

$$z = \frac{\xi_2}{\xi_1}, \quad w = \frac{\xi_3}{\xi_1^2}.$$

Equation (3.13) for Σ becomes

$$0 = \sum_{r,s} C_{r,s} \xi_1^{-r-2s} \xi_2^r \xi_3^s. \quad (3.15)$$

We define the degree of Σ by

$$\tilde{d}(\Sigma) := \max_{r,s} (r + 2s). \quad (3.16)$$

Multiplying (3.15) by $\xi_1^{\tilde{d}(\Sigma)}$ we see that Σ can be represented by an homogeneous polynomial of degree $\tilde{d}(\Sigma)$ in weighted projective space $W\mathbb{C}P^{1,1,2}$.

We will see below that the dimensional reduction of a $D = 4$ twistor amplitude that corresponds to twistors lying on a holomorphic curve of degree d in $\mathbb{C}P^3$ reduces to a mini-twistor amplitude with the mini-twistors restricted to lie on a curve of the form (3.13) and of degree $\tilde{d}(\Sigma) = 2d$. Thus, it follows immediately from the observations in [4] that the $D = 3$ tree-level mini-twistor amplitude with q negative-helicity gluons and $n - q$ positive-helicity gluons is nonzero only if the n mini-twistors lie on a genus 0 curve Σ of degree $\tilde{d}(\Sigma) = 2(q - 1)$ in $T\mathbb{C}P^1$. The Minkowski tree-level amplitude is nonzero if the n mini-twistors lie on an algebraic curve in $T\mathbb{R}P^1 \simeq \mathbb{R}^1 \times S^1$ with the same degree and genus as above.

Suppose that the $D = 4$ tree-level amplitude is supported on a ‘‘complete intersection,’’ which is defined as the zero set of two homogeneous polynomials $f_1(\lambda_\alpha, \mu^{\dot{\alpha}})$ and $f_2(\lambda_\alpha, \mu^{\dot{\alpha}})$, of degrees d_1 and d_2 , respectively. Then the degree of such a curve is $d = d_1 d_2$. The $D = 4$ amplitude can be schematically written as

$$A_4(\tilde{t}_1, \dots, \tilde{t}_n) = \int_{\mathcal{M}} [df_1][df_2] \prod_{i=1}^n \delta(f_1(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}})) \times \delta(f_2(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}})) \mathcal{A}(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}}),$$

where the integration is performed over the moduli space \mathcal{M} of genus 0, degree d curves in $\mathbb{C}P^3$ of this particular complete intersection form [or equivalently, over the space of polynomials $f_1(\lambda_\alpha, \mu^{\dot{\alpha}})$ and $f_2(\lambda_\alpha, \mu^{\dot{\alpha}})$ with specified degrees, after some appropriate identifications]. According to our previous discussion, the $D = 3$ amplitude can then be obtained by integrating the $D = 4$ amplitude over the fibers of the twistors \tilde{t}_i :

$$A_3(t_1, \dots, t_n) = \int_{\mathcal{M}} [df_1][df_2] \int \prod_{i=1}^n dt_i \times \delta(f_1(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})) \times \delta(f_2(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})) \times \mathcal{A}(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)}).$$

Therefore, our $D = 3$ amplitude will not vanish provided there exist two polynomials $f_1(\lambda_\alpha, \mu^{\dot{\alpha}})$ and $f_2(\lambda_\alpha, \mu^{\dot{\alpha}})$ and some value t_i for each i so that both $f_1(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})$ and $f_2(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})$ are nonzero for each i .

Now, for two fixed polynomials $f_1(\lambda_\alpha, \mu^{\dot{\alpha}})$ and $f_2(\lambda_\alpha, \mu^{\dot{\alpha}})$, what is the condition for such values t_i to exist? To answer this question, let us expand the two polynomials $f_1(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})$ and $f_2(\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)})$ and group the terms in decreasing order in t_i . As these polynomials are homogeneous of degrees d_1 and d_2 , we will get two polynomial equations,

$$a_0 t_i^{d_1} + a_1 t_i^{d_1-1} + \dots + a_{d_1} = 0, \quad (3.17)$$

$$b_0 t_i^{d_2} + b_1 t_i^{d_2-1} + \dots + b_{d_2} = 0,$$

where the coefficients a_0, \dots, a_{d_1} and b_0, \dots, b_{d_2} are of degrees d_1 and d_2 , respectively, in $\lambda_\alpha^{(i)}$ and $\mu^{(i)\dot{\alpha}}$. But the condition for the existence of a simultaneous solution t_i for the above two polynomial equations is that the resultant of the polynomials be zero. By explicitly writing down the resultant, one can easily check that each term of the resultant is of degree d_2 in a_0, \dots, a_{d_1} and of degree d_1 in b_0, \dots, b_{d_2} . Hence the condition for the $D = 3$ amplitude to be nonzero is that the twistors $\tilde{T}_i = (\lambda^{(i)}, \mu^{(i)})$ satisfy a certain polynomial equation of degree

$$\tilde{d} = d_1 d_2 + d_2 d_1 = 2d_1 d_2 = 2d.$$

As z is linear and w is quadratic in λ_α and $\mu^{\dot{\alpha}}$ after setting $\lambda_2 = 1$ in (3.6), the degree of the resulting polynomial when expressed in terms of z and w becomes $\tilde{d} = 2d$, according to our definition in (3.16).

We will end this subsection with an analysis of dimensional reduction for Euclidean signature. We have seen at the end of Sec. II B that a harmonic scalar function on \mathbb{R}^3 is mini-twistor transformed to an element of the sheaf cohomology $H^1(T\mathbb{C}\mathbb{P}^1, \tilde{\Omega}^1)$, where $\tilde{\Omega}$ was the pullback of the sheaf $\mathcal{O}(-2)$ over $\mathbb{C}\mathbb{P}^1$. By the arguments of [4], for each external particle the scattering amplitude must be an element of the dual space, which in our case also happens to be $H^1(T\mathbb{C}\mathbb{P}^1, \tilde{\Omega}^1)$. As explained in Sec. 2.5 of [4], for each external particle of helicity h the $D = 4$ amplitude is an element of the sheaf cohomology $H^1(\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1, \mathcal{O}(h - 2))$. We have seen in Sec. III B that $\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1$ is a fibration over $T\mathbb{C}\mathbb{P}^1$ with \mathbb{C} fibers.

Given a $D = 3$ harmonic function, we can lift it to a $D = 4$ harmonic function that is invariant under translations in the 4th direction. What is the corresponding statement for the twistor transforms? The pullback of an element of $H^1(T\mathbb{C}\mathbb{P}^1, \tilde{\Omega}^1)$, with respect to the projection (3.8), is an element of $H^1(\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1, \mathcal{O}(-2))$ that is invariant under translations along the fiber.

E. Explicit examples of amplitudes

We will now give a few examples of dimensionally reduced tree-level amplitudes.

The tree-level maximally helicity-violating (MHV) twistor amplitude with 2 gluons of negative helicity and $(n - 2)$ gluons of positive helicity is given by formula (3.3) of [4],⁴

$$A(\lambda^{(i)}, \mu^{(i)}) = ig_4^{n-2} \int d^4x \prod_{i=1}^n \delta^2(\mu_\alpha^{(i)} + x_{\alpha\dot{\alpha}} \lambda^{(i)\dot{\alpha}}) \times \frac{\langle \lambda^{(r)}, \lambda^{(s)} \rangle^4}{\prod_{i=1}^n \langle \lambda^{(i)}, \lambda^{(i+1)} \rangle}, \quad (3.18)$$

⁴In [31], a nice prescription was given for computing tree-level non-MHV amplitudes from these MHV amplitudes.

where we use the standard notation

$$\langle \lambda^{(i)}, \lambda^{(j)} \rangle := \lambda_\alpha^{(i)} \lambda^{(j)\alpha}, \quad \lambda^{(n+1)} := \lambda^{(1)}.$$

Here, the r th and s th gluons are of negative helicity. To get the $D = 3$ twistor amplitude, we have to replace the $D = 4$ coupling constant g_4 by the $D = 3$ coupling constant g_3 , replace d^4x by d^3x , and integrate over the fibers:

$$ig_3^{n-2} \int d^3x \prod_{i=1}^n dt_i \delta(\mu_1^{(i)} + t\lambda^{(i)1} + x_{\alpha\dot{1}} \lambda^{(i)\dot{\alpha}}) \times \delta(\mu_2^{(i)} + t\lambda^{(i)2} + x_{\alpha\dot{2}} \lambda^{(i)\dot{\alpha}}) \frac{\langle \lambda^{(r)}, \lambda^{(s)} \rangle^4}{\prod_{i=1}^n \langle \lambda^{(i)}, \lambda^{(i+1)} \rangle}.$$

Note that if we first perform the fiber integrations over dt_i and leave the d^3x integral for last, the integrand will be independent of the 4th component of x . This is because momentum conservation in the 4th direction is already enforced, since the dt_i integrations make sure that the 4th component of momentum is zero.

We calculate

$$\int dt \delta(\mu_1 + t\lambda^1 + x_{\alpha\dot{1}} \lambda^{\dot{\alpha}}) \delta(\mu_2 + t\lambda^2 + x_{\alpha\dot{2}} \lambda^{\dot{\alpha}}) = \frac{1}{(\lambda_2)^2} \delta(w + [x^2 - ix^3] - 2x^1z - [x^2 + ix^3]z^2),$$

where we used (3.6) and $z = \lambda_1/\lambda_2$. The argument of the δ function enforces the incidence relation (2.5), which is a polynomial of degree 2. Note also that

$$\langle \lambda^{(i)}, \lambda^{(i+1)} \rangle = (z_{i+1} - z_i)(\lambda_2^{(i)} \lambda_2^{(i+1)}).$$

The $D = 3$ MHV amplitude is therefore

$$ig_3^{n-2} \int d^3x \prod_{i=1}^n \delta(w_i + [x^2 - ix^3] - 2x^1z_i - [x^2 + ix^3]z_i^2) \frac{(z_r - z_s)^4}{\prod_{i=1}^n (z_{i+1} - z_i)}.$$

Here, we used the fact that on the patch where $\lambda_2 \neq 0$ we can scale the factor λ_2 to 1. If, instead, we had included all the λ_2 factors, we would have needed an extra factor of $(\lambda_2^{(r)})^4 (\lambda_2^{(s)})^4 / \prod_{i=1}^n (\lambda_2^{(i)})^4$. This factor indicates that the MHV amplitude in $D = 3$ is homogeneous of degree -4 and 4 for each negative- and positive-helicity particle, respectively. On the patch $\lambda_1 \neq 0$ we define the twistor amplitude as

$$A_3(w'_i, z'_i) = ig_3^{n-2} \int d^3x \prod_{i=1}^n \delta(w'_i + [x_2 - ix_3] - 2x_1 z'_i - [x_2 + ix_3]z_i'^2) \frac{(z'_r - z'_s)^4}{\prod_{i=1}^n (z'_{i+1} - z'_i)}. \quad (3.19)$$

The δ functions enforce the incidence relations. In the geometrical language of Sec. II C, the amplitude is nonzero only if all n mini-twistors, which are oriented lines in \mathbb{R}^3 , intersect at a common point.

Our next example is the “googly” description of the tree-level amplitudes with helicities $---++$. It was shown in [4] that these are supported on genus 0, degree 2 curves in the $D = 4$ twistor space $\mathbb{R}P^3$. It was also shown there that this condition is equivalent to saying that the amplitude is nonzero only if (i) the five points $\tilde{t}_i = (\lambda^{(i)}, \mu^{(i)})$ lie on a common $\mathbb{R}P^2 \subset \mathbb{R}P^3$, and (ii) the five points lie on a common conic section contained in that $\mathbb{R}P^2$. But once the first condition is satisfied, the second one is automatic, because a generic set of five points in $\mathbb{R}P^2$ is contained in a unique conic section. Therefore we can schematically write the $D = 4$ amplitude as⁵

$$A_4(\tilde{t}_1, \dots, \tilde{t}_5) = \int_{\mathcal{M}} [da][db] \prod_{i=1}^5 \delta\left(\sum_{j=1}^4 a_j Z_i^j\right) \times \delta\left(\sum_{I,J=1}^4 b_{IJ} Z_i^I Z_i^J\right) \mathcal{A}(Z_1^I, \dots, Z_5^I), \quad (3.20)$$

where

$$(Z_i^1, Z_i^2, Z_i^3, Z_i^4) = (\lambda_\alpha^{(i)}, \mu^{(i)\dot{\alpha}})$$

are the coordinates of the five twistors \tilde{t}_i , and the integration is to be performed over the moduli space of genus 0, degree 2 algebraic curves in $\mathbb{R}P^3$. (This space is parametrized by the coefficients a_I and b_{IJ} , with appropriate identification.)

As before, dimensional reduction in twistor space is achieved by replacing $\mu^{(i)\dot{\alpha}}$ with $\mu^{(i)\dot{\alpha}} + t_i \lambda_\alpha^{(i)}$ and integrating over t_i for each i . So in the $D = 3$ amplitude, the first delta function for each i in the above expression becomes

$$\delta(a_1 \lambda_1^{(i)} + a_2 \lambda_2^{(i)} + a_3 (\mu^{(i)\dot{1}} + t_i \lambda_1^{(i)}) + a_4 (\mu^{(i)\dot{2}} + t_i \lambda_2^{(i)})).$$

Integration over t_i then amounts to solving the equation

$$a_1 \lambda_1^{(i)} + a_2 \lambda_2^{(i)} + a_3 (\mu^{(i)\dot{1}} + t_i \lambda_1^{(i)}) + a_4 (\mu^{(i)\dot{2}} + t_i \lambda_2^{(i)}) = 0$$

for t_i and plugging it into the t_i in the argument of the second delta function. Therefore, we end up with a product of delta functions whose arguments look like

$$b_{11}(\lambda_1^{(i)})^2 + b_{12} \lambda_1^{(i)} \lambda_2^{(i)} + b_{13} \lambda_1^{(i)} (\mu^{(i)\dot{1}} + t_i \lambda_1^{(i)}) + b_{14} \lambda_1^{(i)} (\mu^{(i)\dot{2}} + t_i \lambda_2^{(i)}) + \dots + b_{44} (\mu^{(i)\dot{2}} + t_i \lambda_2^{(i)})^2,$$

where t_i are to be replaced with the solution of the above equation. After clearing the denominators, these arguments become homogeneous polynomials of degree 4 in $\lambda_\alpha^{(i)}$ and $\mu^{(i)\dot{\alpha}}$. Then from (3.6), we see that when expressed in terms of z_i and w_i , they will become a polynomial of degree 4 in $T\mathbb{R}P^1$ according to our definition of degree in (3.16). We conclude that dimensionally reduced tree-level $---++$ amplitudes are nonvanishing only if the five twistors \tilde{t}_i lie on a common curve of degree 4 in $D = 3$ twistor space. This result agrees with our claim in Sec. III D.

But this condition is actually trivial, because a generic set of five points in $T\mathbb{R}P^1$ always lies on a common curve of degree 4. To see this, it suffices to simply write down the most general form of degree 4 curves in $T\mathbb{R}P^1$:

$$f(z, w) = w^2 + c_1 z^2 w + c_2 z^4 + c_3 z^3 + c_4 z w + c_5 z^2 + c_6 w + c_7 z + c_8 = 0.$$

We have eight parameters c_1, \dots, c_8 at our disposal but have to satisfy only five constraints $f(z_i, w_i) = 0$, so generically such curves exist.

In summary, we have seen that (a) for MHV amplitudes, the $D = 3$ twistor amplitudes after dimensional reduction are nonvanishing only if the twistors lie on a common algebraic curve with genus 0 and degree 2 in $T\mathbb{R}P^1$, while (b) for the googly description of $---++$ amplitudes, there does not exist such nontrivial criterion. The difference between these two cases is easy to understand in geometrical terms. In the MHV case, the $D = 4$ amplitude in (3.18) contains two δ functions for each i . We can think of the first set of δ functions as enforcing the n points $(\lambda^{(i)}, \mu^{(i)} + t_i \lambda^{(i)})$ to lie on a common $\mathbb{R}P^2$. Then the second set of δ functions further demands that the n points lie on a common line contained in that $\mathbb{R}P^2$. We can always pick t_i for each i so that the n points lie on a common $\mathbb{R}P^2$, but after fixing t_i , these points will in general not lie on a common line if $n > 2$. Therefore, we still have a nontrivial criterion for nonvanishing amplitudes after dimensional reduction.

In contrast, for $---++$ amplitudes, the first set of δ functions in (3.20) requires that the five twistor points lie on a common $\mathbb{R}P^2$. Again, this can always be achieved by a judicious choice of t_i . But then the second set of δ functions demands that the five points lie on a common conic section, which is satisfied automatically. So there is no nontrivial criterion for nonvanishing amplitudes.

F. Physical interpretation of the holomorphic curves

In this subsection we will present a geometrical and physical interpretation of the holomorphic curves in

⁵A precise formula for the googly amplitude was derived from the B-model calculation in [32]. For our purposes, this schematic form is enough.

mini-twistor space. There is an interesting connection between holomorphic curves in mini-twistor space $T\mathbb{C}\mathbb{P}^1$ and (real) surfaces of minimal area in the physical space \mathbb{R}^3 . A minimal area surface is constructed from a holomorphic curve $\Sigma \subset T\mathbb{C}\mathbb{P}^1$ as follows [12]. Given a point (z_0, w_0) on Σ , we can write the equation for Σ near (z_0, w_0) as

$$\begin{aligned} w &= w_0 + a_1(z - z_0) + a_2(z - z_0)^2 + O(z - z_0)^3 \\ &= (w_0 - a_1z_0 + a_2z_0^2) + (a_1 - 2a_2z_0)z \\ &\quad + a_2z^2 + O(z - z_0)^3. \end{aligned} \quad (3.21)$$

Dropping the $O(z - z_0)^3$ terms, we can approximate the curve by a quadratic equation and find a vector $\vec{x} \in \mathbb{C}^3$ such that this quadratic equation will look like the incidence relation (2.5). Given the coefficients of 1, z , z^2 in (3.21), we therefore define $\vec{x} \equiv (x_1, x_2, x_3) \in \mathbb{C}^3$ as the unique solution to the linear equations

$$\begin{aligned} -(x_2 - ix_3) &= (w_0 - a_1z_0 + a_2z_0^2), \\ 2x_1 &= (a_1 - 2a_2z_0), \quad x_2 + ix_3 = a_2. \end{aligned} \quad (3.22)$$

Thus, each point on Σ defines a point $\vec{x} \in \mathbb{C}^3$ and the collection of these points defines a holomorphic curve in \mathbb{C}^3 . The projection of this curve to \mathbb{R}^3 , i.e., the collection of points $\text{Re}\vec{x} \equiv (\text{Re}x_1, \text{Re}x_2, \text{Re}x_3)$, is a minimal area (real) surface in \mathbb{R}^3 . Furthermore, there is a one-to-one map from minimal area surfaces in \mathbb{R}^3 to holomorphic curves in $T\mathbb{C}\mathbb{P}^1$. For more details, we refer the reader to the appendix of [12].

We saw in Sec. III D that a physical amplitude defines a holomorphic curve in $T\mathbb{C}\mathbb{P}^1$. From the above discussion it follows that a physical amplitude defines a minimal area surface in \mathbb{R}^3 . What is its significance?

We will address the Minkowski variant of this question. We have seen in Sec. II G that the mini-twistor space of $\mathbb{R}^{2,1}$ is $T\mathbb{R}\mathbb{P}^1$. For $\mathbb{R}^{2,1}$ the coordinates z and w are real, and so are the coefficients a_1, a_2 , in (3.21). We define $\vec{x} \in \mathbb{R}^3$ by comparison with the Minkowski incidence relation (2.25),

$$\begin{aligned} -(x_2 - x_0) &= w_0 - a_1z_0 + a_2z_0^2, & 2x_1 &= a_1 - 2a_2z_0, \\ x_2 + x_0 &= a_2. \end{aligned} \quad (3.23)$$

The solution to (3.23) defines a unique point in $\mathbb{R}^{2,1}$ for every point on the real curve $\Sigma \in T\mathbb{R}\mathbb{P}^1$. The collection of these points form a curve $\mathcal{F}(\Sigma)$ in $\mathbb{R}^{2,1}$. What is the physical significance of $\mathcal{F}(\Sigma)$?

Take a particular amplitude with n mini-twistors t_1, \dots, t_n , and let us consider a particular scattering experiment to which this amplitude would contribute. Thus, we pick $m < n$ twistors t_1, \dots, t_m to describe incoming particles, and assume that t_{m+1}, \dots, t_n describe outgoing particles. We also assume that the number of negative helicities q is fixed and that m and t_1, \dots, t_m are chosen so that there is a unique holomorphic curve, of the corresponding degree $2(q - 1)$, that passes through all the mini-

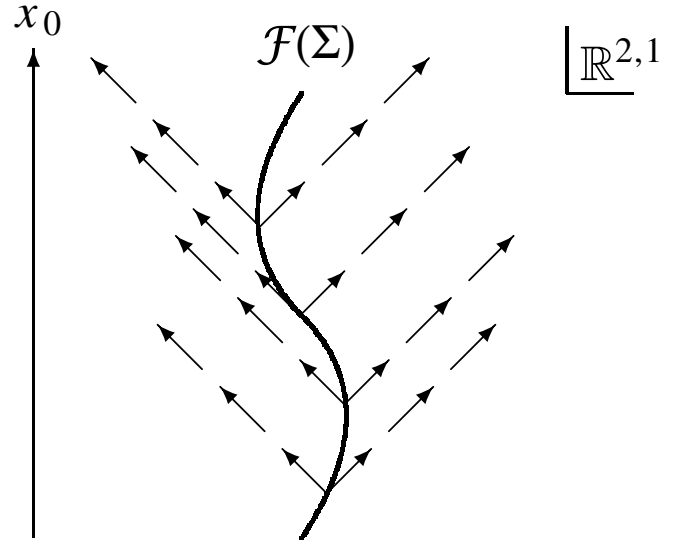


FIG. 2. The outgoing waves of the scattering process can be described as a physical disturbance that is emanating from the “focal curve” $\mathcal{F}(\Sigma)$.

twistors t_1, \dots, t_m . The amplitude will then be nonzero only if t_{m+1}, \dots, t_n lie on that curve.

A mini-twistor $t = (z, w)$ describes an incoming planar shock wave of the form

$$\phi_t(\vec{x}) \sim \delta(w - 2zx^1 + [x^2 - x^0] - [x^2 + x^0]z^2), \quad (3.24)$$

that travels at the speed of light. The scattering process is therefore a collision of m incoming shock waves. What comes out?

For fixed mini-twistors t_{m+2}, \dots, t_n , the outgoing wave function of the $(m + 1)^{\text{st}}$ particle is a linear combination of shock waves of the form (3.24), and in general all mini-twistors t that lie on Σ can contribute. Suppose a particular $t \equiv t_{m+1} \equiv (z_0, w_0)$ contributes to the outgoing wave function. Then nearby mini-twistors $t + \delta t \equiv (z_0 + \delta z, w_0 + \delta w)$ will also contribute provided that they lie on Σ . But near (z_0, w_0) the curve Σ looks like the parabola (3.21). This implies that up to second order in δt , all shock waves $\phi_{t+\delta t}$ are nonzero at the point \vec{x} defined by (3.23). Thus, the outgoing wave function of the $(m + 1)^{\text{st}}$ particle is a linear combination of shock waves, all of which pass through the “focal curve” $\mathcal{F}(\Sigma)$. The outgoing wave function is therefore a disturbance emanating from $\mathcal{F}(\Sigma)$ (See Fig. 2.)

As an example, take the curve Σ given by

$$w = 2z^3.$$

Then from (3.21) and (3.23) we get the parametric equation for $\mathcal{F}(\Sigma)$ in the form

$$\vec{x} \equiv (x_0, x_1, x_2) = (z^3 + 3z, -3z^2, 3z - z^3).$$

Note that the tangent to this curve is null in $\mathbb{R}^{2,1}$. It is easy to see that this is true for a generic curve Σ . In the special case of MHV amplitudes, the focal curve $\mathcal{F}(\Sigma)$ degenerates to a point.

G. Dimensional reduction in B-model twistor string theory

In [4], Witten proposed a reformulation of $D = 4$ super Yang-Mills theory as a B model on supertwistor space $\mathbb{C}\mathbb{P}^{3|4}$. An alternative approach was presented in [17,33]. In this section we will make a few observations about the twistor string theory of $D = 3$ super Yang-Mills theory.

The twistor string theory that describes $D = 3$ super Yang-Mills with $N = 8$ supersymmetry is the topological B model on the $D = 3$ supertwistor space from Sec. II E. It can be obtained from the $D = 4$ twistor space $\mathbb{C}\mathbb{P}^{3|4}$ by dimensional reduction. For the B model, dimensional reduction is implemented by gauging one generator of $SL(4 | 4)^6$ that corresponds to translations in the 4th direction. (This is somewhat reminiscent of the construction of topological σ models in [34].) Let

$$\begin{aligned} Z_1 \equiv \lambda_1, \quad Z_2 \equiv \lambda_2, \quad Z_3 \equiv \mu_1, \\ Z_4 \equiv \mu_2, \quad \theta^1, \dots, \theta^4 \end{aligned} \quad (3.25)$$

be projective coordinates on $\mathbb{C}\mathbb{P}^{3|4}$ with the equivalence relation

$$(Z_1, \dots, Z_4, \theta^1, \dots, \theta^4) \sim (\zeta Z_1, \dots, \zeta Z_4, \zeta \theta^1, \dots, \zeta \theta^4). \quad (3.26)$$

We choose the basis so that the translation generator P_4 in the 4th direction acts as

$$\begin{aligned} P_4: \delta Z_1 = \delta Z_2 = 0, \quad \delta Z_3 = \epsilon Z_1, \\ \delta Z_4 = -\epsilon Z_2, \quad \delta \theta^1 = \dots = \delta \theta^4 = 0. \end{aligned} \quad (3.27)$$

The transformation (3.27) is a symmetry of the B model, since it preserves the complex structure of $\mathbb{C}\mathbb{P}^{3|4}$ and the holomorphic measure.

The resulting topological B model on the $D = 3$ mini-twistor space can also be viewed as a limit of a discrete orbifold of the B model on $\mathbb{C}\mathbb{P}^{3|4}$. To construct this orbifold, pick a constant parameter $r > 0$ and define the group $\Gamma_r \simeq \mathbb{Z}$ generated by $\gamma_r = \exp(2\pi i r P_4)$. It acts on $\mathbb{C}\mathbb{P}^{3|4}$ as

$$\begin{aligned} \gamma_r \equiv e^{2\pi i r P_4}: Z_1 \mapsto Z_1, \quad Z_2 \mapsto Z_2, \\ Z_3 \mapsto Z_3 + 2\pi r Z_1, \quad Z_4 \mapsto Z_4 - 2\pi r Z_2, \\ \theta^A \mapsto \theta^A \quad (A = 1, \dots, 4). \end{aligned} \quad (3.28)$$

This map is compatible with the equivalence relation (3.26), and it also preserves the holomorphic superform

⁶We are now considering the 4D space to have signature $(+ - - +)$ and will reduce along one of the timelike directions.

TABLE I. The fields of the world-sheet B model. Note that the index z on some of the fields refers to the target space coordinate, and should not be confused with the world-sheet coordinate that is implicit.

Field	Variable	World sheet	$SU(4)$ representation
$\tilde{Z}, \tilde{W}, \tilde{\bar{Z}}, \tilde{\bar{W}}$	Commuting	Scalars	1
$\bar{\eta}^z, \bar{\eta}^w$	Anticommuting	Scalars	1
ϑ_z, ϑ_w	Anticommuting	Scalars	1
ρ^z, ρ^w	Anticommuting	1-forms	1
Θ^A	Anticommuting	Scalars	4
$\bar{\Theta}_A$	Anticommuting	Scalars	$\bar{4}$
$\bar{\eta}_A$	Commuting	Scalars	$\bar{4}$
ϑ_A	Commuting	Scalars	4
ρ^A	Commuting	1-forms	4

$\epsilon^{IJKL} Z_I dZ_J \wedge dZ_K \wedge dZ_L \wedge d\theta^1 \wedge \dots \wedge d\theta^4$. The orbifold $\mathbb{C}\mathbb{P}^{3|4}/\Gamma_r$ is therefore a Calabi-Yau superspace. The fixed-point set of Γ_r is the subset $Z_1 = Z_2 = 0$ which is the $\mathbb{C}\mathbb{P}^{1|4}$ that is excised. The physical interpretation of the B model on the Γ_r orbifold is, of course, the twistor string world-sheet theory for $D = 4$ super Yang-Mills theory compactified on a circle of radius r . In the limit $r \rightarrow 0$ we recover the $D = 3$ mini-twistor space. The resulting world-sheet fields of the B model on mini-twistor space are listed in Table I. The Becchi-Rouet-Stora-Tyutin transformation laws are

$$\begin{aligned} \delta \tilde{Z} = \bar{\eta}^z, \quad \delta \tilde{W} = \bar{\eta}^w, \quad \delta \bar{\Theta}_A = \bar{\eta}_A, \\ \delta \rho^z = d\tilde{Z}, \quad \delta \rho^w = d\tilde{W}, \quad \delta \rho^A = d\Theta^A. \end{aligned} \quad (3.29)$$

The transformation laws of the remaining fields are zero.

H. Dimensional reduction in Berkovits's twistor string theory

Dimensional reduction can be performed similarly in Berkovits's model of the twistor string theory [17]. In this model there are separate left- and right-moving world-sheet fields. The $SL(4 | 4)^7$ charged fields are $Z_L^i, Z_R^i, Y_{iL}, Y_{iR}$ ($i = 1, \dots, 4$) and $\Theta_L^A, \Theta_R^A, Y_{AL}, Y_{AR}$ ($A = 1, \dots, 4$), where L (R) denotes a left-moving (right-moving) field. There is an additional $GL(1)$ gauge field A under which Z^i, Θ^A have $+1$ charge and Y_i, Y_A have -1 charge. Also, there are left-moving and right-moving chiral-current algebras that give rise to the space-time $SU(N)$ quantum numbers. The action is [17]

$$\begin{aligned} S = \int d^2 \delta [Y_{Li} \nabla_{\bar{3}} Z_L^i + Y_{LA} \nabla_{\bar{3}} \Theta_L^A + Y_{iR} \nabla_{\bar{3}} Z_R^i \\ + Y_{AR} \nabla_{\bar{3}} \Theta_R^A] + S_C, \end{aligned} \quad (3.30)$$

where $\nabla_{\bar{3}} = \partial_{\bar{3}} - A_{\bar{3}}$ and $\nabla_{\bar{3}} = \partial_{\bar{3}} - A_{\bar{3}}$ are the covariant

⁷We are still working in signature $(+ - - +)$.

derivatives, and S_C is the action of the chiral-current algebras.

It is important to recall that even though (3.30) has cubic gauge interactions it is a conformally invariant theory. In fact, the equations of motion are

$$\begin{aligned} \nabla_{\bar{3}} Z_L^i &= 0, & \nabla_{\bar{3}} Y_{Li} &= 0, & \nabla_{\bar{3}} Z_R^i &= 0, \\ \nabla_{\bar{3}} Y_{Ri} &= 0, & i = 1, \dots, 4, & & \nabla_{\bar{3}} \Theta_L^A &= 0, \\ \nabla_{\bar{3}} Y_{LA} &= 0, & \nabla_{\bar{3}} \Theta_R^A &= 0, & \nabla_{\bar{3}} Y_{RA} &= 0, \\ & & A = 1, \dots, 4, & & & \end{aligned} \quad (3.31)$$

and

$$\begin{aligned} 0 &= \sum_i Z_L^i Y_{Li} + \sum_A Y_{LA} \Theta_L^A, \\ 0 &= \sum_i Z_R^i Y_{Ri} + \sum_A Y_{RA} \Theta_R^A, \end{aligned} \quad (3.32)$$

and the gauge fields can be solved in terms of the other fields. For $Z_L^1 \neq 0$ we can set

$$A_{\bar{3}} = \partial_{\bar{3}} Z_L^1 / Z_L^1.$$

Eliminating $A_{\bar{3}}$ from all the equations of motion (3.31), we find that all left-moving gauge invariant combinations (for example Z_L^2/Z_L^1) are holomorphic, and all right-moving gauge invariant combinations are antiholomorphic. This holomorphicity condition, together with the analytic constraints (3.32), completely captures all the equations of motion. The theory is therefore conformally invariant, since the gauge invariant fields are holomorphic.

As was explained in [33], the path integral splits into sectors that are labeled by an ‘‘instanton number’’ d . This number represents the total $U(1) \subset GL(1)$ flux. Following [33], we gauge fix the Weyl transformations on the world sheet and the $GL(1)$ gauge field by setting the world-sheet metric in such a way that a field of $GL(1)$ charge q and conformal dimension h will be equivalent to a gauge neutral holomorphic field of conformal dimension $h + (d/2)q$. This is a particularly convenient gauge fixing, because as explained in [33], all the left-moving fields are holomorphic and all the right-moving fields are antiholomorphic. We can therefore use conformal field theory operator product expansion (OPE)s to calculate commutators.

(To avoid clutter, unless otherwise specified we will from now on write formulas only for the left movers and suppress the L subscripts.) Dimensional reduction proceeds in a similar fashion as for the B model. Instead of (3.28), we have

$$\begin{aligned} \gamma_r &\equiv e^{2\pi i r P_4}: Z^1 \mapsto Z^1, & Z^2 &\mapsto Z^2, \\ Z^3 &\mapsto Z^3 + 2\pi r Z^1, & Z^4 &\mapsto Z^4 - 2\pi r Z^2, \\ Y_1 &\mapsto Y_1 - 2\pi r Y_3, & Y_2 &\mapsto Y_2 + 2\pi r Y_4, \\ Y_3 &\mapsto Y_3, & Y_4 &\mapsto Y_4, & \Theta^A &\mapsto \Theta^A, \\ Y_A &\mapsto Y_A \quad (A = 1, \dots, 4). \end{aligned} \quad (3.33)$$

In the limit $r \rightarrow 0$ we gauge a continuous symmetry. We can do this by introducing an extra auxiliary gauge field $\tilde{B}_{\bar{3}}$, $\tilde{B}_{\bar{3}}$ and modifying the covariant derivatives of Z^3 , Z^4 to

$$\begin{aligned} \nabla_{\bar{3}} Z^3 &= \partial_{\bar{3}} Z^3 - A_{\bar{3}} Z^3 - \tilde{B}_{\bar{3}} Z^1, \\ \nabla_{\bar{3}} Z^4 &= \partial_{\bar{3}} Z^4 - A_{\bar{3}} Z^4 + \tilde{B}_{\bar{3}} Z^2, \end{aligned} \quad (3.34)$$

and similarly for the right-moving fields. Inserting these covariant derivatives into the action (3.30) and integrating over $\tilde{B}_{\bar{3}}$, we get the constraint

$$Y_3 Z^1 - Y_4 Z^2 = 0. \quad (3.35)$$

Out of Z^1, \dots, Z^4 , $\Theta^1, \dots, \Theta^4$ we can make the following \tilde{B} -gauge invariant combinations: $W := Z^3 Z^2 + Z^4 Z^1$ with $GL(1)$ charge $+2$, and Z^1, Z^2, Θ^A with $GL(1)$ charge $+1$. The constraint (3.35) allows us to define

$$U := \frac{Y_3}{Z^2} = \frac{Y_4}{Z^1}. \quad (3.36)$$

This field U has $GL(1)$ charge -2 . It is well defined provided that either $Z^1 \neq 0$ or $Z^2 \neq 0$, which is indeed always the case. We also define

$$\tilde{Y}_1 = Y_1 - U Z^4, \quad \tilde{Y}_2 = Y_2 - U Z^3. \quad (3.37)$$

The action (3.30) together with the constraint (3.35) becomes

$$\begin{aligned} S &= \int d^2 \bar{\delta} [\tilde{Y}_1 \nabla_{\bar{3}} Z^1 + \tilde{Y}_2 \nabla_{\bar{3}} Z^2 + U \nabla_{\bar{3}} W + Y_A \nabla_{\bar{3}} \Theta^A] \\ &+ (\text{right movers}) + S_C, \end{aligned} \quad (3.38)$$

where S_C is the action of the current algebra and $\nabla_{\bar{3}} := \partial_{\bar{3}} - A_{\bar{3}}$ is the covariant $GL(1)$ derivative.

The resulting theory is easy to interpret if we recall that mini-twistor space is equivalent to $W\mathbb{C}P^{1,1,2}$ [see (3.14)]. The fields Z^1, Z^2 and W correspond to standard projective

TABLE II. The left-moving world-sheet fields of mini-twistor string theory à la Berkovits.

Field	Statistics	World sheet	$GL(1)$ charge
Z^1, Z^2	Commuting	Scalars	1
W	Commuting	Scalar	2
\tilde{Y}_1, \tilde{Y}_2	Commuting	Vectors	-1
U	Commuting	Vector	-2
Θ^A	Anticommuting	Scalars	1
Y^A	Anticommuting	Vectors	-1

coordinates on $W\mathbb{C}P^{1,1,2}$ with weights 1, 1 and 2, respectively. The $GL(1)$ charges of those fields correspond to these weights. In addition, we have four anticommuting fields Θ^A ($A = 1, \dots, 4$) with $GL(1)$ charges 1. Each of the fields above has a canonical conjugate field. \tilde{Y}_1, \tilde{Y}_2 are the conjugates of Z^1, Z^2 , and U is the conjugate of W . The conjugate of Θ^A is Y_A . In addition, there are also left- and right-moving current algebras. As Berkovits and Motl [33] explained, the world-sheet path integral splits into discrete sectors labeled by the flux d of the $GL(1)$ gauge field. The left-moving fields are listed in Table II. The right-moving fields have similar quantum numbers.

The generators of the Poincaré algebra (2.10) and (2.13) can easily be expressed in terms of the fields Z^1, Z^2, W and their conjugates $\tilde{Y}_1, \tilde{Y}_2, U$. The world-sheet currents that correspond to the translation generators P_+, P_- and P_1 are

$$\mathcal{P}_+ = -U(Z^1)^2, \quad \mathcal{P}_1 = UZ^1Z^2, \quad \mathcal{P}_- = U(Z^2)^2. \quad (3.39)$$

The transformation properties can be determined from the OPEs

$$Z^i(\zeta)\tilde{Y}_j(0) \sim \frac{1}{\zeta}, \quad W(\zeta)U(0) \sim \frac{1}{\zeta}, \quad (3.40)$$

$$\begin{aligned} Z^i(\zeta)U(0) &\sim Z^i(\zeta)Z^j(0) \sim \tilde{Y}_i(\zeta)\tilde{Y}_j(0) \sim W(\zeta)W(0) \\ &\sim U(\zeta)U(0) \sim \text{regular}. \end{aligned}$$

The supersymmetry generators (2.12) can also be expressed as currents. The left-moving currents corresponding to the SUSY generators are

$$\begin{aligned} \mathcal{Q}_{A+} &= Z^1 Y_A, & \bar{\mathcal{Q}}_+^A &= -iZ^1 \Theta^A U, \\ \mathcal{Q}_{A-} &= Z^2 Y_A, & \bar{\mathcal{Q}}_-^A &= iZ^2 \Theta^A U. \end{aligned} \quad (3.41)$$

I. Spin(7) R-symmetry

The R-symmetry group of our $D = 3$ super Yang-Mills theory is Spin(7). It acts linearly on the 8 supersymmetry generators, which transform as the spinor representation $\mathbf{8}$ of Spin(7). However, only an $SU(4)$ subgroup is manifest in mini-twistor string theory. This is the subgroup that acts linearly on Θ^A and can be identified with the $D = 4$ R symmetry, before dimensional reduction.⁸

It is not a big surprise that full Spin(7) symmetry is not explicit. Some symmetries are obscure in twistor string theory. A nice example is parity, which is not at all manifest in the B model [35], but was cleverly identified by Berkovits and Motl [33] in the open twistor string model.

⁸We are now working in signature $(-+++)$ again. Had we started instead in signature $(-+-)$ and dimensionally reduced along a timelike direction, we would have had to work with the noncompact R-symmetry group Spin(4,3), which extends the $D = 4$ group Spin(3,3) $\sim SL(4)$.

In this subsection we will identify the full Spin(7) R-symmetry generators in Berkovits's model.

Let us first write down the commutation relations for the Spin(7) R-symmetry generators. We take a basis for the $so(7)$ Lie algebra that consists of $SU(4)$ generators, which we denote by T^A_B , and 6 additional generators, which we denote by $T^{AB} = -T^{BA}$ ($A, B = 1, \dots, 4$). The commutation relations are

$$\begin{aligned} [T^A_B, T^C_D] &= \delta^A_D T^C_B - \delta^C_B T^A_D, \\ [T^A_B, T^{CD}] &= \delta^D_B T^{AC} - \delta^C_B T^{AD}, \\ [T^{AB}, T^{CD}] &= -\epsilon^{ABCE} T^D_E + \epsilon^{ABDE} T^C_E + \epsilon^{CDAE} T^B_E \\ &\quad - \epsilon^{CDBE} T^A_E. \end{aligned} \quad (3.42)$$

To see how the missing R-symmetry generators T^{AB} operate, let us look at a particular example of a Spin(7) multiplet of fields. According to [18], $D = 4$ twistor string theory contains, in addition to the super Yang-Mills theory, a sector that describes conformal supergravity. Our example of a Spin(7) multiplet of fields will comprise the dimensional reduction of some of these $D = 4$ conformal supergravity fields. We will take a multiplet of fields that transform in the irreducible representation $\mathbf{35}$ (antisymmetric 3-tensors) of Spin(7). This irreducible representation decomposes under $SU(4) \subset \text{Spin}(7)$ as

$$\mathbf{35} = \mathbf{15} + \mathbf{10} + \bar{\mathbf{10}}. \quad (3.43)$$

Thus, we can construct our multiplet by combining fields that transform in the three irreducible representations on the right-hand side of (3.43).

The fields of $D = 4$ conformal supergravity are listed in Table 1 in Sec. 4.2 of [18]. We can obtain our multiplet by combining three irreducible $SU(4)_R$ representations from that list. In the notation of [18], we pick the fields $E_{AB} = E_{BA}$, $\bar{E}^{AB} = \bar{E}^{BA}$ and $V_{\mu}^A_B$. The first is a $D = 4$ space-time scalar in the $\bar{\mathbf{10}}$ of $SU(4)_R$; the second is a $D = 4$ space-time scalar in the $\mathbf{10}$ of $SU(4)_R$, and the last one is a $D = 4$ space-time vector in the $\mathbf{15}$ adjoint representation of $SU(4)_R$. After dimensional reduction to $D = 3$, the scalars E_{AB} , \bar{E}^{AB} and the 4th component $V_4^A_B$ of V_{μ} form a Spin(7) multiplet in the irreducible representation $\mathbf{35}$.

Now let us see how Spin(7) acts on E_{AB} , \bar{E}^{AB} and $V_4^A_B$. The action of the $SU(4) \subset \text{Spin}(7)$ generators T^A_B is obvious. The Spin(7) generators T^{AB} must also transform these states to each other. It is, however, difficult to identify the T^{AB} generators in the B-model version of mini-twistor string theory. Part of the problem is that there are no perturbative B-model string vertex operators that correspond to the fields E_{AB} . The vertex operator for the dimensionally reduced \bar{E}^{BA} and the vertex operator for $V_4^A_B$ were constructed in [18], and we will recall them below, in (4.10) and (4.13). But the vertex operators for E_{AB} are absent in the perturbative B model. To understand this, recall that the states of a perturbative B model with target space X

correspond to the sheaf cohomology $H^p(X, \wedge^q TX)$ where TX is the holomorphic tangent bundle. Nonperturbatively, it is conjectured [36] that the B model is S dual to the A model and therefore also has states that correspond to Dolbeault cohomology $H^p(X, \Omega^q X)$ [where $\Omega^q X$ is the sheaf of holomorphic $(q, 0)$ forms]. The states E_{AB} are analogous to the latter. [See also the discussion after Eq. (2.12) of [18].] For recent developments in the non-perturbative formulation of the topological string theory see [37–39].

J. Spin(7) in Berkovits's model

It is easier, however, to identify T^{AB} in Berkovits's model. As usual in a two-dimensional conformal field theory, the symmetry generators correspond to holomorphic and antiholomorphic currents. We will denote these currents by J^{AB} . Let us first identify the world-sheet currents that correspond to the generators $T^A{}_B$. They are easily constructed from the $SU(4)$ transformation properties of Θ^A and Y_A , and we get

$$J^A{}_B := Y_B \Theta^A - \frac{1}{4} \delta_B^A Y_C \Theta^C.$$

We claim that the world-sheet currents that correspond to T^{AB} can be expressed, formally, as

$$J^{AB} := U \Theta^A \Theta^B + \frac{1}{2} U^{-1} \epsilon^{ABCD} Y_C Y_D. \quad (3.44)$$

We determined the generators (3.44) by looking for expressions with total conformal dimension 1, $GL(1)$ charge zero, and the correct residues in their OPEs among themselves and with the Poincaré currents (3.39) and supersymmetry currents (3.41). These residues are determined by the known commutation relations between the R symmetry and super-Poincaré generators. For example, using the residues of the simple poles in the OPEs of the currents J^{AB} and $J^A{}_B$, one can verify the commutation relations (3.42).

The operator U^{-1} that appears in (3.44) has to be defined carefully. It is required to have the OPE

$$U^{-1}(\beta)U(0) = 1 + O(\beta).$$

This operator U^{-1} can be handled by bosonization as follows. The (U, W) system is very similar to the superconformal ghosts (β, γ) of superstring theory, except that the conformal dimensions are shifted. Thus, (U, W) can be bosonized in much the same way as the superconformal ghosts. [See, e.g., Sec. 10.4 of [40]. Bosonization of the (Y, Z^I) fields has also been discussed in [33,41], and see also [42].] Let ϕ be a chiral boson, and let (ξ, η) be anticommuting ghosts, with OPEs

$$\phi(\beta)\partial\phi(0) \sim \frac{1}{\beta}, \quad \eta(\beta)\xi(0) \sim \frac{1}{\beta}.$$

(See Sec. 10.4 of [40].) The bosonization formulas are then

$$W \simeq -e^{-\phi} \partial \xi, \quad U \simeq e^{\phi} \eta. \quad (3.45)$$

We can then take

$$U^{-1} \simeq e^{-\phi} \xi, \quad (3.46)$$

and this has all the properties required of the inverse of U .

We can now use the R-symmetry operators (3.44) to complete partial Spin(7) multiplets in mini-twistor string theory. We will apply this procedure in Sec. IV F, where we will again encounter our operators E_{AB} , \bar{E}^{AB} and $V_4^A{}_B$, from Sec. III I. More details on bosonization can be found in the Appendix.

IV. MASS TERMS

We come now to the main new point of our paper—adding mass terms. We can augment the $D = 3$ super Yang-Mills Lagrangian with mass terms for the scalars and fermions. The $D = 3$ fermions are in the spinor representation $\mathbf{8}$ of the R-symmetry group Spin(7). Fermion mass terms are linear in the mass parameter and correspond to operators of the form $M_{ab} \chi_a^a \chi^{ab}$, where M_{ab} is the mass matrix, and we used the notation of (3.1). These operators are in the symmetric part of the tensor product representation $\mathbf{8} \otimes \mathbf{8}$ of Spin(7). (It decomposes into the irreducible representations $\mathbf{1} + \mathbf{35}$.) The $D = 3$ scalars are in the vector representation $\mathbf{7}$ of Spin(7). Scalar mass terms are quadratic in the mass parameter and correspond to operators in the symmetric part of the tensor product representation $\mathbf{7} \otimes \mathbf{7}$. (It decomposes into irreducible representations $\mathbf{1} + \mathbf{27}$.) The $D = 3$ super-mini-twistor space formalism only exhibits a manifest $SU(4) \subset$ Spin(7) subgroup of the R symmetry. This is the subgroup inherited from $D = 4$, and the anticommuting variables θ^A ($A = 1, \dots, 4$) are in the $\mathbf{4}$ of this $SU(4)$, while the $\partial/\partial\theta^A$ derivatives are in the $\bar{\mathbf{4}}$ conjugate representation. The operators that correspond to $D = 3$ fermion mass terms split into the following $SU(4)$ irreducible representations:

$$\begin{aligned} (\mathbf{8} \otimes \mathbf{8})_S &= (\mathbf{4} \otimes \mathbf{4})_S + (\bar{\mathbf{4}} \otimes \bar{\mathbf{4}})_S + \mathbf{4} \otimes \bar{\mathbf{4}} \\ &= \mathbf{10} + \bar{\mathbf{10}} + [\mathbf{1} + \mathbf{15}]. \end{aligned} \quad (4.1)$$

Here $(\dots)_S$ denotes the symmetric part of a tensor product, and $\mathbf{15}$ is the adjoint representation of $SU(4)$. We will now study the deformations of mini-twistor string theory that correspond to these mass terms. Generating the mass terms in the adjoint representation $\mathbf{15}$ is simpler, and we therefore start with those.

A. Dimensional reduction with twists

A convenient way to achieve a massive theory with mass terms in the $\mathbf{15}$ representation is to modify the procedure of dimensional reduction by adding an R-symmetry twist. In Sec. III G we obtained massless $D = 3$ super Yang-Mills theory from the massless $D = 4$ theory by gauging a translation generator P_4 (in the world-sheet theory). To obtain

massive $D = 3$ super Yang-Mills theory from the massless $D = 4$ theory, we gauge a linear combination of translation and R symmetry, $P_4 - \text{tr}(MR)$, where R is the $SU(4)$ R-symmetry charge (realized as a 4×4 Hermitian traceless matrix) and M is a constant 4×4 Hermitian traceless mass matrix that will end up as the mass matrix of the $D = 3$ fermions. The logic behind this reduction is as follows. Let ψ_α^A ($A = 1, \dots, 4$) be a $D = 4$ Weyl fermion field, which transforms in the fundamental representation $\mathbf{4}$ of the R-symmetry group $SU(4)$. Gauging the combination $P_4 - \text{tr}(MR)$ on the world sheet corresponds to setting

$$i\partial_3\psi_\alpha^A = M^A_B\psi_\alpha^B$$

in physical space. The term $\bar{\psi}_A^\alpha\sigma_{\alpha\dot{\alpha}}^3\partial_3\psi^{A\dot{\alpha}}$ in the $D = 4$ Lagrangian will become a mass term in the $D = 3$ dimensionally reduced Lagrangian. The fermion mass matrix will be M , and the scalar mass matrix squared will be the antisymmetric part of $M \otimes M$, which, in the $\mathbf{6}$ representation of $SU(4)$, is the 6×6 matrix representative of M .

In twistor space, we augment (3.27) to get

$$\begin{aligned} P_4 - \text{tr}(MR): \delta Z_1 = \delta Z_2 = 0, \quad \delta Z_3 = \epsilon Z_1, \\ \delta Z_4 = -\epsilon Z_2, \quad \delta\theta^A = \epsilon M^A_B\theta^B \quad (A, B = 1, \dots, 4). \end{aligned} \quad (4.2)$$

Gauging this translation symmetry can be done along the lines that led to (3.10). We get a similar supermanifold that can be covered by two patches U_1, U_2 with transition functions that are a slight modification of (3.10),

$$z' = \frac{1}{z}, \quad w' = -\frac{w}{z^2}, \quad \theta' = \frac{1}{z}e^{i(w/z)M}\theta \quad (4.3)$$

where θ represents a 4-component vector, and $\exp[i(w/z)M]$ is a 4×4 matrix [in the complexification of $SU(4)$]. This is one of our main results. Equation (4.3) describes the target space of a twistor string theory that is, by conjecture, dual to a mass-deformed $D = 3$ super Yang-Mills theory with a mass matrix M .

Similarly to the discussion at the end of Sec. III G, we can think of (4.2) as the $r \rightarrow 0$ limit of the discrete orbifold of $\mathbb{C}\mathbb{P}^{3|4}$ by the group $\Gamma_{r,M} \simeq \mathbb{Z}$ generated by

$$\begin{aligned} \gamma_{r,M} \equiv e^{2\pi ir(P_4 - \text{tr}(MR))}: Z_1 \mapsto Z_1, \quad Z_2 \mapsto Z_2, \\ Z_3 \mapsto Z_3 + 2\pi r Z_1, \quad Z_4 \mapsto Z_4 - 2\pi r Z_2, \\ \theta \mapsto e^{-2\pi ir M}\theta, \end{aligned} \quad (4.4)$$

where θ again denotes a 4-component vector. Physically, (4.4) corresponds to a circle compactification of $D = 4$ super Yang-Mills theory with an R-symmetry twist. That is, the scalars and fermions have nonperiodic boundary conditions on the circle. Such compactifications have been studied, for example, in [43]. R-symmetry orbifolds of the $D = 4$ theory (without the P_4 generator) have been recently studied in [44]. For recent developments in world-sheet orbifolds see [45].

B. Mini-twistor string theory mass operators

For any value of the mass matrix M , the manifold described by (4.3) is, presumably, the target space of the world-sheet (mini-)twistor string theory that describes the massive deformation studied in Sec. IV A. In particular, for an infinitesimally small M , (4.3) describes a small deformation of the complex structure of the super mini-twistor space \mathbb{T}_3 (defined as $T\mathbb{C}\mathbb{P}^1$ with four extra anticommuting coordinates fibered over it).

In general, a small deformation of the complex structure corresponds to an operator in the world-sheet theory, which is the topological B model with target space \mathbb{T}_3 . For example, an interesting case was studied in [46] where marginal deformations of the $D = 4$ theory [47–49] were associated with certain B-model closed string operators. The operators of the topological B model on a manifold X correspond to the sheaf cohomology classes $H^p(X, \wedge^m T^{(1,0)}X)$ [50]. Locally on X , with a choice of complex coordinates z_i ($i = 1, \dots, \dim_{\mathbb{C}}X$) and their complex conjugates \bar{z}_i , these sheaf cohomology classes can be realized as tensors $V_{\bar{i}_1 \dots \bar{i}_p}^{j_1 \dots j_m}$ that are antisymmetric in the \bar{i}_s and j_t indices. Deformation by the operators with $p = 1$ and $m = 1$ corresponds to a complex structure deformation. (The target space action that describes these deformations is the Kodaira-Spencer action [51].) This correspondence can be extended, presumably, to supermanifolds and we get the cohomology classes $H^{p|q}(X, \wedge^{m|n} T^{(1,0)}X)$. In [46], B-model operators that correspond to elements of $H^{0|2}(X, \wedge^{0|2} T^{(1,0)}X)$, with $X = \mathbb{C}\mathbb{P}^{3|4}$, were identified with cubic deformations of the $D = 4$, $N = 4$ super Yang-Mills superpotential. Furthermore, general deformations of holomorphic vector bundles over weighted projective superspaces were recently studied in [29]. The particular complex structure deformation that we need for the mass term is a special case.

Let us now study the $D = 3$ supercohomology classes. First, let us take the mass deformation in the limit of infinitesimal mass matrix M . Equation (4.3) teaches us that the infinitesimal change in coordinate transition functions on the intersection $U_{12} \equiv U_1 \cap U_2$ of the two patches $U_1 \equiv \{z < \infty\}$ and $U_2 \equiv \{z' < \infty\}$ is

$$\begin{aligned} \delta z' = 0, \quad \delta w' = 0, \\ \delta\theta' = i\frac{w}{z^2}M\theta = -i\frac{w'}{z'}M\theta' + O(M)^2. \end{aligned} \quad (4.5)$$

Here, we think of U_1, U_2 as two patches of the undeformed ($M_B^A = 0$) $D = 3$ twistor supermanifold \mathbb{T}_3 . Equation (4.5) defines a holomorphic vector field on U_{12} whose components are given, in local coordinates, by

$$\begin{aligned} \delta z' \frac{\partial}{\partial z'} + \delta w' \frac{\partial}{\partial w'} + \delta\theta'^T \frac{\partial}{\partial\theta'} &= -i\frac{w'}{z'}\theta'^T M^T \frac{\partial}{\partial\theta'} \\ &= i\frac{w}{z}\theta^T M^T \frac{\partial}{\partial\theta}. \end{aligned} \quad (4.6)$$

This defines an element of the sheaf cohomology $H^{1|0}(X, \wedge^{0|1} T^{(1,0)} X)$ for $X = \mathbb{T}_3$. In fact, $H^{1|0}(X, \wedge^{0|1} T^{(1,0)} X)$ is the space of holomorphic vector fields of the form

$$V^A_B(z, w, \theta_1, \dots, \theta_4) \theta_A \frac{\partial}{\partial \theta_B}$$

that are defined for $0 < z < \infty$ up to the equivalence relation

$$V^A_B \sim V^A_B + (V_{\{1\}})^A_B - (V_{\{2\}})^A_B, \quad (4.7)$$

where $V_{\{1\}}$ is a holomorphic vector field of a similar form that is defined for all $z < \infty$ (including $z = 0$) and $V_{\{2\}}$ is similarly defined for all $0 < z$ (including $z = \infty$). [Recall that in general, given a cover of X by contractible open patches $X = \cup_\alpha U_\alpha$, an element of $H^p(X, T^{(1,0)} X)$ can be described by a collection of local vector fields, one vector field for each nontrivial intersection of $(p+1)$ patches $U_{\alpha_1} \cap \dots \cap U_{\alpha_{p+1}}$, such that a certain linear relation—the cocycle condition—holds on intersections of $(p+2)$ patches. In our case, there are only two patches, U_1 and U_2 , and a vector field that is defined on the intersection $U_1 \cap U_2$ defines an element of $H^1(X, T^{(1,0)} X)$.]

The vector field corresponding to (4.6) is

$$V^A_B = \frac{w}{z} M^A_B = -\frac{w'}{z'} M^A_B.$$

This vector field therefore has poles both at $z = 0$ and $z' = 0$.

More generally, the field

$$\frac{w^r}{z^s} M^A_B = \frac{(-w')^r}{z'^{2r-s}} M^A_B \quad (4.8)$$

has poles both at $z = 0$ and $z' = 0$ only if $0 < s < 2r$. For other values of s , the vector field can be extended to either $z = 0$ or $z = \infty$, and therefore is trivial in cohomology. Let us check for which values of s, r this cohomology class is invariant under translations of the physical \mathbb{R}^3 . The generators of translations were written down in (2.10). Since the field (4.8) is holomorphic, only the holomorphic parts of P_1, P_2, P_3 in (2.10) are relevant. We therefore need to check that $z^k \partial / \partial w$, for $k = 0, 1, 2$, annihilates (4.8) in cohomology. By the discussion above, this is equivalent to requiring that $0 < s - k < 2(r - 1)$ is not satisfied for any of the values $k = 0, 1, 2$. It is not hard to check that the only solution is $s = r = 1$, which is the field from (4.6). Similarly, using Eqs. (2.11) for the rotation generators, it can be checked that (4.6) is rotationally invariant. Thus, the mass deformation (4.6) corresponds to the unique (up to multiplication by a constant) translationally invariant element of $H^{1|0}(X, \wedge^{0|1} T^{(1,0)} X)$.

Alternatively, the cohomology $H^{1|0}(X, \wedge^{0|1} T^{(1,0)} X)$ can be represented by a *global* differential 1-form with coefficients in $\wedge^{0|1} T^{(1,0)} X$. The way to convert a sheaf co-

mology representative to a global form is to write

$$V^A_B = (V_{\{1\}})^A_B - (V_{\{2\}})^A_B,$$

where $V_{\{1\}}$ and $V_{\{2\}}$ are well defined on U_1 and U_2 respectively, but, unlike (4.7), $V_{\{1\}}$ and $V_{\{2\}}$ are not required to be holomorphic (otherwise V^A_B would be trivial in cohomology). Then, the 1-form $\bar{\partial}(V_{\{1\}})^A_B = \bar{\partial}(V_{\{2\}})^A_B$ (where $\bar{\partial}$ is the Dolbeault differential operator) is globally defined and represents the cohomology class. In our case, we can pick an arbitrary differentiable function $f(|z|^2)$ such that $f(0) = 1$ and $f(\infty) = 0$ and set

$$(V_{\{1\}})^A_B = [1 - f(|z|^2)] \frac{w}{z} M^A_B,$$

$$(V_{\{2\}})^A_B = -f(|z|^2) \frac{w}{z} M^A_B.$$

Then,

$$\bar{\partial}(V_{\{1\}})^A_B = \bar{\partial}(V_{\{2\}})^A_B = -f'(|z|^2) M^A_B w d\bar{z} \quad (4.9)$$

is a global 1-form that represents the cohomology class. The corresponding B-model operator is easily constructed from this form. It is

$$V_{(15)} = -f'(\bar{Z} \bar{Z}) \bar{\eta}^{\bar{z}} \partial_A M^A_B \tilde{W} \Theta^B. \quad (4.10)$$

(The world-sheet fields of the B model are listed in Table I.) Can the operator corresponding to (4.9) be related by dimensional reduction to a $D = 4$ operator? We will work in the coordinates (3.26) for $\mathbb{C}\mathbb{P}^{3|4}$, and define the two patches

$$U'_1 := \{Z_1 \neq 0\}, \quad U'_2 := \{Z_2 \neq 0\}.$$

Together, U'_1 and U'_2 cover the $D = 4$ twistor space (given by the condition that Z_1 and Z_2 do not vanish simultaneously). Using (3.6), we calculate

$$\frac{w}{z} = \frac{\mu^1}{\lambda_1} - \frac{\mu^2}{\lambda_2} = \frac{Z_3}{Z_1} - \frac{Z_4}{Z_2}.$$

The element (4.6) can therefore be dimensionally “oxidized” (the inverse of “dimensionally reduced”) to the following (super-)vector field on $U'_1 \cap U'_2$,

$$\begin{aligned} V_m^{(D=4)} &= \left(\frac{Z_3}{Z_1} - \frac{Z_4}{Z_2} \right) \theta^A M^A_B \frac{\partial}{\partial \theta^B} \\ &= \left(\frac{Z_3}{Z_1} \theta^A M^A_B \frac{\partial}{\partial \theta^B} \right) - \left(\frac{Z_4}{Z_2} \theta^A M^A_B \frac{\partial}{\partial \theta^B} \right). \end{aligned} \quad (4.11)$$

In the last equality, we have written V_m as the difference of two terms, the first of which can be extended to a holomorphic function on the entire U'_1 patch, and the second

can be extended to a holomorphic function on the entire U'_2 patch. V_m is therefore exact in sheaf cohomology and corresponds to the zero operator. This was to be expected, since the $D = 3$ mass terms that we consider in this subsection, which are in **15** of $SU(4)$, lift to terms of the form $M^A{}_B A_3 \sigma^3_{\alpha\dot{\alpha}} \psi^{\alpha A} \bar{\psi}^{B\dot{\alpha}} + O(M)^2$ in $D = 4$, where A_3 is the 4th component of the gauge field (counting from 0). These terms can be gauged away by an x_4 -dependent gauge transformation.

C. The **10** and $\overline{\mathbf{10}}$ mass terms

We still have to analyze the remaining mass terms from (4.1). These are the terms in the representations **10**, their complex conjugates in $\overline{\mathbf{10}}$, and the representation **1**. We will discuss the singlet **1** separately, and concentrate on the **10** and $\overline{\mathbf{10}}$ first.

First, let us ask whether these $D = 3$ mass terms can be derived by dimensional reduction of $D = 4$ mass terms. The answer is yes. Looking back at (4.1), we see that, unlike the mass terms in **15** that were discussed in Sec. IVA, the mass terms in **10** and $\overline{\mathbf{10}}$ can indeed be gotten by dimensional reduction of $D = 4$ mass terms. This is because $D = 4$ mass terms are of the form $\epsilon^{\alpha\beta} \psi^A_{\alpha} \psi^B_{\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\alpha}A} \bar{\psi}_{\dot{\beta}B}$ but cannot involve $\psi^A_{\alpha} \bar{\psi}_{\dot{\alpha}B}$, since spinors of different chirality in $D = 4$ cannot be contracted in a Lorentz invariant manner. After dimensional reduction to $D = 3$, however, both spinors become the same representation. Thus, the $D = 3$ mass terms in **10** and $\overline{\mathbf{10}}$ can be derived from dimensionally reduced $D = 4$ mass terms, but the mass terms in **15** have to be derived by twisting as in Sec. IVA.

So what are the $D = 4$ operators that can give such mass terms upon dimensional reduction? These are closed string vertex operators of the B-model twistor string theory of $D = 4$ super Yang-Mills. It has been argued in [4,18,52] that they correspond to fields of $D = 4$ superconformal supergravity. In particular, Table 1 in Sec. 4.2 of [18] lists the physical states of that theory. And indeed, among these fields we find two that have the right quantum numbers. In the notation of [18], these are $E_{(AB)}$ [$A, B = 1, \dots, 4$ are $SU(4)$ indices] in $\overline{\mathbf{10}}$, and $\overline{E}^{(AB)}$ in **10**. The conformal dimensions of these fields can be read off from their $U(1)_R$ charge, which is $Q = -2$ (and is also listed in Table 1 of [18]). Their conformal dimensions are $\Delta = D + (3/2)Q = 1$, and this is consistent with a fermionic mass term.

Now let us write down the vertex operators for the dimensionally reduced $D = 3$ mass terms in the **10** representation. The \overline{E}^{AB} operators correspond to linear combinations of vector fields of the form

$$\frac{w^r}{z^s} M^{AB} \epsilon_{ACDE} \theta^C \theta^D \theta^E \frac{\partial}{\partial \theta^B}, \quad (4.12)$$

defined on the intersection $U_1 \cap U_2$ (i.e., $0 < |z| < \infty$),

where r, s are integers. By the same arguments that follow (4.8) we conclude that only $r = 0$ and $s = 1$ describes a nonzero Poincaré invariant operator. Converting from sheaf to Dolbeault $\bar{\partial}$ cohomology, we find the operator

$$V_{(10)} = -\frac{1}{6} f'(\bar{Z} \bar{Z}) \bar{\eta}^{\bar{z}} M^{AB} \epsilon_{ACDE} \Theta^C \Theta^D \Theta^E \vartheta_B. \quad (4.13)$$

Here, $f(|z|^2)$ is, as in (4.9), an arbitrary function that satisfies $f(0) = 1$ and $f(\infty) = 0$.

D. Chiral $D = 4$ mass terms

Let us make a brief digression to check that \overline{E}^{AB} indeed corresponds to a mass term in $D = 4$. We hope to present a more complete analysis elsewhere.

The mass-deformed $N = 4$ super Yang-Mills theory that, we claim, corresponds to turning on an \overline{E}^{AB} VEV is nonstandard. It is the CPT violating theory that one obtains by giving a mass term only to the positive-helicity spinors of $D = 4$, $N = 4$ super Yang-Mills theory. This is a perturbation of the form $M^{AB} \psi_{\alpha A} \psi^{\alpha B}$, where $\psi_{\alpha A}$ denote, as before, the $D = 4$ fermions in the $\overline{\mathbf{4}}$ of $SU(4)$.

In momentum space, the free massive Dirac equation now reads

$$p_{\alpha\dot{\alpha}} \psi^{\alpha A} = M^{AB} \bar{\psi}_{\dot{\alpha}B}, \quad p_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}A} = 0. \quad (4.14)$$

Thus, the 4-momentum $p_{\alpha\dot{\alpha}}$ is still lightlike, and we can decompose it as in the massless case,

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha} \tilde{\lambda}_{\dot{\alpha}}.$$

The general solution to (4.14) is of the form

$$\bar{\psi}_{\dot{\alpha}A} = \tilde{\lambda}_{\dot{\alpha}} \tilde{\varrho}_A, \quad \psi^A_{\alpha} = \lambda_{\alpha} \varrho^A + M^{AB} \eta_{\alpha} \tilde{\varrho}_B, \quad (4.15)$$

where $\tilde{\varrho}_A(\lambda, \tilde{\lambda})$ and $\varrho^A(\lambda, \tilde{\lambda})$ are arbitrary functions, and η_{α} is any (λ -dependent) solution to the linear equation

$$\eta_{\alpha} \lambda^{\alpha} = 1. \quad (4.16)$$

There is, of course, a family of solutions to this equation. Given a solution η_{α} and an arbitrary function $\zeta(\lambda, \tilde{\lambda})$, the following is also a solution to (4.16):

$$\eta'_{\alpha} := \eta_{\alpha} + \lambda_{\alpha} \zeta. \quad (4.17)$$

If we choose η'_{α} instead of η_{α} as the solution to (4.16), we can preserve the physical wave functions (4.15) by setting

$$\tilde{\varrho}'_A := \tilde{\varrho}_A, \quad \varrho'^A := \varrho^A - M^{AB} \zeta \tilde{\varrho}_B, \quad (4.18)$$

so that

$$\bar{\psi}_{\dot{\alpha}A} = \tilde{\lambda}_{\dot{\alpha}} \tilde{\varrho}'_A, \quad \psi^A_{\alpha} = \lambda_{\alpha} \varrho'^A + M^{AB} \tilde{\varrho}'_B \eta'_{\alpha}.$$

An example of a solution to (4.16) is given by $\eta_1 = 1/\lambda^1$ and $\eta_2 = 0$. It is well defined on the patch of λ space where $\lambda^1 \neq 0$. Similarly, on the patch where $\lambda^2 \neq 0$ we

can take $\eta'_2 = 1/\lambda^2$ and $\eta'_1 = 0$, so that on the intersection of the two patches, where both components of λ are non-zero, we have $\eta'_\alpha = \eta_\alpha - \epsilon_{\alpha\beta} \lambda^\beta / (\lambda^1 \lambda^2)$. If we choose to work with η on the patch $\lambda^1 \neq 0$ and with η' on the patch $\lambda^2 \neq 0$, we need the transition relations

$$\varrho'^A = \varrho^A + \frac{1}{\lambda^1 \lambda^2} M^{AB} \tilde{\varrho}_B, \quad \tilde{\varrho}'_A = \tilde{\varrho}_A.$$

External fermions in scattering amplitudes are described by wave functions of the form (4.15). These wave functions can be twistor transformed as usual,

$$\begin{aligned} \hat{\varrho}^A(\lambda, \mu) &:= \int d^2 \tilde{\lambda} \tilde{e}^{i\mu_\alpha \tilde{\lambda}^\alpha} \varrho^A(\lambda, \tilde{\lambda}), \\ \hat{\tilde{\varrho}}_A(\lambda, \mu) &:= \int d^2 \tilde{\lambda} \tilde{e}^{i\mu_\alpha \tilde{\lambda}^\alpha} \tilde{\varrho}_A(\lambda, \tilde{\lambda}). \end{aligned} \quad (4.19)$$

The freedom (4.18) extends to the twistor transforms

$$\hat{\tilde{\varrho}}'_A := \hat{\tilde{\varrho}}_A, \quad \hat{\varrho}'^A := \hat{\varrho}^A - M^{AB} \hat{\zeta} \cdot \hat{\tilde{\varrho}}_B, \quad (4.20)$$

where $\hat{\zeta} \cdot \hat{\tilde{\varrho}}_A$ denotes the convolution of $\hat{\tilde{\varrho}}_A$ with the twistor transform of ζ , and the convolution is taken with respect to μ_α . Note, however, that the equivalence (4.18) can be generalized to ζ 's that are operator functions of λ_α , $\tilde{\lambda}_\alpha$ and $\partial/\partial \tilde{\lambda}_\alpha$. For the special case that ζ is of the form $\zeta(\lambda_\alpha, i\partial/\partial \tilde{\lambda}_\alpha)$ the convolution becomes an ordinary product, and (4.20) becomes

$$\hat{\tilde{\varrho}}'_A := \hat{\tilde{\varrho}}_A, \quad \hat{\varrho}'^A := \hat{\varrho}^A - M^{AB} \zeta \hat{\tilde{\varrho}}_B, \quad (4.21)$$

where $\zeta(\lambda, \mu)$ is a function on twistor space.

Following similar steps as in the appendix of [4], we take the twistor transforms (4.19), plug them into (4.15), and integrate over λ and $\tilde{\lambda}$ to convert from momentum space back to coordinate space. We perform the λ_α integrals by gauge fixing $\lambda^1 = 1$ and integrating $z \equiv \lambda^2$ over a path C around the origin.

$$\begin{aligned} \bar{\psi}_{\dot{\alpha}A}(x) &= \frac{1}{2\pi} \oint_C d\lambda^2 \int d^2 \tilde{\lambda} \int d^2 \mu e^{ix_{\alpha\dot{\alpha}} \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} - i\mu_\alpha \tilde{\lambda}^\alpha} \tilde{\lambda}_\alpha \hat{\tilde{\varrho}}_A(\lambda, \mu) \\ &= \frac{1}{2\pi i} \oint_C dz \frac{\partial \hat{\tilde{\varrho}}_A}{\partial \mu_{\dot{\alpha}}} \Big|_{(\lambda^\alpha, x_{\alpha\dot{\alpha}} \lambda^\alpha)}, \\ \psi^A_\alpha(x) &= \frac{1}{2\pi} \oint_C d\lambda^2 \int d^2 \tilde{\lambda} \int d^2 \mu e^{ix_{\alpha\dot{\alpha}} \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} - i\mu_\alpha \tilde{\lambda}^\alpha} [\lambda_\alpha \hat{\varrho}^A(\lambda, \mu) + M^{AB} \eta_\alpha(\lambda) \hat{\tilde{\varrho}}_B(\lambda, \mu)] \\ &= \frac{1}{2\pi} \oint_C dz [\lambda_\alpha \hat{\varrho}^A(\lambda^\alpha, x_{\alpha\dot{\alpha}} \lambda^\alpha) + M^{AB} \eta_\alpha(\lambda) \hat{\tilde{\varrho}}_B(\lambda^\alpha, x_{\alpha\dot{\alpha}} \lambda^\alpha)], \end{aligned} \quad (4.22)$$

where

$$(\lambda^1, \lambda^2) \equiv (1, z), \quad (\eta_1, \eta_2) \equiv (1, 0).$$

Now let us analyze the coupling of the closed string B-model mode \bar{E}^{AB} to the open string modes. In general, the coupling of closed string B-model modes to open string modes has been extensively studied. (See [46,53–55] for example.) Turning on a VEV for \bar{E}^{AB} corresponds to a perturbation of the complex structure of the supermanifold target space $\mathbb{C}\mathbb{P}^{3|4}$. We will denote by X^i ($i = 1, \dots, 4$) and Ψ^A ($A = 1, \dots, 4$) the homogeneous coordinates on $\mathbb{C}\mathbb{P}^{3|4}$. (In [4], X^i is denoted by Z^i , but we use the symbols Z^1, Z^2 to denote some of the projective coordinates of $W\mathbb{C}\mathbb{P}^{1,1|2}$.) We will cover the $D = 4$ twistor space $\mathbb{C}\mathbb{P}^{3|4} \setminus \mathbb{C}\mathbb{P}^{1|4}$ with the two patches

$$U_1 := \{X^1 \neq 0\}, \quad \text{and} \quad U_2 := \{X^2 \neq 0\}.$$

On the patch U_1 where $X^1 \neq 0$, we can rescale the projective coordinates and set $X^1 = 1$. Then X^2, X^3, X^4 and the Ψ^A are the independent coordinates. The good coordinates on the patch U_2 , where $X^2 \neq 0$, are then

$$\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2}, \frac{\Psi^1}{X^2}, \dots, \frac{\Psi^4}{X^2}. \quad (4.23)$$

On the patch $X^1 \neq 0$, the VEV $\langle \bar{E}^{AB} \rangle = M^{AB}$ corresponds to the local holomorphic vector field

$$\frac{1}{6X^2} M^{AB} \epsilon_{ACDE} \Psi^C \Psi^D \Psi^E \frac{\partial}{\partial \Psi^B}, \quad (4.24)$$

[see (4.12) with $r = 0$ and $s = 1$]. This vector field, in turn, corresponds to a (super-)complex structure deformation of the B-model target space $\mathbb{C}\mathbb{P}^{3|4}$.

Formula (4.16) of [4] gives us the component expansion of the B-model super 1-form field as follows:

$$\begin{aligned} \mathcal{A}(X, \bar{X}, \Psi) &= d\bar{X}^i (A_{\bar{i}} + \Psi^A \chi_{\bar{i}A} + \frac{1}{2} \Psi^A \Psi^B \phi_{\bar{i}AB} \\ &\quad + \frac{1}{6} \epsilon_{ABCD} \Psi^A \Psi^B \Psi^C \tilde{\chi}_{\bar{i}}^D \\ &\quad + \frac{1}{24} \epsilon_{ABCD} \Psi^A \Psi^B \Psi^C \Psi^D G_{\bar{i}}) \end{aligned} \quad (4.25)$$

where $A_{\bar{i}}, \chi_{\bar{i}A}, \phi_{\bar{i}AB}, \tilde{\chi}_{\bar{i}}^D$ and $G_{\bar{i}}$ are functions of X and \bar{X} . The classical action is given by Eq. (4.18) of [4],

$$I = \frac{1}{2} \int \Omega \wedge \left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \quad (4.26)$$

The equations of motion of this holomorphic Chern-Simons theory, to first order, state that \mathcal{A} is an element of $\bar{\partial}$ cohomology

$$\bar{\partial} \mathcal{A} = 0, \quad \mathcal{A} \sim \mathcal{A} + \bar{\partial} \Lambda,$$

where Λ is an arbitrary function.

For our purposes, however, it is more convenient to let \mathcal{A} be an element of Čech cohomology rather than $\bar{\partial}$ cohomology. For ordinary manifolds these cohomologies are equivalent, but the advantage of Čech cohomology is that we work with holomorphic functions. For supermanifolds the situation is more complicated, and we refer the reader to [28,29,56] for further details.⁹

In Čech cohomology, which is a special case of sheaf cohomology, the B-model field is represented by a holomorphic function \mathcal{A} that is defined on the *intersection* of patches $U_1 \cap U_2$. This is in contrast to $\bar{\partial}$ cohomology for which \mathcal{A} was a (1,0)-form and was not necessarily represented by holomorphic functions, but was defined on the entire manifold $U_1 \cup U_2$. In Čech cohomology, there is an equivalence relation

$$\mathcal{A} \sim \mathcal{A} + \Lambda_1 + \Lambda_2, \quad (4.27)$$

where Λ_1 and Λ_2 are holomorphic functions on U_1 and U_2 , respectively.

How does the infinitesimal complex structure deformation that corresponds to (4.24) change the Čech cohomology class \mathcal{A} ? The infinitesimal complex structure deformation can be interpreted as a change in the holomorphic transition functions between the patch U_1 and the patch U_2 . In the deformed space, (4.23) are no longer good coordinates on U_2 . Instead, a good set of coordinates is

$$\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2}, \frac{\Psi^A}{X^2} + \frac{1}{6(X^2)^2} M^{AB} \epsilon_{BCDE} \Psi^C \Psi^D \Psi^E \quad (4.28)$$

$$A = 1, \dots, 4,$$

where we used (4.24). Thus, the Čech equivalence relation (4.27) has to be modified to

$$\begin{aligned} \mathcal{A}(X^2, X^3, X^4, \{\Psi^A\}) &\sim \mathcal{A}(X^2, X^3, X^4, \{\Psi^A\}) \\ &+ \Lambda_1(X^2, X^3, X^4, \{\Psi^A\}) \\ &+ \Lambda_2 \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2}, \left\{ \frac{\Psi^A}{X^2} + \frac{1}{6(X^2)^2} \right. \right. \\ &\left. \left. \times M^{AB} \epsilon_{BCDE} \Psi^C \Psi^D \Psi^E \right\} \right), \quad (4.29) \end{aligned}$$

where Λ_1 and Λ_2 are holomorphic functions of their variables. Let us expand \mathcal{A} in components, similar to (4.25).

$$\begin{aligned} \mathcal{A} &= A(X^2, X^3, X^4) + \Psi^A \chi_A(X^2, X^3, X^4) \\ &+ \frac{1}{2} \Psi^A \Psi^B \phi_{AB}(X^2, X^3, X^4) \\ &+ \frac{1}{6} \epsilon_{ABCD} \Psi^A \Psi^B \Psi^C \tilde{\chi}^D(X^2, X^3, X^4) \\ &+ \frac{1}{24} \epsilon_{ABCD} \Psi^A \Psi^B \Psi^C \Psi^D G(X^2, X^3, X^4), \quad (4.30) \end{aligned}$$

where $A, \chi_A, \phi_{AB}, \tilde{\chi}^D$ and G are holomorphic functions of X^2, X^3, X^4 and are defined on $U_1 \cap U_2$, that is, for $X^2 \neq 0$ and $X^2 \neq \infty$. (Since confusion is not likely to arise, we use the same notation for the component fields in Čech cohomology as in $\bar{\partial}$ cohomology.) The freedom to add Λ_1 in (4.29) implies that each of the component fields can be augmented by a holomorphic function of X^2, X^3, X^4 that is nonsingular at $X^2 = 0$. Thus, the Čech cohomology classes of the component fields are only sensitive to the singular behavior of the fields at $X^2 = 0$.

The freedom to add Λ_2 in (4.29) is now more complicated. Let the component expansion of Λ_2 be

$$\begin{aligned} \Lambda_2(y^2, y^3, y^4, \{\theta^A\}) &= \varpi + \theta^A \zeta_A + \frac{1}{2!} \theta^A \theta^B \varsigma_{AB} \\ &+ \frac{1}{3!} \epsilon_{ABCD} \theta^A \theta^B \theta^C \nu^D \\ &+ \frac{1}{4!} \epsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \kappa, \quad (4.31) \end{aligned}$$

where $\varpi, \varsigma, \nu, \kappa$ are holomorphic functions of their generic variables y^2, y^3, y^4 . Expanding (4.29) in components, we now find the equivalence relations

$$\begin{aligned} \chi_A(X^2, X^3, X^4) &\sim \chi_A(X^2, X^3, X^4) + \frac{1}{X^2} \zeta_A \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right), \\ \tilde{\chi}^A(X^2, X^3, X^4) &\sim \tilde{\chi}^A(X^2, X^3, X^4) + \frac{1}{(X^2)^3} \nu^A \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right) \\ &- \frac{1}{(X^2)^2} M^{AB} \zeta_B \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right) \quad (4.32) \end{aligned}$$

for the fermions, and

$$\begin{aligned} A(X^2, X^3, X^4) &\sim A(X^2, X^3, X^4) + \varpi \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right), \\ \phi_{AB}(X^2, X^3, X^4) &\sim \phi_{AB}(X^2, X^3, X^4) \\ &+ \frac{1}{(X^2)^2} \varsigma_{AB} \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right), \quad (4.33) \\ G(X^2, X^3, X^4) &\sim G(X^2, X^3, X^4) + \frac{1}{(X^2)^4} \kappa \left(\frac{1}{X^2}, \frac{X^3}{X^2}, \frac{X^4}{X^2} \right) \end{aligned}$$

for the scalars, where we have used the symmetry of M^{AB} in the A, B indices. We see that only the equivalence relation for the field $\tilde{\chi}^A$ is modified. This field is the twistor transform of the $h = -1/2$ helicity spinor ψ_α^A . We would now like to relate the modified equivalence relation to the solution to the massive Dirac equation (4.15).

Let us first recall the origin of the equivalence relations (4.32) in the massless case ($M^{AB} = 0$). It was explained in

⁹We are grateful to Popov, Sämann and Wolf for pointing this out to us.

the appendix of [4] that the physical wave functions are recovered from $\tilde{\chi}^A$ and χ_A by a Cauchy integral along a path C that encircles the origin,

$$\begin{aligned}\psi_\alpha^A(x) &= \frac{1}{2\pi i} \oint_C \lambda_\alpha \tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) dz, \\ \psi_A^\alpha(x) &= \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial x_{1\dot{\alpha}}} \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) dz,\end{aligned}\tag{4.34}$$

$$\begin{aligned}\tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) &\sim \tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) + \frac{1}{z^3} v^A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right), \\ \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) &\sim \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) + \frac{1}{z} \zeta_A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right).\end{aligned}\tag{4.36}$$

These equivalences hold because of the identities

$$\begin{aligned}0 &= \frac{1}{2\pi i} \oint_C \lambda_\alpha \frac{1}{z^3} v^A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right) dz, \\ 0 &= \frac{1}{2\pi i} \oint_C \frac{1}{z^2} \frac{\partial}{\partial x_{2\dot{\alpha}}} \zeta_A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right) dz,\end{aligned}\tag{4.37}$$

that can be derived by deforming the contour of integration into a circle of radius $|z| \rightarrow \infty$. (Note that λ_α behaves at worst like z , and v^A at worst like a constant.) In the second identity in (4.37) we have set $\partial \zeta_A / \partial x_{1\dot{\alpha}} = (\partial \zeta_A / \partial x_{2\dot{\alpha}}) / z$.

For the massless case $M^{AB} = 0$, Eq. (4.34) is the same as (4.22) with

$$\hat{\varrho}^A \equiv \tilde{\chi}^A, \quad \hat{\varrho}_A \equiv \chi_A.\tag{4.38}$$

$$\begin{aligned}\tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) &\sim \tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) + \frac{1}{z^3} v^A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right) \\ &\quad + \frac{1}{z^2} M^{AB} \zeta_B \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right), \\ \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) &\sim \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) + \frac{1}{z} \zeta_A \left(\frac{1}{z}, \frac{x_{1i}}{z} + x_{2i}, \frac{x_{1\dot{2}}}{z} + x_{2\dot{2}} \right).\end{aligned}\tag{4.40}$$

Indeed, these transformations leave the Cauchy integrals (4.39) invariant. Moreover, the Čech equivalence relations (4.32), in the form (4.40), determine the form of the Cauchy integrals (4.39), with η_α determined up to the ambiguity (4.17). Thus we have shown that a VEV for \bar{E}^{AB} corresponds to a chiral mass term.

E. Mass terms in Berkovits's model

We will now study the counterparts of the mass operators (4.10) and (4.13) in Berkovits's twistor string theory. We will work with the world-sheet fields from Sec. III H. Berkovits and Witten explained in [18] how to convert B-model operators to operators in Berkovits's model.

where we have fixed the rescaling freedom by

$$\lambda^1 \equiv 1, \quad \lambda^2 \equiv z.\tag{4.35}$$

[Equations (4.34) are the analogs of Whittaker's formula (2.1) for spinor fields in $D = 4$.] For $M^{AB} = 0$, the equivalence relation (4.32) can be written in this patch as

Now let us turn to the massive case. Rewriting the Dirac wave functions (4.22) in terms of the B-model fields (4.38), we get

$$\begin{aligned}\psi_\alpha^A(x) &= \frac{1}{2\pi i} \oint_C [\lambda_\alpha \tilde{\chi}^A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) \\ &\quad + M^{AB} \eta_\alpha \chi_B(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z)] dz, \\ \psi_A^\alpha(x) &= \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial x_{1\dot{\alpha}}} \chi_A(z, x_{1i} + x_{2i}z, x_{1\dot{2}} + x_{2\dot{2}}z) dz,\end{aligned}\tag{4.39}$$

where

$$(\lambda^1, \lambda^2) \equiv (1, z), \quad (\eta_1, \eta_2) \equiv (1, 0).$$

The equivalences (4.36) now have to be modified to

Adapted to our $D = 3$ setting, the procedure is as follows. We start with a holomorphic vector field on mini-twistor space \mathbb{T}_3 . It can be represented in homogeneous coordinates as

$$\begin{aligned}f^I &:= f^i(Z^1, Z^2, W, \Theta) \frac{\partial}{\partial Z^i} + f^W(Z^1, Z^2, W, \Theta) \frac{\partial}{\partial W} \\ &\quad + f^A(Z^1, Z^2, W, \Theta) \frac{\partial}{\partial \Theta^A},\end{aligned}\tag{4.41}$$

subject to the equivalence relation

$$f^I \simeq f^I + \Lambda(Z^1, Z^2, W, \Theta) \left(Z^i \frac{\partial}{\partial Z^i} + 2W \frac{\partial}{\partial W} + \Theta^A \frac{\partial}{\partial \Theta^A} \right).$$

This vector field is converted to a Berkovits vertex operator

$$\mathcal{V} = Y_i f^i + U f^W + Y_A f^A.$$

In particular, starting with a B-model operator that is represented as a holomorphic vector field

$$f := f^z(z, w, \theta) \frac{\partial}{\partial z} + f^w(z, w, \theta) \frac{\partial}{\partial w} + f^A(z, w, \theta) \frac{\partial}{\partial \theta^A}, \quad (4.42)$$

we can get the world-sheet vertex operator by lifting (4.42) to $\mathbb{C}^{4|4}$ by setting $Z^1 = 1$, $Z^2 = z$ and $W = w$.

$$\begin{aligned} \mathcal{V} = & Y_2 f^z \left(\frac{Z^2}{Z^1}, \frac{W}{(Z^1)^2}, \frac{\Theta}{Z^1} \right) + U f^w \left(\frac{Z^2}{Z^1}, \frac{W}{(Z^1)^2}, \frac{\Theta}{Z^1} \right) \\ & + Y_A f^A \left(\frac{Z^2}{Z^1}, \frac{W}{(Z^1)^2}, \frac{\Theta}{Z^1} \right). \end{aligned}$$

This is an *open* string vertex operator in Berkovits's model. The vector field (4.42) is required to preserve the holomorphic superform on mini-twistor space

$$\Omega = dw \wedge dz \wedge d\theta^1 \cdots d\theta^4.$$

This is a condition that descends from a similar condition in $D = 4$ [18].

Applying this Berkovits-Witten prescription to (4.10) we get our first ansatz for the Berkovits-model mass operators

$$\mathcal{V}_{(15)} \rightarrow \mathcal{V}_{(15)} = \frac{W}{Z^1 Z^2} Y_A M^A{}_B \Theta^B. \quad (4.43)$$

The factor of $Z^1 Z^2$ in the denominator of (4.43) might appear strange at first, but it can be handled by bosonization. We have discussed a similar issue at the end of Sec. IIII, and more details can be found in the Appendix.

To gain more insight, we will now derive (4.43) by directly gauging translations with an R-symmetry twist (4.2) in the world-sheet action (3.30). We therefore augment the covariant derivative (3.34) according to the modified gauge transformation (4.2),

$$\begin{aligned} \nabla_{\tilde{3}} Z_L^3 &= \partial_{\tilde{3}} Z_L^3 - A_{\tilde{3}} Z_L^3 - \tilde{B}_{\tilde{3}} Z_L^1, \\ \nabla_{\tilde{3}} Z_L^4 &= \partial_{\tilde{3}} Z_L^4 - A_{\tilde{3}} Z_L^4 + \tilde{B}_{\tilde{3}} Z_L^2, \\ \nabla_{\tilde{3}} \Theta_L^A &= \partial_{\tilde{3}} \Theta_L^A - i \tilde{B}_{\tilde{3}} M^A{}_B \Theta^B, \end{aligned} \quad (4.44)$$

and similarly for the right-moving fields. Inserting these covariant derivatives into the action (3.30) and integrating out the nondynamical gauge fields $\tilde{B}_{\tilde{3}}$, $\tilde{B}_{\tilde{3}}$, we get the modified constraint

$$j_m(\tilde{3}) := Y_3 Z^1 - Y_4 Z^2 - i M^A{}_B Y_A \Theta^B = 0. \quad (4.45)$$

(Again, we suppress the L, R subscripts on fields.) Note that for $M^A{}_B \neq 0$, the left-moving gauge field $\tilde{B}_{\tilde{3}}$ and the right-moving gauge field $\tilde{B}_{\tilde{3}}$ have to be related by complex conjugation, because the left-moving gauge transformation by itself is anomalous, with an anomaly proportional to $M^A{}_B M^B{}_A$.

According to the previous discussion, and in particular (4.43), the mass deformation corresponds to an open string vertex operator and therefore should manifest itself as a change in the world-sheet boundary conditions. How can we convert (4.45) to a change of boundary conditions?

Let us first recall, on the classical level, what the equations of motion of the Berkovits action are. The charged fields have equations of motion

$$\begin{aligned} \nabla_{\tilde{3}} Z_L^i &= 0, & \nabla_{\tilde{3}} Y_{Li} &= 0, & \nabla_{\tilde{3}} Z_{Ri} &= 0, \\ \nabla_{\tilde{3}} Y_R^i &= 0, & i &= 1, \dots, 4, \end{aligned} \quad (4.46)$$

and similarly for Y_A and Θ^A . The gauge fields $A_{\tilde{3}}$, $A_{\tilde{3}}$ and $\tilde{B}_{\tilde{3}}$, $\tilde{B}_{\tilde{3}}$ do not have dynamical equations of motion, and they can be arbitrary. The equations of motion (4.46) then imply that gauge invariant combinations of left-moving fields are holomorphic, and gauge invariant combinations of right-moving fields are antiholomorphic. This (anti-)analyticity and the constraints

$$\sum_{i=1}^4 Y_{Li} Z_L^i + Y_{LA} \Theta_L^A = 0, \quad \sum_{i=1}^4 Y_{Ri} Z_{Ri} + Y_{RA} \Theta_{RA}^A = 0, \quad (4.47)$$

together with (4.45), are all the restrictions on gauge invariant combinations. For $M^A{}_B = 0$, a full set of \tilde{B} -gauge invariant combinations is given by (3.36) and (3.37). For $M^A{}_B \neq 0$, we need to modify these formulas. We define the following field combinations:

$$\begin{aligned} Z^1, \quad Z^2, \quad W &:= Z^1 Z^4 + Z^2 Z^3, \\ U &:= \frac{Y_4}{Z^1} = \frac{Y_3}{Z^2} - \frac{i}{Z^1 Z^2} M^A{}_B Y_A \Theta^B, \\ \tilde{Y}_1 &:= Y_1 - U Z^4 + i \frac{Z^3}{(Z^1)^2} M^A{}_B Y_A \Theta^B, \\ \tilde{Y}_2 &:= Y_2 - U Z^3, \quad \tilde{\Theta} := \exp\left(i \frac{Z^3}{Z^1} M\right) \Theta, \\ \tilde{Y} &:= \exp\left(-i \frac{Z^3}{Z^1} M\right) Y. \end{aligned} \quad (4.48)$$

They are invariant under the gauge transformations that correspond to \tilde{B} , and using the constraint (4.47), we see that they satisfy

$$0 = \tilde{Y}_1 Z^1 + \tilde{Y}_2 Z^2 + \tilde{Y}_A \tilde{\Theta}^A + 2UW. \quad (4.49)$$

Thus, the equations of motion for Z^1 , Z^2 , \tilde{Y}_1 , \tilde{Y}_2 , W , U , $\tilde{\Theta}$, \tilde{Y} are independent of the mass matrix $M^A{}_B$. The mass matrix must, therefore, enter into the boundary conditions.

Let us consider a world sheet with the topology of a disk. It can be represented as the upper half plane $\text{Im} \tilde{3} > 0$. The boundary conditions at $\text{Im} \tilde{3} = 0$ are [18]

$$\begin{aligned} Z_L^i &= Z_{Ri}^*, & Y_{Li} &= (Y_R^i)^*, & \Theta_L^A &= \Theta_{RA}^*, \\ Y_{LA} &= (Y_R^A)^*. \end{aligned} \quad (4.50)$$

The ‘‘doubling trick’’ (see, e.g., Sec. 2.6 of [40]) is a standard way of treating a conformal field theory on a disk. We extend the definition of the left-moving fields to the full \mathfrak{z} plane by setting

$$\begin{aligned} Z_L^i(\mathfrak{z}) &:= (Z_R^i(\bar{\mathfrak{z}}))^*, & Y_{Li}(\mathfrak{z}) &= (Y_R^i(\bar{\mathfrak{z}}))^*, \\ \Theta_L^A(\mathfrak{z}) &= \Theta_{RA}(\bar{\mathfrak{z}})^*, & Y_{LA}(\mathfrak{z}) &= (Y_R^A(\bar{\mathfrak{z}}))^*, & \text{Im}\mathfrak{z} < 0. \end{aligned} \quad (4.51)$$

$$\begin{aligned} Z^1, \quad Z^2, \quad W &:= Z^1 Z^4 + Z^2 Z^3, & U &:= \frac{Y_3}{Z^2} = \frac{Y_4}{Z^1} + \frac{i}{Z^1 Z^2} M^A{}_B Y_A \Theta^B, & \tilde{Y}_1 &:= Y_1 - U Z^4, \\ \tilde{Y}_2 &:= Y_2 - U Z^3 - i \frac{Z^4}{(Z^2)^2} M^A{}_B Y_A \Theta^B, & \tilde{\Theta} &:= \exp\left(-i \frac{Z^4}{Z^2} M\right) \Theta, & \tilde{Y} &:= \exp\left(i \frac{Z^4}{Z^2} M\right) Y. \end{aligned} \quad (4.52)$$

These fields were determined by the requirement of invariance under \tilde{B} -gauge transformations and the requirement that (4.49) is satisfied.

We still need to specify which of the two sets of formulas, (4.48) or (4.52), to choose for the left-moving and the right-moving sectors. We do not have any compelling reason to prefer one choice over the other, but if we choose (4.52) for the right movers and (4.48) for the left movers we

Including the point $\mathfrak{z} = \infty$, the fields are now defined on a sphere.

Now let us look at the fields from (4.48). They are well defined only if $Z^1 \neq 0$. When $M^A{}_B = 0$, we can use the alternative definition $U = Y_3/Z^2$, which makes sense when $Z^2 \neq 0$. Thus, when $M^A{}_B = 0$, the fields from (4.48) make sense whenever either Z^1 or Z^2 is nonzero. But for $M^A{}_B \neq 0$ we have to use a different set of fields in patches of the world sheet that contain zeros of Z^1 . Such an alternative set of fields is given by

will soon see that we recover the mass operator (4.43). It is not clear to us why we cannot choose the same set of formulas, say (4.48), for both left movers and right movers, but we note that if we do that we need a certain condition, say $Z_L^1 \neq 0$, to hold throughout the doubled world sheet, and this could be too restrictive.

With (4.48) for the left movers and (4.52) for the right movers, we get the boundary conditions

$$\begin{aligned} Z_L^1 &= Z_{R1}^*, & Z_L^2 &= Z_{R2}^*, & W_L &= W_R^*, & U_L &= U_R^* + \frac{i}{Z_L^1 Z_L^2} M^A{}_B Y_{LA} \Theta_L^B, \\ \tilde{Y}_{1L} &= \tilde{Y}_{1R}^* + \frac{i W_L}{(Z_L^1)^2 Z_L^2} M^A{}_B Y_{LA} \Theta_L^B, & \tilde{Y}_{2L} &= \tilde{Y}_{2R}^* - \frac{i W_L}{Z_L^1 (Z_L^2)^2} M^A{}_B Y_{LA} \Theta_L^B, & \Theta_L^A &= \exp\left(i \frac{W_L}{Z_L^1 Z_L^2} M\right) \Theta_{RA}^*, \\ Y_{LA} &= \exp\left(-i \frac{W_L}{Z_L^1 Z_L^2} M\right) (Y_R^A)^*. \end{aligned} \quad (4.53)$$

These boundary conditions can be succinctly described by adding an extra boundary term to the world-sheet action,

$$\begin{aligned} \delta S &= -\frac{i}{2} \int d\mathfrak{z} \frac{W_L}{Z_L^1 Z_L^2} Y_{LA} M^A{}_B \Theta_L^B \\ &+ \frac{i}{2} \int d\bar{\mathfrak{z}} \frac{W_R}{Z_R^1 Z_R^2} Y_R^B M^A{}_B \Theta_{RA}. \end{aligned} \quad (4.54)$$

To first order in M , the full action is given by the bulk world-sheet action (3.30) plus the boundary action (4.54), supplemented with the boundary conditions

$$\begin{aligned} Z_L^1 &= Z_{R1}^*, & Z_L^2 &= Z_{R2}^*, & W_L &= W_R^*, \\ \Theta_L^A &= \exp\left(i \frac{W_L}{Z_L^1 Z_L^2} M\right) \Theta_{RA}^*. \end{aligned}$$

The remaining boundary conditions (4.53) follow [up to $O(M^2)$ corrections] by minimizing the action. To conclude, we note that the boundary term (4.54) is consistent with the mass operator (4.43) that was predicted from the B model.

Similarly to (4.43), we can convert (4.13) to Berkovits’s model. Using the fields from Table II, we get the open

string vertex operator

$$V_{(\mathbf{10})} \rightarrow \frac{1}{Z^1 Z^2} M^{AB} \epsilon_{BCDE} Y_A \Theta^C \Theta^D \Theta^E. \quad (4.55)$$

The mass terms in $\overline{\mathbf{10}}$, which were difficult to identify in the B-model mini-twistor string theory, can be readily identified in Berkovits’s model. Dimensionally reducing the operators E_{AB} from [18], we get the open string vertex operator

$$V_{(\overline{\mathbf{10}})} \rightarrow \frac{1}{Z^1 Z^2} M_{AB} \Theta^A \partial \Theta^B. \quad (4.56)$$

F. Application of Spin(7) R symmetry

In Secs. IV B and IV E we found three kinds of $D = 3$ mass operators. We found the operators (4.55) in the $\mathbf{10}$ of $SU(4)$, the operators (4.56) in the $\overline{\mathbf{10}}$ of $SU(4)$, and the operators (4.43) in the $\mathbf{15}$ of $SU(4)$. The latter were the easiest to analyze, since they could be derived by a twisted dimensional reduction. However, all these operators should be related by the Spin(7) R symmetry that we discussed in

Sec. III I. In this subsection we will apply the R-symmetry generators (3.44) to the operators (4.43). We will discover a surprise: the operators (4.43), (4.55), and (4.56) do not fit into an irreducible representation of Spin(7).

Nevertheless, starting with the operators (4.56) (in the $\overline{\mathbf{10}}$), we can reconstruct a good Spin(7) multiplet of operators by successively applying the Spin(7) generators (3.44). These operators will be different from our previous results (4.43) and (4.55). We can “distill” out of (4.43) the terms that do fall into the irreducible representation **35** and use them to complete the Spin(7) multiplet.

Let us now do this in detail. Set

$$\Xi := \frac{1}{Z^1 Z^2}. \quad (4.57)$$

The various mass terms that we found in Eqs. (4.43), (4.55), and (4.56) are

$$\begin{aligned} V_{(\overline{\mathbf{10}})} &\rightarrow \Xi M_{AB} \Theta^A \partial \Theta^B, & V_{(\mathbf{15})} &\rightarrow \Xi W Y_A M^A_B \Theta^B, \\ V_{(\mathbf{10})} &\rightarrow \Xi M^{AB} \epsilon_{BCDE} Y_A \Theta^C \Theta^D \Theta^E. \end{aligned}$$

These terms are linear combinations of the vertex operators

$$\begin{aligned} \mathcal{V}^{AB} &:= \Xi \Theta^{(A} \partial \Theta^{B)}, \\ \tilde{\mathcal{V}}_B^A &:= \Xi W (Y_B \Theta^A - \frac{1}{4} \delta_B^A Y_C \Theta^C), \\ \tilde{\mathcal{V}}_{AB} &:= \Xi \epsilon_{CDE(B} Y_{A)} \Theta^C \Theta^D \Theta^E. \end{aligned} \quad (4.58)$$

(Here and in the equations below, all operators are understood to be normal ordered.) As we will see below, some of these vertex operators are not quite what we are looking for, and the tilde over \mathcal{V} will remind us that they are about to be modified. We will denote the modified vertex operators by \mathcal{V}^A_B and \mathcal{V}_{AB} . These vertex operators should form an irreducible representation of Spin(7), isomorphic to **35**. We can therefore write down immediately their expected commutation relations with the Spin(7) generators (3.42):

$$\begin{aligned} [T^A_B, \mathcal{V}_{CD}] &= \delta_C^A \mathcal{V}_{BD} + \delta_D^A \mathcal{V}_{BC}, \\ [T^A_B, \mathcal{V}^{CD}] &= -\delta_B^C \mathcal{V}^{AD} - \delta_B^D \mathcal{V}^{AC}, \\ [T^A_B, \mathcal{V}^C_D] &= \delta_D^A \mathcal{V}^C_B - \delta_B^C \mathcal{V}^A_D, \end{aligned} \quad (4.59)$$

and

$$\begin{aligned} [T^{AB}, \mathcal{V}_{CD}] &= \delta_C^A \mathcal{V}^B_D - \delta_C^B \mathcal{V}^A_D + \delta_D^A \mathcal{V}^B_C \\ &\quad - \delta_D^B \mathcal{V}^A_C, \\ [T^{AB}, \mathcal{V}^{CD}] &= \epsilon^{ABCE} \mathcal{V}^D_E + \epsilon^{ABDE} \mathcal{V}^C_E, \\ [T^{AB}, \mathcal{V}^C_D] &= \epsilon^{ABCE} \mathcal{V}_{ED} + \delta_D^A \mathcal{V}^{BC} - \delta_D^B \mathcal{V}^{AC}. \end{aligned} \quad (4.60)$$

Thus, we can calculate the operators \mathcal{V}^A_B from the commutation relations of T^{AB} with \mathcal{V}^{CD} .

We can read off these commutation relations from the coefficient of the simple pole in the OPE of J^{AB} and \mathcal{V}^{CD} . The result is

$$\begin{aligned} [T^{AB}, \mathcal{V}^{CD}] &= \frac{1}{2} \epsilon^{ABCE} \Xi (U^{-1} \partial Y_E \Theta^D + \partial U^{-1} Y_E \Theta^D \\ &\quad - U^{-1} Y_E \partial \Theta^D) + \frac{1}{2} \epsilon^{ABDE} \Xi (U^{-1} \partial Y_E \Theta^C \\ &\quad + \partial U^{-1} Y_E \Theta^C - U^{-1} Y_E \partial \Theta^C). \end{aligned}$$

A comparison with (4.60) teaches us that

$$\begin{aligned} \mathcal{V}^A_B &= \frac{1}{2} \Xi (U^{-1} \partial Y_B \Theta^A + \partial U^{-1} Y_B \Theta^A - U^{-1} Y_B \partial \Theta^A) \\ &\quad - \frac{1}{8} \delta_B^A \Xi (U^{-1} \partial Y_C \Theta^C + \partial U^{-1} Y_C \Theta^C \\ &\quad - U^{-1} Y_C \partial \Theta^C). \end{aligned}$$

We have subtracted the trace to make \mathcal{V}^A_B traceless, as is required of the irreducible representation **15**.

Next, we calculate the commutation relations between the newly found \mathcal{V}^A_B and the Spin(7) generators T^{AB} . A long but straightforward calculation gives

$$\begin{aligned} [T^{AB}, \mathcal{V}^C_D] &= -\delta_D^A \Xi \partial \Theta^B \Theta^C + \delta_D^B \Xi \partial \Theta^A \Theta^C \\ &\quad - \epsilon^{ABCE} \Xi U^{-2} \partial Y_{(E} Y_{D)}. \end{aligned} \quad (4.61)$$

A comparison with (4.60) therefore teaches us that

$$\mathcal{V}_{AB} = U^{-2} \Xi Y_{(A} \partial Y_{B)}, \quad (4.62)$$

where Ξ was defined in (4.57), and using the bosonized fields from Sec. III J, we have defined

$$U^{-2} := -e^{-2\phi} \xi \partial \xi. \quad (4.63)$$

This definition is natural, given the definition of U^{-1} in

TABLE III. The three types of mass operators with their $SU(4)$ irreducible representations, their coupling to the fermions, the corresponding conformal supergravity fields, the B-model complex structure deformations ($\delta\theta^A$ is the extra term in the change of holomorphic variables from the patch $z \neq 0$ to the patch $z \neq \infty$) and the Berkovits-model operators from (4.64), (4.65), and (4.66). The operator $\Xi \equiv 1/Z^1 Z^2$ was defined in (4.57).

Operator	$M_{AB} \mathcal{V}^{AB}$	$M^A_B \mathcal{V}^B_A$	$M^{AB} \mathcal{V}_{AB}$
$SU(4)$ representation	$\overline{\mathbf{10}}$	$\mathbf{15}$	$\mathbf{10}$
Fermion coupling	$M_{AB} \chi^A \chi^B$	$M^A_B \chi^A \chi_B$	$M^{AB} \chi_A \chi_B$
Conformal SUGRA field	E_{AB}	V^A_4	\overline{E}^{AB}
B-model deformation: $\delta\theta^A =$	\dots	$z^{-1} M^A_B \omega^B$	$\frac{1}{6} z^{-1} M^{AB} \epsilon_{BCDE} \theta^C \theta^D \theta^E$
Berkovits model	$\Xi \Theta^{(A} \partial \Theta^{B)}$	$\frac{1}{2} \Xi U^{-1} \partial Y_B \Theta^A + \dots$	$\Xi U^{-2} Y_{(A} \partial Y_{B)}$

TABLE IV. Various symbols used in the text. The subsection column is the place where the symbol first appeared with its given meaning.

Symbol	Subsection	Meaning
a, b, \dots	Sec. III A	Spin(7) spinor R-symmetry indices (1, ..., 8)
A, B, C, \dots	Sec. II E	$SU(4)$ R-symmetry indices (1, ..., 4)
A_μ	Sec. III A	Space-time gauge field (either $D = 3$ or $D = 4$)
$A_{\bar{3}}, A_{\bar{3}}$	Sec. III H	World-sheet $GL(1)$ gauge field
$\tilde{B}_{\bar{3}}, \tilde{B}_{\bar{3}}$	Sec. III H	World-sheet gauge field, used to gauge for translations
E_{AB}, \bar{E}^{AB}	Sec. III I	Space-time conformal supergravity fields from [18]
f, f'	Sec. IV E	Holomorphic vector field on mini-twistor space
\mathcal{F}	Sec. III F	Focal curve
i, j, \dots	Sec. III A	Euclidean space-time indices 1, ..., 3 or 1, ..., 4
I, \dots	Sec. III A	Spin(7) vector index 1, ..., 7
I	Sec. III J	A collective index for one of the eight $D = 4$ Berkovits-model fields
J_\pm, J_1	Sec. II D	$D = 3$ rotation generators
\mathcal{L}	Sec. III A	Space-time Lagrangian
M^A_B, M_{AB}, M^{AB}	Sec. IV A	Mass matrices
P_\pm, P_1	Sec. II D	$D = 3$ translation generators
Q_\pm, \bar{Q}_\pm	Sec. II E	$D = 3$ supersymmetry generators
T^{AB}, T^A_B	Sec. III I	Spin(7) R-symmetry generators
J^{AB}, J^A_B	Sec. III J	Spin(7) R-symmetry currents
\dagger	Sec. II F	$D = 3$ mini-twistor
\mathbb{T}_3	Sec. II E	Mini-twistor space
U_1, U_2	Sec. III C	Patches of mini-twistor space
U	Sec. III H	World-sheet Berkovits-model field dual to W
$\mathcal{V}^A_B, \mathcal{V}^{AB}, \mathcal{V}_{AB}$	Sec. IV F	Berkovits-model mass operators
$\tilde{\mathcal{V}}^A_B, \tilde{\mathcal{V}}^{AB}, \tilde{\mathcal{V}}_{AB}$	Sec. IV F	Berkovits-model mass operators (converted from B model)
\tilde{W}, \bar{W}	Sec. III G	B-model world-sheet field ($D = 3$)
W	Sec. III H	Berkovits-model world-sheet fields ($D = 3$)
w, \bar{w}	Sec. II B	Coordinates on mini-twistor space (for the fiber of $T\mathbb{C}\mathbb{P}^1$)
X^i	Sec. IV D	World-sheet fields in $D = 4$ B model ($i = 1, \dots, 4$)
$x_{\alpha\dot{\alpha}}$	Sec. III B	Space-time coordinate ($D = 3$ or $D = 4$)
$Y_{\underline{i}}$	Sec. III H	Berkovits-model dual field ($i = 1, 2$ in $D = 3$)
\tilde{Z}, \bar{Z}	Sec. III G	B-model world-sheet fields ($D = 3$)
Z^i	Sec. III H	World-sheet fields in $D = 3$ Berkovits model ($i = 1, 2$)
z, \bar{z}	Sec. II B	Coordinate on mini-twistor space (for the base of $T\mathbb{C}\mathbb{P}^1$)
$\beta, \bar{\beta}$	Sec. III H	World-sheet coordinate

(3.46) and the following OPE:

$$U^{-1}(\beta)U^{-1}(0) = -e^{-2\phi(0)}\xi(0)\partial\xi(0) + O(\beta).$$

We can now write down our final result for the three types of mass operators:

$$\mathcal{V}^{AB} = \Xi\Theta^{(A}\partial\Theta^{B)}, \quad (4.64)$$

$$\mathcal{V}_{AB} = \Xi U^{-2}Y_{(A}\partial Y_{B)}, \quad (4.65)$$

$$\begin{aligned} \mathcal{V}^A_B &= \frac{1}{2}\Xi(U^{-1}\partial Y_B\Theta^A - U^{-1}Y_B\partial\Theta^A + \partial U^{-1}Y_B\Theta^A) \\ &\quad - \frac{1}{8}\Xi\delta_B^A(U^{-1}\partial Y_C\Theta^C - U^{-1}Y_C\partial\Theta^C \\ &\quad + \partial U^{-1}Y_C\Theta^C). \end{aligned} \quad (4.66)$$

We have checked that these operators now satisfy all the relations (4.59) and (4.60) and hence constitute an irreduc-

ible representation of the R-symmetry group Spin(7) that is isomorphic to $\mathfrak{35}$.

Our results are summarized in Table III. The operator (4.64) agrees with the previous result (4.56), but the other operators (4.65) and (4.66) do not agree with the operators (4.43) and (4.55) that we found in Sec. IV E. The reason for this apparent discrepancy is not completely clear to us, but we expect the difference between (4.43) to (4.66) to decouple from physical amplitudes. As for the relation between (4.55) and (4.65), we suspect that these operators are in a different ‘‘picture,’’ as we will now explain.

In [33] it was explained that vertex operators in Berkovits’s model come in different pictures. A disk amplitude with $(d + 1)$ negative-helicity gluons and $(n - d - 1)$ positive-helicity gluons requires an insertion of d ‘‘instanton-changing operators,’’ in addition to the n vertex operators that correspond to the physical states. These instanton-changing operators are analogous to the

TABLE V. Various symbols used in the text (Greek letters). The subsection column is the place where the symbol first appeared with its given meaning.

Symbol	Subsection	Meaning
α, β, \dots	Sec. II G	Chiral spinor indices ($D = 3$ or $D = 4$)
$\dot{\alpha}, \dot{\beta}, \dots$	Sec. III A	Dotted spinors ($D = 4$)
Γ^μ	Sec. II G	$D = 3$ Dirac matrices
ζ_A	Sec. IV D	Holomorphic component function on twistor space
ζ	Sec. IV D	Arbitrary function of $\lambda, \tilde{\lambda}$
$\hat{\zeta}$	Sec. IV D	Twistor transform of ζ
η	Sec. III B	Fiber coordinate in the fibration $\mathbb{C}\mathbb{P}^3 \rightarrow T\mathbb{C}\mathbb{P}^1$
η	Sec. III J	World-sheet field used for bosonization of (U, W)
η_α	Sec. IV D	Solution to the equation $\eta_\alpha \lambda^\alpha = 1$
$\bar{\eta}^z, \bar{\eta}^w, \bar{\eta}^A$	Sec. III G	B-model world-sheet fields
$\Theta^A, \bar{\Theta}^A$	Secs. III G and III H	World-sheet fields
$\vartheta_z, \vartheta_w, \vartheta_A$	Sec. III G	B-model world-sheet anticommuting fields
κ	Sec. IV D	Holomorphic component function on twistor space
Λ	Sec. IV D	Arbitrary function on twistor space
λ^α	Sec. III B	Twistor variable
$\tilde{\lambda}_\alpha$	Sec. II G	Appears in the decomposition of null vectors
μ_α	Sec. III B	Twistor variable
μ, ν, \dots	Sec. II G	Space-time indices $0, \dots, 2$ or $0, \dots, 3$
ξ	Sec. III J	World-sheet field used for bosonization of (U, W)
ξ_1, ξ_2, ξ_3	Sec. III D	Projective coordinates on $W\mathbb{C}\mathbb{P}^{1,1,2}$
Ξ	Sec. IV F	Stands for the world-sheet field $1/(Z^1 Z^2)$
π, π^I	Sec. III B	Projections from $D = 4$ twistor space to $D = 3$ mini-twistor space
ϖ	Sec. IV D	Holomorphic component function on twistor space
ρ^z, ρ^w, ρ^A	Sec. III G	B-model world-sheet fields
$\varrho, \tilde{\varrho}$	Sec. IV D	Functions of $\lambda, \tilde{\lambda}$ appearing in the solution to Dirac's equation
$\hat{\varrho}, \tilde{\hat{\varrho}}$	Sec. IV D	Twistor transforms of $\varrho, \tilde{\varrho}$
$\sigma^i_{\alpha\beta}$	Sec. III A	$D = 3$ Pauli matrices
S_{AB}	Sec. IV D	Holomorphic component function on twistor space
Σ	Sec. III D	Curve in twistor space
Σ	Sec. III D	Curve in mini-twistor space
v^A	Sec. IV D	Holomorphic component function on twistor space
Y_A	Sec. III H	Berkovits-model fields dual to Θ^A
ϕ	Sec. III J	World-sheet field used for bosonization of (U, W)
Φ^I	Sec. III A	Space-time scalars
$\chi_A, \tilde{\chi}^A$	Sec. IV D	Component fields of the B model (functions of twistor space)
$\chi^A, \chi_{\alpha A}, \chi^{\dot{\alpha}}_\alpha$	Sec. III A	$D = 3$ fermions (functions of space-time)
$\psi^A_\alpha, \psi^{\dot{\alpha}}_\alpha$	Sec. III A	$D = 4$ fermions (functions of space-time)
Ψ^A	Sec. IV D	Fermionic coordinates in the B model

picture-changing operators of superstring theory. They contain δ functions of the Y_I fields, and are necessary to absorb zero modes of those fields. Similarly to superstring theory, one can get rid of a picture-changing operator by absorbing it in a physical vertex operator, thereby changing the picture of that vertex operator. This observation played a crucial role in identifying parity symmetry [33].

Let us now look at the zero modes of U . Using the bosonization formulas (3.45), we can define the U picture of an operator to be its ϕ momentum, so that an operator that contains an exponent $\exp(p\phi)$ will be in the p picture. It follows from (3.44) that when T^{AB} acts on an operator in the p picture, it produces a sum of operators in the $(p - 1)$ picture and $(p + 1)$ picture.

To see why picture changing is necessary, recall our comment at the end of Sec. III A. There we saw that the notion of helicity in $D = 3$ is not invariant under a general R-symmetry transformation. For example, we saw that a positive-helicity gluon satisfies $\Phi^7 = i\Phi^8$, but one can find an infinitesimal R-symmetry transformation that acts as $\delta\Phi^7 = \epsilon\Phi^6$, say. After this transformation, the gluon state will acquire a 0-helicity component. But replacing a (+1)-helicity state with a 0-helicity state in a scattering amplitude requires, according to [33], an extra instanton-changing operator.

Now take a positive-helicity gluon vertex operator in Berkovits's model. In $D = 3$ it is of the form $\Phi(Z^1, Z^2, W)J_C$, where J_C is a holomorphic current from

the chiral-current algebra component of the world-sheet theory, and Φ is a meromorphic function. Acting with T^{AB} gives

$$[T^{AB}, \Phi] = \frac{\partial \Phi}{\partial W} \Theta^A \Theta^B + \frac{1}{2} \tilde{\Phi}(Z^1, Z^2, U) \epsilon^{ABCD} \Upsilon_C \Upsilon_D, \quad (4.67)$$

where $\tilde{\Phi}$ is the residue of the simple pole in the OPE of Φ with U^{-1} and can be calculated from formula (A3). The first term on the right-hand side of (4.67) describes the above-mentioned 0-helicity state. The second term, however, is in a different U picture and would formally correspond to a (+2)-helicity state. Since no such term exists, it must decouple from physical amplitudes. Similarly, the mass operators (4.55) and (4.65) are obviously in a different picture, since one contains U^{-2} while the other does not.

$$\begin{aligned} \tilde{V}_{AB} &= \frac{1}{6} \epsilon_{CDEA} [T^{CD}, \tilde{V}_B^E] \\ &= U^{-2} \Xi \partial \Upsilon_{(A} \Upsilon_{B)} + \frac{1}{12} \epsilon_{CDE(A} \Theta^C \Theta^D \Theta^E \Upsilon_{B)} + \frac{1}{2} U^{-2} \Xi \partial (\Upsilon_A \Upsilon_B) - \Xi \Upsilon_A \Upsilon_B U^{-2} \partial \phi - \frac{1}{3} \Xi \epsilon_{ABCD} \partial \Theta^C \Theta^D \\ &\quad - \frac{1}{3} \Xi \epsilon_{ABCD} \partial \phi \Theta^C \Theta^D + \frac{1}{12} \epsilon_{CDE[A} \Xi \Theta^C \Theta^D \Theta^E \Upsilon_{B]} + \frac{1}{3} \Xi U^{-2} \Upsilon_A \Upsilon_B \Upsilon_C \Theta^C. \end{aligned} \quad (4.69)$$

But \tilde{V}_{AB} has to be symmetric in the indices A, B , and the last two lines of (4.69) are antisymmetric in A, B . This means that \tilde{V}_{AB} is a mixture of the $SU(4)$ irreducible representations **10** and **15**. It suggests that our starting point, (4.58) for \tilde{V}_B^A , is a mixture of the irreducible $\text{Spin}(7)$ representations **35** and the adjoint **21**. One can similarly check that

$$\begin{aligned} \tilde{V}^{BC} &= \frac{1}{3} [T^{AB}, \tilde{V}_A^C] \\ &= \Xi \partial (\Theta^B \Theta^C) + \frac{1}{6} \epsilon^{CDE(B} \Xi U^{-2} \Upsilon_C \Upsilon_D \Upsilon_E \Theta^C) - \frac{1}{6} \epsilon^{BCDE} \Xi U^{-2} \partial (\Upsilon_D \Upsilon_E) + \frac{1}{3} \epsilon^{BCDE} \Xi U^{-2} \partial \phi \Upsilon_D \Upsilon_E \\ &\quad - \frac{1}{3} \Xi \Theta^B \Theta^C \Upsilon_A \Theta^A + \frac{1}{6} \epsilon^{DEF[B} \Xi U^{-2} \Upsilon_D \Upsilon_E \Upsilon_F \Theta^C] + \frac{1}{2} \Xi \partial (\Theta^B \Theta^C) + \Xi \partial \phi \Theta^B \Theta^C. \end{aligned}$$

This is again not symmetric in A, B and thus contains a mixture of the $SU(4)$ irreducible representations **10** and **6**.

G. The singlet mass term

The last remaining mass term in (4.1) is the singlet **1**. A naive guess for the corresponding B-model operator is to set M in (4.6) to be proportional to the identity matrix. This means that we cannot obtain (4.6) by dimensional reduction with an $SU(4)$ twist, because the identity matrix is not traceless. An even bigger problem is that if M is not traceless, the complex structure deformation to which (4.6) corresponds does not preserve the holomorphic volume form Ω of $\mathbb{C}\mathbb{P}^{3|4}$. Indeed, unless M is traceless,

$$\begin{aligned} \Omega &= \frac{1}{4!^2} \epsilon_{IJKL} Z^I dZ^J \wedge dZ^K \wedge dZ^L \\ &\quad \wedge dZ^L \epsilon_{ABCD} d\psi^A d\psi^B d\psi^C d\psi^D \end{aligned}$$

is not preserved by (4.3). We do not know how to turn on singlet mass terms. [See also the related discussion after Eq. (2.11) in Sec. 2.2 of [18] regarding complex structure deformations that do not preserve Ω .]

We conclude this subsection by demonstrating explicitly that the operators \tilde{V}_B^A are not in the irreducible representation **35**, as we claimed at the beginning of this subsection. If they were, we could complete them to a $\text{Spin}(7)$ multiplet using the commutation relations (4.60), just like we did above with \mathcal{V}^{AB} . But an explicit computation gives

$$\begin{aligned} [T^{AB}, \tilde{V}_C^D] &= 2 \delta_C^{[A} \Xi \partial \Theta^{B]} \Theta^D + 2 \Xi \partial \phi \delta_C^{[A} \Theta^{B]} \Theta^D \\ &\quad + \epsilon^{ABDE} \Xi U^{-2} \partial \Upsilon_E \Upsilon_C \\ &\quad - \epsilon^{ABDE} \Xi U^{-2} \partial \phi \Upsilon_E \Upsilon_C \\ &\quad + \Xi \Theta^A \Theta^B \Theta^D \Upsilon_C \\ &\quad + \frac{1}{2} \epsilon^{ABEF} \Xi U^{-2} \Upsilon_E \Upsilon_F \Upsilon_C \Theta^D. \end{aligned} \quad (4.68)$$

Here $\partial \phi = -:UW:$ is a bosonized current. If (4.60) were satisfied, we could set

H. Comments on the decoupling limit $M \rightarrow \infty$

The discussion in the previous subsections was concerned mainly with infinitesimal mass terms. To first order, these mass terms are related to deformations of the B-model action by closed string vertex operators. Our analysis in Sec. IVA, however, allows us to “integrate” the infinitesimal mass deformations and describe finite mass terms. Specifically, (4.3) describes a super complex structure deformation of $\mathbb{C}\mathbb{P}^{3|4}$ that preserves the holomorphic volume form for any traceless M .

Let m_1, \dots, m_4 be the eigenvalues of M , and set $m_i = c_i m$ for some constants c_i . For generic c_i , the limit $m \rightarrow \infty$ is quite interesting from the physical perspective. If all c_i and all $c_i + c_j$ are nonzero, all fermions and six out of the seven scalars get a large bare mass and decouple. We can preserve $D = 3, N = 2$ supersymmetry if we keep $c_1 = 0$. In this case, one scalar and two $D = 3$ gluinos remain with zero bare mass. We can also preserve $D = 3, N = 4$ supersymmetry if we keep $c_1 = c_2 = 0$. In this case, three scalars and four gluinos remain with zero bare mass.

Let us now comment on the $m \rightarrow \infty$ limit from the perspective of Berkovits’s open twistor string theory. The

mass appears only in the boundary term (4.54). In the limit $m \rightarrow \infty$, the boundary term (4.54) becomes very big. This suggests that the bulk action of Y and Θ can be neglected. We end up with a world-sheet theory that has bulk modes Z^1, Z^2, W and their conjugates Y_1, Y_2, U . The Y and Θ fields live only on the boundary and couple to the bulk fields via the boundary action (4.54). This state of affairs is somewhat reminiscent of the Seiberg-Witten limit of large Neveu-Schwarz–Neveu-Schwarz (NS-NS) 2-form B field that leads to noncommutative geometry [57–59]. (A connection between deformations of twistor string theory and noncommutative geometry, and, in particular, its extension to superspace [60] was also suggested in [46].) This limit in the present context will be studied elsewhere.

V. CONCLUSIONS AND DISCUSSION

We reviewed Hitchin’s construction of mini-twistor space, which relates $D = 3$ mini-twistor space to $D = 4$ twistor space by dimensional reduction. The key point is that the 3(complex)-dimensional $D = 4$ twistor space $\mathbb{C}\mathbb{P}^3 \setminus \mathbb{C}\mathbb{P}^1$ can be written as a fiber bundle with the 2(complex)-dimensional $D = 3$ twistor space $T\mathbb{C}\mathbb{P}^1$ as the base, and the fiber is \mathbb{C} . The structure group is the additive translation group $\sim \mathbb{C}$. We used this fibration to relate tree-level twistor amplitudes of $D = 3$ Yang-Mills theory to $D = 4$ amplitudes. We calculated $D = 3$ tree-level amplitudes from the $D = 4$ ones by integrating over the \mathbb{C} fibers of the above fibration. This immediately implies that Witten’s observations [4] regarding scattering amplitudes and holomorphic curves in twistor space are valid in $D = 3$, at least at tree level. In $D = 3$ there is a known relation between holomorphic curves on mini-twistor space and holomorphic curves in complexified Minkowski space. We used this relation to give a direct physical interpretation to the holomorphic curves in mini-twistor space. At one-loop level and higher, the $D = 3$ and $D = 4$ amplitudes are not directly related by dimensional reduction, and whether or not the conjectures of [4] extend to $D = 3$ remains to be seen. For developments regarding loop amplitudes in $D = 4$ see [61–68].

Our main new results are related to deformations of the $N = 8$ supersymmetric $D = 3$ Yang-Mills theory by mass terms. We proposed a variant of the topological B model that describes massive $D = 3$ Yang-Mills theory. This model can describe 15 out of the 35 different possible mass terms. These mass-deformed $D = 3$ theories can be obtained by twisting the dimensional reduction of massless $D = 4$ theories. As for the other mass terms, we only discussed infinitesimal deformations and conjectured that these deformations correspond to VEVs of certain conformal supergravity fields. In this paper, we gave circumstantial evidence in support of the previous statement. It would be very interesting to show this convincingly by examining the three point functions of two supergravity fields and one gauge field operator. This should be possible

by performing a computation similar to the one in [18] [see Eq. (5.5) of that paper], where the correlation functions of a conformal supergravity vertex operator with an arbitrary number of Yang-Mills vertex operators were calculated in Berkovits’s model. We hope to report on this in another paper.

In $D = 3$ we constructed the Spin(7) R-symmetry currents that transform one type of infinitesimal mass term to another. This allowed us to construct vertex operators for all 35 mass terms in Berkovits’s model. We encountered a surprise when we tried to compare the vertex operators in Berkovits’s model with the topological B model. In Berkovits’s model we found that the 35 mass operators do not quite fit into an irreducible representation of the R-symmetry group Spin(7). We suggested a different set of operators that do fit into an R-symmetry multiplet. It would be interesting to understand the meaning of this in more detail.

Our results provide a way to break the supersymmetry from $N = 8$ to $N = 4, 2, 1, 0$ in $D = 3$ by arbitrary mass terms, in mini-twistor string theory. One direction for further research is to analyze the limit of infinite mass. In this limit we should obtain pure $N = 4, 2, 1$ Yang-Mills theories, and also $N = 0$ with a scalar. Our results also suggest a way to break $N = 4$ in $D = 4$ by infinitesimal mass terms. It would be interesting to try to integrate these infinitesimal deformations to get large mass terms. Understanding R symmetry better may enable us to convert a noninfinitesimal mass term in the representation **15** [described by (4.3) in the B model or (4.45) in Berkovits’s model] into a mass term in the **10** + $\overline{\mathbf{10}}$, and then it might be possible to lift it to $D = 4$. Alternatively, a better understanding of the nonperturbative topological B model, perhaps along the lines suggested recently in [36–38], may shed light on how to turn on noninfinitesimal VEVs for the closed string modes E_{AB} and \overline{E}^{AB} simultaneously.

Another possible direction for further research is to explore the mirror manifold of mini-twistor space. The same techniques of [69], where the mirror manifold of the twistor space $\mathbb{C}\mathbb{P}^{3|4}$ has been constructed, can be applied to mini-twistor space. It was shown in [69] that the mirror of $\mathbb{C}\mathbb{P}^{3|4}$ is a quadric in $\mathbb{C}\mathbb{P}^{3|3} \times \mathbb{C}\mathbb{P}^{3|3}$. One might then be tempted to examine the dimensionally reduced version of the argument in [70], where the authors use S duality in conjunction with mirror symmetry to explain why the amplitudes of $\mathcal{N} = 4$ Yang-Mills are supported on holomorphic curves. Moreover, certain symmetries (e.g. parity) might be more manifest in the mirror. One would expect to lose some of the manifest $SU(4 | 4)$ generators.

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APPENDIX: DEALING WITH INVERSES OF FIELDS

The commuting world-sheet fields of Berkovits's model, (Y_1, Z^1) , (Y_2, Z^2) and (U, W) , have the same OPEs as the superconformal ghosts (β, γ) in superstring theory. Based on this, we bosonized the fields (U, W) in Sec. III J. Actually, we will need to bosonize the three pairs (Y_1, Z^1) , (Y_2, Z^2) and (U, W) , simultaneously. To ensure that fields from different pairs are commuting (rather than anticommuting) it is more convenient to use a bosonization scheme with no anticommuting fields. Recall that any (β, γ) pair of world-sheet fields with OPE

$$\beta(\beta)\gamma(0) \sim -\frac{1}{\beta}$$

can be bosonized in one of two ways:

$$\begin{aligned} \text{either } \{(\text{i}) \quad \beta &= e^{-\phi+\chi}\partial\chi, \quad \gamma = e^{\phi-\chi}\} \\ \text{or } \{(\text{ii}) \quad \beta &= e^{-\phi+\chi}, \quad \gamma = e^{\phi-\chi}\partial\chi\}, \end{aligned} \quad (\text{A1})$$

where $\chi(\beta)$ and $\phi(\beta)$ are chiral bosons. The current in both cases is

$$\beta\gamma = \partial\phi.$$

(See, for example, Sec. 10.4 of [40].)

At the outset, it seems that for each pair of fields (Y_i, Z^i) or (U, W) , we can choose either the bosonization scheme (i) or (ii). How can we decide which scheme to choose?

Before we pick the bosonization scheme, we have to decide which of the β, γ fields is to have an inverse. If it is γ , we should pick scheme (i), because we could then define

$$\gamma^{-1} := e^{-\phi+\chi} \quad \text{in bosonization scheme (i).}$$

If it is β , we should pick scheme (ii) and set

$$\beta^{-1} := e^{\phi-\chi} \quad \text{in bosonization scheme (ii).}$$

The field γ does not have an inverse in scheme (i), and β does not have an inverse in scheme (ii).

For the application to mini-twistor space we would like Z^1, Z^2 and U to have inverses. The inverse of U was needed in (3.44) for the Spin(7) R-symmetry current. The inverses $(Z^1)^{-1}$ and $(Z^2)^{-1}$ were needed for Ξ [defined in (4.57)] and for all the mass operators that contained Ξ . The inverses of the other fields, $Y_1^{-1}, Y_2^{-1}, W^{-1}$ were never needed.

More basically, if we allow $(Z^i)^{-1}$ ($i = 1, 2$) it means that we are restricting to the patch of mini-twistor space where $Z^i \neq 0$. If we proscribe W^{-1} it means that we are including points where $w = 0$. The discussion above compels us to choose the following bosonization scheme:

$$\begin{aligned} Y_i &= e^{-\phi_i+\chi_i}\partial\chi_i, & Z^i &= e^{\phi_i-\chi_i}, \\ (Z^i)^{-1} &= e^{-\phi_i+\chi_i}, & (i = 1, 2), & & U &= e^{-\phi_3+\chi_3}, \\ W &= e^{\phi_3-\chi_3}\partial\chi_3, & U^{-1} &= e^{\phi_3-\chi_3}, \\ j &= Y_1Z^1 + Y_2Z^2 + 2UW = \partial\phi_1 + \partial\phi_2 + 2\partial\phi_3. \end{aligned} \quad (\text{A2})$$

We conclude with a few useful OPEs:

$$\begin{aligned} U(\beta)U^{-1}(0) &\sim 1 + \beta[\partial\phi(0) + \eta(0)\xi(0)] + O(\beta^2), & U(\beta)W(0) &\sim -\frac{1}{\beta} - \partial\phi(0) + O(\beta), \\ U^{-1}(\beta)W(0) &\sim \frac{1}{\beta}U^{-2}(0) - U^{-2}(0)\partial\phi(0) + O(\beta), & U^{-1}(\beta)U^{-1}(0) &\sim U^{-2}(0) + \frac{1}{2}\beta\partial U^{-2} + O(\beta^2), \\ U^{-1}(\beta)U^{-2}(0) &\sim U^{-3}(0) + O(\beta), & U(\beta)\partial U^{-1}(0) &\sim -\partial\phi(0) - \eta(0)\xi(0) + O(\beta), \\ U(\beta)U^{-2}(\beta) &\sim U(0) - \beta\partial(U^{-1}(0)) + \beta^2X(0) + O(\beta^3), & U^{-1}(\beta)\partial U^{-1}(0) &\sim \frac{1}{2}\partial U^{-2}(0) + O(\beta) \end{aligned}$$

where we defined the operators

$$X := e^{-\phi}(\eta(\partial\xi)\xi - \partial\phi\partial\xi + \frac{1}{2}((\partial\phi)^2 + \partial^2\phi)\xi),$$

and

$$U^{-2} := e^{-2\phi}(\partial\xi)\xi, \quad U^{-3} := \frac{1}{2}e^{-3\phi}\partial^2\xi(\partial\xi)\xi.$$

(All products on the right-hand side are assumed to be normal ordered.) Finally, we need the OPEs

$$U^{-1}(\beta)W^n(0) = \frac{1}{\beta}(-1)^n n! U^{-n-1} + O(1). \quad (\text{A3})$$

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