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Static BPS "monopoles" in all even spacetime dimensions

Eugen Radu[1] and D. H. Tchrakian[2]

[1]*Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, Ireland*
[2]*School of Theoretical Physics–DIAS, 10 Burlington Road, Dublin 4, Ireland*
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Two families of $SO(2n)$ Higgs models in $2n$ dimensional spacetimes are presented. The energies corresponding to the monopole solutions of all these models saturate the Bogomol'nyi bound. One of these families arises from the *dimensional reduction* of higher dimensional Yang-Mills systems while the *generic* models are constructed only with a view to saturating Bogomol'nyi-Prasad-Sommerfield monopole (BPS) bounds. The $n = 2$ member of each family coincides with the usual $SU(2)$ Yang-Mills–Higgs system without Higgs potential. All models support BPS "monopole" solutions. While all the monopoles are BPS, only the "dyons" of the *dimensionally descended* models are also BPS, the electrically charged solutions of the *generic* models not saturating a Bogomol'nyi bound.

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I. INTRODUCTION

Field theoretic solitons find much application in various physical models. Most recent applications of these are in the context of extra dimensional theories, e.g. *large extra dimensions* with or without gravity, when gravity and a negative cosmological constant are included in AdS/CFT correspondence, and also in various theories employing Dp-branes. The special case, in which the soliton in question saturates the BPS bound, is particularly pertinent in this last application, a prominent role being played by the BPS dyons of the Yang-Mills–Higgs (YMH) model. The generalizations of the $SU(2)$ YMH model in 4 spacetime dimensions, to all even dimensions, is the objective of this work. A brief discussion of some of their possible applications will be given in the Summary section. Here, we proceed directly to construct the models and their solutions.

When the dimensionality of the space on which the soliton lives is higher than two, then the gauge fields[1] must necessarily be non-Abelian. Thus in any theory in which the number of extra dimensions is larger than 2, the construction of BPS solitons is a pertinent task. The present work does just this, by constructing BPS "monopoles" of non-Abelian Higgs models in arbitrary even dimensions.

By "monopole" in even dimensional spacetime, we mean a static solution to a YMH which is topologically stable. A BPS such monopole is one for which the Bogomol'nyi (topological) lower on the "energy" is saturated.

We present two distinct families of YMH models in all even spacetime dimensions. The first is a class of models descending from Yang-Mills (YM) systems in higher dimensions. The second, *generic* family, is constructed in an *ad hoc* manner, guided only by the requirement that there be a topological lower on the energy, and that it is saturated. (These features are automatically present in the first class of models as a result of the dimensional descent.) What is remarkable in the case of the first class of models, namely, those descended from higher dimensional YM, is that the corresponding dyons are also BPS, directly generalizing this prominent property present in the familiar case of $d = 4$ spacetime.

Subjecting a YM system in even Euclidean dimensions [1] to dimensional reduction results in a residual YMH system. Depending on the specific features of the dimensional descent, the residual YMH system may[2] [2], or may not[3] [3], inherit the topological lower bound[4] of the higher dimensional YM system. Restricting to the first type of residual YMH systems, namely, those supporting topologically stable solitons, those descending from $4p$ Euclidean even dimensions have the particularly simple property that they are characterized with **only one** dimensionful parameter, which is given by the "radius" of compactification, presenting itself as the Higgs vacuum expectation value (VEV). We will henceforth restrict to this simplest type of YMH models in our considerations of the class of "dimensionally descended" models.

In the present work, we consider spherically symmetric BPS monopole solutions of the two types of YMH models just described. Concrete solutions will be constructed nu-

[1]There are solitons of ungauged models, e.g. sigma models or some higher dimensional generalizations of the Goldstone model, but without introducing a gauge field there are no known systems which support BPS solitons *except* in two dimensions.

[2]This is the case when the higher dimensional space is a product space whose extra dimension consists of one compact coset space K^N of N dimensions, or, is the N dimensional torus $S^1 \times S^1 ... \times S^1$, N times.

[3]This is the case when the extra dimension consists of the product of more than one compact symmetric space, e.g. $K_1^p \times K_2^q$, with $p \geq 2, q \geq 2$, and $p + q = N$, in which case the radius of compactification of one of the two $K_{1,2}$ presents itself as a cosmological constant in the residual model.

[4]Assuming that the YM system in even dimensions is one which is stabilized by the corresponding Chern–Pontryagin density.

merically. For the YMH models descended from higher dimensional YM, the corresponding dyons will also be given. In Sec. II we present the models, subject them to spherical symmetry, and display their topological lower bounds and Bogomol'nyi equations. In Sec. III we give the numerical solutions and in Sec. IV we summarize the results and give a brief discussion.

II. THE BPS MODELS

There are two main families of non-Abelian YMH models which can support self-dual solutions that saturate the Bogomol'nyi bound, those which are descended from a higher dimensional YM system [1], and the *generic* models which are constructed in an *ad hoc* manner. The two classes are given in separate subsections below.

Both families, however, share a common feature in their definitions, namely, the gauge group and its representation as well as the multiplet structure of the Higgs fields are the same and are determined only by the dimensionality $d = 2n$ of the spacetime. For this reason we state these at the outset.

In $d = 2n$ dimensional spacetime, the gauge connection A_μ will take its values in the algebra of $SO(d)$. Since d is even, there are two chiral representations of $SO(d)$ and we will take A_μ to be in one or the other of these chiral representations. The elements of the algebra are represented in terms of the gamma matrices Γ_μ in d dimensions and the corresponding chiral operator Γ_{d+1}

$$\Sigma_{\mu\nu}^{(\pm)} = -\frac{1}{4}\left(\frac{1 \pm \Gamma_{d+1}}{2}\right)[\Gamma_\mu, \Gamma_\nu], \quad \Sigma_{\mu\nu}^{(\pm)} = -\frac{1}{4}\Sigma_{[\mu}^{(\pm)}\Sigma_{\nu]}^{(\pm)}, \tag{1}$$

the spacetime index μ running over $0, i$, with the spacelike index i running over $1, 2, \ldots, (d-1)$. The Higgs field will be taken to consist of a real isovector multiplet ϕ_i which can be expressed in terms of the spin matrices (1) as

$$\Phi = \phi_i \Sigma_{i,d}^{(\pm)}.$$

In the following we will be repeatedly making use of the following spinor identities satisfied by the spin matrices (1) in $d = 2(p+q)$ dimensions

$$\Sigma(2p) = \pm(\star\Sigma(2q))(2p), \tag{2}$$

where the $2p$-form $\star\Sigma(2q)$ is the Hodge dual of the $2q$-form $\Sigma(2p)$, and the $\pm$ sign in (2) corresponds to the sign in (1). The $2p$-form spin matrix in (2) is the p-fold totally antisymmetric product of the spin matrices (1),

$$\Sigma(2p) = \Sigma_{\mu_1\mu_2\ldots\mu_{2p}}$$

and the Hodge dual of $\Sigma(2q)$ is

$$(\star\Sigma(2q))(2p) = \frac{1}{(2q)!}\varepsilon_{\mu_1\mu_2\ldots\mu_{2p}\nu_1\nu_2\ldots\nu_{2q}}\Sigma_{\nu_1\nu_2\ldots\nu_{2q}}.$$

Since the present study is restricted to *static spherically symmetric* solutions, we give the fields (A_μ, Φ) subject to this symmetry

$$A_0 = \eta a_0(r)\hat{x}_i\Sigma_{i,d}, \quad A_i = \left(\frac{1-w(r)}{r}\right)\Sigma_{ij}^{(\pm)}\hat{x}_j, \qquad \Phi = \eta h(r)\hat{x}_i\Sigma_{i,d}, \tag{3}$$

where η is a constant with dimension of inverse length and the three functions (w, a_0, h) of $r = \sqrt{x_i x_i}$, are dimensionless. If the model in question is descended from higher dimensions, then η is the VEV of the Higgs field since in that case the Higgs field has the same dimension as the connection. In the other cases, where the Higgs field has other dimensions, we will be modifying the third member of (3).

It should be pointed out here that according to the static spherically symmetric Ansatz (3), all components of the YM connection take their values in the $SO(d-1)$ subgroup of the *chiral* representations of $SO(d)$. Accordingly, as a consequence of symmetry breaking, the asymptotic gauge connections take their values in $SO(d-2)$. This is most clearly seen in the Dirac gauge, in which there will be a line singularity along the positive or negative x_{d-1}-axis, where the asymptotic Higgs isovector points along the positive or negative x_{d-1}-axis [2].

A. Dimensionally descended models

Higgs models arise from the dimensional reduction of YM models in higher dimensions, and their various features depend on the particular mode of dimensional descent, namely, on the extra compact dimension and the gauge group of the YM system in the higher dimension.

A remarkable feature of YMH systems descended from YM models in $4p$ Euclidean dimensions is that for a subclass of these models the Bogomol'nyi bounds can be saturated. This subclass of models consists of those in $4p-1$ and 2 Euclidean dimensions. The Bogomol'nyi equations of all YMH models in the intermediate residual dimensions, $4p-2$ down to 3 are overdetermined [4] and are satisfied only by trivial solutions, so they are excluded from consideration in this work since we insist on BPS systems. The class of models in 2 Euclidean dimensions are generalized Abelian Higgs models [5], which do not interest us here since we are concerned with higher dimensions and hence necessarily non-Abelian systems. Thus the family of $4p-1$ dimensional YMH models just described will be the main focus of our attention. For completeness however, we will also consider all other YMH models in even dimensional spacetimes (in odd dimensional Euclidean spaces) which do not result from the dimensional reduction of higher dimensional YM systems but were constructed in an *ad hoc* manner in [6]. Recently, the spherically symmetric solution to the Bogomol'nyi equations of the 6 dimensional example of these *generic* models [6], was constructed numerically by Kihara *et al.* [7].

This is the case of primary interest in this paper, namely, the family of models descended from the pth YM system on $\mathrm{IR}_{4p-1}\times S^1$ with action

$$\int_{\mathrm{IR}_{4p-1}\times S^1} S = \int_{\mathrm{IR}_{4p-1}\times S^1} \mathrm{Tr}F(2p)^2 = \int_{\mathrm{IR}_{4p-1}\times S^1} \mathrm{Tr}F^2_{M_1M_2\ldots M_{2p}},$$

$M_1 = 1, 2, \ldots, 4p$, etc., which after integration over the coordinate on S^1 yields the residual Higgs model on IR_{4p-1}

$$\begin{aligned} S_{\mathrm{residual}} &= \mathrm{Tr}[F(2p)^2 + 2p(F(2p-2)\wedge D\Phi)^2] \\ &= \mathrm{Tr}[(F_{m_1m_2\ldots m_{2p}})^2 + 2p(F_{[m_1m_2\ldots m_{2p-2}}D_{i_{2p-1}]}\Phi)^2] \end{aligned} \tag{4}$$

$m_1 = 1, 2, \ldots, (2p-1)$, etc., and the square brackets denoting total antisymmetrization of the indices m. In (4), the $2p$-form curvature

$$F(2p) = F_{m_1m_2\ldots m_{2p}}$$

is the totally antisymmetric p-fold product of the YM curvature 2-form $F(2) = F_{m_1m_2}$. This is identical to the notation used in (2), for the $2p$-form spin matrices.

Since the dimensional descent is by one step only[5] the gauge group of the higher dimensional model is not broken as a result of the imposition of symmetry effecting the descent, so it is the same in the residual theory (4). It is also clear that the Higgs field in (4) has the dimensions of inverse length and η is its VEV.

S_{residual} in (4) is a YMH action density defined on a Euclidean space of odd dimensionality $4p-1$. To generate a theory in $4p$ dimensional (flat) Minkowski space, we introduce the time coordinate x_0 by hand, such that the new coordinates are $x_\mu = (x_0, x_i)$, the spacelike coordinates x_i replacing x_m in (4). The Lagrangian of the said model is now defined as

$$\begin{aligned} \mathcal{L}_{4p} &= \mathrm{Tr}\Big[\frac{1}{2(2p)!}F(2p)^2 - \frac{1}{2(2p-1)!}(F(2p-2) \wedge D\Phi)^2\Big] \\ &= \mathrm{Tr}\Big[\frac{1}{2(2p)!}(F_{\mu_1\mu_2\ldots\mu_{2p}})^2 - \frac{1}{2(2p-1)!} \times (F_{[\mu_1\mu_2\ldots\mu_{2p-2}}D_{\mu_{2p-1}]}\Phi)^2\Big] \end{aligned} \tag{5}$$

the spacetime index μ_1, etc. running over $0, i_1$, etc., with $i_1 = 1, 2, \ldots, 4p-1$, etc. Note that for $p = 1$ (5) is just the Lagrangian of the usual four dimensional YMH model.

[5]If the descent was over a larger number of dimensions, e.g. if S^1 were replaced by S^N, $(N > 1)$, the fixed YM curvature on S^N would result in a symmetry breaking leading to a smaller residual gauge group.

The curvature 2-form $F(2) = F_{\mu\nu}$ and the covariant derivative $D_\mu\Phi$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ D_\mu\Phi &= \partial_\mu\Phi + [A_\mu, \Phi] \end{aligned}$$

both take their values in the algebra of the gauge group, but not the higher order forms appearing in (5). This means that the definition of the model depends not only on the gauge group but also on its representation, given by (1).

1. $A_0 = 0$: *Purely magnetic field*

In this case the static Hamiltonian pertaining to (5) is

$$\begin{aligned} \mathcal{H}_{4p} &= \mathrm{Tr}\Big[\frac{1}{2(2p)!}F(2p)^2 + \frac{1}{2(2p-1)!} \times (F(2p-2)\wedge D\Phi)^2\Big] \\ &= \mathrm{Tr}\Big[\frac{1}{2(2p)!}(F_{i_1i_2\ldots i_{2p}})^2 + \frac{1}{2(2p-1)!} \times (F_{[i_1i_2\ldots i_{2p-2}}D_{i_{2p-1}]}\Phi)^2\Big] \end{aligned} \tag{6}$$

which is bounded from below by

$$\mathcal{H}_{4p} \geq \varepsilon_{i_1i_2\ldots i_{2p-3}i_{2p-2}i_{2p-1}}\mathrm{Tr}F_{i_1i_2}\ldots F_{i_{2p-3}i_{2p-2}}D_{i_{2p-1}}\Phi, \tag{7}$$

the right -hand side of which is a total divergence, by virtue of the Bianchi identities, of the "magnetic field"

$$\begin{aligned} \mathbf{B} &= -\frac{1}{N_d}\mathrm{Tr}\Phi F\wedge F\wedge \ldots \wedge F,\ p \quad \text{times} \\ B_{i_1} &= -\frac{1}{N_d}\varepsilon_{i_1i_2\ldots i_{2p-3}i_{2p-2}i_{2p-1}}\mathrm{Tr}\Phi F_{i_2i_3}\ldots F_{i_{2p-2}i_{2p-1}}, \end{aligned} \tag{8}$$

where N_d is the angular volume in the $d-1$ dimensional Euclidean subspace. The general definitions of magnetic fields in arbitrary dimensions given in [8] include (8).

The inequality (7) is saturated by the Bogomol'nyi equations

$$\begin{aligned} F(2p) &= \pm{}^\star(F(2p-2)\wedge D\Phi)F_{i_1i_2\ldots i_{2p}} \\ &= \pm\varepsilon_{i_1i_2\ldots i_{2p}j_1j_2\ldots j_{2p-1}}F_{j_1j_2}F_{j_3j_4}\ldots F_{j_{2p-3}j_{2p-2}}D_{j_{2p-1}}\Phi. \end{aligned} \tag{9}$$

Subjecting (8) to spherical symmetry according to (3), and making use of the spinor identities (2) (with $q = p$ in the case at hand) results in the reduced Bogomol'nyi equations

$$w' \mp \eta wh = 0, \tag{10}$$

$$\eta r[(1-w^2)^{p-1}h]' \pm \frac{(2p-1)}{r}(1-w^2)^p = 0. \tag{11}$$

The analytic proof of existence of the solution to (10) and (11), for the case $p = 2$, was given in [9].

Subjecting the static Hamiltonian (6) to this symmetry yields the one dimensional residual energy density functional

$$\mathcal{E}_{\rm mon} = (1-w^2)^{2(p-1)}\left[2pw'^2 + (2p-1)\frac{(1-w^2)^2}{r^2}\right] + \eta^2\left[\frac{r^2}{2p-1}([(1-w^2)^{p-1}h]')^2 + 2p(1-w^2)^{2(p-1)}w^2h^2\right]. \qquad (12)$$

It instructive to express (12) as

$$\mathcal{E}_{\rm mon} = 2p(1-w^2)^{2(p-1)}(w' \mp \eta wh)^2 + \left(\frac{\eta r}{2p-1}[(1-w^2)^{p-1}h]' \pm \frac{(1-w^2)^p}{r}\right)^2 \pm 2\eta\frac{d}{dr}[(1-w^2)^{2p-1}h]. \qquad (13)$$

Finally, we state the asymptotic behaviors of the solutions to (10) and (11),

in $r \gg 1$ region

$$[h(r)-1] \sim r^{-1} + o(r^{-3}), \qquad w(r) \sim e^{-\eta r} \qquad (14)$$

in $r \ll 1$ region

$$h(r) = 2pbr^{2p-1} + o(r^{2p+1}), \qquad w(r) = 1 - br^{2p} + o(r^{2p+2}). \qquad (15)$$

2. $A_0 \neq 0$: *Dyon fields*

As in the case of the dyons in $d = 4$ spacetime [10], we substitute the full *static spherically symmetric Ansatz* (3) directly into the Lagrangian (5). The consistency of this Ansatz can be readily checked. The resulting static reduced one dimensional action functional analogous to (12) now is

$$\mathcal{E}_{\rm dyon} = (1-w^2)^{2(p-1)}\left[2pw'^2 + (2p-1)\frac{(1-w^2)^2}{r^2}\right] + \eta^2\left[\frac{r^2}{2p-1}([(1-w^2)^{p-1}h]')^2 + 2p(1-w^2)^{2(p-1)}w^2h^2\right] - \eta^2\left[\frac{r^2}{2p-1}([(1-w^2)^{p-1}a_0]')^2 + 2p(1-w^2)^{2(p-1)}w^2a_0^2\right]. \qquad (16)$$

It now follows from the form of the reduced action of the model in $d = 4p$ spacetime, just as for the familiar special case of $p = 1$ [10], that the following substitution

$$h(r) = f(r)\cosh\gamma, \qquad a_0(r) = f(r)\sinh\gamma, \qquad (17)$$

with a constant parameter γ, renders the action functional (16) identical to the energy functional (12), with $h(r)$ in (12) now replaced by $f(r)$.

Since the second order Euler-Lagrange equations pertaining to (12) are solved by the first order Bogomol'nyi equations (10) and (11), the solution $f_{\rm sol}(r)$ of the latter then yields the self-dual dyon solutions to (16) via the replacements (17).

In the absence of *electric-magnetic duality* in spacetimes of dimensions $d > 4$, in all cases with $p \geq 2$ the electric flux might be defined as the flux of the following 1-form field

$$\mathbf{E} = -\frac{1}{N_d}\,{\rm Tr}A_0F\wedge F\wedge\ldots\wedge F, \quad p \quad \text{times}$$
$$E_{i_1} = -\frac{1}{N_d}\varepsilon_{i_1i_2\ldots i_{2p-3}i_{2p-2}i_{2p-1}}\,{\rm Tr}A_0F_{i_2i_3}\ldots F_{i_{2p-2}i_{2p-1}}, \qquad (18)$$

analogous to (8).

B. The *generic* models

These models are defined in all even spacetime dimensions $d = 2n$, and are characterized by the Lagrangians

$$\mathcal{L}_{2n} = {\rm Tr}\left[\frac{1}{2(2p)!}F(2p)^2 - \frac{1}{2(2q-1)!}\times(F(2q-2)\wedge D\Phi)^2\right] \qquad (19)$$

analogous to (5). Note that for $p = q = 1$, in which case $F(0)\wedge D\Phi \equiv D\Phi$, (5) is just the Lagrangian of the usual YMH model.

Here, in (19), $q \neq p$ unlike in (5). Here, our choices will include all possible q within the range $1 \leq q \leq (p-1)$, with $q = p$ omitted since that reverts to the class of models already discussed in the previous subsection. Unlike in the last class of models however, where p is fixed by $4p = d$, the choice of p in (19) is not restricted in that way. It is nonetheless restricted by two other criteria. The first is the requirement that there be first order Bogomol'nyi equations saturating the lower bound of the static Hamiltonian (with $A_0 = 0$) pertaining to (19)

$$\mathcal{H}_{2n} = {\rm Tr}\left[\frac{1}{2(2p)!}(F_{i_1i_2\ldots i_{2p}})^2 + \frac{1}{2(2q-1)!}\times(F_{[i_1i_2\ldots i_{2q-2}}D_{i_{2q-1}]}\Phi)^2\right], \qquad (20)$$

as a result of which p and q are restricted by $p + q = n$. The second criterion is that the integral of the first term in (20) be convergent, i.e.

$$\int_{{\rm IR}_{d-1}}{\rm Tr}F(2p)^2 \sim \int\frac{r^{d-2}}{r^{4p}}dr = \int\frac{dr}{r^{4p-d+2}}, \qquad r \gg 1$$

will be convergent only if

$$4p \geq d = 2n, \tag{21}$$

the minimum acceptable value of p being given by $4p = d$, and the maximum possible value being dictated by the antisymmetry of $F(2p)$, namely $2p = d$.

Another difference of models (20) from the dimensionally descended models (6) is that the Higgs field does not have the same dimensions as the connection.

Subject to these restrictions, there will numerous $SO(2n)$ Higgs models of the types (20) supporting BPS monopoles in spacetime dimensions $d = 2n$, their numbers increasing with n. Amongst the plethora of such models, we will restrict our attention to a subclass with $p = n - 1$ and $q = 1$, the $n = 2$ case yielding the usual BPS monopole, and the $(p = 2, n = 3)$ monopole in $d = 6$ constructed numerically recently in [7]. The Bogomol'nyi equations for these $(p = n - 1, q = 1)$ models are

$$F_{i_1 i_2 \dots i_{2p}} = \pm \varepsilon_{i_1 i_2 \dots i_{2p} j} D_j \Phi, \tag{22}$$

the $p = 2$ member of which was proposed a long time ago in [6]. To subject (22) to spherical symmetry we employ the Ansatz (3), subject to a modification due to the fact that the Higgs field in this class of models has the dimension of length raised to the power of $2n - 3$. We account for this by making the replacement

$$\eta \longrightarrow \eta^{2n-3}$$

in the third member of (3), leaving the other two terms intact. The resulting one dimensional Bogomol'nyi equations are

$$\frac{(1 - w^2)^{n-2}}{r^{n-2}} w' = \mp \eta^{2(2n-3)} r^{n-2} w h, \tag{23}$$

$$\eta^{2(2n-3)} r^{n-1} h' = \pm \frac{(1 - w^2)^{n-1}}{r^{n-1}}. \tag{24}$$

The reduced energy density functional corresponding to (20) with $(p = n - 1, q = 1)$ is

$$\begin{aligned}\mathcal{E}_{\rm mon} = {} & \frac{(1 - w^2)^{2(n-2)}}{r^{2(n-2)}} \left[2(n-1) w'^2 + \frac{(1 - w^2)^2}{r^2} \right] \\ & + \eta^{2(2n-3)} [r^{2(n-1)} h'^2 + 2(n-1) r^{2(n-2)} w^2 h^2]\end{aligned} \tag{25}$$

which can be rewritten as

$$\begin{aligned}\mathcal{E}_{\rm mon} = {} & 2(n-1) \left[\frac{(1 - w^2)^{n-2} w'}{r^{n-2}} \pm \eta^{2n-3} r^{n-2} w h \right]^2 \\ & + \left[\eta^{2(2n-3)} r^{n-1} h' \mp \frac{(1 - w^2)^{n-1}}{r^{n-1}} \right]^2 \\ & \pm 2 \eta^{2(2n-3)} \frac{d}{dr} [(1 - w^2)^{n-1} h],\end{aligned} \tag{26}$$

confirming (23) and (24).

The solutions to (23) and (24) are the BPS monopoles of the class of models (20) with $(p = n - 1, q = 1)$. Again, there will be dyon solutions with $A_0 \neq 0$, but now these will not be given by the BPS functions (evaluated numerically) via the substitution (17). Rather, they will be solutions to the full second order Euler-Lagrange equations. That these dyons are not BPS can be seen directly by examining the reduced action density functional of the Lagrangian (19), with $(p = n - 1, q = 1)$, subject to the Ansatz (3)

$$\begin{aligned}\mathcal{E}_{\rm dyon} = {} & \frac{(1 - w^2)^{2(n-2)}}{r^{2(n-2)}} \left[2(n-1) w'^2 + \frac{(1 - w^2)^2}{r^2} \right] \\ & + \eta^{2(2n-3)} [r^{2(n-1)} h'^2 + 2(n-1) r^{2(n-2)} w^2 h^2] \\ & - \eta^2 [r^2 ([(1 - w^2)^{p-1} a_0]')^2 + 2(n-1)(2n-3) \\ & \times (1 - w^2)^{2(p-1)} w^2 a_0^2]\end{aligned} \tag{27}$$

which simply does not revert to the form of (25), with f replacing h, under the substitution (17).

The asymptotic behaviors of the solutions to (23) and (24) are

in $r \gg 1$ region

$$\begin{aligned}[h(r) - 1] &\sim r^{-(2n-3)} + o(r^{-(2n-3+2)}), \\ w(r) &\sim e^{-(\eta r)^{2n-3}/(2n-3)}\end{aligned} \tag{28}$$

in $r \ll 1$ region

$$h(r) = (2b)^{n-1} r + o(r^3), \qquad w(r) = 1 - b r^2 + o(r^4). \tag{29}$$

Before proceeding with the numerical construction, we examine briefly those generic models (20) which are not subject to the restriction of $q = 1$. In spacetime $d = 10$ $(n = 5)$ there is only one such model, characterized by $(p = 3, q = 2)$. In $d = 12$ $(n = 6)$ there is again only one such model, characterized by $(p = 4, q = 2)$. In $d = 14$ $(n = 7)$ there are two of these, characterized by $(p = 4, q = 2)$ and $(p = 3, q = 2)$, etc., their numbers increasing with d.

For the rest of this section we will restrict our attention to only the simplest example, namely, that in spacetime $d = 10$ $(n = 5)$ with $(p = 3, q = 2)$. The Higgs field in this example has dimension of length raised to the 3rd power. The static, one dimensional energy density functional, with $a_0 = 0$, is

$$\begin{aligned}\mathcal{E}_{\rm mon} = {} & 3 \frac{(1 - w^2)^4}{r^2} \left[w'^2 + \frac{(1 - w^2)^2}{2} r^2 \right] \\ & + \frac{1}{6} \eta^6 r^2 [r^2 ([(1 - w^2) h]')^2 + 18 (1 - w^2)^2 w^2 h^2]\end{aligned} \tag{30}$$

which can be rewritten as

$$\mathcal{E}_{\text{mon}} = 3(1-w^2)^2\left(\frac{(1-w^2)}{r}w' \pm \eta^3 rwh\right)^2 + \frac{1}{6}\left(\eta^3 r^2[(1-w^2)h]' \mp 3\frac{(1-w^2)^3}{r^2}\right)^2 \pm \eta^3 \frac{d}{dr}[(1-w^2)h]. \tag{31}$$

yielding the Bogomol'nyi equations

$$\frac{(1-w^2)}{r}w' = \mp \eta^3 rwh, \tag{32}$$

$$\eta^3 r^2[(1-w^2)h]' = \pm 3\frac{(1-w^2)^3}{r^2}. \tag{33}$$

The asymptotic behaviors of the solutions of (32) and (33) are

in $r \gg 1$ region

$$[h(r) - 1] \sim r^{-3} + o(r^{-5}), \quad w(r) \sim e^{-(\eta r)^3/3} \tag{34}$$

in $r \ll 1$ region

$$h(r) = 4b^2 r + o(r^3), \qquad w(r) = 1 - br^2 + o(r^4). \tag{35}$$

III. NUMERICAL RESULTS

All the Bogomol'nyi equations, (10), (11), (23), (24), (32), and (33) can be expressed as coupled first order ordinary differential equations (ODEs) in the dimensionless variable $\rho = \eta r$.

Both the *dimensionally descended* and the *generic* models have solutions with the correct asymptotics only when the second derivative of the gauge function w evaluated at the origin, $w''(0) \sim b$, takes on a certain value, which is dimension and model dependent. For example for the *dimensionally descended* models we have $b(p=1) = 1/6$, $b(p=2) \simeq 0.055\,209\,6$ and $b(p=4) \simeq 0.017\,615\,4$. For the solutions of *generic* models we find $b(d=6) \simeq 0.722\,803\,9$, $b(d=10) \simeq 0.740\,092\,9$ and $b(d=12) \simeq 0.764\,016\,3$. The corresponding value for the only *hybrid ad hoc* model studied numerically is $b \simeq 0.459\,399\,4$.

In both classes of systems, the models in spacetime dimension $d = 4$ coincide and are identical to the usual $SU(2)$ YMH model in the BPS limit. The Bogomol'nyi equations for this case are the only ones which can be integrated analytically in closed form [11].[6] For all $d \geq 6$, the solutions to these first order equations are constructed numerically. We follow the usual approach and, by using a standard ordinary differential equation solver, we evaluate the initial conditions at $r = 10^{-6}$ for global tolerance 10^{-14}, adjusting for fixed shooting parameter and integrating towards $r \to \infty$.

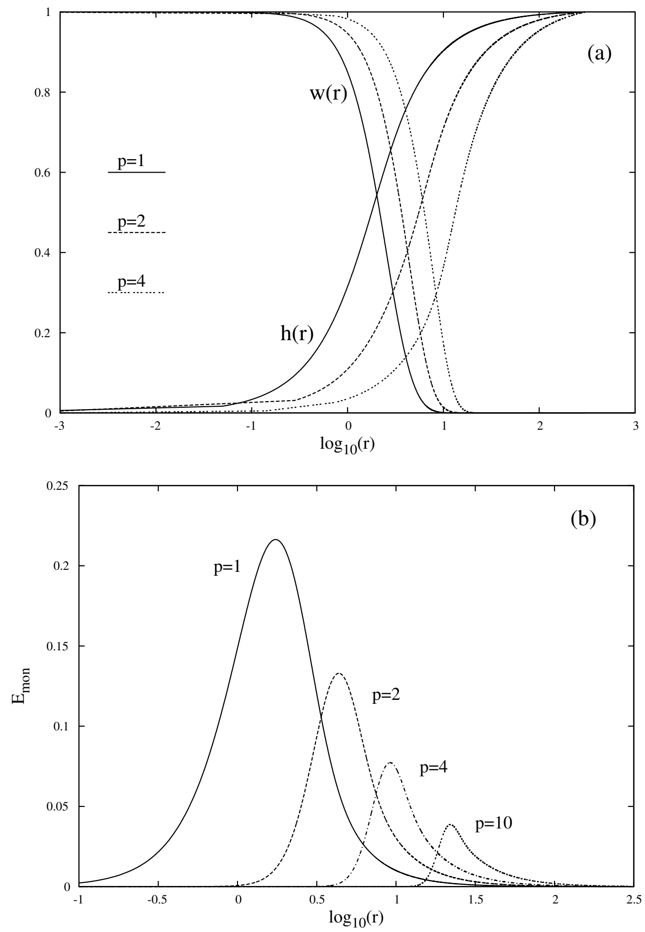

FIG. 1. The profiles of the functions $w(r)$, $h(r)$ and the energy densities $\mathcal{E}_{\text{mon}}$ are shown for several *dimensionally descended* models.

In Figs. 1(a) and 1(b) we present, respectively, the profiles of the functions $w(r)$ and $h(r)$ corresponding to the BPS solutions of Eqs. (10) and (11) of the *dimensionally descended* models, and their energy densities, for several values of p. The same profiles for the BPS equations (23) and (24) and energy densities are plotted in Fig. 2(a) and 2(b) for the *generic* models. For completeness we present in Fig. 3 the profile of the solutions to Eqs. (32) and (33) and its energy density for the ten dimensional monopole of the *hybrid ad hoc* model with $(p = 3, q = 2)$.

The qualitative properties of these solutions are the same as for the well-known $p = 1$, $d = 4$ BPS configurations [11]. The profiles of the functions $w(r)$ and $h(r)$ do not change appreciably for the solutions living in different spacetime dimension. The gauge and Higgs functions interpolate between the asymptotic values, presenting no

[6]The appearance of these nonrational values of b in the expansion at the origin suggests that analytically evaluated solutions of higher p monopoles, if they exist, should be parametrized by a set of functions different from (w, h).

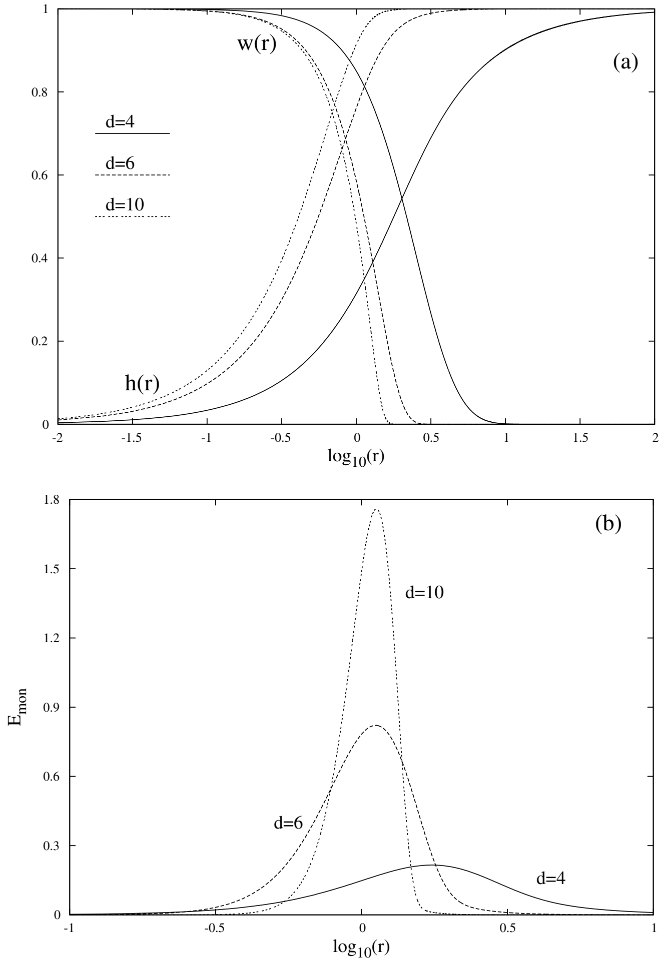

FIG. 2. The profiles of the functions $w(r)$, $h(r)$ and the energy densities $\mathcal{E}_{\rm mon}$ are shown for several *generic* models.

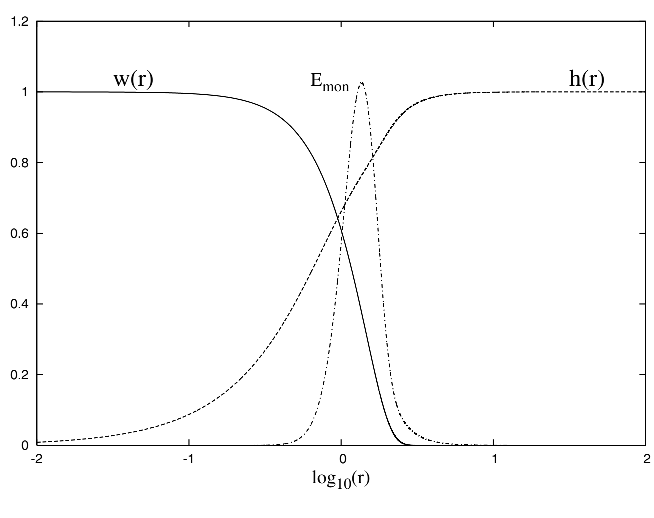

FIG. 3. The profile of the functions $w(r)$, $h(r)$ and the energy density $\mathcal{E}_{\rm mon}$ are shown for the ten dimensional *hybrid ad hoc* model.

local extrema. The energies of these solutions are always concentrated in a small region.

It turns out that in the case of the *dimensionally descended* models, both the profiles of w and h as well as the peaks of the energy densities move out from the origin with increasing dimension $d = 4p$. Also, the heights of those energy density peaks decrease with increasing p, while the areas giving the total energies remain the same, with *unit* normalization due to spherical symmetry.

In the case of the sequence of *generic* models by contrast, the profiles of w and h move in towards the origin with increasing $d = 2n$. The corresponding heights of the energy density peaks increase with increasing n, again with *unit* area unchanged, and the positions of these peaks seem to depend very weakly on n.

IV. SUMMARY AND COMMENTS

We have constructed topologically stable finite energy static solutions to two families of Yang-Mills–Higgs models in all $d = 2n$, even, spacetime dimensions, subject to spherical symmetry in the appropriate dimension.

One of these families is arrived at via the dimensional reduction of the pth member of the Yang-Mills hierarchy on the Euclidean space $d = (4p - 1) \times S^1$ down to $4p - 1$ Euclidean dimensions, whose solutions then appear as the static solutions of the corresponding theory in $4p$ spacetime dimensions. These configurations are solutions of the appropriate family of Bogomol'nyi equations, (9) or (10) and (11), and hence saturate the respective topological lower bounds.

It may be interesting to note a particular feature of the Bogomol'nyi equations (10) and (11): For $p = 1$, the solutions are found [11] in closed form, while for $p = 2$ an analytic proof for the existence of the solution was given in [9]. (This proof [9] can be adapted to all p.) Substituting (10) into (11) eliminates the function w and yields a second order equation for h.

For $p = 1$, this equation corresponds to the Liouville equation resulting from the self-duality equations of 4 dimensional $SU(2)$ YM subject to axial symmetry [12]. Similarly, for $p \geq 2$, the second order equation in h corresponds to the generalization of the Liouville equation resulting from the self-duality equations of $4p$ dimensional $SO_{\pm}(4p)$ YM subject to axial symmetry, presented in [13]. The analytic proof of existence to the latter was given in [14].

The other class of models examined is arrived at in an *generic* manner, the only criterion being that the energies of the static solutions saturate the topological (monopole) lower bounds by the appropriate Bogomol'nyi equations, namely (22) or (23) and (24) and (32) and (33). Typically, the Higgs fields in these models have dimensions different from the inverse of a length.

The first member of both classes of models proposed, namely, those in $d = 4$ dimensional spacetime, coincide with the usual YMH model in the Prasad–Sommerfield (PS) limit whose solutions are known in closed form [11]. The solutions of all the other models in higher spacetime dimensions cannot be constructed in closed form and are evaluated numerically.

What is markedly more interesting about the first, dimensionally descended class of models, is, that like the usual YMH model in the PS limit they also admit dyon solutions obeying first order equations analogous to the Julia-Zee dyons [10] in the PS limit. Of course in spacetime $d > 4$ the corresponding electric field, which we have defined by (18), is not dual to the magnetic field (8), but is nonetheless there as a consequence of solutions with nonvanishing electric potential A_0. This family of solutions generalizing the Julia-Zee dyons obeying first order equations is a feature only of the dimensionally descended models introduced in Sec. II A, and not of the various *generic* models discussed in Sec. II B. In the latter case, there are of course solutions with nonvanishing A_0, but these are subject to the second order Euler–Lagrange equations rather than first order equations.

We conclude by making some brief remarks concerning the potential applicability of the higher dimensional monopoles that we have presented above. For a start, it is always of interest to see how the dimensionality of spacetime affects the physical consequences of a given theory. Also, these types of objects might form in the early Universe when the present three spatial dimensions were not yet separated from others, and a greater number of dimensions were equally important.

The most immediate application, technically, is to proceed to the gravitating case, thus enabling the study of the properties of gravitating monopoles in higher dimensions, with reference to the detailed studies [15,16] in $d = 4$ spacetime.

Another direction to be explored straightforwardly is the construction of the axially symmetric multimonopoles and dyons of the dimensionally descended models. Since all our solutions obey Bogomol'nyi equations, all these solutions are guaranteed to be topologically stable. Such dyonic solutions may be of special interest in light of the lack of electric-magnetic duality in higher dimensions.

Perhaps the most stringent test of our models is their status *vis à vis* supersymmetry. Our Bogomol'nyi equations are first order, but they are not linear in the curvature field strength (except in $d = 4$ spacetime). This contrasts with the BPS equations in 6 and 8 Euclidean dimensions, (not involving Higgs fields) employed in [17]. These are linear in the YM curvature, unlike ours. On the other hand the energies of our models are finite and bounded from below by topological charges. Should a way be found to make our models respect supersymmetry, then they could be candidates for the construction of field theory Supertubes [18], where Higgs fields feature.

ACKNOWLEDGMENTS

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