PHYSICAL REVIEW D **71,** 125007 (2005)

$\mathcal{N} = 4$ "fake" supergravity

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(Received 24 February 2005; published 14 June 2005)

We study curved and flat Bogomol'nyi-Prasad-Sommerfield (BPS)-domain walls in 5D, $\mathcal{N} = 4$ gauged supergravity and show that their effective dynamics along the flow is described by a generalized form of "fake supergravity." This generalizes previous work in $\mathcal{N} = 2$ supergravity and might hint towards a universal behavior of gauged supergravity theories in supersymmetric domain wall backgrounds. We show that BPS-domain walls in 5D, $\mathcal{N} = 4$ supergravity can never be curved if they are supported by the supergravity scalar only. Furthermore, a purely Abelian gauge group or a purely semisimple gauge group can never lead to a curved domain wall, and the flat walls for these gaugings always exhibit a runaway behavior.

DOI: 10.1103/PhysRevD.71.125007 PACS numbers: 04.65.+e

I. INTRODUCTION

Domain wall solutions of $(d + 1)$-dimensional supergravity theories have received a lot of attention during the past few years. This interest was largely driven by applications in the context of holographic renormalization group flows and certain brane world models. In most of these applications, the domain walls of interest preserve a fraction of the original supersymmetry of the supergravity theory they are embedded in. A domain wall of this type can be either Minkowski-sliced or anti-de Sitter (AdS)-sliced,

$$ds^2 = e^{2U(r)} g_{mn}(x) dx^m dx^n + dr^2, \qquad (1.1)$$

depending on whether, respectively, g_{mn} is the metric of d-dimensional Minkowski or anti-de Sitter space. A nontrivial warp factor $U(r)$ [i.e., one that does not give rise to $(d + 1)$-dimensional Minkowski or anti-de Sitter space] requires a nontrivial scalar profile $\phi^x(r)$ $(x = 1, \ldots, m)$, as dictated by the Einstein equations. A domain wall thus defines a curve $\phi^x(r)$ on the scalar manifold.

The allowed scalar manifolds in supergravity theories are in general highly constrained and strongly depend upon the space-time dimension, the amount of supersymmetry, as well as on the type of multiplet the scalars are sitting in. The geometrical constraints on the scalar manifolds also leave their trace in the Bogomol'nyi-Prasad-Sommerfield (BPS) equations of the scalar fields, which are likewise highly space-time-, supersymmetry- and multiplet-dependent.

It came therefore as quite a surprise when it was found in [1] that one can reformulate the BPS conditions for domain walls in 5D, $\mathcal{N} = 2$ supergravity in such a way that their naive strong multiplet dependence effectively disappears. The same is true for the scalar potential, which, in this simplified reformulation, also contains the scalar fields from vector and hypermultiplets in a symmetric way. In order to achieve this simplification, one has to restrict one's attention to the effective dynamics *along the curve* $\phi^x(r)$ of a given BPS-domain wall and properly "integrate out" the orthogonal scalar fields. Interestingly, this also exactly reproduces the equations of "fake supergravity" that were introduced in Ref. [2] to prove the stability of domain walls in certain scalar/gravity theories that, despite some superficial similarities, are not necessarily supersymmetric.[1] The fake supergravity formalism in [2] was tailor-made to describe curved domain walls and generalizes and refines the earlier work [4–6]. It was worked out in [2] in detail for theories with only one scalar field, and it is this scalar field that one has to identify with the scalar direction along the flow curve $\phi^x(r)$ in 5D, $\mathcal{N} = 2$ supergravity. The fake supergravity equations were also generalized to several scalar fields in [2], but only for a very particular type of scalar potential. One of the lessons of [1], however, is that a generalization and covariantization to more than one scalar field can go along various different lines, and it seems that only the effective one-scalar field formulation is universal.

The results of [1] are by no means of only formal interest. On the contrary, it was found that the simplified reformulation of true supergravity à la fake supergravity provides a very handy tool for studying true BPS-domain walls themselves. For example, using the simplified language of "fake" supergravity, it is fairly easy to prove that BPS-domain walls that are only supported by scalars from vector multiplets can at most be *Minkowski-sliced*. An AdS-sliced BPS-domain wall thus must involve nontrivial hypermultiplet scalars. This fact had gone unnoticed before.

In this paper, we will go one step beyond the work of [1] and study domain walls in 5D, $\mathcal{N} = 4$ supergravity along

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[1]Some related work appeared in [3].

similar lines. That is, we will try to similarly recast the BPS equations and the scalar potential in a generalized fake supergravity form. This generalization is highly nontrivial due to the following reasons:

(i) The BPS constraints are stronger, as there are now twice as many supersymmetries to preserve.

(ii) The $\mathcal{N} = 4$ theory is $Usp(4) \cong SO(5)$ instead of $Usp(2) \cong SU(2)$ covariant, i.e., several peculiarities of the group $SU(2)$ no longer hold.

(iii) The scalar manifolds in the $\mathcal{N} = 4$ theory are of the type $SO(1,1) \times SO(5,n)/(SO(5) \times SO(n))$, which are, in general, neither very special nor quaternionic manifolds. Contrary to what happens in rigid supersymmetry, the $\mathcal{N} = 4$ theory can therefore not be viewed as a special case of the $\mathcal{N} = 2$ theory.

Given these differences, it is all the more intriguing that only few features of the $\mathcal{N} = 2$ formulation are identified as $SU(2)$ artifacts and that one finds an exactly analogous picture: The effective BPS equations and the scalar potential can again be brought to a simple, generalized fake supergravity-type form, no matter whether the running scalar field sits in the $\mathcal{N} = 4$ supergravity multiplet or in an $\mathcal{N} = 4$ vector or tensor multiplet. Just as in the $\mathcal{N} = 2$ analogue [1], we can also use this simplified language in order to study the domain walls themselves. It is found that BPS-domain walls that are supported by the supergravity scalar only are necessarily flat. Similarly, if the gauge group is purely Abelian or purely semisimple, the domain wall can at most be flat, no matter by which type of scalar field they are supported. Any flat domain wall for these gaugings, however, has a runaway behavior. These results could prove very useful for studies of holographic renormalization group flows [7] in the setup of, e.g., [8] or for domain walls in gauged supergravities that derive from flux compactifications (see e.g., [9–11] for some recent work in this direction). Moreover, the present work suggests that the language of fake supergravity is far more universal than previously thought and that it might well be applicable to a much wider range of gauged supergravity theories, perhaps, if properly formulated, even to all of them. Fake supergravity might thus turn out to be not that fake after all.

The organization of this paper is as follows: In Sec. II, we briefly recapitulate the structure of BPS-domain walls in 5D, $\mathcal{N} = 2$ gauged supergravity and the relation to the fake supergravity formalism developed in [2]. In Sec. III, we then discuss the structure of 5D, $\mathcal{N} = 4$ gauged and ungauged supergravity and study its 1/2-supersymmetric domain wall solutions. This is done by rewriting the BPS equations and the scalar potential in a generalized, "$\mathcal{N} = 4$" fake supergravity form. In this simplified version, several general statements about possible BPS-domain walls are easily derived. We end with some conclusions in Sec. IV. The appendix proves the equivalence of two flatness conditions.

II. TRUE AND FAKE 5D, $\mathcal{N} = 2$ SUPERGRAVITY

In this section, we briefly summarize the key results of [1] on BPS-domain walls in true and fake, 5D, $\mathcal{N} = 2$ supergravity. For earlier work on (smooth) flat and curved BPS-domain walls in these theories, see [12–22] and [23–27], respectively.

A. 5D, $\mathcal{N} = 2$ gauged supergravity

Five-dimensional, $\mathcal{N} = 2$ supergravity can be coupled to vector-, tensor- and hypermultiplets. The precise form of these theories was derived in the original references [28–34], to which we refer the reader for further details. As was emphasized already in [17,35], all the terms in the theory that are due to the presence of tensor multiplets have to vanish on a BPS-domain wall background, and we can thus restrict ourselves to the coupling of n_V vector multiplets and n_H hypermultiplets to supergravity.

The bosonic field content of such a theory consists of the fünfbein e_μ^m, $(n_V + 1)$ vector fields A_μ^I $(I = 0, 1, \ldots, n_V)$ and $(n_V + n_H)$ real scalar fields (φ^x, q^X), with $x = 1, \ldots, n_V$ and $X = 1, \ldots, 4n_H$. Here, we have combined the graviphoton of the supergravity multiplet with the n_V vector fields of the n_V vector multiplets to form a single $(n_V + 1)$-plet A_μ^I.

The n_V scalar fields φ^x of the vector multiplets parametrize a "very special" real manifold $\mathcal{M}_{\rm VS}$, i.e., an n_V-dimensional hypersurface of an auxiliary $(n_V + 1)$-dimensional space spanned by coordinates h^I $(I = 0, 1, \ldots, n_V)$:

$$\mathcal{M}_{\rm VS} = \{h^I \in \mathbb{R}^{(n_V+1)}: C_{IJK} h^I h^J h^K = 1\}, \qquad (2.1)$$

where the constants C_{IJK} appear in a Chern-Simons-type coupling of the Lagrangian. On $\mathcal{M}_{\rm VS}$, the h^I become functions of the n_V physical scalar fields, φ^x. The metric, g_{xy}, on the very special manifold is determined via

$$g_{xy} = -3C_{IJK}(\partial_x h^I)(\partial_y h^J)h^K. \qquad (2.2)$$

The scalars q^X $(X = 1, \ldots, 4n_H)$ of n_H hypermultiplets, on the other hand, take their values in a quaternionic-Kähler manifold $\mathcal{M}_{\rm Q}$ [36], i.e., a manifold of real dimension $4n_H$ with holonomy group contained in $SU(2) \times USp(2n_H)$. The vielbein on this manifold is denoted by f_X^{iA}, where $i = 1, 2$, and $A = 1, \ldots, 2n_H$ refer to an adapted $SU(2) \times USp(2n_H)$ decomposition of the tangent space. The hypercomplex structure is (-2) times the curvature of the $SU(2)$ part of the holonomy group, denoted as $\mathcal{R}^{rZX}$ $(r = 1, 2, 3)$, so that the quaternionic identity reads

$$\mathcal{R}^{\ r}_{XY}\mathcal{R}^{sYZ} = -\tfrac{1}{4}\delta^{rs}\delta_X^{\ Z} - \tfrac{1}{2}\varepsilon^{rst}\mathcal{R}^{t\ Z}_{X}. \qquad (2.3)$$

The vector fields A_μ^I can be used to gauge up to $(n_V + 1)$

isometries of the quaternionic manifold $\mathcal{M}_Q$ (provided such isometries exist).[2]

The quaternionic Killing vectors, $K_I^X(q)$, that generate these isometries on $\mathcal{M}_Q$ can be expressed in terms of the derivatives of $SU(2)$ triplets of Killing prepotentials (or "moment maps") $P_I^r(q)$ $(r = 1, 2, 3)$ via

$$D_X P_I^r = \mathcal{R}^r_{XY} K_I^Y, \quad \Leftrightarrow \quad \begin{cases} K_I^Y = -\frac{4}{3}\mathcal{R}^{rYX} D_X P_I^r \\ D_X P_I^r = -\varepsilon^{rst}\mathcal{R}^s_{XY} D^Y P_I^t, \end{cases} \tag{2.4}$$

where D_X denotes the $SU(2)$ covariant derivative, which contains the $SU(2)$ connection ω_X^r with curvature $\mathcal{R}^r_{XY}$:

$$\begin{aligned} D_X P^r &= \partial_X P^r + 2\varepsilon^{rst}\omega_X^s P^t, \\ \mathcal{R}^r_{XY} &= 2\partial_{[X}\omega^r_{Y]} + 2\varepsilon^{rst}\omega_X^s\omega_Y^t. \end{aligned} \tag{2.5}$$

The prepotentials have to satisfy the constraint

$$\tfrac{1}{2}\mathcal{R}^r_{XY} K_I^X K_J^Y - \varepsilon^{rst} P_I^s P_J^t + \tfrac{1}{2} f_{IJ}{}^K P_K^r = 0, \tag{2.6}$$

where $f_{IJ}{}^K$ are the structure constants of the gauge group. In this section, we will frequently switch between the above vector notation for $\mathfrak{su}(2)$-valued quantities such as P_I^r, and the usual (2×2) matrix notation,

$$\mathbf{P}_I = (P_{Ii}{}^j), \qquad P_{Ii}{}^j \equiv i\sigma_{ri}{}^j P_I^r, \tag{2.7}$$

where boldface expressions such as $\mathbf{P}_I$ refer to the (2×2) matrices with the indices i, j suppressed. Turning on only the metric and the scalars, the Lagrangian of such a gauged supergravity theory is

$$\begin{aligned} e^{-1}\mathcal{L} = &-\tfrac{1}{2}R - \tfrac{1}{2}g_{xy}\partial_\mu\varphi^x\partial^\mu\varphi^y - \tfrac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y \\ &- g^2\mathcal{V}(\varphi, q), \end{aligned} \tag{2.8}$$

whereas the supersymmetry transformation laws of the fermions are given by

$$\delta\psi_{\mu i} = \nabla_\mu\epsilon_i - \omega_{\mu i}{}^j\epsilon_j - \frac{i}{\sqrt{6}} g\gamma_\mu P_i{}^j\epsilon_j, \tag{2.9}$$

$$\delta\lambda_i^x = -\frac{i}{2}\gamma^\mu(\partial_\mu\varphi^x)\epsilon_i - gP_i{}^{jx}\epsilon_j, \tag{2.10}$$

$$\delta\zeta^A = \frac{i}{2} f_X^{iA}\gamma^\mu(\partial_\mu q^X)\epsilon_i - g\mathcal{N}^{iA}\epsilon_i. \tag{2.11}$$

Here, ψ^i_μ, λ^x_i, ζ^A are the gravitini, gaugini and hyperini, respectively, g denotes the gauge coupling, the $SU(2)$ connection $\boldsymbol{\omega}_\mu$ is defined as $\omega_{\mu i}{}^j = (\partial_\mu q^X)\omega_{Xi}{}^j$, and

$$P^r = h^I(\varphi) P_I^r(q), \tag{2.12}$$

$$P^{rx} = -\sqrt{\tfrac{3}{2}} g^{xy}\partial_y P^r \tag{2.13}$$

$$\mathcal{N}^{iA} = \frac{\sqrt{6}}{4} f_X^{iA}(q) h^I(\varphi) K_I^X(q). \tag{2.14}$$

As usual, the potential is given by the sum of "squares of the fermionic shifts" (the scalar expressions in the above transformations of the fermions):

$$\mathcal{V} = -4P^r P^r + 2P^{xr}P^{yr}g_{xy} + 2\mathcal{N}^{iA}\mathcal{N}^{jB}\varepsilon_{ij}C_{AB}, \tag{2.15}$$

where C_{AB} is the (antisymmetric) symplectic metric of $USp(2n_H)$. Using (2.4) and the quaternionic identity (2.3), the scalar potential for vector and hypermultiplets can be written in the form

$$\mathcal{V}\,\mathbb{1}_2 = 4\mathbf{P}^2 - 3(\partial_x\mathbf{P})(\partial^x\mathbf{P}) - (D_X\mathbf{P})(D^X\mathbf{P}). \tag{2.16}$$

One clearly sees that the scalars of the vector- and hypermultiplets enter the supersymmetry transformations and the scalar potential in a rather different way.

B. Curved and flat BPS-domain walls

In this paper, we are interested in Minkowski-sliced ("flat") and AdS-sliced ("curved") domain walls of the form

$$ds^2 = e^{2U(r)} g_{mn}(x) dx^m dx^n + dr^2 \tag{2.17}$$

with $g_{mn}(x)$ being either the 4D Minkowski metric or a metric of AdS_4 with curvature scale L_4. In a curved domain wall background of the form (2.17), when the scalar fields only depend on the radial coordinate r, the vanishing of the supersymmetry variations (2.9), (2.10), and (2.11) implies

$$\left[\nabla_m^{\mathrm{AdS}_4} + \gamma_m\left(\frac{1}{2}U'\gamma_5 - \frac{ig}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon = 0, \tag{2.18}$$

$$\left[D_r + \gamma_5\left(-\frac{ig}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon = 0, \tag{2.19}$$

$$[\gamma_5\varphi^{x\prime} + ig\sqrt{6}g^{xy}\partial_y\mathbf{P}]\epsilon = 0, \tag{2.20}$$

$$f_X^{iA}[\gamma_5 q^{X\prime} - ig\sqrt{\tfrac{8}{3}}R^{rXY}D_Y P^r]\epsilon_i = 0, \tag{2.21}$$

where

$$D_r\epsilon_i \equiv \partial_r\epsilon_i - q^{X\prime}\omega_{Xi}{}^j\epsilon_j \tag{2.22}$$

has been introduced. The gaugino variation suggests a spinor projector of the form[3]

$$\epsilon_i = -\gamma_5\Theta_i{}^j\epsilon_j \Leftrightarrow (\mathbb{1}_2 + \gamma_5\Theta)\epsilon = 0, \tag{2.23}$$

[2] A non-Abelian gauge group also has to leave the C_{IJK} invariant, which implies that the gauge group also has to be a subgroup of the isometry group of $\mathcal{M}_{VS}$ [29,31,34,37].

[3] As it turns out to be more convenient for the $\mathcal{N} = 4$ case, our Θ differs by a factor i from the one used in [1]: $\Theta^{\mathrm{here}} = i\Theta^{\mathrm{there}}$.

where $\Theta^2 = \mathbb{1}_2 \Leftrightarrow \Theta^r\Theta^r = -1$. Using this projector, the gaugino and hyperino BPS conditions can be brought to the following form [1]:

$$ig_{yx}\varphi^{x\prime}\Theta + \sqrt{6}g\partial_y\mathbf{P} = 0, \tag{2.24}$$

$$ig_{YX}q^{X\prime}\Theta + iq^{X\prime}[\mathbf{R}_{YX}, \Theta] + \sqrt{6}gD_Y\mathbf{P} = 0. \tag{2.25}$$

The hyperino BPS equation (2.25) can be written in the equivalent form

$$\sqrt{6}gK_Y + 2iq^{X\prime}\{\mathbf{R}_{YX}, \Theta\} = 0 \tag{2.26}$$

by contracting (2.25) with the $SU(2)$ curvature.

Contracting now (2.24) and (2.25) with, respectively, $\varphi^{y\prime}$ and $q^{Y\prime}$, one can solve for the projector Θ [23,25,26]:

$$\Theta = ig\sqrt{6}\frac{\varphi^{x\prime}\partial_x\mathbf{P}}{\varphi^{y\prime}\varphi^{z\prime}g_{yz}} \tag{2.27}$$

$$\Theta = ig\sqrt{6}\frac{q^{X\prime}D_X\mathbf{P}}{q^{Y\prime}q^{Z\prime}g_{YZ}}. \tag{2.28}$$

When both vector multiplet scalars and hyperscalars are nontrivial, consistency of (2.27) and (2.28) obviously requires

$$\frac{q^{X\prime}D_X\mathbf{P}}{q^{Y\prime}q^{Z\prime}g_{YZ}} = \frac{\varphi^{x\prime}\partial_x\mathbf{P}}{\varphi^{y\prime}\varphi^{z\prime}g_{yz}}. \tag{2.29}$$

Squaring (2.27) and (2.28) finally yields the equations of motion for the scalar fields,

$$\varphi^{x\prime}\varphi^{y\prime}g_{xy} = \pm g\sqrt{6}\sqrt{-(\varphi^{x\prime}\partial_x\mathbf{P})^2} \tag{2.30}$$

$$q^{X\prime}q^{Y\prime}g_{XY} = \pm g\sqrt{6}\sqrt{-(q^{X\prime}D_X\mathbf{P})^2}. \tag{2.31}$$

As for the warp factor $U(r)$, a first order equation can be obtained from the integrability condition of (2.18), which yields

$$(U')^2 = -\frac{e^{-2U}}{L_4^2} - \frac{2}{3}g^2\mathbf{P}^2. \tag{2.32}$$

However, the compatibility condition of (2.18) and (2.23) also implies a first order equation for $U(r)$:

$$U' = -\frac{ig}{\sqrt{6}}\{\Theta, \mathbf{P}\}. \tag{2.33}$$

Consistency of (2.32) and (2.33) then implies an algebraic equation for the warp factor:

$$\frac{6e^{-2U}}{g^2L_4^2}\mathbb{1}_2 = \{\Theta, \mathbf{P}\}^2 - 4\mathbf{P}^2. \tag{2.34}$$

This is an important equation, because it tells us that the domain wall is flat (corresponding to $L_4 \to \infty$) if and only if $\mathbf{P}$ and Θ are proportional to one another, $\mathbf{P} = c\Theta$.

There is yet one other important consistency condition, which follows from the compatibility of (2.19) and (2.23). It reads

$$[\Theta, D_r\Theta + i\sqrt{\tfrac{2}{3}}g\mathbf{P}] = 0. \tag{2.35}$$

Since Eqs. (2.27) and (2.28) imply that Θ is proportional to $D_r\mathbf{P}$:

$$D_r\mathbf{P} \equiv \varphi^{\prime x}\partial_x\mathbf{P} + q^{\prime X}D_X\mathbf{P} = -\frac{i}{\sqrt{6}g}g_{\Lambda\Sigma}\phi^{\Lambda\prime}\phi^{\Sigma\prime}\Theta, \tag{2.36}$$

where $\phi^\Lambda = \{\varphi^x, q^X\}$, the consistency condition (2.35) can be rewritten in the form

$$[D_r\mathbf{P}, D_rD_r\mathbf{P} + \tfrac{1}{3}g_{\Lambda\Sigma}\phi^{\Lambda\prime}\phi^{\Sigma\prime}\mathbf{P}] = 0. \tag{2.37}$$

Obviously, (2.37) is a constraint on the possible field dependence of $\mathbf{P}$ on a supersymmetric domain wall solution. As was shown in [1], this constraint is only partially compatible with the geometric constraints from very special geometry. More precisely, if the domain wall is supported only by vector multiplet scalars, (2.37) can only be satisfied if Θ and $\mathbf{P}$ are proportional to one another. But, according to (2.34), this means that the domain wall then has to be *flat*. Thus, any BPS-domain wall that is supported by vector multiplet scalars only has to be flat, and curved domain walls require nontrivial hyperscalar profiles [1].

C. The relation to ($\mathcal{N} = 2$) fake supergravity

The BPS-domain wall solutions reviewed in the previous subsection are classically stable solutions of the underlying gauged supergravity theories. This follows from standard arguments based on the existence of Killing spinors and the first order form of the field equations along the lines of [38,39]. In [4–6], these stability arguments were formalized and generalized to flat domain wall solutions of a broader class of theories which, while having some superficial similarities with true supergravity theories, do not necessarily have to be supersymmetric and can live in any space-time dimension $D = (d + 1)$. In Ref. [2], such theories were dubbed fake supergravity theories, and the formalism was further generalized and refined to include also curved domain walls. More precisely, the theories studied in Ref. [2] are gravitational theories with a single scalar field ϕ and an action

$$S = \int d^{d+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big], \tag{2.38}$$

with a scalar potential $V(\phi)$ given by

$$V(\phi) = \frac{2(d-1)^2}{\kappa^2}\Big(\frac{1}{2}\,\mathrm{Tr}\Big)\Big[\frac{1}{\kappa^2}(\partial_\phi\mathbf{W})^2 - \frac{d}{d-1}\mathbf{W}^2\Big]. \tag{2.39}$$

Here, $\mathbf{W}(\phi)$ is an $\mathfrak{su}(2)$-valued (2×2) matrix, which implies that quadratic expressions such as $\mathbf{W}^2$, $(\partial_\phi \mathbf{W})^2$ or $\{\mathbf{W}, \partial_\phi \mathbf{W}\}$ are proportional to the unit matrix. This allows one to write the potential in an equivalent form without explicitly taking the trace:

$$V(\phi)\mathbb{1}_2 = \frac{2(d-1)^2}{\kappa^2}\left[\frac{1}{\kappa^2}(\partial_\phi \mathbf{W})^2 - \frac{d}{d-1}\mathbf{W}^2\right]. \quad (2.40)$$

The matrix $\mathbf{W}$ also enters some fake Killing spinor equations for an $SU(2)$-doublet spinor ϵ,

$$[\nabla_m^{\mathrm{AdS}_d} + \gamma_m(\tfrac{1}{2}U'\gamma_5 + \mathbf{W})]\epsilon = 0, \quad (2.41)$$

$$[\partial_r + \gamma_5 \mathbf{W}]\epsilon = 0, \quad (2.42)$$

$$\left[\gamma_5 \phi' - \frac{2(d-1)}{\kappa^2}\partial_\phi \mathbf{W}\right]\epsilon = 0. \quad (2.43)$$

In this expression, $U(r)$ is the warp factor of a $(d+1)$-dimensional metric of the form (2.17), and $\nabla_m^{\mathrm{AdS}_d}$ denotes the covariant derivative for the AdS_d background metric $g_{mn}(x)$. The prime means a derivative with respect to r, which we have chosen, for all d, to be the fifth coordinate x^5. These fake Killing spinor equations can be thought of as arising from some fake supersymmetry transformation rules in a domain wall background (2.17),

$$[\nabla_\mu + \gamma_\mu \mathbf{W}]\epsilon = 0,$$
$$\left[\gamma^\mu \nabla_\mu \phi - \frac{2(d-1)}{\kappa^2}\partial_\phi \mathbf{W}\right]\epsilon = 0, \quad (2.44)$$

where $\nabla_\mu \epsilon = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{\nu\rho}\gamma_{\nu\rho})\epsilon$.

It is shown in [2] that the system (2.41), (2.42), and (2.43) reproduces the second order field equations for the warp factor $U(r)$ and the scalar field $\phi(r)$ that follow from (2.38) and (2.39) with

$$\frac{e^{-2U(r)}}{L_d^2} = \frac{2\,\mathrm{Tr}\mathbf{W}^2\,\mathrm{Tr}(\partial_\phi \mathbf{W})^2 - \mathrm{Tr}\{\mathbf{W}, \partial_\phi \mathbf{W}\}^2}{\mathrm{Tr}(\partial_\phi \mathbf{W})^2} \quad (2.45)$$

(where $L_d^2 = -12/R_{\mathrm{AdS}}$ is determined by the scalar curvature of the AdS space) *provided that* the "superpotential" $\mathbf{W}(\phi)$ satisfies the constraint

$$\left[\partial_\phi \mathbf{W}, \frac{d-1}{\kappa^2}\partial_\phi\partial_\phi \mathbf{W} + \mathbf{W}\right] = 0, \quad (2.46)$$

which is a compatibility condition of (2.42) and (2.43).

As there are some obvious similarities with the analogous equations in Secs. II A and II B, one might wonder what exactly the relation between fake and real supergravity is, and how far-reaching the similarities are. As was found in [1], the answer to this question turns out to be surprisingly simple. In order to see this, three cases should be distinguished:

(i) The domain wall is supported only by scalar fields φ^x that sit in vector multiplets.

(ii) The domain wall is supported only by scalar fields q^X that sit in hypermultiplets.

(iii) The domain wall is supported by both types of scalar fields, φ^x and q^X.

Let us first consider case (i). In this case, a supersymmetric domain wall solution is given by profile functions $U(r)$ and $\varphi^x(r)$ that solve the BPS equations (2.18), (2.19), and (2.20), where now $D_r\epsilon_i = \partial_r\epsilon_i$, because $q^{X\prime} = 0$:

$$\left[\nabla_m^{\mathrm{AdS}_4} + \gamma_m\left(\frac{1}{2}U'\gamma_5 - \frac{ig}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon = 0, \quad (2.47)$$

$$\left[\partial_r + \gamma_5\left(-\frac{ig}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon = 0, \quad (2.48)$$

$$[\gamma_5 \varphi^{x\prime} + ig\sqrt{6}g^{xy}\partial_y \mathbf{P}]\epsilon = 0. \quad (2.49)$$

Obviously, the two gravitino equations (2.47) and (2.48) are now exactly of the fake supergravity form (2.41) and (2.42) if we identify

$$\mathbf{W} = -\frac{ig}{\sqrt{6}}\mathbf{P}. \quad (2.50)$$

Upon this identification, the gaugino equation (2.49) also assumes the form (2.43), the only difference being the different number of scalar fields in these two expressions. There are now two attitudes one could take. One could, for example, simply view (2.49) as a suggestion for a generalized form of fake supergravity which involves several scalar fields. As we will see, however, running hypermultiplet scalars in cases (ii) and (iii) suggest quite a different generalization to several scalar fields. We will therefore, at this point, choose the interpretation adopted in [1] and bring (2.43) and (2.49) to exact agreement, by reducing (2.49) effectively to an equation for one scalar field. In order to do this, one recalls that a given domain wall solution defines a curve on the scalar manifold $\mathcal{M}$, which in the case at hand lies entirely in $\mathcal{M}_{VS}$. As the coordinates φ^x on $\mathcal{M}_{VS}$ can be chosen at will, one can, at least locally, choose "adapted" coordinates $\varphi^x(r) = (\varphi(r), \varphi^{\hat{x}})$, where $\varphi(r)$ is aligned with the flow curve, and the other scalars $\varphi^{\hat{x}}$ correspondingly do not depend on r. It is convenient (and locally always possible) to choose these r-independent coordinates $\varphi^{\hat{x}}$ to be orthogonal to the coordinate φ, at least on the flow curve $\varphi(r)$ itself (or on a sufficiently short segment of it). This is illustrated in Fig. 1. On the flow curve, the scalar field metric g_{xy} then takes the form

$$g_{xy} = \begin{pmatrix} g_{\varphi\varphi} & 0 \\ 0 & g_{\hat{x}\hat{y}} \end{pmatrix}. \quad (2.51)$$

By a suitable rescaling of φ, one can, on the curve $(\varphi(r), \varphi^{\hat{x}})$, also achieve $g_{\varphi\varphi} = 1$. The φ component of the gaugino equation (2.49) now coincides with the fake supergravity version (2.43), and the orthogonal compo-

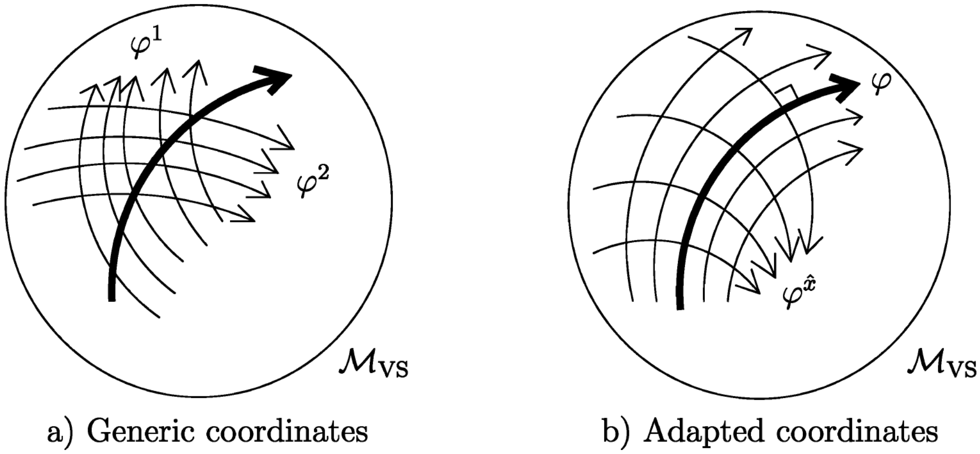

FIG. 1. A given domain wall defines a flow curve (thick arrow) on the scalar manifold $\mathcal{M}_{\rm VS}$. In (a), the thin arrows correspond to a generic coordinate system $\varphi^x = (\varphi^1, \varphi^2)$. In (b), the coordinate system $\varphi^x = (\varphi, \varphi^{\hat{x}})$ is adapted to the flow curve, i.e., the flow curve coincides with a coordinate line of φ and intersects the coordinate lines $\varphi^{\hat{x}}$ at right angles.

nents of (2.49) imply

$$\partial_{\hat{x}}\mathbf{P} = 0. \tag{2.52}$$

As $q^{X\prime} = 0$ also implies $D_X\mathbf{P} = 0$ via (2.25), the effective scalar potential (2.16) on the domain wall assumes a simple form,

$$\begin{aligned} V\mathbb{1}_2 = g^2\mathcal{V}\mathbb{1}_2 &= 4g^2\mathbf{P}^2 - 3g^2(\partial_\varphi\mathbf{P})^2 \\ &= -24\mathbf{W}^2 + 18(\partial_\varphi\mathbf{W})^2, \end{aligned} \tag{2.53}$$

which precisely matches (2.40) for $d = 4$, $\kappa = 1$ and $\phi = \varphi$. Thus, once the $(n_V - 1)$ orthogonal BPS equations (2.52) have determined the line of flow on the scalar manifold, the effective dynamics of the supporting scalar field and the warp factor are precisely described by single-field fake supergravity equations à la [2].

Let us now turn to case (ii) and assume the domain wall is supported by hypermultiplet scalars only. In that case, the gaugino BPS equation (2.20) implies

$$\partial_x\mathbf{P} = 0, \tag{2.54}$$

because we now have $\varphi^{x\prime} = 0$. The gravitino equations (2.18) and (2.19) are again of the same form as the corresponding fake supergravity equations (2.41) and (2.42) provided that we again make the identification (2.50) and gauge away the $SU(2)$ connection along the flow line:

$$q^{X\prime}\omega_{Xi}{}^{j} = 0 \qquad [SU(2) \text{ gauge choice}], \tag{2.55}$$

which is locally always possible, as explained in [1]. However, due to the explicit appearance of the $SU(2)$ curvature tensor, the hyperino BPS condition (2.21), or its equivalent version (2.25),

$$ig_{YX}q^{X\prime}\Theta + iq^{X\prime}[\mathbf{R}_{YX}, \Theta] + \sqrt{6}gD_Y\mathbf{P} = 0, \tag{2.56}$$

clearly differs from the corresponding fake supergravity analogue (2.43). Likewise, the scalar potential (2.16) now reads [remembering (2.54)]

$$\mathcal{V}\mathbb{1}_2 = 4\mathbf{P}^2 - (D_X\mathbf{P})(D^X\mathbf{P}), \tag{2.57}$$

which seems to have the "wrong" prefactor in front of the derivative term when compared to (2.40). In order to make contact between the two formulations, one again has to interpret fake supergravity as the effective theory along the flow line. More precisely, for a given domain wall solution, one again chooses adapted coordinates $q^X(r) = (q(r), q^{\hat{X}})$ such that, on the flow curve,

$$g_{XY} = \begin{pmatrix} g_{qq} & 0 \\ 0 & g_{\hat{X}\hat{Y}} \end{pmatrix}, \tag{2.58}$$

where $g_{qq}(q(r), q^{\hat{X}})$ can again be chosen to be equal to one. The supersymmetry condition (2.56) now splits into two equations:

$$q'\Theta - ig\sqrt{6}D_q\mathbf{P} = 0, \tag{2.59}$$

$$q'[\mathbf{R}_{\hat{X}q}, \Theta] - ig\sqrt{6}D_{\hat{X}}\mathbf{P} = 0. \tag{2.60}$$

In view of (2.23), the first equation (2.59) is easily seen to be equivalent to the fake supergravity equation (2.43) provided the $SU(2)$ gauge (2.55) is imposed. The second equation should again be viewed as a set of constraint equations that determines the position of the flow curve in the full scalar manifold $\mathcal{M}_Q$ as a codimension one hypersurface. Note that (2.60) is different from the analogous equation (2.52) in the case of running vector multiplet scalars, as it no longer implies that the hatted derivatives of $\mathbf{P}$ have to vanish. In fact, one can show that (2.60) implies that, on a BPS-domain wall solution [1],

$$D_{\hat{X}}\mathbf{P}D^{\hat{X}}\mathbf{P} = 2D_q\mathbf{P}D^q\mathbf{P}, \tag{2.61}$$

showing that at least some components of $D_{\hat{X}}\mathbf{P}$ have to be nonzero. Luckily, this is precisely as it should be, because (2.61) exactly corrects the wrong prefactor (-1) in the potential (2.57) to (-3), so that, using the $SU(2)$ gauge (2.55),

$$V\mathbb{1}_2 = g^2\mathcal{V}\mathbb{1}_2 = 4g^2\mathbf{P}^2 - 3g^2(\partial_q\mathbf{P})^2 = -24\mathbf{W}^2 + 18(\partial_q\mathbf{W})^2, \tag{2.62}$$

i.e., exactly as in fake supergravity.

Let us finally mention the last case (iii) of running vector- and hypermultiplet scalars. The flow curve now has nontrivial projections $\varphi^x(r)$ and $q^X(r)$ on both scalar manifold components, $\mathcal{M}_{VS}$ and $\mathcal{M}_Q$. One can then, in a first step, choose separate adapted coordinates $\varphi^x(r) = (\varphi(r), \varphi^{\hat{x}})$ and $q^X(r) = (q(r), q^{\hat{X}})$ on $\mathcal{M}_{VS}$ and $\mathcal{M}_Q$, respectively. In the $SU(2)$ gauge (2.55), the BPS equations and the scalar potential then look the same for both types of scalars φ and q. One can then employ a coordinate transformation in the (φ, q) plane,

$$(\varphi(r), q(r)) \rightarrow (\phi(r), \hat{\phi}), \tag{2.63}$$

such that, locally, $\partial_r = q'\partial_q + \varphi'\partial_\varphi = \phi'\partial_\phi$. In this new, totally adapted coordinate system, the BPS equation for ϕ and the scalar potential as a function of $\phi(r)$ then take the standard fake supergravity form [1].

The lesson we learn from this is that a generalization of the single-field formalism of fake supergravity to several scalar fields is not so straightforward, as the prefactors in the scalar potential can be different and nontrivial connections and curvatures might come into play. However, interpreting single-field fake supergravity as an effective theory along the flow curve seems to make sense in all cases. It is this latter interpretation that we will now try to generalize to the $\mathcal{N} = 4$ case.

III. BPS-DOMAIN WALLS IN $\mathcal{N} = 4$ FAKE AND TRUE SUPERGRAVITY

In this section, we study curved and flat BPS-domain walls in 5D, $\mathcal{N} = 4$ gauged supergravity and verify to what extend one can generalize "$\mathcal{N} = 2$" fake supergravity to $\mathcal{N} = 4$ fake supergravity. We begin with a brief summary of 5D, $\mathcal{N} = 4$ ungauged [40] and gauged [41,42] supergravity. Our notation follows that of Ref. [42], to which the reader is referred for further details.

A. Ungauged 5D, $\mathcal{N} = 4$ supergravity

In the previous section, the index $i = 1, 2$ was used to denote the fundamental representation of the R-symmetry group $Usp(2) \cong SU(2)$ of the 5D, $\mathcal{N} = 2$ Poincaré superalgebra. In this section,

$$i = 1, \ldots, 4 \tag{3.1}$$

will instead denote the fundamental representation of the $\mathcal{N} = 4$ R-symmetry group $Usp(4)$, which is locally isomorphic to $SO(5)$.

In *ungauged* 5D supergravity, vector fields and antisymmetric tensor fields are equivalent, and the most general ungauged $\mathcal{N} = 4$ theory describes the coupling of n vector multiplets to supergravity.

The supergravity multiplet,

$$(e_\mu{}^m, \psi^i_\mu, A^{ij}_\mu, a_\mu, \chi^i, \sigma), \tag{3.2}$$

contains the graviton $e_\mu{}^m$, four gravitini ψ^i_μ, six vector fields (A^{ij}_μ, a_μ), four spin 1/2 fermions χ^i and one real scalar field σ. The vector field a_μ is $USp(4)$ inert, whereas the vector fields A^{ij}_μ transform in the **5** of $USp(4)$, i.e.,

$$A^{ij}_\mu = -A^{ji}_\mu, \qquad A^{ij}_\mu\Omega_{ij} = 0, \tag{3.3}$$

with Ω_{ij} being the symplectic metric of $USp(4)$.

An $\mathcal{N} = 4$ vector multiplet is given by

$$(A_\mu, \lambda^i, \varphi^{ij}), \tag{3.4}$$

where A_μ is a vector field, λ^i denotes four spin 1/2 fields, and the φ^{ij} are scalar fields transforming in the **5** of $USp(4)$, similar to Eq. (3.3). Coupling n vector multiplets to supergravity, the field content of the theory can then be summarized as follows:

$$(e_\mu{}^m, \psi^i_\mu, A^{\tilde{I}}_\mu, a_\mu, \chi^i, \lambda^{ia}, \sigma, \varphi^x). \tag{3.5}$$

Here, $a = 1, \ldots, n$ counts the number of vector multiplets whereas $\tilde{I} = 1, \ldots, (5 + n)$ collectively denotes the A^{ij}_μ and the vector fields of the vector multiplets. Similarly, $x = 1, \ldots, 5n$ is a collective index for all the scalar fields in the vector multiplets. We will further adopt the following convention to raise and lower $USp(4)$ indices:

$$T^i = \Omega^{ij}T_j, \qquad T_i = T^j\Omega_{ji}, \tag{3.6}$$

whereas a, b are raised and lowered with δ^{ab}. Before we proceed, we note that, in a more familiar language, quantities such as $A_{\mu i}{}^j$ in the **5** of $Usp(4) \cong SO(5)$ can be expressed as $A_{\mu i}{}^j = A^\alpha_\mu\Gamma_{\alpha i}{}^j$, where $\alpha = 1, \ldots, 5$, and $\Gamma_{\alpha i}{}^j$ denote $SO(5)$ gamma matrices,

$$\Gamma_{\alpha i}{}^j\Gamma_{\beta j}{}^k + (\alpha \leftrightarrow \beta) = 2\delta_{\alpha\beta}\delta^k_i. \tag{3.7}$$

As was shown in [40], the manifold spanned by the $(5n + 1)$ scalar fields is

$$\mathcal{M} = \frac{SO(5, n)}{SO(5) \times SO(n)} \times SO(1, 1), \tag{3.8}$$

where the $SO(1, 1)$ part corresponds to the $USp(4)$ singlet σ of the supergravity multiplet. The theory therefore has a *global* symmetry group $SO(5, n) \times SO(1, 1)$ and a *local composite* $SO(5) \times SO(n)$ invariance. The coset part of the scalar manifold $\mathcal{M}$ can be described in two different ways:

(i) *Standard sigma model description.*—As in (3.5) one can simply choose a parametrization in terms of $5n$

independent real coordinates φ^x on the coset space. The vielbeins on the coset space can then be chosen to be of the form

$$f_x^{ija} = -f_x^{jia}, \qquad f_x^{ija}\Omega_{ij} = 0, \tag{3.9}$$

where $[ij]$ and a refer to the natural $(\mathbf{5}, \mathbf{n})$ tangent space decomposition with respect to the holonomy group $\mathcal{H} = SO(5) \times SO(n)$. The inverse vielbein, f^x_{aij}, is defined by

$$f_x^{aij} f_{kl}^{xb} = 4(\delta_k^{[i}\delta_l^{j]} - \tfrac{1}{4}\Omega^{ij}\Omega_{kl})\delta^{ab}. \tag{3.10}$$

The nonlinear σ-model metric g_{xy} on the coset part of $\mathcal{M}$ is then given by

$$g_{xy} = \tfrac{1}{4} f_x^{ija} f^a_{yij}, \tag{3.11}$$

and the kinetic term for the scalar fields takes the standard form $\frac{1}{2} g_{xy}\partial_\mu \varphi^x \partial^\mu \varphi^y$. This way of describing $\mathcal{M}$ is particularly useful for discussing geometrical properties of the theory.

(ii) *Coset representatives.*—The parametrization that makes the symmetries of the theory as manifest as possible is in terms of coset representatives, i.e., $(5+n)\times(5+n)$ matrices $L_{\tilde{I}}{}^A \subset \mathcal{G} \equiv SO(5,n)$ that are subject to local ("composite") $\mathcal{H} = SO(5)\times SO(n)$ transformations (acting on the index A) and admit the action of global $\mathcal{G} = SO(5,n)$ transformations (acting on the index $\tilde{I}$). The index $A = 1, \ldots, (5+n)$ naturally decomposes into $A = (ij, a)$, and so do the coset representatives, $L_{\tilde{I}}{}^A = (L_{\tilde{I}}^{ij}, L_{\tilde{I}}^a)$, where $L_{\tilde{I}}^{ij}$ transforms in the $\mathbf{5}$ of $SO(5)$, just as in (3.3). Denoting the inverse of $L_{\tilde{I}}{}^A$ by $L_A{}^{\tilde{I}}$ (i.e., $L_{\tilde{I}}{}^A L_B{}^{\tilde{I}} = \delta_B^A$), the vielbeins on $\mathcal{G}/\mathcal{H}$ and the composite $\mathcal{H}$ connections are determined from the $\mathcal{G}$-invariant 1-form:

$$L^{-1}dL = Q^{ab}\mathfrak{T}_{ab} + Q^{ij}\mathfrak{T}_{ij} + P^{aij}\mathfrak{T}_{aij}, \tag{3.12}$$

where $(\mathfrak{T}_{ab}, \mathfrak{T}_{ij})$ are the generators of the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$, and $\mathfrak{T}_{aij}$ denotes the generators of the coset part of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. More precisely,

$$Q^{ab} = L^{\tilde{I}a} dL_{\tilde{I}}{}^b \quad \text{and} \quad Q^{ij} = L^{\tilde{I}ik} dL_{\tilde{I}k}{}^j \tag{3.13}$$

are the composite $SO(n)$ and $USp(4)$ connections, respectively, and

$$P^{aij} = L^{\tilde{I}a} dL_{\tilde{I}}{}^{ij} = -\tfrac{1}{2} f_x^{aij} d\varphi^x \tag{3.14}$$

describes the space-time pullback of the $\mathcal{G}/\mathcal{H}$ vielbein. Note that Q^{ab}_μ is antisymmetric in the $SO(n)$ indices, whereas Q^{ij}_μ is symmetric in i and j. Denoting by D_x the corresponding $USp(4)$ and $SO(n)$ covariant derivative, one has the following differential relations for the coset representatives [40]:

$$D_x L_{\tilde{I}ij} = -\tfrac{1}{2} L^a_{\tilde{I}} f^a_{xij}, \tag{3.15}$$

$$D_x L^{\tilde{I}}_{ij} = \tfrac{1}{2} L^{\tilde{I}a} f^a_{xij}, \tag{3.16}$$

$$D_x L^a_{\tilde{I}} = -\tfrac{1}{2} f^a_{xij} L^{ij}_{\tilde{I}}, \tag{3.17}$$

$$D_x L^{\tilde{I}a} = \tfrac{1}{2} f_x^{ija} L^{\tilde{I}}_{ij}, . \tag{3.18}$$

We finally note the identities (see [40,42])

$$\begin{aligned} \delta_{\tilde{I}}{}^{\tilde{J}} &= L^{ij}_{\tilde{I}} L^{\tilde{J}}_{ij} + L^a_{\tilde{I}} L^{\tilde{J}a}, \\ C_{\tilde{I}\tilde{J}} &= L^{ij}_{\tilde{I}} L_{\tilde{J}ij} - L^a_{\tilde{I}} L^a_{\tilde{J}}, \end{aligned} \tag{3.19}$$

where $C_{\tilde{I}\tilde{J}}$ is the (constant) $SO(5,n)$ metric.

In the following, we will frequently switch between these two formulations, which is easily done using Eq. (3.14). The Lagrangians and supersymmetry transformation rules can be found in [40,42].

B. 5D, $\mathcal{N} = 4$ gauged supergravity

The above ungauged supergravity theories cannot support domain walls, because their scalar potentials vanish identically, as enforced by supersymmetry. As is typical for extended supergravity, nontrivial scalar potentials are related to nontrivial local gauge groups, K. These gauge groups cannot be chosen at will, but have to be subgroups of the global symmetry group $G = SO(1,1) \times SO(5,n)$ of the ungauged supergravity. As is explained in more detail in [42], the $SO(1,1)$ factor in G cannot be gauged, and all gauge groups, K, actually have to be suitable subgroups of $\mathcal{G} = SO(5,n)$. Under $\mathcal{G} = SO(5,n)$, the vector fields $A^{\tilde{I}}_\mu$ transform in the defining representation $(\mathbf{5} + \mathbf{n})$, whereas a_μ is $SO(5,n)$-inert. If some of these vector fields are promoted to gauge fields of a local gauge group $K \subset SO(5,n)$ under which some of the other fields are charged, the general equivalence between vector and tensor fields is broken [42]. Instead, one now has to distinguish carefully between vector and tensor fields and pay attention to their transformation properties under the gauge group $K \subset SO(5,n)$. The result of the analysis in Ref. [42] is as follows[4]:

(i) If the gauge group K is a direct product of an Abelian factor K_A and a (possibly trivial) non-Abelian factor K_S, the Abelian factor K_A has to be one-dimensional [i.e., either $U(1)$ or $SO(1,1)$, but no higher powers/products thereof]. The gauge field

[4]In [42], particular attention was paid to gauge groups of the form $K =$ Abelian $\times$ semisimple, but all results of [42] equally apply to all gauge groups $K \subset SO(5,n)$ under which the $(\mathbf{5} + \mathbf{n})$ of $SO(5,n)$ decomposes into a completely reducible representation so that tensor fields and vector fields are not connected by K transformations (see also [34]). We only consider such gauge groups in this paper. They include, in particular, the gauge groups encountered in [43,44].

of this Abelian factor is a_μ. Decomposing the vector fields $A_\mu^{\tilde{I}}$ into K_A singlets, A_μ^I, and nonsinglets, A_μ^M, the nonsinglets A_μ^M have to be converted to tensor fields $B_{\mu\nu}^M$ for the gauging to be possible:

$$A_\mu^{\tilde{I}} \to (A_\mu^I, B_{\mu\nu}^M). \tag{3.20}$$

(ii) A possible non-Abelian factor, K_S, is gauged by the remaining vector fields A_μ^I. The tensor fields $B_{\mu\nu}^M$ are inert under K_S.

Turning on only the metric and the scalars, the corresponding Lagrangian is of the form

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}R - \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}g_{xy}\partial_\mu\varphi^x\partial^\mu\varphi^y - V \tag{3.21}$$

whereas the supersymmetry transformations of the fermions are

$$\delta\psi_{\mu i} = D_\mu\epsilon_i - i\gamma_\mu(g_A U_i{}^j + g_S S_i{}^j)\epsilon_j \tag{3.22}$$

$$\delta\chi_i = -\frac{i}{2}\partial\!\!\!/\sigma\epsilon_i + 3\partial_\sigma(g_A U_i{}^j + g_S S_i{}^j)\epsilon_j \tag{3.23}$$

$$\delta\lambda_i^a = \frac{i}{2}f_{xi}^{a\ j}(\partial\!\!\!/\varphi^x)\epsilon_j - (g_A V_i^{aj} + g_S T_i^{aj})\epsilon_j. \tag{3.24}$$

Here, g_A and g_S are the gauge couplings of, respectively, the Abelian and the non-Abelian gauge group, and

$$U_{ij} = U_{ji} = \frac{\sqrt{2}}{6}e^{2\sigma/\sqrt{3}}\Lambda_M^N L_{Nik}L^{Mk}{}_j \tag{3.25}$$

$$S_{ij} = S_{ji} = -\frac{2}{9}e^{-\sigma/\sqrt{3}}L_{ik}^J f_{JI}^K L_K^{kl} L_{lj}^I, \tag{3.26}$$

$$V_{ij}^a = -V_{ji}^a = \frac{1}{\sqrt{2}}e^{2\sigma/\sqrt{3}}\Lambda_M^N L_{Nij}L^{Ma} \tag{3.27}$$

$$T_{ij}^a = T_{ji}^a = -e^{-\sigma/\sqrt{3}}L^{Ja}L_i^{Kk}f_{JK}^I L_{Ikj}, \tag{3.28}$$

denote the "fermionic shifts" with the structure constants, f_{JI}^K, of K_S and the K_A transformation matrix, Λ_M^N, of the tensor fields $B_{\mu\nu}^M$. The fermionic shifts also enter the scalar potential,

$$V = \tfrac{1}{2}[g_A^2 V_{ij}^a V^{aij} - 36 g_A g_S U_{ij}S^{ij} + g_S^2(T_{ij}^a T^{aij} - 9S_{ij}S^{ij})], \tag{3.29}$$

which is obtained from the trace of the "Ward identity" [42]

$$\tfrac{1}{4}\delta_i^j V = \tfrac{1}{2}g_A^2 V_i^{ak}V_k^{aj} + g_A g_S[9(S_i{}^k U_k{}^j + U_i{}^k S_k{}^j) + \tfrac{1}{2}(V_i^{ak}T_k^{aj} - T_i^{ak}V_k^{aj})] - \tfrac{1}{2}g_S^2[T_i^{ak}T_k^{aj} - 9S_i{}^k S_k{}^j]. \tag{3.30}$$

C. BPS-domain walls

Our goal is to study domain walls of the form (2.17) that are supported by nontrivial scalar profiles $\sigma(r)$ and/or $\varphi^x(r)$ and preserve one-half of the $\mathcal{N} = 4$ supersymmetry. This analysis would be considerably simplified if one could bring the BPS equations and the scalar potential into a fake supergravity form similar to (2.39), (2.41), (2.42), and (2.43) for the $\mathcal{N} = 2$ case. In $\mathcal{N} = 2$ supergravity, the gravitino shift $\mathbf{W} = -(ig/\sqrt{6})\mathbf{P}$ was a $\mathfrak{usp}(2) \cong \mathfrak{su}(2)$-valued (2×2) matrix [cf. Eq. (2.50)]. For $\mathcal{N} = 4$ supergravity, the gravitino shift is a $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$-valued (4×4) matrix, $-i(g_A U_i{}^j + g_S S_i{}^j)$ [see Eq. (3.22)]. In analogy with the $\mathcal{N} = 2$ case, we choose to call this gravitino shift $W_i{}^j$:

$$W_i{}^j := -i(g_A U_i{}^j + g_S S_i{}^j). \tag{3.31}$$

Furthermore, we will, from now on, suppress the $\mathfrak{usp}(4)$ indices $i, j = 1, \ldots, 4$ by using boldface expressions such as

$$\mathbf{W} = W_i{}^j, \qquad \mathbf{WW} = W_i{}^j W_j{}^k, \qquad \text{etc.}, \tag{3.32}$$

just as we did in Sec. II for the analogous (2×2) matrices. Note that, in this boldface notation, the position of the indices is always assumed to be of the form shown in (3.32), which differs, for example, by a minus sign from expressions such as $W_{ij}W^{jk}$ due to the convention (3.6). In a domain wall background, the gravitino and dilatino BPS equations (3.22) and (3.23) then take the form

$$[\nabla_m^{\mathrm{AdS}_4} + \gamma_m(\tfrac{1}{2}U'\gamma_5 + \mathbf{W})]\epsilon = 0, \tag{3.33}$$

$$[D_r + \gamma_5\mathbf{W}]\epsilon = 0, \tag{3.34}$$

$$[\gamma_5\sigma' - 6\partial_\sigma\mathbf{W}]\epsilon = 0, \tag{3.35}$$

which are precisely of the same form as the fake supergravity equations (2.41), (2.42), and (2.43), except, that $\mathbf{W}$ is now a (4×4) matrix instead of a (2×2) matrix. Adding $0 = 6g_A^2\mathbf{U}^2 - \frac{9}{2}g_A^2(\partial_\sigma\mathbf{U})^2$ to the right-hand side of (3.30), the scalar potential reads

$$\tfrac{1}{4}V\mathbb{1}_4 = -6\mathbf{W}^2 + \tfrac{9}{2}(\partial_\sigma\mathbf{W})^2 + \tfrac{1}{2}[g_A^2\mathbf{V}^a\mathbf{V}^a + g_A g_S[\mathbf{V}^a, \mathbf{T}^a] - g_S^2\mathbf{T}^a\mathbf{T}^a]. \tag{3.36}$$

Thus, if the domain wall is supported by $\sigma(r)$ only [i.e., if $\varphi^{x\prime} = 0$ and hence $\mathbf{V}^a = \mathbf{T}^a = 0$ via Eq. (3.24)], the BPS equations (3.33), (3.34), and (3.35) and the scalar potential (3.36) generalize the $\mathcal{N} = 2$ fake supergravity equations (2.40), (2.41), (2.42), and (2.43) to what one might call $\mathcal{N} = 4$ fake supergravity. The interesting question now is: Can a nontrivial profile $\varphi^x(r)$ also be incorporated in this formalism? This obviously requires two things:

(i) The gaugino/tensorino BPS condition (3.24) has to be brought into a form in which $\mathbf{V}^a$ and $\mathbf{T}^a$ are expressed in terms of derivatives of $\mathbf{W}$ with respect to φ^x with the same relative prefactors as in (3.35).

(ii) The term in brackets in the scalar potential (3.36) should likewise be reexpressed in terms of φ^x derivatives of $\mathbf{W}$ with the prefactor $9/2$, just as for the $(\partial_\sigma \mathbf{W})^2$ term.

As we will see, the rewriting of the vector and tensor multiplet sector along these lines bears many similarities with the hypermultiplet sector of $\mathcal{N}=2$ supergravity, but also shows some novel features. Let us start with the BPS equation (3.24). In the domain wall background (2.17) it reads, after using a projector of the form (2.23) [now with (4×4) matrices],

$$-\frac{i}{2}\varphi^{x\prime}\mathbf{f}_x^a\Theta - g_A\mathbf{V}^a - g_S\mathbf{T}^a = 0. \tag{3.37}$$

Multiplying by $\mathbf{f}_y^a$ from the left and using [40,42]

$$\mathbf{f}_y^a\mathbf{f}_x^a = g_{yx}\mathbb{1}_4 + 4\mathbf{R}_{yx}, \tag{3.38}$$

this becomes

$$-\frac{i}{2}\varphi^{x\prime}g_{yx}\Theta - 2i\varphi^{x\prime}\mathbf{R}_{yx}\Theta - \mathbf{f}_y^a(g_A\mathbf{V}^a + g_S\mathbf{T}^a) = 0. \tag{3.39}$$

Decomposing (3.39) into symmetric and antisymmetric part, one obtains

$$-\frac{i}{2}\varphi^{x\prime}g_{yx}\Theta - i\varphi^{x\prime}[\mathbf{R}_{yx},\Theta] - \frac{g_A}{2}[\mathbf{f}_y^a,\mathbf{V}^a] - \frac{g_S}{2}\{\mathbf{f}_y^a,\mathbf{T}^a\} = 0 \tag{3.40}$$

$$-i\varphi^{x\prime}\{\mathbf{R}_{yx},\Theta\} - \frac{g_A}{2}\{\mathbf{f}_y^a,\mathbf{V}^a\} - \frac{g_S}{2}[\mathbf{f}_y^a,\mathbf{T}^a] = 0. \tag{3.41}$$

Using (3.15), (3.16), (3.17), (3.18), and (3.19) and the invariance conditions for the structure constants f_{IJ}^K and the transformation matrices Λ_M^N,

$$C_{IJ}f_{KL}^I + C_{IL}f_{KJ}^I = 0 \tag{3.42}$$

$$\Lambda_M^P C_{PN} + \Lambda_N^P C_{MP} = 0, \tag{3.43}$$

one derives [42]

$$D_y\mathbf{U} = \tfrac{1}{6}[\mathbf{f}_y^a,\mathbf{V}^a] \tag{3.44}$$

$$D_y\mathbf{S} = \tfrac{1}{6}\{\mathbf{f}_y^a,\mathbf{T}^a\}, \tag{3.45}$$

so that (3.40) becomes

$$\varphi^{x\prime}g_{yx}\Theta + 2\varphi^{x\prime}[\mathbf{R}_{yx},\Theta] + 6D_y\mathbf{W} = 0. \tag{3.46}$$

If one now switches to adapted coordinates $(\varphi(r),\varphi^{\hat{x}})$, with $\varphi^{\hat{x}}$ constant and perpendicular to φ along a given flow curve, one obtains for the (canonically normalized) φ component of (3.46)

$$\varphi'\Theta + 6D_\varphi\mathbf{W} = 0. \tag{3.47}$$

Gauging away the composite $Usp(4)$ connection,

$$\varphi^{x\prime}Q_{xi}{}^j = 0 \qquad [Usp(4)\text{ gauge choice}] \tag{3.48}$$

and taking into account (2.23), this assumes the desired fake supergravity form [cf. (3.35)]. The other BPS equations can again be viewed as constraints that determine the position of the flow curve on the full scalar manifold. Let us now turn to the scalar potential. Using (3.44) and (3.45) as well as (3.10), one derives

$$D_x\mathbf{U}D^x\mathbf{U} = -\tfrac{4}{9}\mathbf{V}^a\mathbf{V}^a \tag{3.49}$$

$$\{D_x\mathbf{S},D^x\mathbf{U}\} = \tfrac{1}{3}[\mathbf{T}^a,\mathbf{V}^a] \tag{3.50}$$

$$D_x\mathbf{S}D^x\mathbf{S} = \tfrac{1}{9}[\mathbf{T}^a\mathbf{T}^a + \tfrac{1}{2}\mathrm{Tr}(\mathbf{T}^a\mathbf{T}^a)\mathbb{1}_4]. \tag{3.51}$$

These relations allow one to reexpress the term in brackets in (3.36) in terms of derivatives of $\mathbf{U}$ and $\mathbf{S}$:

$$\begin{aligned}\tfrac{1}{4}V\mathbb{1}_4 = {}& -6\mathbf{W}^2 + \tfrac{9}{2}(\partial_\sigma\mathbf{W})^2 - \tfrac{9}{8}g_A^2D_x\mathbf{U}D^x\mathbf{U}\\ &- \tfrac{3}{2}g_Ag_S\{D_x\mathbf{S},D^x\mathbf{U}\} - \tfrac{9}{2}g_S^2(D_x\mathbf{S}D^x\mathbf{S}\\ &- \tfrac{1}{6}\mathrm{Tr}(D_x\mathbf{S}D^x\mathbf{S})\mathbb{1}_4).\end{aligned} \tag{3.52}$$

We note three interesting features of this expression:

(i) In contrast to the $\mathcal{N}=2$ analogue (2.16), the Ward identity (3.52) can, in general, not be written without taking some traces.

(ii) Even after taking the trace of (3.52), the prefactor of the $(D_x\mathbf{U})^2$ term is different from the prefactor of the $\{D_x\mathbf{S},D^x\mathbf{U}\}$ term and the $(D_x\mathbf{S})^2$ term. This means that, as long as g_A and g_S are both nonvanishing, one cannot write these terms as something proportional to $(D_x\mathbf{W})^2$, i.e., in terms of derivatives of the *full* gravitino shift $\mathbf{W}$. Again, this is different from the $\mathcal{N}=2$ case (2.16).

(iii) If $g_S=0$, or if $g_A=0$, the scalar potential *can* be written as the full gravitino shift and its derivatives, but in neither of these two special cases, the φ^x derivatives appear with the "right" coefficient $9/2$ required by fake supergravity.

Properties (i) and (ii) are clearly different from the $\mathcal{N}=2$ case. These differences can in part be traced to the fact that the adjoint of $Usp(4)$ is no longer equivalent to the vector representation of $SO(5)$, as was the case for $SU(2)$ and $SO(3)$. This implies, in particular, that symmetric products of $\mathfrak{usp}(4)$-valued matrices such as $D_x\mathbf{S}D^x\mathbf{S}$ are no longer automatically proportional to the unit matrix, as was the case for $\mathfrak{su}(2)$-valued matrices such as $D_X\mathbf{P}D^X\mathbf{P}$ in the $\mathcal{N}=2$ case due to the anticommutation properties of the Pauli matrices, i.e., the Clifford algebra of $SO(3)$. On the other hand, even the $\mathcal{N}=2$ hypermultiplet sector did not fall into the $\mathcal{N}=2$ fake supergravity framework before the BPS equations and adapted coordinates were imposed [see Eqs. (2.16) vs (2.62)]. Thus, there is still some hope that imposing the BPS equations (3.40) and (3.41) and using the adapted coordinates $\varphi^x(r) = (\varphi(r), \varphi^{\hat{x}})$ miraculously transforms the last three terms in (3.52) into $9/2(D_\varphi \mathbf{W})^2$ and removes at least some of the above-mentioned differences to the $\mathcal{N} = 2$ case. To see whether this works, let us go back to the BPS condition (3.46) and its φ component (3.47), which we rearrange as (normalizing $g_{\varphi\varphi} = 1$)

$$D_y \mathbf{W} = -\tfrac{1}{6}\varphi^{x\prime} g_{xy}\Theta - \tfrac{1}{3}\varphi^{x\prime}[\mathbf{R}_{yx}, \Theta] \tag{3.53}$$

$$\longrightarrow D_\varphi \mathbf{W} = -\tfrac{1}{6}\varphi'\Theta \qquad (y = \varphi). \tag{3.54}$$

Squaring (3.54) gives

$$D_\varphi \mathbf{W} D_\varphi \mathbf{W} = \tfrac{1}{36}(\varphi')^2 \mathbb{1}_4. \tag{3.55}$$

On the other hand, squaring (3.53) and using (3.41) as well as the identity

$$\mathbf{R}_{xy}\mathbf{R}^x{}_z = -\tfrac{1}{4} g_{yz}\mathbb{1}_4 - \tfrac{3}{4}\mathbf{R}_{yz}, \tag{3.56}$$

one derives

$$\begin{aligned} D_y \mathbf{W} D^y \mathbf{W} = {}& \frac{5}{36}\varphi^{x\prime}\varphi^{z\prime} g_{xz}\mathbb{1}_4 - \frac{g_A^2}{36}\{\mathbf{f}_y^a, \mathbf{V}^a\}\{\mathbf{f}^{by}, \mathbf{V}^b\} \\ & - \frac{g_A g_S}{36}(\{\mathbf{f}_y^a, \mathbf{V}^a\}[\mathbf{f}^{by}, \mathbf{T}^b] + [\mathbf{f}_y^a, \mathbf{T}^a]\{\mathbf{f}^{by}, \mathbf{V}^b\}) \\ & - \frac{g_S^2}{36}[\mathbf{f}_y^a, \mathbf{T}^a][\mathbf{f}^{by}, \mathbf{T}^b]. \end{aligned} \tag{3.57}$$

Using (3.10), the vielbeins can be eliminated, and (3.57) becomes

$$\begin{aligned} D_y \mathbf{W} D^y \mathbf{W} = {}& \frac{5}{36}\varphi^{x\prime}\varphi^{z\prime} g_{xz}\mathbb{1}_4 - \frac{g_A^2}{9}\mathbf{V}^a\mathbf{V}^a \\ & - \frac{g_A g_S}{9}[\mathbf{V}^a, \mathbf{T}^a] + \frac{g_S^2}{18}\operatorname{Tr}(\mathbf{T}^a\mathbf{T}^a)\mathbb{1}_4. \end{aligned} \tag{3.58}$$

However, $(D_x \mathbf{W})^2$ can be computed directly from (3.49), (3.50), and (3.51) and the definition (3.31):

$$\begin{aligned} D_y \mathbf{W} D^y \mathbf{W} = {}& \frac{4}{9} g_A^2 \mathbf{V}^a\mathbf{V}^a - \frac{g_A g_S}{3}[\mathbf{T}^a, \mathbf{V}^a] \\ & - \frac{g_S^2}{9}\left(\mathbf{T}^a\mathbf{T}^a + \frac{1}{2}\operatorname{Tr}(\mathbf{T}^a\mathbf{T}^a)\mathbb{1}_4\right). \end{aligned} \tag{3.59}$$

Consistency of (3.58) and (3.59) then implies

$$\begin{aligned} \tfrac{5}{4}\varphi^{x\prime}\varphi^{z\prime} g_{xz}\mathbb{1}_4 = {}& 5 g_A^2 \mathbf{V}^a\mathbf{V}^a + 4 g_A g_S[\mathbf{V}^a, \mathbf{T}^a] \\ & - g_S^2(\mathbf{T}^a\mathbf{T}^a + \operatorname{Tr}(\mathbf{T}^a\mathbf{T}^a)\mathbb{1}_4), \end{aligned} \tag{3.60}$$

or, after taking the trace,

$$\varphi^{x\prime}\varphi^{z\prime} g_{xz} = g_A^2 \operatorname{Tr}(\mathbf{V}^a\mathbf{V}^a) - g_S^2 \operatorname{Tr}(\mathbf{T}^a\mathbf{T}^a). \tag{3.61}$$

Switching to adapted coordinates, (3.55) then becomes

$$D_\varphi \mathbf{W} D_\varphi \mathbf{W} = \tfrac{1}{36}(g_A^2 \operatorname{Tr}(\mathbf{V}^a\mathbf{V}^a) - g_S^2 \operatorname{Tr}(\mathbf{T}^a\mathbf{T}^a))\mathbb{1}_4, \tag{3.62}$$

and we finally obtain for the scalar potential (3.36)

$$V = -6\operatorname{Tr}\mathbf{W}^2 + \tfrac{9}{2}\operatorname{Tr}(\partial_\sigma \mathbf{W})^2 + \tfrac{9}{2}\operatorname{Tr}(D_\varphi \mathbf{W})^2. \tag{3.63}$$

Thus, after employing the $Usp(4)$ gauge choice (3.48), the φ sector and the σ sector enter the theory symmetrically and with the right prefactors. If both φ and σ are running, one can, just as in $\mathcal{N} = 2$ supergravity, go over to a total adapted coordinate $\phi(r)$ with $\sigma'\partial_\sigma + \varphi'\partial_\varphi = \phi'\partial_\phi$ so as to obtain $\mathcal{N} = 4$ single-field fake supergravity equations [see the discussion around (2.63)].

D. Consistency conditions and domain wall curvature

Let us summarize what we have shown so far. In 5D, $\mathcal{N} = 4$ supergravity, the gravitino BPS equations in a $\frac{1}{2}$-supersymmetric domain wall background read

$$[\nabla_m^{\mathrm{AdS}_4} + \gamma_m(\tfrac{1}{2}U'\gamma_5 + \mathbf{W})]\epsilon = 0, \tag{3.64}$$

$$[D_r + \gamma_5 \mathbf{W}]\epsilon = 0. \tag{3.65}$$

Subjecting the spinor ϵ to

$$\gamma_5\epsilon = -\Theta\epsilon, \tag{3.66}$$

the dilatino equation becomes

$$\sigma'\Theta + 6\partial_\sigma \mathbf{W} = 0, \tag{3.67}$$

and the gaugino/tensorino BPS equation can be decomposed as follows:

$$\varphi^{x\prime} g_{yx}\Theta + 2\varphi^{x\prime}[\mathbf{R}_{yx}, \Theta] + 6 D_y \mathbf{W} = 0 \tag{3.68}$$

$$2\varphi^{x\prime}\{\mathbf{R}_{yx}, \Theta\} - i g_A\{\mathbf{f}_y^a, \mathbf{V}^a\} - i g_S[\mathbf{f}_y^a, \mathbf{T}^a] = 0. \tag{3.69}$$

In adapted coordinates, $\varphi^x(r) = (\varphi(r), \varphi^{\hat{x}})$, the scalar potential reads

$$V = -6\operatorname{Tr}\mathbf{W}^2 + \tfrac{9}{2}\operatorname{Tr}(\partial_\sigma \mathbf{W})^2 + \tfrac{9}{2}\operatorname{Tr}(D_\varphi \mathbf{W})^2, \tag{3.70}$$

and (3.68) splits further into

$$\varphi'\Theta + 6 D_\varphi \mathbf{W} = 0 \tag{3.71}$$

$$2\varphi'[\mathbf{R}_{\hat{y}\varphi}, \Theta] + 6 D_{\hat{y}} \mathbf{W} = 0. \tag{3.72}$$

Equations (3.67) and (3.68) can be solved for Θ:

$$\Theta = -6\frac{\partial_\sigma \mathbf{W}}{\sigma'} \tag{3.73}$$

$$\Theta = -6\frac{\varphi^{y\prime} D_y \mathbf{W}}{\varphi^{x\prime}\varphi^{z\prime} g_{xz}} = -6\frac{D_\varphi \mathbf{W}}{\varphi'}. \tag{3.74}$$

If both σ and φ^x are running, consistency of these two expressions requires

$$\frac{\partial_\sigma \mathbf{W}}{\sigma'} = \frac{\varphi^{y\prime} D_y \mathbf{W}}{\varphi^{x\prime}\varphi^{z\prime} g_{xz}} = \frac{D_\varphi \mathbf{W}}{\varphi'}. \tag{3.75}$$

Equations (3.73) and (3.74) also imply

$$D_r\mathbf{W} \equiv (\sigma'\partial_\sigma + \varphi^{x\prime}D_x)\mathbf{W} = -\tfrac{1}{6}((\sigma')^2 + \varphi^{x\prime}g_{xy}\varphi^{y\prime})\Theta. \tag{3.76}$$

Squaring (3.73) and (3.74) gives the first order equation for the scalars,

$$\sigma' = \pm 6\sqrt{(\partial_\sigma \mathbf{W})^2} \tag{3.77}$$

$$\varphi^{x\prime}g_{xy}\varphi^{y\prime} = \pm 6\sqrt{(\varphi^{x\prime}D_x\mathbf{W})^2} \Rightarrow \varphi' = \pm 6\sqrt{(D_\varphi \mathbf{W})^2}. \tag{3.78}$$

Let us now turn to the warp factor. Just as in Sec. II B, the integrability condition of (3.64), yields

$$(U')^2 = 4\mathbf{W}^2 - \frac{e^{-2U}}{L_4^2}. \tag{3.79}$$

On the other hand, the compatibility condition between (3.64) and (3.66) gives

$$U' = \{\Theta, \mathbf{W}\}, \tag{3.80}$$

which, when combined with (3.79), gives

$$-\frac{e^{-2U}}{L_4^2}\mathbb{1}_4 = \{\Theta, \mathbf{W}\}^2 - 4\mathbf{W}^2, \tag{3.81}$$

just as in (2.34). Hence, a BPS-domain wall is flat if and only if

$$\{\Theta, \mathbf{W}\}^2 = 4\mathbf{W}^2 \quad \text{(flatness condition).} \tag{3.82}$$

It is shown in the appendix that this is equivalent to

$$[\Theta, \mathbf{W}] = 0 \Leftrightarrow [D_r\mathbf{W}, \mathbf{W}] = 0 \quad \text{(equivalent flatness condition),} \tag{3.83}$$

where we have used (3.76). Finally, there is the $\mathcal{N} = 4$ analogue of Eq. (2.35), namely, the compatibility condition of (3.65) and (3.66),

$$[\Theta, D_r\Theta - 2\mathbf{W}] = 0, \tag{3.84}$$

or, using (3.76),

$$[D_r\mathbf{W}, D_rD_r\mathbf{W} + \tfrac{1}{3}((\sigma')^2 + \varphi^x\varphi^y g_{xy})\mathbf{W}] = 0. \tag{3.85}$$

E. Special cases

In this subsection, we will take a closer look at the implications of the equations listed in Sec. III D and apply them to a number of special cases.

Case 1.—The gauging is purely Abelian:

$$g_S = 0. \tag{3.86}$$

In this case, $\mathbf{W} = -ig_A\mathbf{U}$, and (3.67) implies

$$\sigma'\Theta = -4\sqrt{3}\mathbf{W}. \tag{3.87}$$

Thus, $\sigma' = 0$ would imply $\mathbf{W} = 0$ along the flow, and, hence, because of (3.80), also $U(r) = \text{const}$, i.e., a trivial domain wall. In other words, for a nontrivial Abelian BPS-domain wall, σ' has to be nonzero. Now, a running $\sigma(r)$, however, means that (3.87) can be solved for Θ:

$$\Theta = -4\sqrt{3}\frac{\mathbf{W}}{\sigma'}. \tag{3.88}$$

This implies that the domain wall has to be *flat* because of (3.83). On the other hand, due to the simple σ dependence, (3.77) can be readily solved:

$$\sigma(r) = a\ln(r - r_o) + b \tag{3.89}$$

with some constants a and b. Hence, $\sigma(r)$ always approaches infinity, which is not surprising in view of the simple runaway behavior of the scalar potential in the σ direction when $g_S = 0$.

Case 2.—The gauging is purely non-Abelian:

$$g_A = 0. \tag{3.90}$$

In this case, $\mathbf{W} = -ig_S\mathbf{S}$, and the simple exponential behavior of $\mathbf{W}$ leads to similar conclusions as in the purely Abelian case: Just as in (3.87), one concludes that $\sigma(r)$ has to have a nontrivial r dependence for the domain wall to be nontrivial. This, however, also implies that Θ is always proportional to $\mathbf{W}$, and any domain wall must be flat, due to (3.83). Again, the runaway behavior of the potential leads to a logarithmic r dependence of the scalar field $\sigma(r)$.

Case 3.—The mixed gauging:

$$g_Ag_S \neq 0. \tag{3.91}$$

When both the Abelian part and the non-Abelian part are gauged, the σ dependence of $\mathbf{W}$ no longer factors out and neither does the scalar potential have a simple runaway behavior in the σ direction. Nevertheless, the structure of domain walls that are solely supported by the supergravity scalar σ are still quite limited. In order to see this, consider the consistency condition (3.85), which, for $\varphi^{x\prime} = 0$, simplifies to

$$[\partial_\sigma\mathbf{W}, \partial_\sigma^2\mathbf{W} + \tfrac{1}{3}\mathbf{W}] = 0. \tag{3.92}$$

Using (3.31) and the σ dependence of $\mathbf{U}$ and $\mathbf{S}$, (3.25) and (3.26), one easily sees that (3.92) implies

$$[\mathbf{U}, \mathbf{S}] = 0. \tag{3.93}$$

But this also implies $[\partial_\sigma\mathbf{W}, \mathbf{W}] = 0$, and hence, via (3.83), the flatness of the domain wall. Thus, taking into account our observations in the purely Abelian and purely non-

Abelian case, we find that a domain wall supported by $\sigma(r)$ only can never be curved. Obviously, the condition $[\mathbf{U}, \mathbf{S}] = 0$ is automatically satisfied if either g_A or g_S vanishes. One way to satisfy $[\mathbf{U}, \mathbf{S}] = 0$ for $g_A g_S \neq 0$ is as follows: Suppose, the gauge group K contains the obvious $SO(3) \times SO(2)$ subgroup of $SO(5) \subset SO(5, n)$ as a factor. More generally, one could take the $SO(2)$ factor of the gauge group to be a diagonal subgroup of the $SO(2) \subset SO(5)$ and some product of $SO(2)$'s that are contained in $SO(n) \subset SO(5, n)$ such that, under K_A transformations, the tensors charged under $SO(2) \subset SO(5)$ do not mix with the tensors charged under the other $SO(2)$ subgroups of $SO(n)$. These gaugings are precisely the ones that occur in the $\mathcal{N} = 4$ orbifold compactifications [45] in the AdS/conformal field theory correspondence [8]. In such a gauging, the five vector fields $A_\mu^{1,\ldots,5}$ of the ungauged supergravity multiplet split into a triplet of $SO(3)$-gauge fields, which we take to be $A_\mu^{1,2,3}$, and a doublet of tensor fields, $B_{\mu\nu}^{4,5}$, charged under the $SO(2)$. Suppose further that the scalar fields φ^x of the vector and tensor multiplets all sit at the "origin" of the symmetric space $SO(5,n)/SO(5) \times SO(n)$: $L_I^A = \delta_I^A$. Since, by assumption, the tensor field transformation matrix Λ_M^N does not mix the supergravity tensors $B_{\mu\nu}^{4,5}$ with the other tensor fields (provided such additional tensor fields exist), and since the $SO(3)$ part of the gauge group is supposed to be a direct factor, it is easy to see that V_{ij}^a and T_{ij}^a from Eqs. (3.27) and (3.28) vanish at this critical point, which is consistent with the BPS conditions (3.40) and (3.41) and $\varphi^{x\prime} = 0$. Using $SO(5)$-gamma matrices [see Eq. (3.7)] to convert $Usp(4)$ indices $i, j = 1, \ldots, 4$ into $SO(5)$ indices $\alpha, \beta = 1, \ldots 5$, it is easy to see that

$$\mathbf{U} \sim e^{2\sigma/\sqrt{3}}\Gamma_{45} \tag{3.94}$$

$$\mathbf{S} \sim e^{-\sigma/\sqrt{3}}\Gamma_{123} \sim e^{-\sigma/\sqrt{3}}\Gamma_{45} \tag{3.95}$$

and hence $[\mathbf{U}, \mathbf{S}] = 0$. In fact, if the scalars φ^x are frozen as described, the model becomes effectively the Romans model [41]. The BPS conditions for the general case of running scalars $\varphi^x(r)$ and $\sigma(r)$ can be analyzed along similar lines using Eqs. (3.44) and (3.45) as well as [42]

$$D_x\mathbf{V}^a = \left(\frac{e^{2\sigma/\sqrt{3}}}{2\sqrt{2}}\Lambda_M^N L_N^a L^{Mb}\right)\mathbf{f}_x^b + \frac{3}{2}[\mathbf{f}_x^a, \mathbf{U}] \tag{3.96}$$

$$D_x\mathbf{T}^a = \left(\frac{e^{-\sigma/\sqrt{3}}}{2} f_{JK}^I L^{Ja} L_I^b\right)[\mathbf{L}^K, \mathbf{f}_x^b] + \frac{3}{2}\{\mathbf{f}_x^a, \mathbf{S}\}, \tag{3.97}$$

but the results are in general model-dependent and beyond the scope of this paper.

IV. CONCLUSIONS

In this paper, we have shown that the BPS equations and the scalar potentials for 1/2-BPS-domain walls in 5D, $\mathcal{N} = 4$ gauged supergravity can be cast into a generalized form of fake supergravity. In many respects, this parallels the situation in $\mathcal{N} = 2$ supergravity in five dimensions, but there are also important differences. Most importantly, the gravitino shift is now a $\mathfrak{usp}(4)$-valued (4×4) matrix instead of a $\mathfrak{su}(2)$-valued (2×2) matrix. This means that some peculiarities of the group $SU(2)$ are no longer valid, which makes it all the more surprising that the simple form of the fake supergravity equations remains largely unaltered (in fact, the only immediately visible difference is that the scalar potential in $\mathcal{N} = 4$ fake supergravity can no longer be written in terms of the gravitino shift $\mathbf{W}$ without taking the trace). Furthermore, due to the doubled amount of supersymmetry, there are now twice as many BPS conditions to satisfy. It should also be noted that the scalar manifolds are no longer of the type encountered in $\mathcal{N} = 2$ supergravity, but are subject to completely different geometrical constraints. The fact that nevertheless the full dynamics along the flow line is captured by almost identical equations suggests that possibly all BPS-domain walls in all space-time dimensions and for all amounts of supersymmetry can be described in terms of an appropriately generalized form of fake supergravity. This could, for instance, be due to some general properties of gauged supergravities in the spirit of [46]. The results of this paper should help distinguish $\mathcal{N} = 2$ artifacts from this general formulation.

Recasting domain wall equations of supergravity theories into a fake supergravity language greatly simplifies their study. This was already demonstrated in [1], and in this paper we saw that also $\mathcal{N} = 4$ domain walls can be studied quite efficiently in this language. For example, we could easily rule out curved BPS-domain walls if the gauge group is purely Abelian or purely semisimple. In both of these cases, domain walls furthermore show a runaway behavior in the σ direction. In the mixed gauging, curved domain walls are ruled out if they are supported by $\sigma(r)$ only.

It might be interesting to apply the results of this paper to the study of holographic renormalization group flows along the lines of [8] or in the context of domain wall solutions in flux compactifications, which single out certain types of gauge groups [9–11]. Another interesting further direction concerns a generalization to 1/4-BPS-domain walls [8], which are impossible in $\mathcal{N} = 2$ supergravity. One might also wonder whether other types of solutions such as charged black holes or cosmic strings have a similar description in terms of some other form of fake supergravity, in which all scalars are treated equally, independently of the space-time dimension or the amount of supersymmetry.

ACKNOWLEDGMENTS

I have benefited from earlier discussions with A. Celi, A. Ceresole, G. Dall'Agata, C. Herrmann and A. Van Proeyen on the material of Refs. [1,42]. This work is supported by the German Research Foundation (DFG) within the Emmy Noether Program (ZA 279/1-1).

APPENDIX: THE FLATNESS CONDITION

As shown in Sec. III C, a 1/2-BPS-domain wall in 5D, $\mathcal{N} = 4$ supergravity is flat if and only if [cf. Eq. (3.82)]

$$\{\Theta, \mathbf{W}\}^2 = 4\mathbf{W}^2. \tag{A1}$$

A sufficient condition for this to hold is obviously

$$[\Theta, \mathbf{W}] = 0. \tag{A2}$$

We will now show that this condition is also necessary. Both Θ and $\mathbf{W}$ are $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$-valued, so they can be expressed in terms of the $SO(5)$ gamma matrices (3.7) via

$$\Theta = \Theta^{\alpha\beta}\Gamma_{\alpha\beta}, \qquad \mathbf{W} = W^{\alpha\beta}\Gamma_{\alpha\beta}. \tag{A3}$$

Without loss of generality, we can assume $\Theta = 2\Theta^{12}\Gamma_{12}$. Using

$$\{\Gamma_{\alpha\beta}, \Gamma_{\gamma\delta}\} = 2\Gamma_{\alpha\beta\gamma\delta} + 2\delta_{\alpha\delta}\delta_{\beta\gamma} - 2\delta_{\beta\delta}\delta_{\alpha\gamma}, \tag{A4}$$

one finds that $\Theta^2 = \mathbb{1}_4$ implies $(\Theta^{12})^2 = -1/4$. (A4) now implies

$$\{\Theta, \mathbf{W}\}^2 = [2\Theta^{\alpha\beta}W^{\gamma\delta}\Gamma_{\alpha\beta\gamma\delta} - 4\Theta^{\alpha\beta}W^{\alpha\beta}\mathbb{1}_4]^2. \tag{A5}$$

Isolating the part of (A5) that is proportional to the unit matrix, one easily sees that this is [remembering $\Theta = 2\Theta^{12}\Gamma_{12}$ and $(\Theta^{12})^2 = -1/4$]

$$-16[(W^{12})^2 + (W^{34})^2 + (W^{35})^2 + (W^{45})^2]. \tag{A6}$$

In $4\mathbf{W}^2$, on the other hand, the part proportional to the unit matrix is easily seen to be $-8W^{\alpha\beta}W^{\alpha\beta}$, so that (A1) implies that all components except $W^{12}, W^{34}, W^{45}, W^{35}$ have to vanish. This implies (A2).

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