Nonequilibrium dynamics of moving mirrors in quantum fields: Influence functional and the Langevin equation

Chun-Hsien Wu* and Da-Shin Lee†

Department of Physics, National Dong-Hwa University, Hualien, Taiwan, Republic of China

(Received 24 January 2005; published 10 June 2005)

We employ the Schwinger-Keldysh formalism to study the nonequilibrium dynamics of the mirror with perfect reflection moving in a quantum field. In the case where the mirror undergoes the small displacement, the coarse-grained effective action is obtained by integrating out the quantum field with the method of influence functional. The semiclassical Langevin equation is derived, and is found to involve two levels of backreaction effects on the dynamics of mirrors: radiation reaction induced by the motion of the mirror and backreaction dissipation arising from fluctuations in quantum field via a fluctuation-dissipation relation. Although the corresponding theorem of fluctuation and dissipation for the case with the small mirror's displacement is of model independence, the study from the first principles derivation shows that the theorem is also *independent* of the regulators introduced to deal with short-distance divergences from the quantum field. Thus, when the method of regularization is introduced to compute the dissipation and fluctuation effects, this theorem must be fulfilled as the results are obtained by taking the short-distance limit in the end of calculations. The backreaction effects from vacuum fluctuations on moving mirrors are found to be hardly detected while those effects from thermal fluctuations may be detectable.

DOI: 10.1103/PhysRevD.71.125005 PACS numbers: 03.70.+k, 05.40.−a, 11.10.Wx, 42.50.Lc

I. INTRODUCTION

Zero-point fluctuations due to the imposition of the boundary conditions can lead to an impact on macroscopic physics. One of the most celebrated examples is the attractive Casimir force between two parallel conducting plates [1]. However, the dynamics of fluctuations subject to the moving boundary may also be detectable, sometimes referred to as the dynamical Casimir effects [2–9]. Consider a perfectly reflecting mirror moving in quantum fields. The boundary conditions on quantum fields corresponding to perfect reflection result in the interaction of the mirror with the fields. The motion of the mirror, which leads to the moving boundary, can create quantum radiation that in turn damps out the motion of the mirror as a result of the motion-induced radiation reaction force. In fact, as required by Lorentz invariance of quantum fields, this radiation reaction force vanishes for a motion with uniform velocity. In a motion of uniform acceleration, the mirror suffers from the same fluctuations as if it was at rest in a thermal bath due to the Unruh effects [10], also leading to the zero dissipative radiation reaction force. Fulling and Davies have computed this force for a moving mirror in a massless scalar field in the $1 + 1$ dimensional spacetime. It turns out that the induced dissipative force is proportional to the third time derivative of the mirror's position [7]. In $3 + 1$ dimensional spacetime, the problem has been studied by Ford and Vilenkin in terms of a first order approximation of the mirror's displacement. The corresponding dissipative force then is given by the fifth time derivative of the position in the nonrelativistic limit [8]. However, as we know, all quantum fields exhibit fluctuations that manifest themselves through the fluctuating forces on the mirror such as fluctuations of Casimir forces [11–14]. Thus, through a fluctuation and dissipation relation as in the case of Brownian motion, in addition to motion-induced radiation reaction, the mirror must experience the backreaction dissipation effect arising from the force fluctuations [2–5]. In this paper, a first principles derivation is provided to study the dynamics of the moving mirror by taking account of the backreaction effects from quantum fields consistently within the context of the Schwinger-Keldysh formalism. Coarse graining the degrees of freedom of quantum fields results in the coarse-grained effective action with the method of influence functional. This approach can naturally lead to the Langevin equation in the semiclassical approximation, and allows us to obtain the corresponding fluctuation and dissipation theorem from a microscopic point of view.

The problem addressed in this paper can be viewed as a special case of the larger problem of radiation reaction arising from vacuum or/and thermal fluctuations [15–17]. Especially, in vacuum, this problem can probe the nature of vacuum fluctuations and viscosity in relation with the backreaction of cosmological particle creation.

This paper is organized as follows: The theory to describe the interaction between the mirror and quantum fields is discussed in Sec. II. In Sec. III the Langevin equation in the semiclassical approximation is obtained by integrating out quantum fields. The backreaction forces are computed in Sec. VI. In Sec. V we derive the corresponding fluctuation and dissipation theorem, and discuss its applications. The calculations to obtain the dynamics of

*Electronic address: chunwu@phys.sinica.edu.tw
†Electronic address: dslee@mail.ndhu.edu.tw

the moving mirror involving backreaction effects from quantum fields in vacuum and at finite temperature, respectively, are presented in Sec. VI. We then draw the conclusions in Sec. VII.

II. FIELD PERTURBATION DRIVEN BY SMALL MIRROR'S DISPLACEMENT

We consider a mirror with perfect reflection moving in a quantum field given by a massless, minimally coupled scalar field. As a result, the corresponding boundary condition on the scalar field is as follows:

$$\phi|_S = 0. \tag{1}$$

The mirror of mass m and area A is oriented parallel to the $z = 0$ plane. We assume that the mirror has a small displacement $\delta q(t)$ along the z direction from the origin which can be obtained, for example by applying the classical external force. Then, the boundary condition above can be expressed in the specific form

$$\phi(x, y, \delta q(t), t) = 0. \tag{2}$$

To first order in $\delta q(t)$, we obtain

$$[\phi(x, y, 0, t) + \delta q(t)\partial_z \phi(x, y, 0, t) + \cdots] = 0. \tag{3}$$

We then further assume that the mirror's surface S has small perturbations induced from the motion of the mirror. This means that the quantum field ϕ can be written as

$$\phi = \phi_0 + \delta\phi, \tag{4}$$

where the field ϕ_0 corresponds to the field fluctuations with respect to the unperturbed surface S_0 at the $z = 0$ plane, while the field $\delta\phi$ is the induced fluctuations on the surface S driven by the motion of the mirror, and is of order $\delta q(t)$. Thus, together with Eqs. (3) and (4), and the vanishing boundary condition of the field ϕ_0 on S_0,

$$\phi_0(x, y, 0) = 0, \tag{5}$$

the perturbed field $\delta\phi$, to first order in $\delta q(t)$, is given by

$$\delta\phi(x, y, 0, t) = -\delta q(t)\partial_z \phi_0(x, y, 0, t). \tag{6}$$

The force acting on both sides of the mirror is given by the area integral of the $z - z$ component of the stress tensor in terms of field operators:

$$F(t) = F(0^-, t) - F(0^+, t) = \int_A dxdy[T_{zz}(x, y, 0^-, t) - T_{zz}(x, y, 0^+, t)], \tag{7}$$

where

$$T_{zz} = \frac{1}{2}[(\partial_t\phi)^2 + (\partial_z\phi)^2 - (\partial_x\phi)^2 - (\partial_y\phi)^2]. \tag{8}$$

To first order in $\delta q(t)$, we can write

$$T_{zz} = T_{0,zz} + \delta T_{zz}, \tag{9}$$

where

$$T_{0,zz} = \frac{1}{2}[(\partial_t\phi_0)^2 + (\partial_z\phi_0)^2 - (\partial_x\phi_0)^2 - (\partial_y\phi_0)^2], \tag{10}$$

$$\delta T_{zz} = \frac{1}{2}[\partial_t\phi_0\partial_t\delta\phi + \partial_t\delta\phi\partial_t\phi_0 + \partial_z\phi_0\partial_z\delta\phi + \partial_z\delta\phi\partial_z\phi_0 - \partial_x\phi_0\partial_x\delta\phi - \partial_x\delta\phi\partial_x\phi_0 - \partial_y\phi_0\partial_y\delta\phi - \partial_y\delta\phi\partial_y\phi_0]. \tag{11}$$

It leads to the following effective force term:

$$F(t) = F_0(t) + \delta q(t)\frac{\delta F}{\delta q}(t), \tag{12}$$

where

$$F_0(t) = F_0(0^-, t) - F_0(0^+, t) = \int_A dxdy[T_{0,zz}(x, y, 0^-, t) - T_{0,zz}(x, y, 0^+, t)], \tag{13}$$

$$\delta q(t)\frac{\delta F}{\delta q}(t) = \delta q(t)\left[\frac{\delta F}{\delta q}(0^-, t) + \frac{\delta F}{\delta q}(0^+, t)\right] = \delta q(t)\int_A dx\, dy\left[\frac{\delta T_{zz}}{\delta q}(x, y, 0^-, t) + \frac{\delta T_{zz}}{\delta q}(x, y, 0^+, t)\right]. \tag{14}$$

Notice that the $+$ sign for the perturbed force in Eq. (14) is due to the fact that there is a sign difference for the mirror's displacement seen from the forces in the opposite sides of the mirror. The δq in Eq. (14) is defined to be the mirror's displacement with respect to the force from $z = 0^-$.

Thus, the Lagrangian can be expressed as

$$L[\delta q, \phi_0] = \frac{1}{2}m(\delta\dot{q})^2 - V(\delta q) + \delta q(t)F_0(t) + \frac{1}{2}\delta q^2(t) \times \frac{\delta F}{\delta q}(t) + \int d^3\mathbf{x}\left[\frac{1}{2}(\partial_t\phi_0)^2 - \frac{1}{2}(\vec{\nabla}\phi_0)^2\right], \tag{15}$$

which is subject to the boundary condition on the field ϕ_0 given by

$$\phi_0(x, y, 0, t) = 0. \tag{16}$$

The classical external force is also considered to apply to the mirror with potential energy $V(\delta q)$. Units with $\hbar = c = 1$ are used, and factors $\hbar$ and c will be restored in our main results.

Notice that the first term in the right hand side of Eq. (12) given by the homogeneous background scalar field is evaluated at the unperturbed surface S_0, where by symmetry, the mean pressure force vanishes as the forces are cancelled from both sides of the mirror, i.e.,

$$\langle F_0(t)\rangle = \langle F_0(0^-, t)\rangle - \langle F_0(0^+, t)\rangle = 0. \tag{17}$$

However, this force undergoes fluctuations about its mean value due to quantum and/or thermal effects, and will influence the dynamics of the mirror. In addition, $\langle \delta F/\delta q\rangle \delta q(t)$ is the force arising from the motion of the mirror. This motion-induced radiation reaction force has been extensively studied in the case of the background scalar field in vacuum as well as in thermal equilibrium respectively [7,8,14].

Here we employ the Schwinger and Keldysh formalism to obtain the influence functional on the moving mirror by integrating out the scalar field with the Lagrangian given by Eq. (15). Recall that Eq. (6) is based upon the fact that the mirror undergoes the small displacement where the Lagrangian in Eq. (15) is correct up to order $\mathcal{O}(\delta q^2)$. We then implement the semiclassical approximation by assuming that the quantum fluctuations coming from the mirror itself can be ignored to obtain its semiclassical Langevin equation. It can be justified by the fact that the typical size of the mirror is much larger than its Compton wavelength. Under this semiclassical approximation, the dynamics of the mirror is governed by the coarse-grained effective action involving the influence functional. However, for a general interacting field theory, one cannot obtain the influence functional that includes all of the quantum loop effects by integrating out the scalar field. However, here we obtain the influence functional including all quantum effects up to order $\mathcal{O}(\delta q^2)$ consistent with the approximation in the Lagrangian [Eq. (15)] we mention above. It is then expected that in addition to the classical dynamical equation given by the external potential $V(\delta q)$, the obtained semiclassical Langevin equation will involve the backreaction force terms arising from the quantum effects of the scalar field where the terms are kept up to δq. The terms we ignore are say, $\delta \dot{q}^2$, $\delta q \delta \ddot{q}$, so on and so forth. On top of that, the noise force with the Gaussian correlation function will be introduced to mimic the stochastic dynamics from the scalar field fluctuations.

III. INFLUENCE FUNCTIONAL AND LANGEVIN EQUATION

We now consider the case where an initial density matrix for the mirror plus the scalar field at $t = t_i$ is factorized as

$$\hat{\rho}(t_i) = \hat{\rho}_{\rm mirror}(t_i) \otimes \hat{\rho}_{\phi_0}(t_i), \tag{18}$$

where we have assumed that the mirror and the scalar field are initially uncoupled. The mirror initially is assumed to be in its position eigenstate with the eigenvalue δq_i given by

$$\hat{\rho}_{\rm mirror}(t_i) = |\delta q_i, t_i\rangle\langle \delta q_i, t_i|. \tag{19}$$

However, the scalar field is in thermal equilibrium at temperature $T = 1/\beta$ with the density matrix

$$\hat{\rho}_{\phi_0}(t_i) = e^{-\beta H_{\phi_0}}, \tag{20}$$

where H_{ϕ_0} is the Hamiltonian for the free scalar field given from Eq. (15). The zero-temperature limit corresponding to the initial vacuum state for the scalar field can be studied by taking $T \to 0$. The interaction between the mirror and the scalar field is considered to switch on at $t = t_i$. Then, in the Schrödinger picture, the density matrix evolves in time as

$$\hat{\rho}(t_f) = U(t_f, t_i)\hat{\rho}(t_i)U^{-1}(t_f, t_i) \tag{21}$$

with $U(t_f, t_i)$, the time evolution operator. Thus, the nonequilibrium partition function can be defined as

$$\mathcal{Z} = {\rm Tr}(U(t_f, t_i)\hat{\rho}(t_i)U^{-1}(t_f, t_i)). \tag{22}$$

We then insert an identity in terms of a complete set of the mirror plus field eigenstates,

$$\int dq\, d\phi |q, \phi\rangle\langle q, \phi| = 1, \tag{23}$$

between all time evolution operators where the mirror plus field state denoted as $|q, \phi\rangle$ is given by the direct product of the state of the mirror and that of the scalar field, namely, $|q, \phi\rangle = |q\rangle \otimes |\phi\rangle$. Then, the nonequilibrium partition function becomes

$$\begin{aligned}\mathcal{Z} &= \int d\delta q_1 d\phi_1 \int d\delta q_2 d\phi_2 \int d\delta q_3 d\phi_3 \langle \delta q_1, \phi_1|U(t_f, t_i)|\delta q_2, \phi_2\rangle\langle \delta q_2, \phi_2|\hat{\rho}(t_i)|\delta q_3, \phi_3\rangle\langle \delta q_3, \phi_3|U^{-1}(t_f, t_i)|\delta q_1, \phi_1\rangle \\ &= \int d\delta q_1 \int \mathcal{D}\delta q^+ \mathcal{D}\delta q^- \int d\phi_1 d\phi_2 d\phi_3 \int \mathcal{D}\phi_0^+ \mathcal{D}\phi_0^- \exp\left\{i\int_{t_i}^{t_f} dt[L[\delta q^+, \phi_0^+] - L[\delta q^-, \phi_0^-]]\right\} \\ &\quad \times \langle \phi_2|\hat{\rho}_{\phi_0}(t_i)|\phi_3\rangle, \end{aligned} \tag{24}$$

with the boundary conditions: $\phi_0^+(\mathbf{x}, t_f) = \phi_0^-(\mathbf{x}, t_f) = \phi_1(\mathbf{x})$, $\phi_0^+(\mathbf{x}, t_i) = \phi_2(\mathbf{x})$, $\phi_0^-(\mathbf{x}, t_i) = \phi_3(\mathbf{x})$ as well as $\delta q^+(t_f) = \delta q^-(t_f) = \delta q_1$, $\delta q^+(t_i) = \delta q^-(t_i) = \delta q_i$. This method for studying nonequilibrium phenomena has been developed by Schwinger and Keldysh [18]. In recent years, it has been applied in particle physics and cosmology by one of us [19,20].

Then, we can obtain the coarse-grained effective action from the nonequilibrium partition function

$$\mathcal{Z} = \int dq_1 \int \mathcal{D}\delta q^+ \mathcal{D}\delta q^- \exp iS[\delta q^+, \delta q^-] \quad (25)$$

that involves the influence functional $\mathcal{F}[\delta q^+, \delta q^-]$ by integrating out the degrees of freedom of the scalar field given by

$$S[\delta q^+, \delta q^-] = \left\{\left[\frac{1}{2}m(\delta\dot{q}^+)^2 - V(\delta q^+)\right] - \left[\frac{1}{2}m(\delta\dot{q}^-)^2 - V(\delta q^-)\right]\right\} - i\ln\mathcal{F}[\delta q^+, \delta q^-]. \quad (26)$$

In the semiclassical approximation where we ignore the quantum fluctuations from the mirror itself, the dynamics of the mirror is governed by the above coarse-grained effective action $S[\delta q^+, \delta q^-]$.

To obtain the influence functional, we now construct the real-time Green's functions for the scalar field ϕ_0 with the boundary condition in Eq. (16). The field can be expanded in terms of the creation and annihilation operators which obey the commutation relation with the proper choice of the mode functions:

$$\phi_0(\mathbf{x}, t) = \int \frac{dk_\perp}{(2\pi)} \int \frac{d^2\mathbf{k}_\parallel}{(2\pi)^2} \frac{i\sin(k_\perp z)}{\sqrt{k}} \times [(a_\mathbf{k} e^{-ikt} + a^\dagger_{-\mathbf{k}} e^{ikt})\Theta(z) + (b_\mathbf{k} e^{-ikt} + b^\dagger_{-\mathbf{k}} e^{ikt})\Theta(-z)] e^{i\mathbf{k}_\parallel\cdot\mathbf{x}_\parallel} \quad (27)$$

with $\mathbf{x} = (\mathbf{x}_\parallel, z)$, and $\mathbf{k} = (\mathbf{k}_\parallel, k_\perp)$, $k = |\mathbf{k}|$ for a massless scalar field. We have assumed that the area of the mirror A is large as compared with the relevant length scales under consideration so that the scalar field can be expanded with respect to an infinite area. However, the area A can be obtained as the quantum effects of the scalar field on the mirror are included from all over the mirror's surface. As we will see, the results we define to measure are in general for per unit area. The mirror of perfect reflection, which is thus of impermeability to the quantum scalar field, means that the fluctuations from opposite sides of the mirror have no correlation, thus leading to the commutability between $a_\mathbf{k}, a^\dagger_\mathbf{k}$ and $b_\mathbf{k}, b^\dagger_\mathbf{k}$. The essential ingredients to perturbative calculations are the following Green's functions where $\mathbf{x}, \mathbf{x}'$ are in the same side of the mirror:

$$\begin{aligned} G_0^{++}(\mathbf{x}, \mathbf{x}'; t, t') &= G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t')\Theta(t-t') + G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t')\Theta(t'-t), \\ G_0^{--}(\mathbf{x}, \mathbf{x}'; t, t') &= G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t')\Theta(t'-t) + G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t')\Theta(t-t'), \\ G_0^{+-}(\mathbf{x}, \mathbf{x}'; t, t') &= G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t'), \\ G_0^{-+}(\mathbf{x}, \mathbf{x}'; t, t') &= G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t'); \\ G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t') &= \langle \phi_0(\mathbf{x}, t)\phi_0(\mathbf{x}', t')\rangle = \mathrm{Tr}(\hat{\rho}_\phi \phi_0(\mathbf{x}, t)\phi_0(\mathbf{x}', t')), \\ G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t') &= \langle \phi_0(\mathbf{x}', t')\phi_0(\mathbf{x}, t)\rangle = \mathrm{Tr}(\hat{\rho}_\phi \phi_0(\mathbf{x}', t')\phi_0(\mathbf{x}, t)). \end{aligned} \quad (28)$$

Using the field expansion in Eq. (27), the Green's functions can be expressed as

$$\begin{aligned} G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t') &= G^{>}(\mathbf{x}-\mathbf{x}'; t-t') - G^{>}(\mathbf{x}-\bar{\mathbf{x}}'; t-t'), \\ G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t') &= G^{<}(\mathbf{x}-\mathbf{x}'; t-t') - G^{<}(\mathbf{x}-\bar{\mathbf{x}}'; t-t'), \end{aligned} \quad (29)$$

where the Green's functions in the right hand side of the above expressions are the corresponding functions in free space given by

$$\begin{aligned} G^{>}(\mathbf{x}-\mathbf{x}'; t-t') &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle \phi_\mathbf{k}(t)\phi_{-\mathbf{k}}(t')\rangle e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2k}[(1+n_\mathbf{k})e^{-ik(t-t')} + n_\mathbf{k} e^{ik(t-t')}]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}; \\ G^{<}(\mathbf{x}-\mathbf{x}'; t-t') &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle \phi_{-\mathbf{k}}(t')\phi_\mathbf{k}(t)\rangle e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2k}[n_\mathbf{k} e^{-ik(t-t')} + (1+n_\mathbf{k})e^{ik(t-t')}]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \end{aligned} \quad (30)$$

The point $\bar{\mathbf{x}} = (x, y, -z)$ is the mirror image of the point $\mathbf{x} = (x, y, z)$ with respect to the unperturbed mirror's surface S_0 at the $z = 0$ plane. From Eq. (29), we can derive the following useful identities:

$$\partial_{t'} G_0^{<(>)}(\mathbf{x}, \mathbf{x}'; t, t')|_{z'=0} = \partial_{x' \text{ or } y'} G_0^{<(>)}(\mathbf{x}, \mathbf{x}'; t, t')|_{z'=0} = 0, \qquad \partial_{z'} G_0^{<(>)}(\mathbf{x}, \mathbf{x}'; t, t')|_{z'=0} = 2\partial_{z'} G^{<(>)}(\mathbf{x}, \mathbf{x}'; t, t')|_{z'=0}, \quad (31)$$

where the Green's functions are evaluated on the mirror's surface S_0. The momentum integral can be carried out to obtain the Green's functions in terms of space and time as

$$
\begin{aligned}
G^{>}(\mathbf{x}-\mathbf{x}';t-t') &= \mathrm{Re}[G(\mathbf{x}-\mathbf{x}';t-t')] + i\mathrm{Im}[G(\mathbf{x}-\mathbf{x}';t-t')],\\
G^{<}(\mathbf{x}-\mathbf{x}';t-t') &= \mathrm{Re}[G(\mathbf{x}-\mathbf{x}';t-t')] - i\mathrm{Im}[G(\mathbf{x}-\mathbf{x}';t-t')],
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
\mathrm{Re}\,[G(\mathbf{x}-\mathbf{x}';t-t')] &= \frac{\pi k_{\mathrm{B}}T}{8\pi^2|\mathbf{x}-\mathbf{x}'|}\{\coth[\pi k_{\mathrm{B}}T(t-t'+|\mathbf{x}-\mathbf{x}'|)] - \coth[\pi k_{\mathrm{B}}T(t-t'-|\mathbf{x}-\mathbf{x}'|)]\},\\
\mathrm{Im}[G(\mathbf{x}-\mathbf{x}';t-t')] &= \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}'|}\{\delta[t-t'+|\mathbf{x}-\mathbf{x}'|] - \delta[t-t'-|\mathbf{x}-\mathbf{x}'|]\}.
\end{aligned} \tag{33}
$$

Up to order $\mathcal{O}(\delta q^2)$ consistent with the approximation on the Lagrangian in Eq. (15), the influence functional is given by

$$
\begin{aligned}
\mathcal{F}[\delta q^+, \delta q^-] = \exp\bigg\{ & i\int dt\bigg[\frac{1}{2}(\delta q^+(t))^2\Big\langle\frac{\partial F}{\partial q}\Big\rangle(t) - \frac{1}{2}(\delta q^-(t))^2\Big\langle\frac{\partial F}{\partial q}\Big\rangle(t)\bigg]\\
& - \frac{1}{2}\int dt\int dt'[\delta q^+(t)\langle F_0^+(t)F_0^+(t')\rangle\delta q^+(t') + \delta q^-(t)\langle F_0^-(t)F_0^-(t')\rangle\delta q^-(t') - \delta q^+(t)\langle F_0^+(t)F_0^-(t')\rangle\\
& \times \delta q^-(t') - \delta q^-(t)\langle F_0^-(t)F_0^+(t')\rangle\delta q^+(t')]\bigg\}.
\end{aligned} \tag{34}
$$

The nonequilibrium force-force correlation functions are defined as follows:

$$
\begin{aligned}
\langle F_0^+(t)F_0^+(t')\rangle &= \langle F_0(t)F_0(t')\rangle\Theta(t-t') + \langle F_0(t')F_0(t)\rangle\Theta(t'-t),\\
\langle F_0^-(t)F_0^-(t')\rangle &= \langle F_0(t)F_0(t')\rangle\Theta(t'-t) + \langle F_0(t')F_0(t)\rangle\Theta(t-t'),\\
\langle F_0^+(t)F_0^-(t')\rangle &= \langle F_0(t')F_0(t)\rangle \equiv \mathrm{Tr}(\hat{\rho}_\phi F_0(t')F_0(t)),\\
\langle F_0^-(t)F_0^+(t')\rangle &= \langle F_0(t)F_0(t')\rangle \equiv \mathrm{Tr}(\hat{\rho}_\phi F_0(t)F_0(t')).
\end{aligned} \tag{35}
$$

Together with Eqs. (7)–(14), the force Green's functions can be written in terms of that of the scalar field.

To obtain the semiclassical Langevin equation, it is more convenient to change variables to the average and relative coordinates:

$$
\delta q = \frac{1}{2}(\delta q^+ + \delta q^-), \qquad \delta r = \delta q^+ - \delta q^-. \tag{36}
$$

The coarse-grained effective action defined in Eq. (26) with the influence functional in Eq. (34) then becomes

$$
\begin{aligned}
S[\delta q, \delta r] = & \int dt\,\delta r(t)\bigg[-m\delta\ddot{q}(t) - \frac{\delta V}{\delta q}(t) + \Big\langle\frac{\partial F}{\partial q}\Big\rangle\delta q(t) + \int dt'\chi_{FF}(t-t')\delta q(t')\bigg]\\
& + \frac{i}{2}\int dt\int dt'\,\delta r(t)\sigma_{FF}(t-t')\delta r(t') + \mathcal{O}(\delta r^3),
\end{aligned} \tag{37}
$$

where

$$
\chi_{FF}(t-t') = i\Theta(t-t')\langle[F_0(t), F_0(t')]\rangle, \tag{38}
$$

$$
\sigma_{FF}(t-t') = \frac{1}{2}\langle\{F_0(t), F_0(t')\}\rangle. \tag{39}
$$

We then further introduce an auxiliary quantity $\eta(t)$, the noise force, with the distribution function in terms of the Gaussian form:

$$
P[\eta(t)] = \exp\bigg\{-\frac{1}{2}\int dt\int dt'\,\eta(t)\sigma_{FF}^{-1}(t-t')\eta(t')\bigg\}. \tag{40}
$$

In terms of the noise force $\eta(t)$, the above coarse-grained action S can be written as the field integration over $\eta(t)$ given by

$$\exp iS = \int \mathcal{D}\eta P[\eta(t)] \exp iS_{\rm eff}[\delta q, \delta r, \eta], \tag{41}$$

with the effective action $S_{\rm eff}$:

$$S_{\rm eff}[\delta q, \delta r, \eta] = \int dt \delta r(t)\bigg[-m\delta\ddot{q}(t) - \frac{\delta V}{\delta q}(t) + \bigg\langle\frac{\partial F}{\partial q}\bigg\rangle\delta q(t) + \int dt' \chi_{FF}(t-t')\delta q(t') + \eta(t)\bigg] + \mathcal{O}(\delta r^3). \tag{42}$$

The semiclassical approximation requires to extremize the effective action $\delta S_{\rm eff}/\delta r$ with respect to a particular trajectory of the mirror. The lowest order equation of motion for $\delta q(t)$ where the terms beyond order $\mathcal{O}(\delta r^3)$ are ignored, can be obtained as follows:

$$m\delta\ddot{q}(t) + \frac{\delta V}{\delta q}(t) - \bigg\langle\frac{\partial F}{\partial q}\bigg\rangle\delta q(t) - \int dt' \chi_{FF}(t-t')\delta q(t') = \eta(t). \tag{43}$$

The noise force correlation function is of the Gaussian form given by Eq. (40):

$$\langle \eta(t)\eta(t')\rangle = \sigma_{FF}(t-t'). \tag{44}$$

This is a typical Langevin equation. It contains all of quantum corrections arising from the scalar field which are linear in δq. The terms we ignored above involve the coupling between δr and δq. A consistent improvement over this semiclassical Langevin equation will involve a perturbation expansion in these terms.

Here we would like to point out that this Langevin equation reveals two levels of backreaction effects on the dynamics of the mirror. They are radiation reaction induced by the motion of the mirror as well as backreaction dissipation arising from fluctuations in quantum fields via a fluctuation-dissipation theorem. Both of them are valid in a first order expansion in the mirror's displacement. In fact, the term for motion-induced radiation reaction is given by the variation of the *mean pressure force* from the quantum field that responds to the small displacement of the mirror. The backreaction dissipation effect involving the nonlocal kernel obtained from the *force correlations* that reflects the general non-Markovian nature of the pressure forces is balanced by the force fluctuations. The kernel of the dissipative force can be obtained from the commutator of the forces in Eq. (38), and however, the autocorrelation function for the noise forces is given by the anticommutator of the forces in Eq. (39). Thus, the balance between the effects from dissipation and fluctuation can be encoded in the underlying fluctuation-dissipation theorem which we can compute explicitly in this work. In general, when the full dynamics between the mirror and quantum fields is considered, the above two backreaction effects have to be treated in a self-consistent way [2–5] where one may find the dissipation effect via a fluctuation-dissipation relation on the uniform accelerated particle in which radiation reaction vanishes.

IV. BACKREACTION FORCES

We now try to compute the backreaction forces. Here we mainly follow the approach developed by Ford and Vilenkin to obtain motion-induced radiation reaction [8]. From Eq. (14), we have

$$\bigg\langle\frac{\delta F}{\delta q}\bigg\rangle(0^+, t)\delta q(t) = \bigg\langle\frac{\delta F}{\delta q}\bigg\rangle(0^-, t)\delta q(t) \tag{45}$$

by symmetry. Thus, the motion-induced force from one side of the mirror needs to be computed. Technically speaking, it is known that the expectation values of stress tensors and stress tensor correlation functions will confront short-distance divergences in the coincidence limit. The method of the point splitting will be adopted to regularize these quantities where we take the fields in all products of the stress tensor at different points, and the same point limit is taken after doing renormalization. To do so, the expectation value of the motion-induced force obtained from Eqs. (11) and (14) now becomes

$$\bigg\langle\frac{\delta F}{\delta q}\bigg\rangle\delta q(t) = 2\int_A d^2\mathbf{x}_\parallel \bigg\langle\frac{\delta T_{zz}}{\delta q}\bigg\rangle(\mathbf{x}_\parallel, 0^-, t)\delta q(t) \tag{46}$$

with

$$\bigg\langle\frac{\delta T_{zz}}{\delta q}\bigg\rangle(\mathbf{x}_\parallel, z, t)\delta q(t) = \frac{1}{4}(\partial_t\partial_{t'} + \partial_z\partial_{z'} - \partial_x\partial_{x'} - \partial_y\partial_{y'})[\langle\phi_0(\mathbf{x}, t)\delta\phi(\mathbf{x}', t')\rangle + \langle\phi_0(\mathbf{x}', t')\delta\phi(\mathbf{x}, t)\rangle + \langle\delta\phi(\mathbf{x}, t)\phi_0(\mathbf{x}', t')\rangle + \langle\delta\phi(\mathbf{x}', t')\phi_0(\mathbf{x}, t)\rangle]|_{\mathbf{x}'_\parallel\to\mathbf{x}_\parallel, z'\to z, t'\to t+\epsilon}, \tag{47}$$

where ϵ is introduced for the point-splitting method. The limit of $\epsilon \to 0$ will be taken, and the motion-induced force expects to be finite in this limit [8]. The perturbed field due to the motion of the mirror in Eq. (6) can be written involving the retarded Green's function as (see Ref. [8] for details):

$$\delta\phi(\mathbf{x}_{\|}, z, t)|_{z\to 0^-} = -\int dt' \int_A d^2\mathbf{x}'_{\|}\partial_{z'}G_0^{\rm Ret}(\mathbf{x}, \mathbf{x}'; t, t')\partial_{z'}\phi_0(\mathbf{x}', t')\delta q(t')|_{z',z\to 0^-}. \tag{48}$$

The retarded Green's function is defined to be

$$G_0^{\rm Ret}(\mathbf{x}, \mathbf{x}'; t, t') \equiv i\Theta(t-t')\langle[\phi_0(\mathbf{x}, t), \phi_0(\mathbf{x}', t')]\rangle = i\Theta(t-t')[G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t') - G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t')]. \tag{49}$$

Putting all together, the motion-induced force term becomes

$$\begin{aligned}\left\langle\frac{\delta F}{\delta q}\right\rangle\delta q(t) = &-\int^t dt'\delta q(t')\bigg\{8\int_A d^2\mathbf{x}_{\|}\int_A d^2\mathbf{x}'_{\|}[\partial_t\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]\partial_t\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t')]\\ &+\partial_z\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]\partial_z\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t')] - \partial_x\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]\partial_x\partial_{z'}\\ &\times{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t')] - \partial_y\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]\partial_y\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t')]\\ &+\partial_t\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t')]\partial_t\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)] + \partial_z\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t')]\partial_z\partial_{z'}\\ &\times{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)] - \partial_x\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t')]\partial_x\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]\\ &-\partial_y\partial_{z'}{\rm Im}[G(\mathbf{x}-\mathbf{x}'; t-t')]\partial_y\partial_{z'}{\rm Re}[G(\mathbf{x}-\mathbf{x}'; t-t'+\epsilon)]]\bigg\}\bigg|_{z',z\to 0^-}^{\epsilon\to 0},\end{aligned} \tag{50}$$

where we have used Eqs. (31) and (32). The retardation effect is included as the time integration in t' runs to the time t. In addition, the force above will be evaluated at the surface of the mirror by taking the limits of $z', z \to 0^-$. It will suffer from short-distance divergences that we will discuss later [8].

The force-force correlation function evaluated on the unperturbed mirror's surface S_0 at rest can be expressed as

$$\langle F_0(t)F_0(t')\rangle = \langle F_0(t)F_0(t')\rangle - \langle F_0(t)\rangle\langle F_0(t')\rangle = 2[\langle F_0(0^-, t)F_0(0^-, t')\rangle - \langle F_0(0^-, t)\rangle\langle F_0(0^-, t')\rangle], \tag{51}$$

where we have used the fact that for a static mirror the mean pressure force vanishes. In addition, the force-force correlations of each side of the mirror are the same by symmetry, and there is no correlation between the forces from opposite sides of the mirror, namely,

$$\begin{aligned}\langle F_0(0^-, t)F_0(0^-, t')\rangle - \langle F_0(0^-, t)\rangle\langle F_0(0^-, t')\rangle &= \langle F_0(0^+, t)F_0(0^+, t')\rangle - \langle F_0(0^+, t)\rangle\langle F_0(0^+, t')\rangle,\\ \langle F_0(0^{\mp}, t)F_0(0^{\pm}, t')\rangle &= \langle F_0(0^{\mp}, t)\rangle\langle F_0(0^{\pm}, t')\rangle.\end{aligned} \tag{52}$$

However, one may expect that motion-induced radiation reaction on opposite sides of the mirror might have correlations as they both arise due to the motion of the mirror even though the induced forces on opposite sides of the mirror can not communicate with each other. This correlation effect will contribute to the Langevin equation where it is of order $\mathcal{O}(\delta q^3)$, and can be neglected here [21].

Using Eq. (10), we can write the above correlation functions in terms of the Green's functions of the scalar field given by

$$\begin{aligned}\langle F_0(t)F_0(t')\rangle &= 2\int_A d^2\mathbf{x}_{\|}\int_A d^2\mathbf{x}'_{\|}[\langle T_{zz}(\mathbf{x}_{\|}, 0^-; t)T_{zz}(\mathbf{x}'_{\|}, 0^-; t')\rangle - \langle T_{zz}(\mathbf{x}_{\|}, 0^-; t)\rangle\langle T_{zz}(\mathbf{x}'_{\|}, 0^-; t')\rangle],\\ \langle F_0(t')F_0(t)\rangle &= 2\int_A d^2\mathbf{x}_{\|}\int_A d^2\mathbf{x}'_{\|}[\langle T_{zz}(\mathbf{x}'_{\|}, 0^-; t')T_{zz}(\mathbf{x}_{\|}, 0^-; t)\rangle - \langle T_{zz}(\mathbf{x}'_{\|}, 0^-; t')\rangle\langle T_{zz}(\mathbf{x}_{\|}, 0^-; t)\rangle],\end{aligned} \tag{53}$$

where

$$\begin{aligned}&\langle T_{zz}(\mathbf{x}_{\|}, z; t)T_{zz}(\mathbf{x}'_{\|}, z'; t')\rangle - \langle T_{zz}(\mathbf{x}_{\|}, z; t)\rangle\langle T_{zz}(\mathbf{x}'_{\|}, z'; t')\rangle\\ &= \frac{1}{4}\{(\partial_t\partial_{t''} + \partial_z\partial_{z''} - \partial_x\partial_{x''} - \partial_y\partial_{y''})(\partial_{t'}\partial_{t'''} + \partial_{z'}\partial_{z'''} - \partial_{x'}\partial_{x'''} - \partial_{y'}\partial_{y'''})[G_0^{>}(\mathbf{x}, \mathbf{x}'; t, t')G_0^{>}(\mathbf{x}'', \mathbf{x}'''; t'', t''')\\ &\quad + G_0^{>}(\mathbf{x}, \mathbf{x}'''; t, t''')G_0^{>}(\mathbf{x}'', \mathbf{x}'; t'', t')]\}|_{\mathbf{x}'''_{\|}\to\mathbf{x}'_{\|}, z'''\to z', t'''\to t'+\epsilon''}^{\mathbf{x}''_{\|}\to\mathbf{x}_{\|}, z''\to z, t''\to t+\epsilon'},\\ &\langle T_{zz}(\mathbf{x}'_{\|}, z'; t')T_{zz}(\mathbf{x}_{\|}, z; t)\rangle - \langle T_{zz}(\mathbf{x}'_{\|}, z'; t')\rangle\langle T_{zz}(\mathbf{x}_{\|}, z; t)\rangle\\ &= \frac{1}{4}\{(\partial_t\partial_{t''} + \partial_z\partial_{z''} - \partial_x\partial_{x''} - \partial_y\partial_{y''})(\partial_{t'}\partial_{t'''} + \partial_{z'}\partial_{z'''} - \partial_{x'}\partial_{x'''} - \partial_{y'}\partial_{y'''})[G_0^{<}(\mathbf{x}, \mathbf{x}'; t, t')G_0^{<}(\mathbf{x}'', \mathbf{x}'''; t'', t''')\\ &\quad + G_0^{<}(\mathbf{x}, \mathbf{x}'''; t, t''')G_0^{<}(\mathbf{x}'', \mathbf{x}'; t'', t')]\}|_{\mathbf{x}'''_{\|}\to\mathbf{x}'_{\|}, z'''\to z', t'''\to t'+\epsilon''}^{\mathbf{x}''_{\|}\to\mathbf{x}_{\|}, z''\to z, t''\to t+\epsilon'}.\end{aligned} \tag{54}$$

The limits of $\epsilon', \epsilon'' \to 0$ due to the point splitting will be taken. We then evaluate the correlation functions at the surface of the mirror by taking the limits of $z, z' \to 0^-$. The above expressions can be simplified with Eq. (31) as

$$\begin{aligned}
\langle F_0(t)F_0(t')\rangle = 2\int_A d^2\mathbf{x}_\parallel \int_A d^2\mathbf{x}'_\parallel [&\partial_t\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t')\partial_t\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon'')) \\
&+\partial_z\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t')\partial_z\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon''))-\partial_x\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t')\partial_x\partial_{z'} \\
&\times G^>(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon''))-\partial_y\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t')\partial_y\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon'')) \\
&+\partial_t\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_t\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')+\partial_z\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_z\partial_{z'} \\
&\times G^>(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')-\partial_x\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_x\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+\epsilon') \\
&-\partial_y\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_y\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')]|^{\epsilon',\epsilon''\to 0}_{z',z''\to 0}, \\
\langle F_0(t')F_0(t)\rangle = 2\int_A d^2\mathbf{x}_\parallel \int_A d^2\mathbf{x}'_\parallel [&\partial_t\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t')\partial_t\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon'')) \\
&+\partial_z\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t')\partial_z\partial_{z'}G^>(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon''))-\partial_x\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t')\partial_x\partial_{z'} \\
&\times G^<(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon''))-\partial_y\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t')\partial_y\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'+(\epsilon'-\epsilon'')) \\
&+\partial_t\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_t\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')+\partial_z\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_z\partial_{z'} \\
&\times G^<(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')-\partial_x\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_x\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'+\epsilon') \\
&-\partial_y\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'-\epsilon'')\partial_y\partial_{z'}G^<(\mathbf{x}-\mathbf{x}';t-t'+\epsilon')]|^{\epsilon',\epsilon''\to 0}_{z',z\to 0},
\end{aligned} \tag{55}$$

In this stage, we can compute the commutator, χ_{FF} and the anticommutator, σ_{FF} of the forces in Eqs. (38) and (39), respectively, which allow us to discuss the issue of the fluctuation-dissipation theorem below. In particular, it is a straightforward calculation to show that

$$\left\langle \frac{\delta F}{\delta q} \right\rangle \delta q(t) = \int dt' \chi_{FF}(t-t')\delta q(t'), \tag{56}$$

using the Green's functions of the scalar field in Eqs. (32) and (33). The above relation holds only for the mirror with the small displacement where the coupling of the mirror to the quantum field is quadratic in field variables. Similar result that relates these two backreaction effects has been found in Ref. [9] in the $1+1$ dimensional spacetime. Even though the fluctuation-dissipation relation we will discuss later can link the dissipation effect obtained from force correlations to the force fluctuations, one cannot conclude that radiation reaction due to the motion of the mirror is balanced by the force fluctuations. Notice that it has been recently mentioned by Hu [2–4] that there are incorrect claims in which radiation reaction is balanced by the force fluctuations. We would like to emphasize that Eq. (56) does not hold for the general situations of couplings. For example, one can consider the coupling between the mirror and the quantum field which is proportional to field variables to the nth power. For an odd number of the power, it is obvious that diagrammatically the above radiation reaction vanishes as the dissipation effect from force fluctuations gives the nonzero contribution to the Langevin equation. However, as for an even number of the power, since the effect of radiation reaction is given by the $(n/2)$-loop integral while the dissipation effect is given by the $(n-1)$-loop integral, two backreaction effects can possibly be equal only for $n = 2$.

V. FLUCTUATION-DISSIPATION THEOREM

Fluctuation-dissipation theorem plays a vital role in balancing between these two effects to dynamically stabilize a nonequilibrium Brownian motion in the presence of external fluctuation forces. In the case of classical Brownian motion, the nonequilibrium dynamics of the Brownian object moving in a stationary fluid can be described by a phenomenological Langevin equation. Incessant collisions from the molecules of the fluid with the Brownian object produce both resistance to the motion of the object and fluctuations in its trajectory. The Langevin equation can account for these two effects by introducing friction and dissipation as well as a stochastic force as below:

$$\ddot{q}(t) + \gamma\dot{q}(t) = \eta(t), \tag{57}$$

where the dissipative force is given by the time derivative of the position with the damping coefficient γ, and $\eta(t)$ stands for a stochastic force that mimics random kicks of the molecules on the Brownian object with white noise properties:

$$\langle\eta(t)\rangle = 0; \qquad \langle\eta(t)\eta(t')\rangle = 2\gamma k_B T\delta(t-t'). \tag{58}$$

k_B is Boltzmann constant and the average is taken with respect to the thermal ensemble of fluctuations of the fluid at temperature T. The dissipation and fluctuation kernels can be defined, respectively, as

$$\gamma\dot{q}(t) = -\int dt'\mu(t-t')q(t'),$$
$$\mu(t-t') = -\gamma\frac{d}{dt}\delta(t-t'); \qquad \langle\eta(t)\eta(t')\rangle = \nu(t-t'),$$
$$\nu(t-t') = 2\gamma k_B T\delta(t-t'). \tag{59}$$

The fluctuation-dissipation theorem is to relate the dissipation kernel to the fluctuation kernel of the form

$$\mu(t-t') = -\frac{1}{2k_B T}\frac{d}{dt}\nu(t-t'), \tag{60}$$

which is independent of the spectrum density of thermal fluid and the coupling strength of the Brownian object with the molecules in a fluid.

A very clear microscopic description to the Langevin equation within the context of one-particle quantum mechanics coupled to a bath of harmonic oscillators has been presented by Caldeira and Leggett [22]. Using the Feynman-Vernon influence functional, their study reveals that in general the dissipation term arises from a local approximation to the non-Markovian kernel for a particular choice of the density of states of the heat bath, and as a result, the noise forces become uncorrelated over macroscopic time scales larger than the typical scales determined by the bath. They are thus related by the above classical fluctuation and dissipation theorem. Recently, the studies have been devoted to this issue where the Brownian object is coupled to quantum fields. The coupling between the Brownian object and quantum fields is assumed to be linear or nonlinear in terms of the variable of the Brownian object, and it is linear in terms of the field variable in which the field in momentum space can be treated as a bath of harmonic oscillators [2–6]. However, the case we consider is more complicated since the coupling of the mirror to quantum fields is given by the area integral of the stress tensor which is quadratic in fields [9,14]. In the presence of a perfectly reflecting mirror, we impose an idealized boundary condition on quantum fields where the fields vanish on the surface of the mirror. This unrealistic boundary condition in fact leads to a troublesome result: the stress tensor is divergent when it is evaluated on the surface of the mirror [8,12]. Hence we need to introduce a cutoff on z, the distance to the mirror, thus resulting in some complications as we try to derive the corresponding fluctuation-dissipation theorem from a microscopic point of view.

To obtain the corresponding fluctuation and dissipation theorem, we take the Fourier transform of $\chi_{FF}(t-t')$ and $\sigma_{FF}(t-t')$ as

$$\chi_{FF}(t-t') = \int\frac{dw}{2\pi}\chi_{FF}(\omega)e^{-i\omega(t-t')},$$
$$\sigma_{FF}(t-t') = \int\frac{dw}{2\pi}\sigma_{FF}(\omega)e^{-i\omega(t-t')}. \tag{61}$$

Then, we introduce the spectral density $\rho(\omega)$ of quantum field defined to be

$$\chi_{FF}(\omega) = \int\frac{dw'}{2\pi}\frac{\rho(\omega')}{w-w'+i\delta}, \tag{62}$$

where the $i\delta$ prescription is introduced to account for the retardation effect as the limit of $\delta\rightarrow 0^+$ is taken. Thus, substituting Eq. (62) into Eq. (61) leads to

$$\chi_{FF}(t-t') = -i\Theta(t-t')\int\frac{d\omega}{2\pi}\rho(\omega)e^{-i\omega(t-t')}. \tag{63}$$

The Fourier transform of the Green's functions ($G^>, G^<$) in Eq. (30) are given by

$$G^{>(<)}(\mathbf{x}-\mathbf{x}'; t-t') = \int\frac{dw}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3}g^{>(<)}(\mathbf{k},\omega) \times e^{-i\omega(t-t')}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{64}$$

where

$$g^>(\mathbf{k},\omega) = \frac{1}{2k}[(1+n_k)\delta(\omega-k)+n_k\delta(\omega+k)],$$
$$g^<(\mathbf{k},\omega) = \frac{1}{2k}[(1+n_k)\delta(\omega+k)+n_k\delta(\omega-k)], \tag{65}$$

with $k=|\mathbf{k}|$, and $n_k=(e^{\beta k}-1)^{-1}$, the Bose-Einstein distribution function. Then, they obey the KMS relation [23] given by

$$g^<(\mathbf{k},\omega) = e^{-\beta\omega}g^>(\mathbf{k},\omega). \tag{66}$$

Using Eq. (63), the spectral density can be obtained from $\chi_{FF}(t-t')$ given by the commutator of the forces from Eqs. (38) and (55) as

$$\begin{aligned}\rho(\omega) = -2A\int\frac{d^2\mathbf{k}_\parallel}{(2\pi)^2}\int\frac{dk_\perp}{2\pi}\frac{dk'_\perp}{2\pi}\int\frac{d\omega'}{2\pi} &\times [k_\perp k'_\perp(\omega'(\omega-\omega')+k_\perp k'_\perp+\mathbf{k}_\parallel^2)] \\ &\times g^>(\mathbf{k}_\parallel,k_\perp,\omega-\omega')g^>(-\mathbf{k}_\parallel,k'_\perp,\omega') \\ &\times [1-e^{-\beta\omega}]e^{i(k_\perp+k'_\perp)(z-z')} \\ &\times e^{-i\omega'(\epsilon'-\epsilon'')}(1+e^{i\omega\epsilon''})|^{\epsilon',\epsilon''\rightarrow 0}_{z',z\rightarrow 0^-}.\end{aligned} \tag{67}$$

The fluctuation-dissipation theorem can be obtained before taking the short-distance limits. It is then a straightforward calculation to obtain the relation between the Fourier transform of the anticommutator of the forces $\sigma_{FF}(\omega)$ from Eqs. (39) and (55) and the spectral density $\rho(\omega)$ above. The fluctuation and dissipation theorem is to link the Fourier transform of the fluctuation kernel, the anticommutator of the forces $\sigma_{FF}(\omega)$, to the imaginary part of the dissipation kernel, the commutator of the forces $\chi_{FF}(\omega)$, as follows:

$$\sigma_{FF}(\omega) = -\frac{1}{2}\rho(\omega)\coth\left[\frac{\beta\omega}{2}\right] = \mathrm{Im}[\chi_{FF}(\omega)]\coth\left[\frac{\beta\omega}{2}\right]. \tag{68}$$

The above relation relies on the fact that

$$\mathrm{Im}[\chi_{FF}(\omega)] = -\frac{1}{2}\rho(\omega), \tag{69}$$

as a result of Eq. (62). The high-T limit can be taken and then the fluctuation and dissipation theorem in this limit reduces to

$$\mathrm{Im}[\chi_{FF}(\omega)] = \frac{\omega}{2k_{\mathrm{B}}T}\sigma_{FF}(\omega). \tag{70}$$

As expected, it corresponds to the classical Brownian motion which can be seen by taking the Fourier transform of Eq. (60).

The fluctuation and dissipation effects driven by quantum fields in vacuum on a microscopic object are of great interest in regard to imposing fundamental limits on the uncertainty of the position and velocity of an object. In vacuum, the Fourier transforms of the Green's functions ($G^>$, $G^<$) are found to satisfy the following relation:

$$g^<(\mathbf{k}, \omega) = g^>(\mathbf{k}, -\omega), \tag{71}$$

from taking the limit of $T \to 0$ in Eq. (65). Then, it leads to the fluctuation and dissipation theorem in vacuum given by

$$\begin{aligned}\sigma_{FF}(\omega) &= -\frac{1}{2}\rho(\omega)[\Theta(\omega) - \Theta(-\omega)] \\ &= \mathrm{Im}[\chi_{FF}(\omega)][\Theta(\omega) - \Theta(-\omega)].\end{aligned} \tag{72}$$

This result can also be obtained by taking the limit of $T \to 0$ directly from Eq. (68).

Although it is generally expected that the theorem of fluctuation and dissipation is of model independence for the case with the small mirror's displacement in the vacuum and/or thermal states of the field such that this theorem has been used to study the dynamics of moving mirrors in quantum fields on various situations of couplings [9], it is still worth noticing that the study from the first principles derivation reveals that the obtained theorem is also *independent* of the short-distance regulators introduced to deal with divergences from quantum fields. The theorem relates these two effects in vacuum and/or in a thermal bath regardless of the details of short-distance divergences associated with the underlying microscopic dynamics. Thus, when the method of regularization is introduced to compute the dissipation and fluctuation effects, this theorem must be fulfilled as the results are obtained by taking the short-distance limit in the end of calculations. It seems to play a role as the Ward identity derived from underlying symmetry in quantum field theory where the introduction of regularization and renormalization to deal with divergences must respect this identity. This theorem also allows us to compute the dissipation kernel from the obtained fluctuation kernel and vice versa which we will adopt to obtain the Langevin equation later.

VI. MOVING MIRRORS DYNAMICS

We are now to study the dynamics of moving mirrors in quantum fields driven by either vacuum or thermal fluctuations, respectively. We will take advantage of the fluctuation-dissipation theorem derived above to obtain the Langevin equation and to solve it consistently. The same Langevin equation can be obtained by computing the effects of fluctuation and dissipation separately where the corresponding fluctuation-dissipation theorem must be fulfilled as the short-distance limit is taken.

A. Vacuum fluctuations

We compute the dissipation kernel from Eqs. (50) and (56). To do so, the Green's functions of the scalar field in the limit of $T \to 0$ are obtained from Eq. (33) as follows:

$$\begin{aligned}\mathrm{Re}[G(\mathbf{x} - \mathbf{x}'; t - t')] &= \frac{-1}{4\pi^2[(t - t')^2 - |\mathbf{x} - \mathbf{x}'|^2]}, \\ \mathrm{Im}[G(\mathbf{x} - \mathbf{x}'; t - t')] &= \frac{-1}{8\pi^2|\mathbf{x} - \mathbf{x}'|}\{\delta[t - t' - |\mathbf{x} - \mathbf{x}'|]\},\end{aligned} \tag{73}$$

where $\mathrm{Im}[G(\mathbf{x} - \mathbf{x}', t - t')]$ has included the retardation effect. The area integration over $\mathbf{x}'_\parallel$ in Eq. (50) gives the factor A, area of the mirror. Taking advantage of the δ function in $\mathrm{Im}[G(\mathbf{x} - \mathbf{x}', t - t')]$ allows us to carry out the area integral on $\mathbf{x}_\parallel$ where we assume that the mirror is of a disk. Then, after a lengthy calculation, the dissipative force term ends up with

$$\begin{aligned}\int dt' \chi_{FF}(t - t')\delta q(t') = \frac{A}{480\pi^2}\int_0^{t-z} dt' \bigg[&\frac{75\delta q(t')}{(t - t')^6} + \frac{75\delta q'(t')}{(t - t')^5} + \frac{30\delta q''(t')}{(t - t')^4} + \frac{5\delta q^{[3]}(t')}{(t - t')^3} \\ &- z^2\bigg(\frac{1575\delta q(t')}{(t - t')^8} + \frac{1575\delta q'(t')}{(t - t')^7} + \frac{675\delta q''(t')}{(t - t')^6} + \frac{150\delta q^{[3]}(t')}{(t - t')^5} + \frac{15\delta q^{[4]}(t')}{(t - t')^4}\bigg) \\ &+ z^4\bigg(\frac{1890\delta q(t')}{(t - t')^{10}} + \frac{1890\delta q'(t')}{(t - t')^9} + \frac{840\delta q''(t')}{(t - t')^8} + \frac{210\delta q^{[3]}(t')}{(t - t')^7} + \frac{30\delta q^{[4]}(t')}{(t - t')^6} + \frac{2\delta q^{[5]}(t')}{(t - t')^5}\bigg)\bigg],\end{aligned} \tag{74}$$

where the limit of $\epsilon \to 0$ has been taken. We now perform the remaining time integral and use the relation

$$\int_0^{t+(1/\Lambda)} dt'(t-t')^{-n}\delta q^{(m)}(t') = \frac{(-\Lambda)^{n-1}\delta q^{(m)}(t+\frac{1}{\Lambda})}{(n-1)} - \frac{1}{n-1}\int_0^{t+(1/\Lambda)} \frac{dt'}{(t-t')^{n-1}}\delta q^{(m+1)}(t') \tag{75}$$

by dropping out the terms evaluated at an initial time which is equivalent to introducing an adiabatical switch-on interaction. Apparently, the force cannot be evaluated infinitesimally close to the surface of the mirror by taking the limit of $z \to 0^-$ due to short-distance divergences. This is mainly due to an unrealistic perfectly reflecting condition imposed on the mirror. It can be solved by introducing either a fluctuating boundary in $3+1$ dimensions [13] or a nonperfectly reflecting boundary in $1+1$ dimensions [6,9]. The latter condition seems to be not sufficient to solve the divergence problem in $3+1$ dimensions [24]. The introduced energy cutoff Λ is to set a cutoff on $z \approx 1/\Lambda$ due to fluctuations of the mirror's surface. Then, a local approximation can be made as the time scales we consider are such that $t \gg 1/\Lambda$. In vacuum, the local dissipative force can be obtained as

$$\int dt' \chi_{FF}(t-t')\delta q(t') = \frac{A}{48\pi^2}\left(\Lambda^3 \delta\ddot{q}(t) - \frac{\Lambda}{10}\delta q^{[4]}(t) - \frac{1}{15}\delta q^{[5]}(t) + \mathcal{O}\left(\frac{1}{\Lambda}\right)\right). \tag{76}$$

Then, the dissipation kernel can be read off as

$$\chi_{FF}(t-t') = \frac{A}{48\pi^2}\left(\Lambda^3 \delta^{[2]}(t-t') - \frac{\Lambda}{10}\delta^{[4]}(t-t') - \frac{1}{15}\delta^{[5]}(t-t')\right), \tag{77}$$

where the derivatives of the δ function are involved, and the terms of order $\mathcal{O}(1/\Lambda)$ are ignored. Using Eq. (72), the fluctuation-dissipation theorem in vacuum, we can obtain $\sigma_{FF}(t-t')$ by taking the Fourier transform of $\sigma_{FF}(\omega)$ as

$$\sigma_{FF}(t-t') = \int \frac{d\omega}{2\pi} \mathrm{Im}[\chi_{FF}(\omega)][\Theta(\omega) - \Theta(-\omega)]e^{-i\omega(t-t')} = \frac{A}{720\pi^2}\int \frac{d\omega}{2\pi}\omega^5 \cos[\omega(t-t')]. \tag{78}$$

The backreaction dissipation effect above is related to the force fluctuations via a fluctuation-dissipation relation as in the case of Brownian motion. In addition, motion-induced radiation reaction due to nonuniform acceleration of the moving mirror can be obtained from Eq. (56) consistent with the result from Ref. [8].

Notice that the known problems of the runaway solution and preacceleration are in the Lorentz-Dirac theory of radiation reaction on the motion of point charges in quantum electromagnetic fields. The motion-induced radiation reaction force is given by the third time derivative of the position, and is to accelerate point charges [15–17]. The recent studies in Refs. [2–5] have found that the non-Markovian nature of the dissipation kernel from quantum fields plays a key role to obtain the causal equations with free of runaway solutions within a context of the fully nonequilibrium open system dynamics. However, in the case with the small mirror's displacement, as we will see, the obtained Langevin equation below even including the Markovian backreaction force terms of the higher derivatives (e.g. $\delta q^{[n]}$, $n > 2$) can be solved consistently with the ordinary Newtonian initial data.

Then, the corresponding Langevin equation including all backreaction effects becomes

$$m\delta\ddot{q}(t) + \frac{\delta V}{\delta q}(t) + \left[\frac{A}{24\pi^2}\left(-\Lambda^3\delta\ddot{q}(t) + \frac{\Lambda}{10}\delta q^{[4]}(t) + \frac{1}{15}\delta q^{[5]}(t)\right)\right] = \eta(t), \tag{79}$$

with the Gaussian force correlations given by Eq. (78) as

$$\langle \eta(t)\eta(t')\rangle = \frac{A}{720\pi^2}\int \frac{d\omega}{2\pi}\omega^5 \cos[\omega(t-t')]. \tag{80}$$

The first two terms of the backreaction effects in the Langevin equation will modify the dispersive part of the mirror while the third term is a dominant dissipative force term to slow down the motion of the mirror. In fact, the first term above can be absorbed into the renormalization of mass given by

$$m_{\mathrm{R}} = m - \frac{A}{24\pi^2}\left(\frac{\Lambda}{\hbar c}\right)^2 \frac{\Lambda}{c^2}, \tag{81}$$

where the energy cutoff Λ is chosen for having positive renormalized mass so as to avoid the runaway solution. The renormalized mass is a parameter here to be determined from experiment.

We now try to solve the equation by first of all, taking the average of the above equation to understand its relaxational dynamics. Consider the case where the mirror is attached to a spring and undergoes oscillations with a natural frequency ω_0. Then, the equation can be written as

$$m\delta\ddot{q}(t) + m\omega_0^2\delta q + \left[\frac{A}{24\pi^2}\left(\frac{\Lambda}{10}\delta q^{[4]}(t) + \frac{1}{15}\delta q^{[5]}(t)\right)\right] = 0. \tag{82}$$

To see the quantum effects from the scalar field on the dynamic of the mirror driven by the classical external potential, we write the solution of the equation as

$$\delta q(t) = \delta q_c(t) + \delta q_\hbar(t), \tag{83}$$

where $\delta q_c(t)$ is a solution of the equation for harmonic oscillations, and $\delta q_\hbar(t)$ is derivation from its classical trajectory induced from vacuum fluctuations due to the presence of forth and fifth time derivatives of $\delta q_c(t)$.

Thus, they obey the following equations respectively:

$$m\delta\ddot{q}_c(t) + m\omega_0^2\delta q_c(t) = 0,$$
$$m\delta\ddot{q}_\hbar(t) + m\omega_0^2\delta q_\hbar(t) = -\left[\frac{A}{24\pi^2}\left(\frac{\Lambda}{10}\delta q_c^{[4]}(t) + \frac{1}{15}\delta q_c^{[5]}(t)\right)\right]. \quad (84)$$

The equations can be solved iteratively in terms of the retarded Green's function:

$$G_{\rm ret}(t-t') = \Theta(t-t')\left[\frac{1}{\omega_0}\sin[\omega_0(t-t')]\right]. \quad (85)$$

Thus, the solution to Eq. (84) is given by

$$\delta q_c(t) = l_0\cos[\omega_0(t-\theta_0)],$$
$$\delta q_\hbar(t) = -\frac{A}{24\pi^2 m\omega_0}\int_{t_0}^t dt'\sin[\omega_0(t-t')]\left(\frac{\Lambda}{10}\delta q_c^{[4]}(t') + \frac{1}{15}\delta q_c^{[5]}(t')\right)$$
$$= l_0\left[-\frac{A}{720\pi^2 m}\omega_0^4(t-t_0)\cos[\omega_0(t-\theta_0)] - \frac{A}{240\pi^2 m}\omega_0^3\Lambda(t-t_0)\sin[\omega_0(t-\theta_0)]\right] + \text{nonsecular terms}. \quad (86)$$

Then the $\delta q(t)$ is obtained as

$$\delta q(t) = l_0\left\{\left[1 - \frac{A}{720\pi^2 m}\omega_0^4(t-t_0)\right]\cos[\omega_0(t-\theta_0)] - \frac{A}{240\pi^2 m}\omega_0^3\Lambda(t-t_0)\sin[\omega_0(t-\theta_0)]\right\} + \text{nonsecular terms}. \quad (87)$$

The initial time is set at t_0 and the parameters, l_0 and θ_0, can be determined by the initial conditions. Note that the naive perturbation contains the secular terms that grow linearly in time while the terms denoted by nonsecular terms are finite at all times. It indicates that the perturbation breaks down at late times. In order to obtain the solution with the correct damping behavior, the method of dynamical renormalization group will be invoked to resum these secular terms consistently [25]. The dynamical renormalization can be achieved by introducing an arbitrary time scale τ, splitting $t - t_0$ as $t - \tau + \tau - t_0$, and absorbing the terms containing $\tau - t_0$ into renormalization of the amplitude $l(\tau)$ and the phase $\theta(\tau)$ respectively. We then relate l_0 and θ_0 to $l(\tau)$ and $\theta(\tau)$ as follows:

$$l_0 = Z_l(\tau)l(\tau); \qquad \theta_0 = \theta(\tau) + Z_\theta(\tau), \quad (88)$$

where Z_l and Z_θ are renormalization constants for multiplicative amplitude renormalization and additive phase renormalization, respectively. They are given by

$$Z_l(\tau) = 1 + a(\tau) + \cdots, \qquad Z_\theta(\tau) = b(\tau) + \cdots. \quad (89)$$

The $\cdots$ means the terms to be involved while the approximation under consideration goes beyond the small displacement approximation. Substituting Eqs. (88) and (89) into Eq. (87) leads us to choose

$$a(\tau) = \frac{A}{720\pi^2 m}\omega_0^4(\tau - t_0),$$
$$b(\tau) = \frac{A}{240\pi^2 m}\omega_0^2\Lambda(\tau - t_0), \quad (90)$$

so as to remove the secular terms containing $\tau - t_0$. After doing renormalization, the solution is given by Eq. (87) as l_0, θ_0 and t_0 are replaced by $l(\tau)$, $\theta(\tau)$ and τ respectively. The independence of the time scale τ on l_0 and θ_0 can lead to the renormalization group equations by taking the τ derivative on Eq. (88), which are of the form

$$\frac{d}{d\tau}l(\tau) = -\frac{A}{720\pi^2 m}\omega_0^4 l(\tau),$$
$$\frac{d}{d\tau}\theta(\tau) = -\frac{A}{240\pi^2 m}\omega_0^2\Lambda, \quad (91)$$

with the solutions:

$$l(\tau) = l_0 e^{[-(A/720\pi^2 m)]\omega_0^4(\tau - t_0)},$$
$$\theta(\tau) = \theta_0 - \frac{A}{240\pi^2 m}\omega_0^2\Lambda(\tau - t_0). \quad (92)$$

A change of the renormalization point τ is compensated by a change in the renormalized amplitude $l(\tau)$ and phase $\theta(\tau)$. Substituting the solutions above to the renormalized solution and setting $\tau = t$, we obtain

$$\delta q(t) = l_0 e^{[-(A/720\pi^2 m)]\omega_0^4 t}\left\{\cos\left[\omega_0\left(1 + \frac{A}{240\pi^2 m}\omega_0^2\Lambda\right)\times\left[t - \theta_0\left(1 - \frac{A}{240\pi^2 m}\omega_0^2\Lambda\right)\right]\right]\right\}, \quad (93)$$

where the initial time has been set at $t_0 = 0$.

Obviously, the term of forth time derivative in Eq. (82) modifies the dispersive part of the mirror by changing the oscillation frequency as well as shifting the phase. The relaxation time scales are mainly determined from the term of fifth time derivative given by

$$t_{\rm relax} \simeq 720\pi^2\left(\frac{c^2}{A\omega_0^2}\right)\left(\frac{mc^2}{\hbar\omega_0}\right)\frac{1}{\omega_0}. \quad (94)$$

$mc^2 \gg \hbar\omega_0$ holds for a macroscopic mirror. Typically, the emitted quanta driven by a nonuniform accelerated mirror is with a frequency which is the same as the oscillation frequency of a mirror. This condition means that the energy loss from emitted quanta is far much less than the rest mass energy of a microscopic mirror. Thus, the recoiled effect of the mirror for this process is small where one can provide a prescribed motion of the mirror, and then find its correction arising from the effects of quantum fields. The validity of the small displacement approximation imposes the condition of $l_0\omega_0 \ll 1$. Then, the order of magnitude of the relaxation time scales can be obtained as

$$t_{\rm relax} \gg 10^4\left(\frac{c}{l_0\omega_0}\right)^2\frac{1}{\omega_0} \gg 10^4\frac{1}{\omega_0}, \tag{95}$$

where $A \approx l_0^2$ has been assumed. Thus, the very long time scales for having at least much more than 10^4 oscillations are needed to detect tiny damping on the amplitude of the oscillating mirror [8].

We now study the fluctuations effects from quantum fields on the mirror. The vacuum fluctuations are of great importance in early times, say $t \ll t_{\rm relax}$, as the dissipation effects can be ignored. The equation of the mirror then reduces to

$$m\delta\ddot{q}(t) + m\omega_0^2\delta q(t) = \eta(t). \tag{96}$$

Its solution is obtained as

$$\delta v(t) = \int_0^t dt' \cos[\omega_0(t-t')]\frac{\eta(t')}{m}, \tag{97}$$

leading to the velocity fluctuations given by

$$\begin{aligned}\Delta\delta v^2(t) &= \langle\delta v^2(t)\rangle - \langle\delta v(t)\rangle^2 = \frac{1}{m^2}\int_0^t dt_1\int_0^t dt_2 \cos[\omega_0(t-t_1)]\cos[\omega_0(t-t_2)][\langle\eta(t_1)\eta(t_2)\rangle - \langle\eta(t_1)\rangle\langle\eta(t_2)\rangle]\\ &= \frac{1}{360\pi^2}\frac{A}{m^2}\int_0^t dt_1\int_0^t dt_2\int_0^\infty\frac{d\omega}{2\pi}\omega^5\cos[\omega_0(t-t_1)]\cos[\omega_0(t-t_2)]\cos[\omega(t_1-t_2)],\end{aligned} \tag{98}$$

where we have used the fact that the forces from vacuum fluctuations are Gaussian with correlations given by Eq. (80). We change variables of integration as $u = t_1 - t_2$, $v = t_1 + t_2$, and the integral above in terms of u, v is of the form

$$\Delta\delta v^2(t) = \frac{A}{1440\pi^2 m^2}\int_0^\infty\frac{d\omega}{2\pi}\omega^5\left\{\int_{-t}^0 du\int_{-u}^{u+2t} + \int_0^t du\int_u^{2t-u}\right\}\{\cos[\omega_0(2t-v)] + \cos[\omega_0 u]\}\cos[\omega u]. \tag{99}$$

For the time t, say $1/\omega_0 \ll t \ll t_{\rm relax}$, we find that the velocity fluctuations grow linearly in t as

$$\begin{aligned}\Delta\delta v^2(t) &\simeq \frac{A}{720\pi^2 m^2}t\int_0^\infty\frac{d\omega}{2\pi}\omega^5\int_0^\infty du[\cos[(\omega+\omega_0)u] + \cos[(\omega-\omega_0)u]]\\ &\simeq \frac{A}{720\pi^2 m^2}t\int_0^\infty\frac{d\omega}{2\pi}\omega^5\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)] \simeq \frac{A}{1440\pi^2 m^2}\omega_0^5 t.\end{aligned} \tag{100}$$

It can be seen that the typical frequency of quanta absorbed by the moving mirror to increase its velocity fluctuations is the frequency of the oscillating mirror. The energy gained from vacuum fluctuations for each oscillation can be obtained as

$$E \simeq \frac{1}{1440\pi^2}\left(\frac{A\omega_0^2}{c^2}\right)\left(\frac{\hbar\omega_0}{mc^2}\right)\hbar\omega_0, \tag{101}$$

with the order of magnitude given by

$$E \ll 10^{-4}\left(\frac{l_0\omega_0}{c}\right)^2\hbar\omega_0 \ll 10^{-4}\hbar\omega_0, \tag{102}$$

where again $A \approx l_0^2$ and $mc^2 \gg \hbar\omega_0$ have been used. Thus, roughly about fewer than 10^{-4} quanta with frequency ω_0 is absorbed by a mirror per oscillation per area l_0^2. Thus, the effects from vacuum fluctuations can hardly be detected. The largely nonuniform acceleration of a microscopic object can possibly amplify vacuum fluctuations where the treatment to tackle this issue beyond the small displacement approximation is required.

B. Thermal fluctuations

This section will be devoted to understanding the dynamics of moving mirrors in thermal fields. The large time and high temperature limits give rise to $|t-t'| \gg l \gg \tau_{\rm B}$ where $A = \pi l^2$, area of the mirror, and $\tau_{\rm B} \equiv 1/(\pi k_{\rm B}T)$, a characteristic thermal correlation length scale. Then, the Green's function for scalar fields in Eqs. (30) and (33) can be approximated by

$$G^{(>,<)}(\mathbf{x}-\mathbf{x}';t-t') \simeq \frac{1}{4\pi^2\tau_{\rm B}|\mathbf{x}-\mathbf{x}'|}\left[e^{(-2/\tau_{\rm B})(|t-t'|+|\mathbf{x}-\mathbf{x}'|)} - e^{(-2/\tau_{\rm B})(|t-t'|-|\mathbf{x}-\mathbf{x}'|)}\right]. \tag{103}$$

Thus, the force correlations including thermal effects can be obtained from Eqs. (39) and (55) as

$$\sigma_{FF}(t-t') \simeq \frac{16l^2}{\pi^2\tau_B^6}\left\{\left(1+\frac{1}{4(\frac{l}{\tau_B})}-\frac{1}{32(\frac{l}{\tau_B})^4}\right)e^{(-4/\tau_B)|t-t'|}-\left(\frac{1}{16(\frac{l}{\tau_B})^3}-\frac{1}{64(\frac{l}{\tau_B})^4}\right)e^{(-4/\tau_B)(|t-t'|-l)} + \left(\frac{1}{16(\frac{l}{\tau_B})^3}+\frac{1}{64(\frac{l}{\tau_B})^4}\right)e^{(|t-t'|+l)}\right\} \simeq \frac{16l^2}{\pi^2\tau_B^6}e^{(-4/\tau_B)|t-t'|}, \tag{104}$$

which can be further approximated by

$$\sigma_{FF}(t-t') \simeq \frac{8l^2}{\pi^2\tau_B^5}\delta(t-t') \tag{105}$$

using the fact that

$$\lim_{\alpha\to\infty}\frac{\alpha}{2}e^{-\alpha|x|} = \delta(x). \tag{106}$$

It reveals that the high temperature fluctuations are of uncorrelated white noise. Using Eq. (70), the fluctuation-dissipation theorem in the high-T limit, one can determine the imaginary part of $\chi_{FF}(\omega)$ that leads to the dominant effect on dissipation with the term proportional to the mirror's velocity. It is due to the force fluctuations. The real part of $\chi_{FF}(\omega)$ renormalizes the oscillation frequency as well as the mass of the mirror with temperature corrections. However, the corresponding temperature correction to the oscillation frequency, which describes a position dependent static force, vanishes since the mean pressure force from thermal scalars on the mirror is zero by the symmetry argument [9]. The mass will acquire the temperature correction which is subdominant as its correction is suppressed by a factor of $\hbar\omega_0/k_BT$ comparing with the damping term. From Eq. (56), the high-T motion-induced force can be obtained from the corresponding dissipative force, and is found to be also proportional to the mirror's velocity. It arises from the Doppler shift of thermal scalars.

Thus, involving the dominant thermal effects, the Langevin equation now becomes

$$m\delta\ddot{q}(t) + \gamma_T\delta\dot{q}(t) + m\omega_0^2\delta q(t) = \eta(t) \tag{107}$$

with the white noise correlations

$$\langle\eta(t)\eta(t')\rangle = 8\pi^2c^3A\left(\frac{k_BT}{\hbar c}\right)^3\left(\frac{k_BT}{c^2}\right)^2\delta(t-t'). \tag{108}$$

The damping coefficient can be found to be [14]

$$\gamma_T \simeq 8\pi^2cA\left(\frac{k_BT}{\hbar c}\right)^3\left(\frac{k_BT}{c^2}\right). \tag{109}$$

The relaxation time scales, $t_{\rm relax} \simeq (\gamma_T/m)^{-1}$, are the time scales when dissipation effects become important. To obtain the maximal fluctuations for the mirror, we now consider the time scales, say $t_{\rm relax} \gg t \gg l$, where dissipation effects can be ignored. We find that

$$\Delta\delta v^2(t) \simeq 4\pi^2c^3A\left(\frac{k_BT}{\hbar c}\right)^3\left(\frac{k_BT}{mc^2}\right)^2 t. \tag{110}$$

The maximal velocity fluctuations can be achieved by roughly setting the time scales, $t = t_{\rm relax}$, as follows:

$$\Delta\delta v^2_{\rm max}(t) \simeq c^2\left(\frac{k_BT}{mc^2}\right). \tag{111}$$

Thus, it leads to

$$\frac{\Delta l_{\rm max}}{l_0} \simeq \frac{\Delta\delta v_{\rm max}}{\delta v} \simeq \left(\frac{c}{l_0\omega_0}\right)\left(\frac{k_BT}{mc^2}\right)^{1/2} \simeq 10^{-8}\left(\frac{10\ {\rm cm}}{l_0}\right)\left(\frac{1\ {\rm s}^{-1}}{\omega_0}\right)\left(\frac{1\ {\rm kg}}{m}\right)^{1/2}\left(\frac{T}{1{\rm kev}}\right)^{1/2} \tag{112}$$

with the corresponding relaxation time scales given by

$$t_{\rm relax} \simeq 10^{-2}\ {\rm s}\left(\frac{100\ {\rm cm}^2}{A}\right)\left(\frac{m}{1\ {\rm kg}}\right)\left(\frac{1\ {\rm kev}}{T}\right)^4, \tag{113}$$

where l_0 and ω_0 are the typical oscillation amplitude and frequency of the mirror. As long as the temperature of thermal fields is of order kev, the amplitude fluctuations of the oscillating mirror are of order $10^{-8}l_0$ within the time scales of 10^{-2} s, which can be detectable. The mass correction from thermal effects can be obtained from Eq. (81) by replacing the energy cutoff Λ with the typical thermal energy k_BT given by [14] :

$$\Delta m_T \simeq -A\left(\frac{k_BT}{\hbar c}\right)^2\left(\frac{k_BT}{mc^2}\right)m \simeq -10^{-16}m \tag{114}$$

with the above value of the parameters. This extremely small mass correction can be ignored in our calculations.

VII. CONCLUSIONS

In this paper, we present a general framework for describing the dynamics of moving mirrors in quantum fields in the case where the mirror undergoes the small displacement. The mirror of perfect reflection imposes the boundary conditions on field fluctuations, and leads to the coupling between the mirror and fields. The force on the mirror is given by the area integral of the stress tensor of the fields. Using the Schwinger-Keldysh formalism, coarse-graining quantum fields leads to the stochastic behavior in the mirror's trajectory encoded in the coarse-grained effective action with the method of influence functional. In the semiclassical regime, the Langevin equation can be derived involving backreaction effects. We find that the Langevin equation reveals two levels of backreaction effects on the dynamics of the mirror: radiation reaction induced by the motion of the mirror as well as backreaction dissipation arising from fluctuations of quantum fields via a fluctuation-dissipation relation. The corresponding fluctuation-dissipation theorem is derived for quantum fields in vacuum and at finite temperature, respectively. We find that, although the theorem of fluctuation and dissipation for the case with the small mirror's displacement is of model independence, the obtained theorem from the first principles derivation reveals that it is also *independent* of the regulators introduced to deal with short-distance divergences from quantum fields. Thus, when the method of regularization is introduced to compute the dissipation and fluctuation effects, this theorem must be fulfilled as the results are obtained by taking the short-distance limit in the end of calculations. This theorem also allows us to compute the dissipation kernel from the obtained fluctuation kernel and vice versa.

Consider a situation where the mirror is attached to a spring and undergoes oscillations with a natural frequency ω_0. In vacuum, we find that the relaxation time scales for having much more than 10^4 oscillations are needed to detect tiny damping on the oscillation amplitudes of the mirror due to the backreaction effects. The energy gain of the mirror from vacuum fluctuations is by absorbing fewer than 10^{-4} quanta for each oscillation with frequency ω_0. Thus, these vacuum fluctuations can hardly be detected. The largely nonuniform acceleration of a microscopic object can possibly amplify the effects of vacuum fluctuations where the treatment to tackle this issue beyond the small displacement approximation is required. On the contrary, at finite temperature, as long as the temperature of thermal fields is of order kev, the ratio of the amplitude fluctuations to the amplitude of the oscillating mirror are of order 10^{-8} within the time scales of 10^{-2}s, leading to the detectable effects.

ACKNOWLEDGMENTS

We would like to thank Bei-Lok Hu and Larry H. Ford for stimulating discussions. Part of this work was done when D. S. L. visited the Theory Group at the Blackett Laboratory, Imperial College in London, U.K. D. S. L. would like to thank R. J. Rivers and T. S. Evans for their hospitality. This work was supported in part by the National Science Council, R.O.C., under Grant No. NSC92-2112-M-259-007.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[2] P. R. Johnson and B L. Hu, quant-ph/0012137; Phys. Rev. D **65**, 065015 (2002).
[3] P. R. Johnson and B. L. Hu, in *Proceedings of the Second Conference on Quantum Aspects of Beam Physics, 2000*, edited by Pisin Chen (World Scentific, Singapore, 2001); B. L. Hu and A. Raval, quant-ph/0012134; B. L. Hu and P. R. Johnson, "Beyond Unruh Effect: Nonequilibrium Quantun Dynamics of Moving Charges" (unpublished).
[4] C. R. Galley, B. L. Hu, and P. R. Jonson, quant-ph/0402002.
[5] P. R. Anderson and B. L. Hu, Phys. Rev. D **69**, 064039 (2004); B. L. Hu, A. Roura, and S. Shresta, Quantum Semiclass. Opt. **6**, S698 (2004).
[6] M.-T. Jaekel and S. Reynaud, J. Phys. I (France) **3**, 339 (1993).
[7] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. A **348**, 393 (1976); **356**, 237 (1977).
[8] L. H. Ford and A. Vilenkin, Phys. Rev. D **25**, 2569 (1982).
[9] M.-T. Jaekel and S. Reynaud, Quantum Opt. **4**, 391 (1992).
[10] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[11] G. Barton, J. Phys. A **24**, 5533 (1991).
[12] C.-H. Wu and L. H. Ford, Phys. Rev. D **64**, 045010 (2001); C.-H. Wu and D.-S. Lee, Phys. Lett. A **318**, 303 (2003).
[13] L. H. Ford and N. F. Svaiter, Phys. Rev. D **58**, 065007 (1998).
[14] L. A. S. Machado, P. A. Maia Neto, and C. Farina, Phys. Rev. D **66**, 105016 (2002).
[15] E. J. Moniz and D. H. Sharp, Phys. Rev. D **10**, 1133 (1974); **15**, 2850 (1977); H. Levine, E. J. Moniz, and D. H. Sharp, Am. J. Phys. **45**, 75 (1977).
[16] C. Cohen-Tannoudji, *Atom-Photon Interactions* (Wiley, New York, 1992).
[17] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. Lett. **55**, 2273 (1985).
[18] J. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961); L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965); K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. (N.Y.) **4**, 1 (1963); **4**, 12 (1963); K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985); J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).
[19] D.-S. Lee and K.-W. Ng, Phys. Lett. B **449**, 274 (1999); **492**, 303 (2000); Y.-Y. Charng, K.-W. Ng, C.-Y. Lin, and D.-S. Lee, Phys. Lett. B **548**, 175 (2002).
[20] D.-S. Lee and K.-W. Ng, Phys. Rev. D **61**, 085003 (2000); D.-S. Lee, W.-L. Lee, and K.-W. Ng, Phys. Lett. B **542**, 1 (2002); W.-L. Lee, Y.-Y. Charng, D.-S. Lee, and L.-Z. Fang, Phys. Rev. D **69**, 123522 (2004); D.-S. Lee, L.-Z.

Fang, W.-L. Lee, and Y.-Y. Charng, J. Korean Phys. Soc. **45**, S203 (2004).
[21] Larry H. Ford (private communication).
[22] A. O. Caldeira and A. J. Leggett, Physica (Amsterdam) **121A**, 587 (1983).
[23] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[24] V. Sopova and L. H. Ford, Phys. Rev. D **66**, 045026 (2002).
[25] S. Y. Wang, D. Boyanovsky, H. J. de Vega, and D. S. Lee, Phys. Rev. D **62**, 105026 (2000); D. Boyanovsky, H. J. de Vega, and S. Y. Wang, Phys. Rev. D **67**, 065022 (2003); D. Boyanovsky and H. J. de Vega, Ann. Phys. (N.Y.) **307**, 335 (2003).