Stellar oscillations in scalar-tensor theory of gravity

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We derive the perturbation equations for relativistic stars in scalar-tensor theories of gravity and study the corresponding oscillation spectrum. We show that the frequency of the emitted gravitational waves is shifted proportionally to the scalar field strength. Scalar waves which might be produced from such oscillations can be a unique probe for the theory, but their detectability is questionable if the radiated energy is small. However, we show that there is no need for a direct observation of scalar waves: The shift in the gravitational wave spectrum could unambiguously signal the presence of a scalar field.

I. INTRODUCTION

Scalar-tensor theories of gravity are an alternative or generalization of Einstein's theory of gravity, where, in addition to the tensor field, a scalar field is present. The theory has been proposed in its earlier form about a half century ago [1–3], and it is a viable theory of gravity for a specific range of the coupling strength of the scalar field to gravity [4–6]. Actually, the existence of scalar fields is crucial (e.g., in inflationary and quintessence scenarios) to explain the accelerated expansion phases of the Universe. In addition, scalar-tensor theories of gravity can be obtained from the low-energy limit of string theory and/or other gauge theories. Experimentally, the existence of a scalar field has not yet been probed, but a number of experiments in the weak field limit of general relativity set severe limits on the existence and coupling strengths of scalar fields [6,7].

A basic assumption is that the scalar and gravitational fields φ and $g_{\mu\nu}$ are coupled to matter via an "effective metric" $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$. The Fierz-Jordan-Brans-Dicke [1–3] theory assumes that the "coupling function" has the form $A(\varphi) = \alpha_0\varphi$; i.e., it is characterized by a unique free parameter $\alpha_0^2 = (2\omega_{BD} + 3)^{-1}$, and all its predictions differ from those of general relativity by quantities of order α_0^2 [8]. Solar system experiments set strict limits in the value of the Brans-Dicke parameter ω_{BD}, i.e., $\omega_{BD} \gtrsim 40\,000$, which suggests a very small $\alpha_0^2 < 10^{-5}$ (see [7,9]).

In the early 1990s, based on a simplified version of scalar-tensor theory where $A(\varphi) = \alpha_0\varphi + \beta\varphi^2/2$, Damour and Esposito-Farèse [8,10] found that for certain values of the coupling parameter β the stellar models develop some strong field effects, which induce significant deviations from general relativity. This sudden deviation from general relativity for specific values of the coupling constants has been named "spontaneous scalarization."

Harada [11] studied in more detail models of nonrotating neutron stars in the framework of the scalar-tensor theory and he reported that spontaneous scalarization is possible for $\beta \lesssim -4.35$. DeDeo and Psaltis suggested that the effects of scalar fields might be apparent in the observed redshifted lines of x rays and γ rays observed by Chandra and XMM-Newton [12] and in quasiperiodic oscillations [13].

Recently, we have examined the possibility to obtain the information for the presence of the scalar field via gravitational wave observations of oscillating neutron stars ([14], hereafter paper I). This previous study has been done using the so-called Cowling approximation. In this approximation, one studies the fluid oscillations freezing the perturbations of the spacetime and of the scalar field. Even under these assumptions, the effect of the scalar field on the fluid perturbations can be significant. We showed that for values of $\beta \lesssim -4.35$ the oscillation frequencies of the fluid change drastically, and the observation of such oscillations can verify or rule out the spontaneous scalarization phenomenon.

It has been suggested that stellar oscillations can provide a unique tool for estimating the parameters of the star, i.e., mass, radius, rotation rate, magnetic field, and equation of state. These ideas have been developed in the last decade in a series of papers [15–21], where the properties of the various families of oscillation modes have been used to probe the stellar parameters. The modes which are mainly excited during the formation of a neutron star or during starquakes and emit detectable gravitational waves are the fluid f and p modes and the w modes, which are associated to oscillations of the spacetime [22].

The effect of the scalar field on the f and p modes has been examined in paper I (in the Cowling approximation). In this article, we derive the full set of equations describing the oscillations of a relativistic star, i.e., the perturbations of the fluid, the spacetime, and the scalar field. Since the stellar models are spherically symmetric, the oscillations can be classified as axial or polar depending on their parity,

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and we can derive perturbation equations for each class of perturbations. In the polar case, we show that the wave equations describing the perturbations of the fluid and spacetime couple to the wave equation describing the perturbations of the scalar field. In other words, the polar perturbations are affected not only by the presence of the scalar field in the background but also by the coupling with the wave equation describing the perturbations of the scalar field. In the axial case, we find a single equation describing the perturbation of the spacetime. This equation is not coupled to perturbations of the fluid and of the scalar field: The scalar field affects only the dynamics through its influence on the background.

The paper is organized as follows. In the next section, we establish our notation and briefly introduce the theoretical framework for the scalar-tensor theories of gravity we consider. In Sec. III we derive the perturbation equations which will be used for the numerical estimation of the oscillation frequencies. In Sec. IV we describe the methods that have been used to derive the axial w modes in scalar-tensor theory of gravity and show the results. In the final section V we discuss the results and their implications. We have included in three appendices the details of the various analytic and numerical calculations. In Appendix A we describe the perturbations of the energy-momentum tensors for the fluid and the scalar field, while in the next Appendix B we provide the analytic forms of the perturbed Einstein equations. Finally, in the last Appendix C we describe the numerical techniques that have been used to calculate the quasinormal modes. In this paper, we adopt the unit of $c = G = 1$, where c and G denote the speed of light and the gravitational constant, respectively, and the metric signature of $(-, +, +, +)$.

II. STELLAR MODELS IN SCALAR-TENSOR THEORIES OF GRAVITY

In this section, we will study neutron star models in scalar-tensor theory of gravity with one scalar field. This is a natural extensions of Einstein's theory, in which gravity is mediated not only by a second rank tensor (the metric tensor $g_{\mu\nu}$) but also by a massless long-range scalar field φ. The action is given by [4]

$$S = \frac{1}{16\pi G_*} \int \sqrt{-g_*}(R_* - 2g_*^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})d^4x + S_m[\Psi_m, A^2(\varphi)g_{*\mu\nu}],$$ (1)

where all quantities with asterisks are related to the "Einstein metric" $g_{*\mu\nu}$; then R_* is the curvature scalar for this metric, and G_* is the bare gravitational coupling constant. Ψ_m represents collectively all matter fields, and S_m denotes the action of the matter represented by Ψ_m, which is coupled to the "Jordan-Fierz metric tensor" $\tilde{g}_{\mu\nu}$. The field equations are usually written in terms of the Einstein metric, but all nongravitational experiments mea-

sure the Jordan-Fierz or "physical metric." The Jordan-Fierz metric is related to the Einstein metric via the conformal transformation,

$$\tilde{g}_{\mu\nu} = A^2(\varphi)g_{*\mu\nu}.$$ (2)

Hereafter, we denote by a tilde quantities in the "physical frame" and by an asterisk those in the "Einstein frame." From the variation of the action S, we get the field equations in the Einstein frame

$$G_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + T_{*\mu\nu}^{(\varphi)},$$ (3)

$$\Box_* \varphi = -4\pi G_* \alpha(\varphi)T_*,$$ (4)

where $T_{*\mu\nu}^{(\varphi)}$ is the energy momentum of the massless scalar field, i.e.,

$$T_{*\mu\nu}^{(\varphi)} \equiv 2(\varphi_{,\mu}\varphi_{,\nu} - \tfrac{1}{2}g_{*\mu\nu}g_*^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta})$$ (5)

and $T_*^{\mu\nu}$ is the energy-momentum tensor in the Einstein frame which is related to the physical energy-momentum tensor $\tilde{T}_{\mu\nu}$ as follows:

$$T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}}\frac{\delta S_m}{\delta g_{*\mu\nu}} = A^6(\varphi)\tilde{T}^{\mu\nu}.$$ (6)

The scalar quantities T_* and $\alpha(\varphi)$ are defined as $T_* \equiv T_{*\mu}^{\ \mu} = T_*^{\mu\nu}g_{*\mu\nu}$ and $\alpha(\varphi) \equiv d\ln A(\varphi)/d\varphi$. It is apparent that $\alpha(\varphi)$ is the only field-dependent function which couples the scalar field with matter; for $\alpha(\varphi) = 0$ the theory reduces to general relativity. Finally, the law of energy-momentum conservation $\tilde{\nabla}_\nu \tilde{T}_\mu^{\ \nu} = 0$ is transformed into

$$\nabla_{*\nu} T_{*\mu}^{\ \nu} = \alpha(\varphi)T_* \nabla_{*\mu}\varphi,$$ (7)

and we set φ_0 as the cosmological value of the scalar field at infinity. In this paper, we adopt the same form of conformal factor $A(\varphi)$ as in Damour and Esposito-Farèse [8], which is

$$A(\varphi) = e^{1/2\beta\varphi^2},$$ (8)

i.e., $\alpha(\varphi) = \beta\varphi$, where β is a real number. In the case $\beta = 0$, this scalar-tensor theory reduces to general relativity, while spontaneous scalarization occurs for $\beta \lesssim -4.35$ [11].

We will model the neutron stars as self-gravitating perfect fluid of cold degenerate matter in equilibrium. Then the metric describing an unperturbed, nonrotating, spherically symmetric neutron star can be written as

$$ds_*^2 = g_{*\mu\nu}dx^\mu dx^\nu$$
$$= -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$ (9)

where $e^{-2\Lambda} = 1 - 2\mu(r)/r$, while for the calculation of the mass function $\mu(r)$ and the "potential" function $\Phi(r)$, the reader should refer to paper I. Finally, the stellar matter is assumed to be a perfect fluid

$$\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{U}_\mu \tilde{U}_\nu + \tilde{P}\tilde{g}_{\mu\nu}, \qquad (10)$$

where $\tilde{U}_\mu$ is the four-velocity of the fluid, $\tilde{\rho}$ is the total energy density, and $\tilde{P}$ is the pressure in the physical frame.

III. BASIC PERTURBATION EQUATIONS

In this section, we present the equations describing perturbations of the spacetime, scalar field, and fluid in a spherically symmetric background. The equations we pro-

vide describe the nonradial oscillations of spherically symmetric neutron stars in scalar-tensor theories. We assume, in the physical frame, using the Regge-Wheeler gauge [23], the following form of the perturbed metric tensor:

$$\tilde{h}_{\mu\nu} = \tilde{h}_{\mu\nu}^{(-)} + \tilde{h}_{\mu\nu}^{(+)}, \qquad (11)$$

where $\tilde{h}_{\mu\nu}^{(-)}$ denotes the *axial* (or *odd parity*) part of metric perturbations

$$\tilde{h}_{\mu\nu}^{(-)} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \begin{pmatrix} 0 & 0 & -h_{0,lm}\sin^{-1}\theta \partial_\phi & h_{0,lm}\sin\theta \partial_\theta \\ 0 & 0 & -h_{1,lm}\sin^{-1}\theta \partial_\phi & h_{1,lm}\sin\theta \partial_\theta \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} Y_{lm}, \qquad (12)$$

and $\tilde{h}_{\mu\nu}^{(+)}$ denotes the *polar* (or *even parity*) part of metric perturbations

$$\tilde{h}_{\mu\nu}^{(+)} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \begin{pmatrix} H_{0,lm}e^{2\Phi} & H_{1,lm} & 0 & 0 \\ * & H_{2,lm}e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 K_{lm} & 0 \\ 0 & 0 & 0 & r^2 K_{lm}\sin^2\theta \end{pmatrix} Y_{lm}. \qquad (13)$$

The functions $h_{0,lm}$, $h_{1,lm}$, $H_{0,lm}$, $H_{1,lm}$, $H_{2,lm}$, and K_{lm} describing the spacetime perturbations have only radial and temporal dependence, while $Y_{lm} = Y_{lm}(\theta, \phi)$ is the spherical harmonic function.

Following the previous definitions, the perturbed metric tensor $h_{*\mu\nu}$ in the Einstein frame has the form:

$$h_{*\mu\nu} = \frac{1}{A^2}\tilde{h}_{\mu\nu} - \frac{2}{A}g_{*\mu\nu}\delta A \qquad (14)$$

$$= \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{A^2} \begin{pmatrix} (H_{0,lm} + 2A\delta A)e^{2\Phi} & H_{1,lm} & -h_{0,lm}\sin^{-1}\theta \partial_\phi & h_{0,lm}\sin\theta \partial_\theta \\ * & (H_{2,lm} - 2A\delta A)e^{2\Lambda} & -h_{1,lm}\sin^{-1}\theta \partial_\phi & h_{1,lm}\sin\theta \partial_\theta \\ * & * & (K_{lm} - 2A\delta A)r^2 & 0 \\ * & * & 0 & (K_{lm} - 2A\delta A)r^2\sin^2\theta \end{pmatrix} Y_{lm}, \quad (15)$$

where $\delta A \equiv A\beta_\varphi \delta\varphi$ is the perturbation of the conformal factor A and $\delta\varphi$ is the perturbation of the scalar field, where $\delta\varphi$ is a function of t and r only. The above definition of $h_{*\mu\nu}$ will be used to derive the perturbation equations: In other words, we will work out the perturbations in the Einstein frame and we will transform back to the physical frame whenever we need it.

By defining the new set of perturbation functions $\hat{H}_0$, $\hat{H}_1$, $\hat{H}_2$, $\hat{K}$, $\hat{h}_0$, and $\hat{h}_1$ as follows:

$$\hat{H}_{0,lm} = \frac{1}{A^2}(H_{0,lm} + 2A\delta A), \qquad \hat{H}_{1,lm} = \frac{1}{A^2}H_{1,lm}, \qquad (16)$$

$$\hat{H}_{2,lm} = \frac{1}{A^2}(H_{2,lm} - 2A\delta A), \qquad (17)$$

$$\hat{K}_{lm} = \frac{1}{A^2}(K_{lm} - 2A\delta A),$$

$$\hat{h}_{0,lm} = \frac{1}{A^2}h_{0,lm}, \qquad \hat{h}_{1,lm} = \frac{1}{A^2}h_{1,lm}, \qquad (18)$$

the perturbed metric $h_{*\mu\nu}$, in the Einstein frame, is simplified considerably and reduced to the "standard" Regge-Wheeler form of a perturbed spherical metric. We should notice that the scalar perturbations δA are linked only with the polar perturbation functions $H_{0,lm}$, $H_{1,lm}$, $H_{2,lm}$, and K_{lm}. The axial perturbation functions $h_{0,lm}$ and $h_{1,lm}$ are affected only by the contribution of the scalar field to the background.

The perturbation equations will be derived by taking the variation of Eqs. (3) and (4)

$$\delta G_{*\mu\nu} = 8\pi G_* \delta T_{*\mu\nu} + \delta T_{*\mu\nu}^{(\varphi)}, \qquad (19)$$

$$\Box_* \delta\varphi = -4\pi G_* \delta[\alpha(\varphi)T_*]. \qquad (20)$$

The various components of $\delta T_{*\mu\nu}^{(\varphi)}$ are expressed as linear combinations of $\delta\varphi$ and $\tilde{h}_{\mu\nu}$. In the Einstein frame, the perturbed energy-momentum tensor describing the matter fields $\delta T_{*\mu\nu}$ is some linear combination of the velocity variation of the fluid $\delta\tilde{U}^i \sim (WY_{lm}, V\partial_\theta Y_{lm} - u\sin^{-1}\theta \partial_\phi Y_{lm}, V\partial_\phi Y_{lm} + u\sin\theta \partial_\theta Y_{lm})$, the density and

pressure variations ($\delta\tilde{\rho}$ and $\delta\tilde{P}$) together with the variation of the scalar field ($\delta\varphi$ or δA), and the metric perturbation $h_{*\mu\nu}$. The explicit form of the energy-momentum tensor is given in Appendix A.

The linearized Einstein equations (19) will be written as follows. From the tt, tr, rr components and the sum of the components $\theta\theta$ and $\phi\phi$, we get

$$\sum_{l,m} A_{lm}^{(I)} Y_{lm} = 0 \qquad (I = 0\text{--}3), \qquad (21)$$

where the four expressions $A_{lm}^{(I)} = 0$ are given in Appendix B. They contain combinations of $\hat{H}_0$, $\hat{H}_1$, $\hat{H}_2$, $\hat{K}$, W, $\delta\tilde{P}$, $\delta\tilde{\rho}$, $\delta\varphi$, and their temporal and spatial derivatives. It is worth noticing that all four equations above are describing only polar perturbations. In a similar way, from the $t\theta$, $t\phi$, $r\theta$, and $r\phi$ components, we get four more equations

$$\sum_{l,m} \left\{ \alpha_{lm}^{(J)} \partial_\theta Y_{lm} + \beta_{lm}^{(J)} \frac{1}{\sin\theta} \partial_\phi Y_{lm} \right\} = 0 \qquad (J = 0, 1),$$
$$(22)$$

$$\sum_{l,m} \left\{ \beta_{lm}^{(J)} \partial_\theta Y_{lm} - \alpha_{lm}^{(J)} \frac{1}{\sin\theta} \partial_\phi Y_{lm} \right\} = 0 \qquad (J = 0, 1).$$
$$(23)$$

Here the expressions $\alpha_{lm}^{(J)}$ are some linear combinations of polar perturbation functions, while, on the other hand, the expression $\beta_{lm}^{(J)}$ is a combination of only axial perturbation functions (see Appendix B).

Furthermore, from the $\theta\phi$ component and the subtraction of $\theta\theta$ and $\phi\phi$ components, we get two more equations

$$\sum_{l,m} \{ s_{lm} X_{lm} - t_{lm} \sin\theta W_{lm} \} = 0, \qquad (24)$$

$$\sum_{l,m} \{ t_{lm} X_{lm} + s_{lm} \sin\theta W_{lm} \} = 0, \qquad (25)$$

where s_{lm} and t_{lm} describe polar and axial type perturbations, respectively, while X_{lm} and W_{lm} are defined as

$$X_{lm} = 2\partial_\phi \left(\partial_\theta - \frac{\cos\theta}{\sin\theta} \right) Y_{lm} \quad \text{and}$$
$$W_{lm} = \left(\partial_\theta^2 - \frac{\cos\theta}{\sin\theta} \partial_\theta - \frac{1}{\sin^2\theta} \partial_\phi^2 \right) Y_{lm}.$$
$$(26)$$

Taking the product of Eqs. (21)–(25) with $\bar{Y}_{lm}$, integrating over the solid angle, and looking at components with fixed values of l and m, we get ten partial differential equations in the variables t and r:

$$A_{lm}^{(I)} = 0, \qquad \alpha_{lm}^{(J)} = 0, \qquad s_{lm} = 0,$$
$$(I = 0\text{--}3 \quad \text{and} \quad J = 0, 1)$$
$$(27)$$

$$\beta_{lm}^{(J)} = 0, \qquad t_{lm} = 0 \qquad (J = 0, 1). \qquad (28)$$

Equations (27) describe the polar perturbations, and Eqs. (28) describe the axial perturbations. The analytic expressions for Eqs. (28), i.e., Eqs. (B7), (B8), and (B10), do not couple to the perturbations of the scalar field δA or $\delta\varphi$. Therefore, the perturbed scalar field is coupled only to the polar perturbations.

A. Axial perturbations

It is quite easy to derive a wave equation for the axial perturbations by combining Eqs. (B8) and (B10)

$$\ddot{X} - e^{\Phi-\Lambda} (e^{\Phi-\Lambda} X')'$$
$$+ e^{2\Phi} \left(\frac{l(l+1)}{r^2} - \frac{6\mu}{r^3} + 4\pi G_*(\tilde{\rho} - \tilde{P})A^4 \right) X = 0, \quad (29)$$

where we introduce the new function $X(t, r)$ defined as $\hat{h}_1 = e^{\Lambda-\Phi} X r$. This equation does not couple to the scalar field perturbations, as we have mentioned earlier, and the effects of the scaler field will enter only via the background terms. Thus, for $\beta = 0$, i.e., $A = 1$, it reduces to the standard wave equation describing axial perturbations [24]. Finally, from Eq. (B8) we get the following relation:

$$\dot{u} = -e^{-2\Phi} \dot{\hat{h}}_0, \qquad (30)$$

which suggests that there are no axial oscillatory fluid motions; i.e., they have zero frequency and represent stationary currents. Thus, axial perturbations are described only by a single wave equation (29), which does not couple neither to polar fluid or spacetime perturbations nor to the perturbed scalar field and it can be studied independently.

B. Polar perturbations

The equations describing polar perturbations can be simplified, introducing a new set of perturbation functions. Introducing these functions, we can reformulate the seven equations describing polar perturbations as a pair of wave equations and a constraint equation, using a procedure similar to Ref. [25]. The new metric perturbation functions will be

$$F(t, r) = r\hat{K} \quad \text{and} \quad S(t, r) = \frac{e^{2\Phi}}{r}(\hat{H}_0 - \hat{K}), \qquad (31)$$

while the fluid perturbations can be described by variation of the enthalpy function, i.e.,

$$H(t, r) = \frac{\delta\tilde{P}}{\tilde{\rho} + \tilde{P}}. \qquad (32)$$

The system of equations describing the polar perturbations can be reduced to the following pair of wave equations:

$$\ddot{F} - e^{2(\Phi-\Lambda)}F'' - e^{2\Phi}\left[4\pi G_*r(\tilde{P}-\tilde{\rho})A^4 + \frac{2\mu}{r^2}\right]F' + 2r(\tilde{\rho}+\tilde{P})\left(1-\frac{1}{C_s^2}\right)A^4e^{2\Phi}H$$

$$+ e^{2\Phi}\left[\frac{l(l+1)}{r^2} - \frac{2\mu}{r^3} - 4\pi G_*(3\tilde{\rho}+\tilde{P})A^4 - 2e^{-2\Lambda}\Psi^2\right]F + 2e^{-2\Lambda}[1-r^2\Psi^2 - 4\pi G_*r^2e^{-2\Lambda}(\tilde{\rho}+\tilde{P})A^4]S$$

$$+ 8e^{2\Phi}[\Psi e^{-2\Lambda} + 4\pi G_*\alpha r(\tilde{P}-\tilde{\rho})A^4]\delta\varphi = 0, \quad (33)$$

and

$$\ddot{S} - e^{2(\Phi-\Lambda)}S'' - e^{2\Phi}\left[4\pi G_*r(\tilde{P}-\tilde{\rho})A^4 + \frac{2\mu}{r^2}\right]S' + e^{2\Phi}\left[\frac{l(l+1)}{r^2} - \frac{2\mu}{r^3} - 4\pi G_*(3\tilde{P}+\tilde{\rho})A^4 + 2e^{-2\Lambda}\Psi^2\right]S$$

$$+ \frac{4e^{4\Phi}}{r^5}\left\{\Phi'^2r^3e^{-2\Lambda} + 4\pi G_*\tilde{\rho}Ar^3 - 3\mu + \frac{4\pi G_*}{r}(\tilde{\rho}-3\tilde{P})\alpha A^4\Psi + [4\pi G_*(\tilde{\rho}-\tilde{P})r^5A^4 - 2r^3]\Psi^2\right\}F$$

$$+ 4e^{4\Phi}\left\{\Psi^3 e^{-2\Lambda} + \left[8\pi G_*(2\tilde{P}-\tilde{\rho})A^4 + \frac{10\mu}{r^3} - \frac{2}{r^2}\right]\Psi\right\}\delta\varphi = 0. \quad (34)$$

From the *tt* component of the perturbed Einstein equations, we get the Hamiltonian constraint:

$$F'' + \left[\frac{e^{2\Lambda}}{r^2}(\mu - 4\pi G_*r^3\tilde{\rho}A^4) - \frac{1}{2}r\Psi^2\right]F' + \frac{e^{2\Lambda}}{r^3}\left[12\pi G_*\tilde{\rho}r^3A^4 - \mu - rl(l+1) + \frac{1}{2}r^3\Psi^2e^{-2\Lambda}\right]F$$

$$- re^{-2\Phi}S' + e^{-2\Phi+2\Lambda}\left[\Psi^2r^2e^{-2\Lambda} + 8\pi G_*r^2(\tilde{\rho}+\tilde{P})A^4 - 2e^{-2\Lambda} - \frac{l(l+1)}{2}\right]S + \frac{8\pi G_*r}{C_s^2}(\tilde{\rho}+\tilde{P})e^{2\Lambda}A^4H$$

$$+ 2r\Psi\delta\varphi' + 32\pi G_*rA^4e^{2\Lambda}\tilde{\rho}\alpha\delta\varphi = 0. \quad (35)$$

Furthermore, for the special form of the conformal factor, i.e., $A = \exp(\beta\varphi^2/2)$ or $\alpha = \beta\varphi$, from Eq. (20) we obtain a wave equation for the perturbed scalar field $\delta\varphi$:

$$\delta\ddot{\varphi} - e^{\Phi-\Lambda}(e^{\Phi-\Lambda}\delta\varphi')' - \frac{2}{r}e^{2\Phi-2\Lambda}\delta\varphi' + e^{2\Phi}\left[\frac{l(l+1)}{r^2} + 4e^{-2\Lambda}\Psi^2 - 4\pi G_*A^4(\tilde{\rho}-3\tilde{P})(4\alpha^2+\beta)\right]\delta\varphi$$

$$= \left[r^2e^{-2\Lambda}\Psi^3 + \frac{2\mu\Psi}{r} + 4\pi G_*rA^4\{2r\Psi\tilde{P} - \alpha(\tilde{\rho}-3\tilde{P})\}\right]S$$

$$+ e^{2\Phi}\left[e^{-2\Lambda}\Psi^3 + \frac{2\mu\Psi}{r^3} + 4\pi G_*A^4\left\{2\Psi\tilde{P} - \frac{\alpha}{r}(\tilde{\rho}-3\tilde{P})\right\}\right]F - 4\pi G_*A^4\alpha e^{2\Phi}(\tilde{\rho}+\tilde{P})\left(\frac{1}{C_s^2}-3\right)H, \quad (36)$$

where $C_s^2 = d\tilde{P}/d\tilde{\rho}$.

Finally, by linearizing Eq. (7), i.e., the energy-momentum conservation equations, we get a system of evolution equations for the components of the perturbed velocity and the density perturbation

$$W' + \frac{1}{C_s^2}e^{2\Lambda-2\Phi}\dot{H} + e^{2\Lambda-2\Phi}\left(3\alpha\delta\dot{\varphi} + \frac{r}{2}e^{-2\Phi}\dot{S} + \frac{3}{2r}\dot{F}\right) - \frac{l(l+1)}{r^2}e^{2\Lambda}V$$

$$+ \left[2\Phi' - \Lambda' + \frac{2}{r} + 3\alpha\Psi - \frac{1}{C_s^2}(\Phi'+\alpha\Psi)\right]W = 0, \quad (37)$$

$$\dot{W} + H' + \alpha\delta\varphi' + \frac{r}{2}e^{-2\Phi}S' - \frac{1}{2r}F' + e^{-2\Phi}\left(\frac{1}{2} - r\Phi' + r^2\Psi^2 + \frac{2\mu}{r}e^{2\Lambda} + 8\pi G_*r^2A^4e^{2\Lambda}\tilde{P}\right)S$$

$$+ \left(\Psi^2 + \frac{2\mu}{r^3}e^{2\Lambda} + \frac{1}{2r^2} + 8\pi G_*A^4e^{2\Lambda}\tilde{P}\right)F + (\beta-4)\Psi\delta\varphi = 0, \quad (38)$$

$$\dot{V} - \frac{r}{2}e^{-2\Phi}S - \frac{1}{2r}F + H + \alpha\delta\varphi = 0. \quad (39)$$

Concluding, the dynamics of the polar perturbation is described by the system of evolution equations (33), (34), and (36) together with the constraint equation (35). The rest of the functions describing the spacetime and fluid perturbations will be computed by taking proper combinations of F, S, H, and $\delta\varphi$.

IV. SPACETIME PERTURBATIONS IN SCALAR-TENSOR GRAVITY

In paper I we have studied stellar perturbations in scalar-tensor theories of gravity freezing the spacetime and scalar field perturbations. This is the so-called Cowling approximation, in which we consider only perturbations of the fluid. In practice, we worked with a system of equations similar to (37)–(39), but setting $H_1 = H_0 = H_2 = K = \delta\varphi = 0$. Even under this approximation, spontaneous scalarization has a remarkable effect on the oscillation spectra of the f and p modes. Based on this observation, we proposed in paper I that a successful detection of gravitational waves from oscillating stars will provide us with a tool to constrain the phenomenon of spontaneous scalarization.

The quasinormal modes of the fluid perturbations described in paper I will be affected by the coupling to the spacetime and scalar field perturbations. This is an interesting problem on its own. However, since we have already shown that the effect of spontaneous scalarization is quite strong when we limit consideration to perturbations of the fluid, in this paper we will examine the effect of the scalar field on the quasinormal modes describing the pure spacetime oscillations, i.e., the so-called w modes [26]. The w modes are similar to quasinormal modes of black holes. They have higher frequencies and shorter damping times than the f modes, typical frequencies being of order 7–12 kHz and damping times of order of 0.1 ms. These properties of the w modes are common to axial and polar perturbations [17]. In the case of polar perturbations, the w modes are associated to small fluid motions, while in the axial case there is no coupling with the fluid at all. This is

the reason why here we choose to study the effect of the scalar field considering only the axial perturbations, described by the single wave equation (29). We expect the effect of the scalar field on the axial and polar w modes to be of the same order of magnitude. It should be pointed out here that, according to recent collapse calculations [27], the w modes are significantly excited. This adds further motivation to our study of the effects of scalar fields on w mode oscillations.

The equations needed to construct the equilibrium stellar configurations as well as the equations of state (EOS) used are described in paper I. In paper I we also considered cases where the asymptotic value of the scalar field $\varphi_0 \neq 0$; here, for simplicity, we deal only with scalar fields with $\varphi_0 = 0$.

To compute the quasinormal frequencies of the axial w modes, we will use two different techniques. In the first approach, we carry out time evolutions of Eq. (29) and Fourier transform the signal at infinity; in the second approach, we assume a harmonic time dependence of the perturbations and the corresponding boundary value problem.

A. Evolving the axial perturbation equation

The time evolution of the $1 + 1$ equation (29) is rather simple to obtain. We set some arbitrary initial data (for example, a Gaussian pulse) in Eq. (29) and evolve these data for some time. Then we compute the oscillation frequencies by taking the Fourier transform of the signal emitted at infinity.

In Fig. 1 we show the waveforms observed at a distance of about 300 km from a neutron star with Arnowitt-Deser-Misner (ADM) mass $1.4M_\odot$. It is noticeable that the arrival time of wave for different values of β is not the same, because the effective potential due to central neutron star changes as a function of β. In this figure, we can see that the waveforms for $\beta = 0$ and for $\beta = -4$ are identical. This result can be understood as follows. For the axial perturbation, the gravitational wave is not coupled with the perturbation of the matter and the scalar field. So the presence of the scalar field is realized only due to the
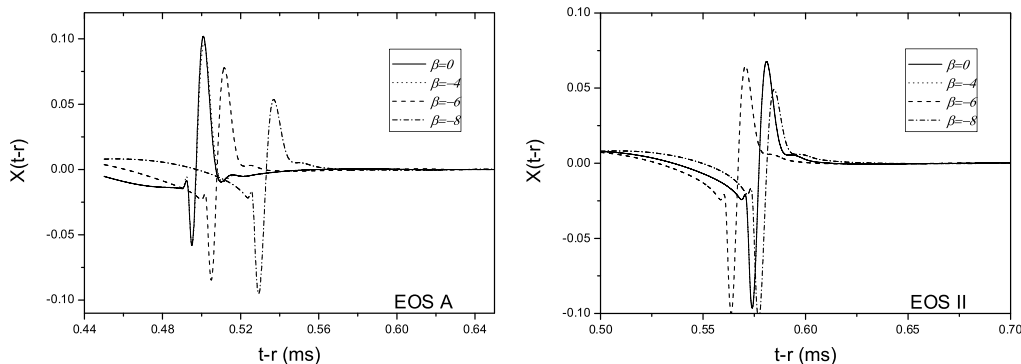

FIG. 1.   Waveforms describing the evolution of X for EOS A (left panel) and EOS II (right panel) generated for stellar models with $M_{\mathrm{ADM}} = 1.4M_\odot$ for $\beta = 0$, -4, -6, and -8. The cases $\beta = 0$ and $\beta = -4$ are indistinguishable.

modified background. On the other hand, with $\varphi_0 = 0$, the central value of scalar field φ_c is zero for any $\beta > -4.35$ (see [14]). Thus, for values of $\beta > -4.35$ the effect of the scalar field is insignificant and it will affect the results only for $\beta < -4.35$.

B. Boundary value method

Our second approach to calculate the quasinormal frequencies of the axial w modes is more involved than simple time evolutions. However, using time evolutions we can identify only those modes that are significantly excited by a certain set of initial data. For example, using time evolutions it is not easy to identify quasinormal modes that damp out very fast. The approach we present here allows us to calculate both slowly and highly damped quasinormal modes. We Fourier expand the wave equation (29) as $X(t, r) = X(r) \exp(i\omega t)$ and get

$$X'' + \frac{2\mu}{r(r - 2\mu)}X' + \left(1 - \frac{2\mu}{r}\right)^{-1}$$
$$\times \left[\omega^2 e^{-2\Phi} - \frac{l(l+1)}{r^2} + \frac{6\mu}{r^3} - 4\pi G_*(\tilde{\rho} - \tilde{P})A^4 \right] X = 0.$$
$$(40)$$

In this way, we obtain an eigenvalue problem: The complex quasinormal modes ω can be obtained by imposing appropriate boundary conditions. In our case, the boundary conditions are that $X(r)$ should be regular at the stellar center and that there are no incoming waves at infinity. Inside the star, we can just integrate the above differential equation; outside the star, we use appropriate asymptotic expansions to ensure that there is no incoming radiation. Here we adopt a variant of Leaver's continued fraction method [28] that has been originally used for the calcula-

tion of quasinormal modes of black holes. The procedure is described in detail in Appendix C.

C. Results

The results of the two methods we described agree very well, providing a good consistency check on our calculations. In Figs. 2–4, we present the eigenvalues. Our results suggest that the presence of a spontaneous scalarization can be inferred from the w modes emitted by a newly born, oscillating neutron star.

In Fig. 2 we show the eigenfrequencies of the w modes for neutron star models with $M_{\text{ADM}} = 1.4M_\odot$. The plot is reminiscent of earlier calculation of these modes (see, e.g., Fig. 3 in Ref. [29]). The modes that might be relevant for gravitational wave detectors are the lowest w modes [26]. The w_{II} modes damp out [30] roughly twice as fast as the w modes, but having lower frequencies they could also be relevant for detection by Earth-based interferometers. The higher-frequency w modes ($w_2, w_3, w_4, \dots$) are difficult, if not impossible, to detect.

In the study of w modes as a tool for asteroseismology [15–17,22], it has been suggested that a proper normalization for $Re(\omega)$ is to multiply it with the radius R of the star and to scale it as a function of the compactness M/R. This phenomenological argument has been recently verified analytically by Tsui and Leung [31]. Introducing $f = Re(\omega)/2\pi$, it is clear that Rf scales linearly as a function of the compactness M/R. This applies to both w_{II} and w_1 modes (and even to the higher overtones). The linear relations that can be derived from Fig. 3 are

$$f_{\omega_1\text{-mode}}(\text{kHz}) = \frac{1}{\bar{R}}\left(\alpha_1 - \beta_1 \frac{\bar{M}}{\bar{R}}\right) \quad \text{and}$$

$$f_{\omega_{\text{II}}\text{-mode}}(\text{kHz}) = \frac{1}{\bar{R}}\left(\alpha_{\text{II}} + \beta_{\text{II}} \frac{\bar{M}}{\bar{R}}\right), \qquad (41)$$
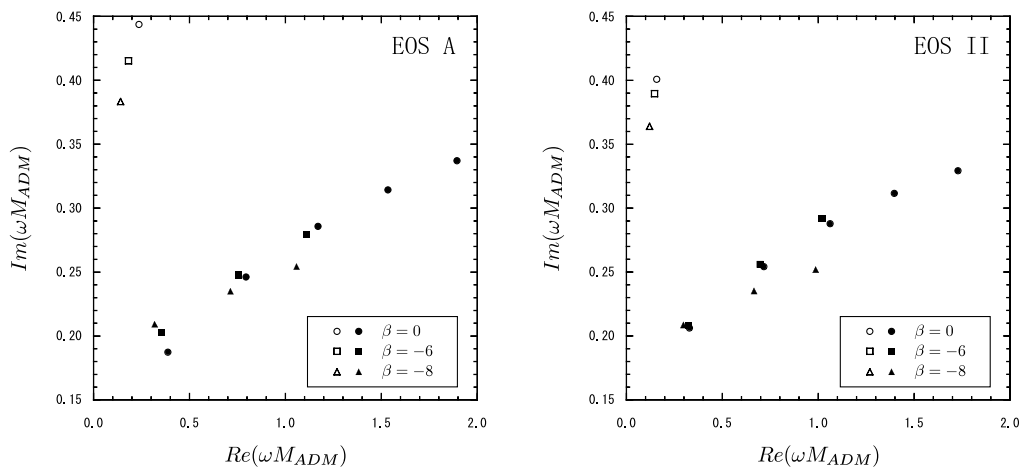

FIG. 2. The frequencies of the axial w modes for $l = 2$, here we have considered stellar models with $M_{\text{ADM}} = 1.4M_\odot$. The left panel corresponds to EOS A and the right panel to EOS II. We show both the "ordinary" w modes ($w_1, w_2, w_3, \dots$) and the w_{II} modes (upper left corner in the plot). Open diamonds, squares, and triangles correspond to the w_{II} modes with $\beta = 0$, $\beta = -6$, and $\beta = -8$, respectively. Similarly, solid symbols refer to the fundamental w mode and its overtones.
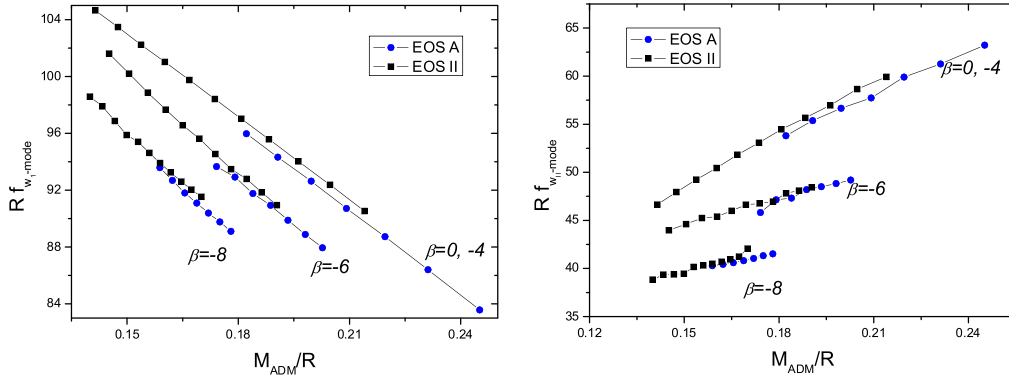
FIG. 3 (color online). The w_1 modes (left panel) and the w_{II} modes (right panel) for EOS A and II and for values of $\beta = 0, -4, -6$, and -8.
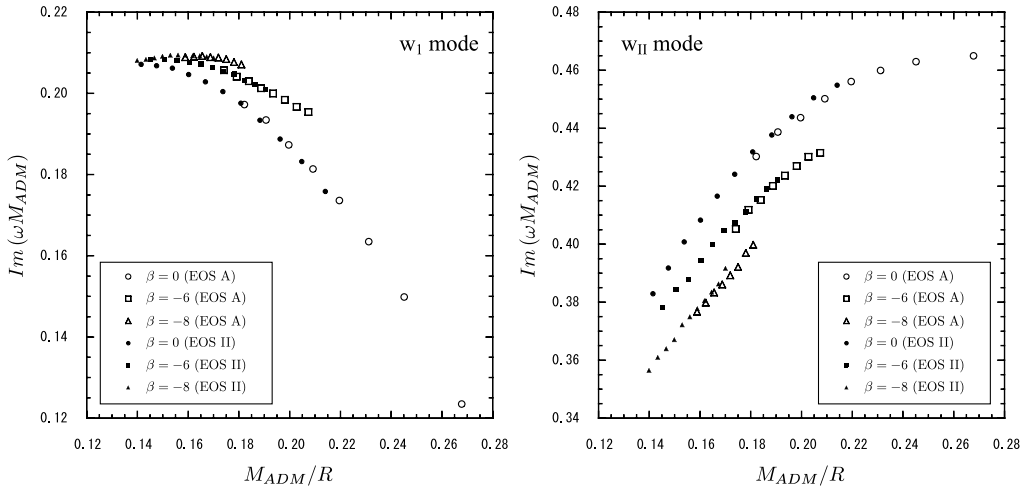

FIG. 4. Dependence of the imaginary part of w_1 and w_{II} mode on the stellar compactness M_{ADM}/R. The left panel is for the w_1 mode and the right for the w_{II} mode. It is apparent that the imaginary part for w_{II} mode can decrease by up to 30% in the presence of a scalar field. For the w_1 modes, on the contrary, the damping time becomes shorter.

where the constants α_1, β_1, α_{II}, and β_{II} are listed in Table I.

Another reason why it is harder to detect high-damped quasinormal modes such as the w_{II} modes for compact stars is that the effective amplitude scales as the square root of the number of oscillations [22]. Typically, we can hardly observe more than $2-3$ cycles for highly damped quasinormal modes of black holes and for the w_{II} modes of compact stars. Spontaneous scalarization might help in this direction. Figure 4 shows that the damping time of the w_{II} mode for stars with $\beta \lesssim -4.35$ is significantly longer than

for typical stars in general relativity. The reason is that the presence of a scalar field increases the maximum mass of the stars and their compactness. Since the damping scales with compactness, the w_{II} modes live considerably longer. On the contrary, the damping times of the w_1 modes become shorter as the compactness increases (left panel in Fig. 4).

V. CONCLUSION

We derived the equations describing stellar perturbations in scalar-tensor theories of gravity. The presence of a scalar field affects the equilibrium model and, consequently, the oscillation spectrum. The scalar field perturbations couple with the polar perturbations of the spacetime and fluid, but they do not couple with the axial perturbations. Since the spacetime modes of polar and axial perturbations have the same qualitative behavior, we have chosen to study the effect of the scalar field on the axial perturbations.

TABLE I. The coefficients for the fitting factors of Eqs. (41).

β	α_1	β_1	α_{II}	β_{II}
0	13.35	4.20	2.48	3.32
-6	13.57	4.89	3.09	1.88
-8	13.36	5.17	2.89	1.50

The results show that, in the presence of spontaneous scalarization, a scalar field reduces the oscillation frequency of the w_1 modes by about 10% (i.e., by about 1 kHz). The decrease in frequency for the w_{II} modes is about 25% the frequency (i.e., about 1.5 kHz). The effect on the damping time is even more pronounced. The damping of w_{II} modes can decrease by as much as 30%, while it can increase by as much as 50% for the w_1 modes. Detectors operating at these high frequencies are under development. Through a detection of the w mode spectrum, they could provide a unique proof for the existence of scalar fields with $\beta \lesssim -4.35$.

A more detailed model of the effect of the scalar field on the oscillation spectra requires the inclusion of a larger set of equations of state. Another open problem is the study of polar oscillations, which couple directly to the scalar field. Work in these directions is in progress.

ACKNOWLEDGMENTS

APPENDIX A: THE PERTURBED ENERGY-MOMENTUM TENSOR

In this appendix, we show the explicit form of the various components of the perturbed energy-momentum tensor (of the fluid and of the scalar field) appearing in the perturbation equations. We will use primes for spatial derivatives and dots for temporal derivatives. For simplicity, we will omit the subscript lm in the various perturbed quantities.

The components of the perturbed energy-momentum tensor $T_{*\mu\nu}^{(\varphi)}$ for the scalar field have the form

$$\delta T_{*tt}^{(\varphi)} = e^{2\Phi-2\Lambda}[2\Psi\delta\varphi' - (\hat{H}_0 + \hat{H}_2)\Psi^2]Y_{lm}, \quad (A1)$$

$$\delta T_{*tr}^{(\varphi)} = [2\Psi\delta\dot{\varphi} - e^{-2\Lambda}\Psi^2\hat{H}_1]Y_{lm}, \quad (A2)$$

$$\delta T_{*t\theta}^{(\varphi)} = e^{-2\Lambda}\Psi^2\hat{h}_0\frac{1}{\sin\theta}\partial_\phi Y_{lm}, \quad (A3)$$

$$\delta T_{*t\phi}^{(\varphi)} = -e^{-2\Lambda}\Psi^2\hat{h}_0\sin\theta\partial_\theta Y_{lm}, \quad (A4)$$

$$\delta T_{*rr}^{(\varphi)} = 2\Psi\delta\varphi'Y_{lm}, \quad (A5)$$

$$\delta T_{*r\theta}^{(\varphi)} = \left[2\Psi\delta\varphi\partial_\theta + e^{-2\Lambda}\Psi^2\hat{h}_1\frac{1}{\sin\theta}\partial_\phi\right]Y_{lm}, \quad (A6)$$

$$\delta T_{*r\phi}^{(\varphi)} = [2\Psi\delta\varphi\partial_\phi - e^{-2\Lambda}\Psi^2\hat{h}_1\sin\theta\partial_\theta]Y_{lm}, \quad (A7)$$

$$\delta T_{*\theta\theta}^{(\varphi)} = r^2 e^{-2\Lambda}[-2\Psi\delta\varphi' + (\hat{H}_2 - \hat{K})\Psi^2]Y_{lm}, \quad (A8)$$

$$\delta T_{*\phi\phi}^{(\varphi)} = r^2 e^{-2\Lambda}[-2\Psi\delta\varphi' + (\hat{H}_2 - \hat{K})\Psi^2]\sin^2\theta Y_{lm}. \quad (A9)$$

In order to get the components of the perturbed energy-momentum tensor for the fluid, we define, in the physical frame, the variations of pressure $\delta\tilde{P} = \delta\tilde{P}Y_{lm}$, energy density $\delta\tilde{\rho} = \delta\tilde{\rho}Y_{lm}$, and the components of the perturbed 4-velocity (in the physical frame)

$$\delta\tilde{U}^t = \frac{1}{2A^3}e^{-\Phi}H_0 Y_{lm}, \quad (A10)$$

$$\delta\tilde{U}^r = \frac{1}{A}e^{\Phi-2\Lambda}WY_{lm}, \quad (A11)$$

$$\delta\tilde{U}^\theta = \frac{1}{Ar^2}e^{\Phi}\left(V\partial_\theta Y_{lm} - u\frac{1}{\sin\theta}\partial_\phi Y_{lm}\right), \quad (A12)$$

$$\delta\tilde{U}^\phi = \frac{1}{Ar^2\sin^2\theta}e^{\Phi}(V\partial_\phi Y_{lm} + u\sin\theta\partial_\theta Y_{lm}), \quad (A13)$$

where the perturbation functions $\delta\tilde{P}$, $\delta\tilde{\rho}$, W, V, and u defined in the previous relations depend only on t and r. Using the above definition, the components of the perturbed energy-momentum tensor $\delta T_{*\mu\nu}$ take the form

$$\delta T_{*tt} = A^4 e^{2\Phi}\left[\delta\tilde{\rho} - \tilde{\rho}\hat{H}_0 + \frac{4\tilde{\rho}}{A}\delta A\right]Y_{lm}, \quad (A14)$$

$$\delta T_{*tr} = -A^4 e^{2\Phi}[(\tilde{\rho} + \tilde{P})W + e^{-2\Phi}\tilde{\rho}\hat{H}_1]Y_{lm}, \quad (A15)$$

$$\delta T_{*t\theta} = -A^4 e^{2\Phi}(\tilde{\rho} + \tilde{P})V\partial_\theta Y_{lm} + A^4 e^{2\Phi}[(\tilde{\rho} + \tilde{P})u + e^{-2\Phi}\tilde{\rho}\hat{h}_0]\frac{1}{\sin\theta}\partial_\phi Y_{lm}, \quad (A16)$$

$$\delta T_{*t\phi} = -A^4 e^{2\Phi}(\tilde{\rho} + \tilde{P})V\partial_\phi Y_{lm} - A^4 e^{2\Phi}[(\tilde{\rho} + \tilde{P})u + e^{-2\Phi}\tilde{\rho}\hat{h}_0]\sin\theta\partial_\theta Y_{lm}, \quad (A17)$$

$$\delta T_{*rr} = A^4 e^{2\Lambda}\left[\delta\tilde{P} + \tilde{P}\hat{H}_2 + \frac{4\tilde{P}}{A}\delta A\right]Y_{lm}, \quad (A18)$$

$$\delta T_{*r\theta} = -A^4\tilde{P}\hat{h}_1\frac{1}{\sin\theta}\partial_\phi Y_{lm}, \quad (A19)$$

$$\delta T_{*r\phi} = A^4\tilde{P}\hat{h}_1\sin\theta\partial_\theta Y_{lm}, \quad (A20)$$

$$\delta T_{*\theta\theta} = A^4 r^2 \left[\delta\tilde{P} + \tilde{P}\,\hat{K} + \frac{4\tilde{P}}{A}\,\delta A \right] Y_{lm}, \qquad \text{(A21)}$$

$$\delta T_{*\phi\phi} = A^4 r^2 \left[\delta\tilde{P} + \tilde{P}\,\hat{K} + \frac{4\tilde{P}}{A}\,\delta A \right] \sin^2\theta\, Y_{lm}. \qquad \text{(A22)}$$

APPENDIX B: THE COMPONENTS OF THE LINEARIZED EINSTEIN EQUATIONS

Here we provide the explicit form of the various expressions used in Eqs. (21)–(25) for the description of the perturbed Einstein equations (19). We have chosen to use the same notation as Kojima [32] to facilitate comparison.

$$
\begin{aligned}
A_{lm}^{(0)} = {}& -\hat{K}'' + e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 + \frac{5\mu}{r^2} - \frac{3}{r} + 4\pi G_* rA^4\tilde{\rho}\right)\hat{K}' + \frac{1}{r}\hat{H}_2' + \frac{(l-1)(l+2)}{2r^2}e^{2\Lambda}\hat{K} \\
& + e^{2\Lambda}\left(\frac{l(l+1)}{2r^2} + \frac{1}{r^2} - 8\pi G_* A^4\tilde{\rho}\right)\hat{H}_2 - 2\Psi\delta\varphi' - 8\pi G_* A^4 e^{2\Lambda}(\delta\tilde{\rho} + 4\tilde{\rho}\alpha\delta\varphi),
\end{aligned} \tag{B1}
$$

$$
A_{lm}^{(1)} = -\dot{\hat{K}} + e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 + \frac{3\mu}{r^2} - \frac{1}{r} + 4\pi G_* rA^4\tilde{P}\right)\dot{\hat{K}} + \frac{1}{r}\dot{\hat{H}}_2 + \frac{l(l+1)}{2r^2}\hat{H}_1 - 2\Psi\delta\dot{\varphi} + 8\pi G_* A^4 e^{2\Phi}(\tilde{\rho} + \tilde{P})W, \tag{B2}
$$

$$
\begin{aligned}
A_{lm}^{(2)} = {}& -e^{-2\Phi+2\Lambda}\ddot{\hat{K}} + e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 - \frac{\mu}{r^2} + \frac{1}{r} + 4\pi G_* rA^4\tilde{P}\right)\hat{K}' + \frac{2}{r}e^{-2\Phi}\dot{\hat{H}}_1 - \frac{1}{r}\hat{H}_0' - \frac{(l-1)(l+2)}{2r^2}e^{2\Lambda}\hat{K} \\
& + \frac{l(l+1)}{2r^2}e^{2\Lambda}\hat{H}_0 - e^{2\Lambda}\left(\frac{1}{r^2} + 8\pi G_* A^4\tilde{P}\right)\hat{H}_2 - 2\Psi\delta\varphi' - 8\pi G_* A^4 e^{2\Lambda}(\delta\tilde{P} + 4\tilde{P}\alpha\delta\varphi),
\end{aligned} \tag{B3}
$$

$$
\begin{aligned}
A_{lm}^{(3)} = {}& \hat{K}'' - \hat{H}_0'' - e^{-2\Phi+2\Lambda}(\ddot{\hat{K}} + \ddot{\hat{H}}_2) + 2e^{-2\Phi}\dot{\hat{H}}_1' - e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 + \frac{r+\mu}{r^2} + 4\pi G_*(2\tilde{P} - \tilde{\rho})rA^4\right)\hat{H}_0' \\
& - e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 - \frac{\mu}{r^2} + \frac{1}{r} + 4\pi G_* rA^4\tilde{P}\right)\hat{H}_2' + e^{2\Lambda}\left(4\pi G_*(\tilde{P} - \tilde{\rho})rA^4 + \frac{2(r-\mu)}{r^2}\right)\hat{K}' \\
& - 2e^{-2\Phi+2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 + \frac{\mu-r}{r^2} + 4\pi G_* rA^4\tilde{\rho}\right)\dot{\hat{H}}_1 + \frac{l(l+1)}{2r^2}e^{2\Lambda}\hat{H}_0 - e^{2\Lambda}\left(16\pi G_* A^4\tilde{P} + \frac{l(l+1)}{2r^2}\right)\hat{H}_2 \\
& + 4\Psi\delta\varphi' - 16\pi G_* A^4 e^{2\Lambda}(\delta\tilde{P} + 4\tilde{P}\alpha\delta\varphi),
\end{aligned} \tag{B4}
$$

$$
\alpha_{lm}^{(0)} = \frac{1}{2}e^{-2\Lambda}\left[\hat{H}_1' - e^{2\Lambda}(\dot{\hat{H}}_2 + \dot{\hat{K}}) + e^{2\Lambda}\left(4\pi G_*(\tilde{P} - \tilde{\rho})rA^4 + \frac{2\mu}{r^2}\right)\hat{H}_1\right] + 8\pi G_* A^4 e^{2\Phi}(\tilde{\rho} + \tilde{P})V, \tag{B5}
$$

$$
\begin{aligned}
\alpha_{lm}^{(1)} = {}& \frac{1}{2}\left[\hat{H}_0' - \hat{K}' - e^{-2\Phi}\dot{\hat{H}}_1 + e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 + \frac{3\mu}{r^2} - \frac{1}{r} + 4\pi G_* rA^4\tilde{P}\right)\hat{H}_0 \right. \\
& \left. + e^{2\Lambda}\left(\frac{1}{2}re^{-2\Lambda}\Psi^2 - \frac{\mu}{r^2} + \frac{1}{r} + 4\pi G_* rA^4\tilde{P}\right)\hat{H}_2\right] - 2\Psi\delta\varphi,
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
\beta_{lm}^{(0)} = {}& \frac{1}{2}e^{-2\Lambda}(\hat{h}_0'' - \dot{\hat{h}}_1') - \left\{2\pi G_*(\tilde{\rho} + \tilde{P})rA^4 + \left(\frac{r}{2} - \mu\right)\Psi^2\right\}(\hat{h}_0' - \dot{\hat{h}}_1) - \frac{1}{r}e^{-2\Lambda}\dot{\hat{h}}_1 \\
& - \frac{1}{2r^3}\{l(l+1)r - 4\mu + 8\pi G_*(\tilde{\rho} + \tilde{P})r^3 A^4 - 2r^3 e^{-2\Lambda}\Psi^2\}\hat{h}_0 - 8\pi G_* A^4 e^{2\Phi}(\tilde{\rho} + \tilde{P})u,
\end{aligned} \tag{B7}
$$

$$
\beta_{lm}^{(1)} = \frac{1}{2}e^{-2\Phi}(\dot{\hat{h}}_0' - \ddot{\hat{h}}_1) - \frac{1}{r}e^{-2\Phi}\dot{\hat{h}}_0 - \frac{(l-1)(l+2)}{2r^2}\hat{h}_1, \tag{B8}
$$

$$
t_{lm} = e^{-2\Phi}\dot{\hat{h}}_0 - e^{-2\Lambda}\hat{h}_1' - \frac{1}{r^2}\{2\mu + 4\pi G_*(\tilde{P} - \tilde{\rho})r^3 A^4\}\hat{h}_1, \tag{B10}
$$

$$
s_{lm} = \frac{1}{2}(\hat{H}_0 - \hat{H}_2), \tag{B9}
$$

where the equations are simplified by virtue of Eqs. (13)–(17) in paper I.

APPENDIX C: NUMERICAL TECHNIQUES

In this appendix, we present two numerical techniques to determine quasinormal modes. The first is the direct evolution of the time dependent, axial perturbation equation (29). In the second approach, we assume a harmonic decomposition for the perturbation function X of the form $X(r, t) = X(r)e^{i\omega t}$ and solve Eq. (40) as the eigenvalue problem. We shall consider Eqs. (40) in the interior and the exterior of the star; then we will find the eigenvalues (quasinormal modes) by matching the interior and exterior solutions.

The largest numerical error in the interior solution occurs at stellar surface, where the pressure is zero. In order to avoid this difficulty, we integrate the perturbation equation (40) from both sides, i.e., from the stellar center $r = 0$ and from the stellar surface $r = R$. Then we match the solutions at some intermediate point, e.g., $r = R/2$ (see, for example, [26,33]). In order to deal with the boundary condition at infinity, we adopt the continued fraction method, originally used for black-hole perturbations by Leaver [28]. To use this method, we must know the forms of the coefficient in the perturbation equation as functions of $1/r$. Because of the presence of a scalar field, we do not know the exact forms of these coefficients. Therefore, we just use the asymptotic forms of the coefficients and derive a five-term recurrence relation. We believe that the quasinormal modes obtained using these asymptotic forms are accurate enough, because the difference between the value of μ at the stellar surface and at infinity is not so large.

1. Interior region of the star

The numerical integration of Eq. (40) inside the star will be split (for numerical reasons) into two parts. First, we will integrate Eq. (40) from the center towards $R/2$, and then we will integrate from the surface towards the same point. The matching of the two solutions will provide a unique solution valid throughout the star.

Near the center, it can be shown that X has a behavior of the form

$$X = X_c r^{l+1}(1 + O(r^2)), \qquad (C1)$$

where X_c is some arbitrary constant. Using this boundary condition (C1) and by integrating Eq. (40) from $r = 0$ to the matching point $r = R/2$, one can obtain the values of $X(r)$ and $X'(r)$. For convenience, we represent the two functions X and X' in the vector form $\mathbf{Y} = (X, X')$, and we will call $\mathbf{Y}_0(r)$ the solution in the range $0 \leq r \leq R/2$. The next step will be to integrate Eq. (40) from the stellar surface towards $R/2$ with a set of boundary conditions at $r = R$ such as $(X(R), X'(R)) = (1, 0)$ and $(X(R), X'(R)) = (0, 1)$. In this way, we get two independent solutions $\mathbf{Y}_1(r)$ and $\mathbf{Y}_2(r)$ corresponding to each one of the previous boundary conditions. Thus, the solution of the perturbation equation (40) is

$$\mathbf{Y}(r) = \mathbf{Y}_0(r) \quad \text{for } 0 \leq r \leq R/2, \qquad (C2)$$

$$\mathbf{Y}(r) = a\mathbf{Y}_1(r) + b\mathbf{Y}_2(r) \quad \text{for } R/2 \leq r \leq R, \qquad (C3)$$

where a and b are some constant, which will be determined from the junction condition at $r = R/2$:

$$\mathbf{Y}_0(R/2) = a\mathbf{Y}_1(R/2) + b\mathbf{Y}_2(R/2). \qquad (C4)$$

The determination of the two constants specifies uniquely the solution in the interior of the star for a given value of the frequency ω and the constant X_c. At the stellar surface the values of $X(R)$, $X'(R)$ are simply $X(R) = a$ and $X'(R) = b$.

2. Exterior region of the star

The functions describing the stellar background simplify considerably outside the star. This leads to a corresponding simplification of the wave equation (40). In the exterior, the equations describing the background reduce to

$$\mu' = \frac{1}{2} r e^{-2\Lambda} \Psi^2, \qquad (C5)$$

$$\Phi' = \frac{1}{2} r \Psi^2 + \frac{\mu}{r^2} e^{2\Lambda}, \qquad (C6)$$

$$\varphi' = \Psi, \qquad (C7)$$

$$\Psi' = -\frac{2}{r^2} (r - \mu) e^{2\Lambda} \Psi. \qquad (C8)$$

Therefore, the asymptotic forms of the above background quantities are

$$\mu = M_{\text{ADM}} + \frac{\mu_1}{r} + O\left(\frac{1}{r^2}\right), \qquad (C9)$$

$$\Phi = -\frac{M_{\text{ADM}}}{r} + O\left(\frac{1}{r^2}\right), \qquad (C10)$$

$$\varphi = \varphi_0 + \frac{\omega_A}{r} + O\left(\frac{1}{r^2}\right), \qquad (C11)$$

where $\mu_1 = -\omega_A^2/2$ and $\omega_A = -M_{\text{ADM}} \Psi_s / \Phi_s'$.

The perturbation equation (40) in view of the above relations, outside the star, get the form

$$\left(1 - \frac{2\mu}{r}\right) X'' + \frac{2\mu}{r^2} X' + \left[\omega^2 e^{-2\Phi} - \frac{l(l+1)}{r^2} + \frac{6\mu}{r^3}\right] X = 0,$$

$$(C12)$$

which is similar (in the absence of a scalar field, identical) to the Regge-Wheeler [23] equation describing the axial

perturbations in the exterior of a spherically symmetric spacetime (either a black hole or a neutron star). Using as boundary values for the integration the values of X and X' at the surface given by the two relations $X(R) = a$ and $X'(R) = b$, one can integrate Eq. (C12) together with (C5)–(C8) from the stellar surface towards infinite. The numerical integration will obviously stop at some distance $r = r_a$, where we will have to match the numerical solution with the appropriate asymptotic boundary conditions (in this case, the absence of incoming radiation).

In order to find the asymptotic form of the solution of Eq. (C12), we can assume a solution of the form

$$X(r) = \left(\frac{r}{2\hat{M}} - 1\right)^{-2i\hat{M}\omega} e^{-i\omega r} \sum_{n=0}^{\infty} a_n \left(1 - \frac{r_a}{r}\right)^n, \quad \text{(C13)}$$

where $\hat{M} = M_{\text{ADM}}$. By substituting this form of the solution into the perturbation equation (C12) and taking the leading orders for μ and Φ, i.e., keeping only the terms up to order $1/r$, from Eqs. (C9) and (C10), we obtain a five-term recurrence relation for the expansion coefficients a_n ($n \geq 1$),

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} + \epsilon_n a_{n-3} = 0, \quad \text{(C14)}$$

where the coefficients of the recurrence relation are given by the following formulas:

$$\alpha_n = c_0 n(n + 1), \quad \text{(C15)}$$

$$\beta_n = d_0 n + c_1 n(n - 1), \quad \text{(C16)}$$

$$\gamma_n = e_0 + d_1(n - 1) + c_2(n - 1)(n - 2), \quad \text{(C17)}$$

$$\delta_n = e_1 + d_2(n - 2) + c_3(n - 2)(n - 3), \quad \text{(C18)}$$

$$\epsilon_n = e_2 + d_3(n - 3) + c_4(n - 3)(n - 4). \quad \text{(C19)}$$

The coefficients c_i, d_i and e_i are functions of the background quantities and have the form

$$c_0 = 1 - \frac{2\hat{M}}{r_a} - \frac{2\mu_1}{r_a^2}, \quad \text{(C20)}$$

$$c_1 = -2 + \frac{6\hat{M}}{r_a} + \frac{8\mu_1}{r_a^2}, \quad \text{(C21)}$$

$$c_2 = 1 - \frac{6\hat{M}}{r_a} - \frac{12\mu_1}{r_a^2}, \quad \text{(C22)}$$

$$c_3 = \frac{2\hat{M}}{r_a} + \frac{8\mu_1}{r_a^2}, \quad \text{(C23)}$$

$$c_4 = -\frac{2\mu_1}{r_a^2}, \quad \text{(C24)}$$

$$d_0 = -2i\omega r_a - 2 + \frac{6\hat{M}}{r_a} + \frac{4i\omega\mu_1}{r_a} + \frac{8i\omega\hat{M}\mu_1}{r_a^2} + \frac{6\mu_1}{r_a^2}, \quad \text{(C25)}$$

$$d_1 = 2 - \frac{12\hat{M}}{r_a} - \frac{8i\omega\mu_1}{r_a} - \frac{24i\omega\hat{M}\mu_1}{r_a^2} - \frac{18\mu_1}{r_a^2}, \quad \text{(C26)}$$

$$d_2 = \frac{2}{r_a}\left(3\hat{M} + 2i\omega\mu_1 + \frac{12i\omega\hat{M}\mu_1}{r_a} + \frac{9\mu_1}{r_a}\right), \quad \text{(C27)}$$

$$d_3 = -\frac{2\mu_1}{r_a^2}(3 + 4i\omega\hat{M}), \quad \text{(C28)}$$

$$e_0 = -l(l + 1) + 2\omega^2\mu_1 + \frac{6\hat{M}}{r_a} + \frac{8\omega^2\hat{M}\mu_1}{r_a} - \frac{2i\omega\mu_1}{r_a} - \frac{8i\omega\hat{M}\mu_1}{r_a^2} + \frac{6\mu_1}{r_a^2}, \quad \text{(C29)}$$

$$e_1 = \frac{2}{r_a}\left(-3\hat{M} - 4\omega^2\hat{M}\mu_1 + i\omega\mu_1 + \frac{8i\omega\hat{M}\mu_1}{r_a} - \frac{6\mu_1}{r_a}\right), \quad \text{(C30)}$$

$$e_2 = \frac{2\mu_1}{r_a^2}(3 - 4i\omega\hat{M}). \quad \text{(C31)}$$

The first four terms of the recurrence relation (C14) a_{-2}, a_{-1}, a_0, and a_1 are provided by the values of X and X' at $r = r_a$, i.e.,

$$a_{-2} = a_{-1} = 0, \qquad a_0 = \frac{X(r_a)}{\Xi(r_a)}, \quad \text{and}$$

$$a_1 = \frac{r_a}{\Xi(r_a)}\left[X'(r_a) + \frac{i\omega r_a}{r_a - 2\hat{M}}X(r_a)\right], \quad \text{(C32)}$$

where

$$\Xi(r) = \left(\frac{r}{2\hat{M}} - 1\right)^{-2i\hat{M}\omega} e^{-i\omega r}. \quad \text{(C33)}$$

The five-term recurrence relations have, in principle, four possible solutions. A high order recurrence relation can generally be reduced to a three-term recurrence relation, in which case convergence criteria for the solution can be easily applied, and we can identify the solution describing

only outgoing radiation [28]. To obtain a three-term recurrence relation, we define new coefficients $\hat{\alpha}_n$, $\hat{\beta}_n$, $\hat{\gamma}_n$, and $\hat{\delta}_n$ as

$$\hat{\alpha}_n = \alpha_n, \qquad \hat{\beta}_n = \beta_n, \qquad \hat{\gamma}_n = \gamma_n \quad \text{for } n = 1, 2 \tag{C34}$$

and for $n \geq 3$

$$\hat{\alpha}_n = \alpha_n, \qquad \hat{\beta}_n = \beta_n - \frac{\hat{\alpha}_{n-1}\epsilon_n}{\hat{\delta}_{n-1}},$$

$$\hat{\gamma}_n = \gamma_n - \frac{\hat{\beta}_{n-1}\epsilon_n}{\hat{\delta}_{n-1}}, \qquad \hat{\delta}_n = \delta_n - \frac{\hat{\gamma}_{n-1}\epsilon_n}{\hat{\delta}_{n-1}}. \tag{C35}$$

The original five-term recurrence relation becomes a four-term recurrence relation:

$$\hat{\alpha}_n a_{n+1} + \hat{\beta}_n a_n + \hat{\gamma}_n a_{n-1} + \hat{\delta}_n a_{n-2} = 0. \tag{C36}$$

Note that, for the case of $\mu_1 = 0$, that is, the case of the standard neutron star obtained by Einstein's theory for gravity ($\beta = 0$), the recurrence relation for a_n has four terms [17,34,35].

The final step will be to define another set of coefficients $\tilde{\alpha}_n$, $\tilde{\beta}_n$, and $\tilde{\gamma}_n$:

$$\tilde{\alpha}_1 = \hat{\alpha}_1, \qquad \tilde{\beta}_1 = \hat{\beta}_1, \qquad \tilde{\gamma}_1 = \hat{\gamma}_1, \tag{C37}$$

and for $n \geq 2$

$$\tilde{\alpha}_n = \hat{\alpha}_n, \qquad \tilde{\beta}_n = \hat{\beta}_n - \frac{\tilde{\alpha}_{n-1}\hat{\delta}_n}{\tilde{\gamma}_{n-1}}, \quad \text{and}$$

$$\tilde{\gamma}_n = \hat{\gamma}_n - \frac{\tilde{\beta}_{n-1}\hat{\delta}_n}{\tilde{\gamma}_{n-1}}. \tag{C38}$$

The four-term recurrence relation (C36) is thus reduced to a three-term relation of the form

$$\tilde{\alpha}_n a_{n+1} + \tilde{\beta}_n a_n + \tilde{\gamma}_n a_{n-1} = 0. \tag{C39}$$

Using this three-term recurrence relation, the boundary condition can be expressed as a continued fraction relation between $\tilde{\alpha}_n$, $\tilde{\beta}_n$, and $\tilde{\gamma}_n$:

$$\frac{a_1}{a_0} = \frac{-\tilde{\gamma}_1}{\tilde{\beta}_1-} \frac{\tilde{\alpha}_1\tilde{\gamma}_2}{\tilde{\beta}_2-} \frac{\tilde{\alpha}_2\tilde{\gamma}_3}{\tilde{\beta}_3-}\dots, \tag{C40}$$

that can be rewritten as

$$0 = \tilde{\beta}_0 - \frac{\tilde{\alpha}_0\tilde{\gamma}_1}{\tilde{\beta}_1-} \frac{\tilde{\alpha}_1\tilde{\gamma}_2}{\tilde{\beta}_2-} \frac{\tilde{\alpha}_2\tilde{\gamma}_3}{\tilde{\beta}_3-}\dots \equiv f(\omega), \tag{C41}$$

where $\tilde{\beta}_0 \equiv a_1/a_0$, $\tilde{\alpha}_0 \equiv -1$. The eigenfrequency ω of a quasinormal mode can be obtained solving the equation $f(\omega) = 0$.

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