PHYSICAL REVIEW D **71,** 124035 (2005)

Planckian scattering effects and black hole production in low M_{Pl} scenarios

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(Received 22 October 2004; published 27 June 2005)

We reanalyze the question of black hole creation in high energy scattering via shock wave collisions. We find that string corrections tend to increase the scattering cross section. We analyze corrections in a more physical setting, of Randall-Sundrum type and of higher dimensionality. We also analyze the scattering inside anti–de Sitter backgrounds.

DOI: 10.1103/PhysRevD.71.124035 PACS numbers: 04.50.+h, 04.70.Dy, 11.25.Mj, 12.10.Dm

I. INTRODUCTION

The problem of black hole creation in high energy scattering is one of significant importance, for two possible reasons. One is that one can have a low gravitational scale, as in the large extra dimensions [1,2] or Randall-Sundrum [3,4] scenarios. Thus the possibility of black hole creation at accelerators has been explored at length in a number of papers (e.g. [5–13]).

Another reason is that, via gauge-gravity dualities, the high energy scattering in a gravity theory can be related to high energy scattering in QCD, or gauge theories in general [14–16]. Simply put, high energy scattering in QCD can be described in terms of a conformal field theory with a cutoff, and that is dual to a two-brane Randall-Sundrum scenario. But then black hole creation that happens in high energy gravity scattering has to have some implications for the QCD side. In fact, in [15] it was argued that black hole creation, when the black hole size is comparable to the size of the gravity dual = AdS slice, is responsible for the much sought-for Froissart behavior (saturation of the unitarity bound). We will revisit these questions in a future paper [17], but we will still set up some of the calculations needed for that case in here.

In particular, we will analyze the case of high energy scattering with black hole formation inside anti–de Sitter (AdS) space.

We will focus instead on the actual black hole creation at high energy $s \sim M_{\mathrm{Pl}}^2$ with the idea of applying it to theories with a low fundamental scale.

Giddings and Thomas [5] and a number of other people [6,8–10,12] (see also earlier work in [7]) have proposed that the cross section for black hole creation in flat space at high energy is just proportional to the geometric horizon area of a black hole of mass equal to the total center of mass energy, i.e.,

$$\sigma \simeq \pi r_H^2; \qquad r_H = 2G\sqrt{s} \qquad (D = 4). \tag{1.1}$$

There has been a considerable amount of debate over whether this assumption is correct (see, e.g., [9,10,12,13,18,19]).

In an attempt to prove it, Eardley and Giddings [11] have treated the high energy collision according to a recipe proposed some time ago by 't Hooft [20]. The process is well described by the collision of two gravitational shock waves of Aichelburg-Sexl type. Even though one cannot calculate precisely the metric in the future of the collision except perturbatively [21], one can use a trick due to Penrose that just uses the properties of Einstein gravity to calculate a lower bound on the area of the horizon that will form in the collision.

In $D = 4$ [11] were able to extend Penrose's method to collision at nonzero impact parameter b of the two Aichelburg-Sexl waves, and prove that the cross section for black hole scattering is indeed of the order of magnitude of the geometric cross section of the classical black hole.

In this paper we will try to refine this calculation, and answer some of the criticisms addressed to the calculation and the geometric cross-section result. One such criticism was that string corrections will significantly lower this result (see [22] for example) We will try to analyze string corrections explicitly via two methods.

There are two modifications of the Aichelburg-Sexl metric that were shown to reproduce string scattering results (effective metrics). The one in [23] analyzes specifically the scattering at impact parameter b, and gives an effective metric for large b ($> R_s$, the gravitational radius for black hole formation). It is therefore unsuited for our purposes, yet with some approximations one can find that the head-on collision of two such waves (each having a parameter b) will have an increased horizon area of the formed black hole, with respect to the Aichelburg-Sexl case. The second modification [24] corresponds to string-corrected 't Hooft scattering in an Aichelburg-Sexl metric. We will show that scattering of two modified shock waves will again increase the horizon area of the formed black hole.

Another possible caveat to the calculation in [11] is that it was done in flat $D = 4$. We will analyze the case of the more realistic Randall-Sundrum scenario and find that we just get small corrections to the flat $D = 4$ case. We will also offer a method of estimating the cross section in the arbitrary D case.

We should note that we will use the term black hole to describe an object with a horizon, which can radiate particles thermally, even if that object is small (comparable with, but bigger than, Planck size). In that case clearly the formed object cannot be treated classically anymore [25], but one should find a smooth interpolation between the quantum process of creation and decay of the small "black hole" and the formation of the large classical black hole and its subsequent thermal decay. This is the essence of the black hole information "paradox," for which we implicitly assume that there is a resolution, i.e. one can somehow recover the quantum information from the black hole decay, even if we do not know how. We should also mention that other attempts to verify the black hole production cross-section formula were made in [26].

The paper is organized as follows. In Sec. II we will review the Aichelburg-Sexl wave and 't Hooft's scattering calculation, and generalize it to higher dimension. In Sec. III we will review the analysis of [11] and set it up for generalization to any shock waves and any dimension. We will also analyze the collision of sourceless waves, which should describe graviton-graviton scattering, and present a puzzle. In Sec. IV, we will analyze string corrections via the effective metrics in [23,24]. In Sec. V we analyze the case of Randall-Sundrum background and calculate corrections. In Sec. VI we will write down a solution for an Aichelburg-Sexl wave inside AdS and do a 't Hooft scattering analysis.

II. THE AICHELBURG-SEXL WAVE AND 'T HOOFT SCATTERING AT HIGH ENERGY

't Hooft [20] has proposed that an (almost) massless particle at high energies $s \sim M_{\text{Pl}}^2 \gg t$ behaves like a plane gravitational wave—a shock wave—and its only interactions are given by massless particles, with the gravitational interactions described by deflection in the gravitational shock wave corresponding to the massless particle. That shock wave solution is due to Aichelburg and Sexl [27].

In this section we will review this procedure of gravitational interaction and generalize it to higher dimensions.

The Aichelburg-Sexl solution is of the pp wave type. A pp wave (plane fronted gravitational waves) has the general form in d dimensions:

$$ds^2 = -dx^+dx^- + (dx^+)^2 H(x^+, x^i) + \sum_{i=1}^{d-2} (dx_i)^2, \quad (2.1)$$

and has Ricci tensor

$$R_{++} = -1/2\partial_i^2 H(x^+, x^i) \quad (2.2)$$

and the rest are zero. Horowitz and Steif [28] showed that there are no quantum (α') corrections to the (purely gravitational and NS-NS background) pp wave solutions, since all the gravitational invariants made from Ricci and Riemann tensors vanish on this solution. The inverse metric is given by

$$g^{\mu\nu}\partial_\mu\partial_\nu = -4\partial_+\partial_- - 4H\partial_-^2 + \partial_i^2 \quad (2.3)$$

and so for instance

$$R^{(2)} \equiv R_{\mu\nu}R^{\mu\nu} = R_{\mu\nu}R_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \quad (2.4)$$

does not contain $(R_{++})^2$, and is thus zero.

In particular, a class of purely gravitational (sourceless) solutions (of $R_{++} = 0$) are given by

$$H = \sum_{ij} A_{ij}x^i x^j, \qquad \text{tr}\, A = 0 \quad (2.5)$$

and preserve 1/2 supersymmetry (SUSY) $\Gamma_-\epsilon = 0$.

The Aichelburg-Sexl solution is a solution for a point particle (delta function source), moving at the speed of light. It is obtained by boosting the black hole solution to the speed of light, and taking its mass M to zero, while keeping $Me^\beta = p = \text{const.}$ (β = boost parameter). But a simpler way to get it is to boost the energy-momentum tensor and then solve the Einstein equations for the resulting pp wave (thus we have to assume the pp wave ansatz, which however turns out to be consistent with the energy-momentum tensor).

A black hole at rest has

$$T_{00} = m_0 \delta^{d-2}(x^i)\delta(y) \quad (2.6)$$

and the rest zero. Boosted, one gets

$$T_{00} = \frac{m_0}{\sqrt{1-v^2}} \delta^{d-2}(x^i)\delta(y - vt) \quad (2.7)$$

and corresponding T_{10} and T_{11}. At the limit, one has

$$T_{++} = p\delta^{d-2}(x^i)\delta(x^+). \quad (2.8)$$

This means that $H(x^+, x^i) = \delta(x^+)\Phi(x^i)$, where (since Einstein's equation is $R_{++} = 8\pi G T_{++}$)

$$\partial_i^2 \Phi(x^i) = -16\pi G p \delta^{d-2}(x^i) \quad (2.9)$$

(Φ is harmonic with source).

For 4D gravity, $\Phi = -8pG\ln\rho$, and

$$ds^2 = -dudv - 4pG\ln\rho^2\delta(u)du^2 + dx^2 + dy^2 \quad (2.10)$$

in the notation of [29] ($\rho^2 = x^2 + y^2$), but the result is easily generalizable to any dimension d higher than four:

$$\Phi = \frac{16\pi G}{\Omega_{d-3}(d-4)} \frac{p}{\rho^{d-4}}, \qquad d > 4. \quad (2.11)$$

Particles following geodesics in the A-S metric are subject to two effects [29]:

It is found that geodesics going along u at fixed v are straight except at $u = 0$ where there is a discontinuity

$$\Delta v = \Phi = -4Gp\ln\frac{\rho^2}{l_{\text{Pl}}^2}, \quad (2.12)$$

where the Planck constant l_{Pl} in the ln is conventional (only relative shifts, $\Delta v_1 - \Delta v_2$, have physical meaning). That means that one basically has two portions of flat space glued together along $u = 0$ with a Δv shift. The shift can be easily understood by the fact that, after a singular coordinate transformation, defined later in (3.4), the metric becomes continuous. So geodesics are continuous in (u, v) coordinates, which means they are discontinuous in $(\bar{u}, \bar{v})$ coordinates, with the above Δv.

The second effect is a "refraction" (or gravitational deflection, rather), where the angles α and β made by the incoming and outgoing waves with the plane $u = 0$ at an impact parameter $\rho = b$ from the origin in transverse space satisfy

$$\cot\alpha + \cot\beta = \frac{4Gp}{b} \tag{2.13}$$

(here p is the momentum of the photon creating the A-S wave), and at small deflection angles (near normal to the plane of the wave) we have

$$\Delta\theta \simeq \frac{4Gp}{b}. \tag{2.14}$$

We can understand this also by using the singular coordinate transformation in (3.4), as

$$\Delta\left(\frac{\partial\bar{\rho}}{\partial\bar{u}}\right) = \Delta\left(\frac{\partial\rho}{\partial u}\right) + \frac{\partial_\rho\Phi}{2} \tag{2.15}$$

and $\Delta(\frac{\partial\rho}{\partial u}) = 0$ [no refraction in (ρ, u, v) coordinates], so

$$\Delta\left(\frac{\partial\bar{\rho}}{\partial\bar{u}}\right) = \frac{\partial_\rho\Phi}{2}. \tag{2.16}$$

One can then describe the scattering of two massless particles of very high energy [20] ($m_{1,2} \ll M_P$, $Gs \sim 1$, yet $Gs < 1$) by saying that particle two creates a massless shock wave of momentum $p_\mu^{(2)}$ and particle one follows a massless geodesic in that metric. In covariant notation ($v = x^0 - x^1 \equiv x^-$, $\tilde{x}^2 \equiv \rho^2 = x^2 + y^2$),

$$\Delta x_\mu = -2Gp_\mu^{(2)} \log(\tilde{x}^2/C). \tag{2.17}$$

Then particle one comes in with a free wave function,

$$\psi_{(-)}^{(1)} = e^{i\tilde{p}^{(1)}\tilde{x} + ip_-^{(1)}v + ip_+^{(1)}u}, \tag{2.18}$$

and becomes (at $u = 0$, just after the shock wave)

$$\psi_{(+)}^{(1)} = e^{i\tilde{p}^{(1)}\tilde{x} + ip_-^{(1)}(v - 4Gp^{(2)}\log(\tilde{x}^2/C))}. \tag{2.19}$$

Then by definition the scattering amplitude is the Fourier transform of this wave function

$$\begin{aligned}\mathcal{A}(k_-, \tilde{k}) &= \frac{1}{(2\pi)^3}\int d^2\tilde{x}dv e^{-i\tilde{k}^{(1)}\tilde{x} - ik_-^{(1)}v}\psi_{(+)}^{(1)} \\ &= \delta(k_-^{(1)} - p_-^{(1)})\int\frac{d^2\tilde{x}}{(2\pi)^2}e^{i\tilde{x}(\tilde{p}^{(1)} - \tilde{k}^{(1)}) - iGs\log\tilde{x}^2} \\ &= -i\delta(k_-^{(1)} - p_-^{(1)})\int\frac{d^2\vec{b}}{(2\pi)^2}e^{i\vec{q}\,\vec{b}}e^{i\delta(b,s)},\end{aligned} \tag{2.20}$$

where we have expressed $\mathcal{A}(s, t)$ via an impact parameter transform to an eikonal form, with $\delta(b, s) = p_+^{(1)}\Delta v = -Gs\log b^2$ and after doing the $d^2\vec{b} = bdbd\theta$ integration one gets 't Hooft's result:

$$\mathcal{A} = \frac{1}{4\pi}\delta(k_-^{(1)} - p_-^{(1)})\frac{\Gamma(1 - iGs)}{\Gamma(iGs)}\left[\frac{4}{(\tilde{p} - \tilde{k})^2}\right]^{1 - iGs}. \tag{2.21}$$

But

$$\begin{aligned}&4Gp_+^{(1)}p^{(2)} = Gs \quad \text{and} \quad (\tilde{p} - \tilde{k})^2 = -t; \\ &\mathcal{A}(k_+, \tilde{k}) = \delta(k_+^{(1)} - p_+^{(1)})U(s, t),\end{aligned} \tag{2.22}$$

and then we get the differential cross section,

$$\begin{aligned}&U(s, t) = \frac{1}{4\pi}\left(\frac{4}{-t}\right)^{1 - iGs}\frac{\Gamma(1 - iGs)}{\Gamma(iGs)} \\ &\frac{d^2\sigma}{d^2k} = \frac{4}{s}\frac{d\sigma}{d\Omega} = 4\pi^2|U(s, t)|^2 = \frac{4}{t^2}(Gs)^2,\end{aligned} \tag{2.23}$$

which is like Rutherford scattering, as if a single graviton is exchanged (with the effective gravitational coupling Gs replacing $\alpha = e^2/4\pi$ of QED).

The argument is that graviton exchange dominates the amplitude in this limit, for massive particles it takes an infinite time to interact. Indeed, at large impact parameter there is the natural exponential decay of the massive interactions, whereas at small impact parameter the harmonic function $\Phi(r)$ diverges, and as the time shift Δv is proportional to Φ, it diverges as well. Other massless particles can be introduced easily: for example, Maxwell interactions are taken into account just by having a shift (such that at $Gs = 0$ we recover Rutherford scattering of QED):

$$Gs \to Gs + q^{(1)}q^{(2)}/4\pi. \tag{2.24}$$

For trans-Planckian scattering, $Gs \gg 1$, one should take both particles as creating shock waves, and these shock waves should interact and create a black hole.

The generalization to higher dimensions is now pretty straightforward. Let us first notice, as Amati and Klimcik did also [24], that a shock wave metric

$$ds^2 = -dudv + \Phi(x)\delta(u)du^2 + d\vec{x}^2 \tag{2.25}$$

would shift the geodesics at $u = 0$ by $\Delta v = \Phi$ and the S matrix was described by 't Hooft by the Fourier transform of the shifted wave function, giving essentially

$$S = e^{ip_v\Delta v} \equiv e^{ip_-\Phi}. \tag{2.26}$$

What we mean is that we can perform an impact parameter transform as in $D = 4$ and get

$$i\mathcal{A} = \int\frac{d^{D-2}\vec{b}}{(2\pi)^{D-2}}e^{i\vec{q}\,\vec{b}}(e^{i\delta(b,s)} - 1) \tag{2.27}$$

with ($\mu = \sqrt{s}/2 = p =$ photon energy and $p_-^{(1)} = \mu$ also)

$$\delta(b,s) = p_-^{(1)}\Phi(b) = \frac{aGs}{b^{D-4}};\qquad \Phi(b) = \frac{16\pi G\mu}{\Omega_{D-3}(D-4)b^{D-4}}, \tag{2.28}$$

so $a = 4\pi/(\Omega_{D-3}(D-4))$. Then one obtains (with $q^2 = t$)

$$\begin{aligned} i\mathcal{A} &= \frac{\Omega_{D-4}\Gamma(\frac{D-3}{2})\sqrt{\pi}2^{(D-4)/2}}{(2\pi)^{D-2}q^{D-2}} \times \int_0^\infty dz z^{(D-2)/2}(e^{i\alpha/z^{D-4}}-1)J_{(D-4)/2}(z) \\ &\equiv \frac{A}{q^{D-2}}\int_0^\infty dz z^{(D-2)/2}(e^{i\alpha/z^{D-4}}-1)J_{(D-4)/2}(z), \end{aligned} \tag{2.29}$$

where the q dependence of the integral comes from $\alpha = aGsq^{D-4} = aGst^{(D-4)/2}$ and $z = qb$, and the exponential is $e^{i\delta}$ in general, so for small δ the bracket in the integral is $i\delta$. The integral can also be rewritten as

$$\int_0^\infty \frac{du}{4-D}u^{-\{(3D-8)/[2(D-4)]\}}(e^{i\alpha u}-1)J_{(D-4)/2}(u^{-[1/(D-4)]}), \tag{2.30}$$

but we can find no analytic expression for it. At most one can make an expansion in α which gives for the integral $= i\alpha c$, $c = 2^{(6-D)/2}/\Gamma((D-4)/2)$, and so

$$\mathcal{A} \simeq \frac{Gs}{t}\left(\frac{ac\Omega_{D-4}\Gamma(\frac{D-3}{2})\sqrt{\pi}2^{(D-4)/2}}{(2\pi)^{D-2}}\right) = \frac{Gs}{\pi t}\frac{1}{(2\pi)^{D-4}}. \tag{2.31}$$

But this is an expansion in $Gst^{(D-4)/2}$ and so in $D = 10$ we have $Gst^3 \ll 1$, or $g_s(\alpha' s)(\alpha' t)^3 \ll 1$, certainly satisfied. Note also that this result matches in $D = 4$ what one obtains by expanding in Gs.

III. BLACK HOLE PRODUCTION VIA AICHELBURG-SEXL WAVE SCATTERING

Let us now analyze black hole production in the high energy collision of particles ($Gs \gg 1$). We will analyze the collision of two massless particles in flat space, in $D = 4$ and $D > 4$, first reviewing the treatment of Eardley and Giddings [11]. As noted by 't Hooft and analyzed by [11], in this regime we have to take into account the gravitational field created by both particles, so one has to analyze the scattering of two A-S waves. For an estimate of the gravitational energy being radiated away in the high energy collision, see [30].

As one can imagine, in general, the collision of two gravitational waves is a highly nonlinear and nontrivial process, and as such it is hard to say anything about the collision region. If we denote by I the region $u < 0$, $v < 0$ before the collision, by II the region $u > 0$, $v < 0$ (after the wave at $u = 0$ has passed), III for $u < 0$, $v > 0$ (after the wave at $v = 0$ has passed), and IV for $u > 0$, $v > 0$ (the interacting region, after both waves have passed), the solution in region IV was calculated in [21] only perturbatively in the distance away from the interaction point $u = v = 0$.

In the case of sourceless waves (pure gravitational waves), Khan and Penrose [31] and Szekeres [32] have found complete interacting solutions, but they do not represent the collision of photons. We will discuss them in a next subsection. A general treatment of collision of pure gravitational waves can be found in [33], as well as in [34,35]

A. Review

Coming back to the case of the collision of two A-S waves, there is an observation, first due to Penrose and extended by Eardley and Giddings, which permits one to say that there will be a black hole in the future of the collision without actually calculating the gravitational field. One can prove the existence of a trapped surface, and then one knows that the future of the solution will involve a black hole whose horizon will be outside the trapped surface.

An apparent horizon is the outermost marginally trapped surface. The existence of a marginally trapped surface thus implies an apparent horizon outside it. A marginally trapped surface is defined as a closed spacelike $D-2$ surface, the outer null normals (in both future-directed directions) of which have zero convergence. In physical terms, what this means is that there is a closed surface whose normal null geodesics (light rays) do not diverge, so are trapped by gravity. For a Schwarzschild black hole, the marginally trapped surface is a sphere around the singularity, that happens to coincide with the horizon.

Convergence is easier to define in the case of a congruence of timelike geodesics. For a congruence of timelike geodesics characterized by the tangent vector ξ^a, $\xi^a\xi_a = -1$, defining $B_{ab} = \nabla_b\xi_a$ and the projector onto the subspace orthogonal to ξ^a, $h_{ab} = g_{ab} + \xi_a\xi_b$ (induced metric), the convergence is $\theta = B^{ab}h_{ab}$.

But we need the case of null geodesics, which is more involved. We have to first define the affine parameter λ along the curve C such that

$$\frac{D}{d\lambda}\left(\frac{\partial}{\partial\lambda}\right)_C = \frac{D}{d\lambda}\xi^a = \xi^a_{;b}\xi^b = 0. \tag{3.1}$$

Then we define a ("pseudo-orthonormal") basis for the tangent space, E_1, E_2, E_3, E_4, such that $E_4^a = \xi^a$, and $E_3^a = L^a$ is another null vector: $E_3 \cdot E_3 = E_4 \cdot E_4 = 0$, $E_1 \cdot E_1 = E_2 \cdot E_2 = 1$ and $E_{1,2}$ orthogonal to $E_{3,4}$, but $E_3^a\xi^b g_{ab} = -1$. If m, n takes the values 1,2 in the above basis, then $\theta = \xi_{m;n}g^{mn}$. If the geodesics are null, one cannot find an orthonormal basis (as in the timelike case), one can only find this pseudo-orthonormal basis. Also note that, by definition, the null geodesics defined by ξ^a are normal to the 2D surface spanned by E_1, E_2, and we are taking the derivative of ξ just in those directions.

So to calculate the existence of a marginally trapped surface, we first need to find the null geodesics normal to the surface, and then impose that their convergence is zero.

To calculate the convergence, take the approach from [36]. The convergence is

$$\theta = h^{ab} D_a \xi_b; \qquad \xi = \xi^a \partial_a = \frac{dx^a}{d\lambda}\partial_a \tag{3.2}$$

for a congruence of null geodesics ξ_μ normal to the surface B, and h_{ab} is the induced metric on B. B is spanned by the E_1, E_2 of before, and contracting with the induced metric is equivalent to contracting with g^{mn} in the above basis.

We see that we need to get the form of the geodesics $x^a(\lambda)$ to proceed. We can impose the fact that the geodesics are null, so $(\xi, \xi) = 0$, normal to the generators of the surface, K_i, so $(\xi, K_i) = 0$, and also the normalization $(\xi, \partial_t) = -E$ (which can be chosen to be $= -1$ for simplicity). Note that in [36] B is a sphere, so the generators are ∂_{ϕ_i}. Then one calculates $x^a(\lambda)$ and then $\xi(\lambda(x^a)) = \xi(x^a)$, and then $h_{mn} = \partial_m X^a \partial_n X^b g_{ab}$ (where X^a are coordinates on the surface B), and finally $\theta = h^{mn} D_m \xi_n$.

Let us apply this procedure to the metric of two colliding general shock waves (without specifying for the moment the Aichelburg-Sexl solution for Φ_i), one moving in the u direction, and the other in the v direction.

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{x}^{i2} + \Phi_1(\bar{x})\delta(\bar{u})d\bar{u}^2 + \Phi_2(\bar{x})\delta(\bar{v})d\bar{v}^2. \tag{3.3}$$

After the coordinate transformation,

$$\begin{aligned} \bar{u} &= u + \Phi_2\theta(v) + v\theta(v)\frac{(\nabla\Phi_2)^2}{4} \\ \bar{v} &= v + \Phi_1\theta(u) + u\theta(u)\frac{(\nabla\Phi_1)^2}{4} \\ \bar{x}^i &= x^i + \frac{u}{2}\partial_i\Phi_1(x)\theta(u) + \frac{v}{2}\partial_i\Phi_2(x)\theta(v), \end{aligned} \tag{3.4}$$

it becomes

$$ds^2 = -du dv + [H^{(1)}_{ik}H^{(1)}_{jk} + H^{(2)}_{ik}H^{(2)}_{jk} - \delta_{ij}]dx^i dx^j, \tag{3.5}$$

where

$$\begin{aligned} H^{(1)}_{ij} &= \delta_{ij} + \frac{1}{2}\partial_i\partial_j\Phi^{(1)}u\theta(u) \\ H^{(2)}_{ij} &= \delta_{ij} + \frac{1}{2}\partial_i\partial_j\Phi^{(2)}v\theta(v). \end{aligned} \tag{3.6}$$

At zero impact parameter ($b = 0$), and for A-S shock waves in $D = 4$, we have

$$\Phi_1 = \Phi_2 = -8G\mu \ln\bar{\rho}; \qquad \bar{\rho} = \sqrt{\bar{x}^i\bar{x}^i}. \tag{3.7}$$

In general D, but for an A-S wave at $b = 0$, there is a $D - 2$ dimensional trapped surface consisting of two disks (balls), parametrized by $\bar{x}$, of radius ρ_c in $\bar{\rho}$.

In complete generality, the surface S is defined as follows. Take the union of the two null hypersurfaces $v \le 0$, $u = 0$ and $u \le 0$, $v = 0$ with a $D - 2$ dimensional intersection $u = v = 0$, that intersects on its turn S on a $D - 3$ surface C (*a priori*, two $D - 2$ surfaces intersect on a $D - 4$ surface though, more on that later). Then S is composed of

disk 1—$\{v = -\Psi_1(\vec{x}), u = 0\}$, ($\Psi_1 = 0$ on C),

disk 2—$\{u = -\Psi_2(\vec{x}), v = 0\}$ ($\Psi_2 = 0$ on C).

As we will show, the condition of zero convergence implies that

$$\nabla^2(\Psi_1 - \Phi_1) = 0 \tag{3.8}$$

interior to C. We will see that in the $b = 0$ A-S case, we can actually choose $\Psi_1 = \Phi_1$, $\Psi_2 = \Phi_2$ which, with the definition $\theta(0) = 1$, means that both disks correspond to $\bar{u} = \bar{v} = 0$. So we would not see the topology in the bar coordinates, we need to go to the unbarred ones to get explicit formulas.

On the first disk, we have

$$\begin{aligned} ds^2 &= -du dv + d\vec{x}^2 + \frac{1}{2}u\theta(u)(\partial_i\partial_j\Phi)dx^i dx^j \\ &\quad + \frac{u^2}{4}\theta(u)\partial_i\partial_k\Phi\partial_j\partial_k\Phi dx^i dx^j \\ &= -du dv + dx^i dx^j g_{ij} \end{aligned} \tag{3.9}$$

and the null geodesics through $\{v = -\Psi(\vec{x}), u = 0\}$ are defined by

$$\xi = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{x}^i\partial_i. \tag{3.10}$$

The tangent generators of the surface are

$$K^\mu_j = \frac{\partial X^\mu}{\partial x^j}, \tag{3.11}$$

where x^i are the coordinates on S and X^μ the coordinates on the space, but we choose $x^i = X^i$ and so

$$K^\mu_j = (0, -\partial_j\Psi, \delta^i_j) \to K^\mu_j\partial_\mu = -\partial_j\Psi\partial_v + \partial_j. \tag{3.12}$$

We have to impose the condition that ξ is null $(\xi, \xi) = 0$, transverse to all the generators: $(\xi, K_i) = 0$, and we have to define the time direction [in [36], that was $(\xi, \partial_t) = -E$, where E can be scaled to 1], in this case $(\xi, \partial_v) = -1$.

These conditions together fix

$$\dot{u} = 2; \qquad \dot{x}^i = -g^{ij}\partial_j\Psi; \qquad \dot{v} = \frac{1}{2}\partial_i\Psi\partial_j\Psi g^{ij}, \tag{3.13}$$

and then we calculate

$$\xi = -dv - \frac{1}{4}g^{ij}\partial_i\Psi\partial_j\Psi du - \partial_i\Psi dx^i \tag{3.14}$$

and thus

$$\theta|_{u=0} = -\nabla^2(\Psi - \Phi) \tag{3.15}$$

as advertised.

Actually, what we have found is that by imposing $(\xi_1, \partial_v) = -1$, we get

$$\xi_1 = -dv - \frac{1}{4}(\nabla\Psi_1)^2 du - \partial_i \Psi_1 dx^i, \tag{3.16}$$

but similarly, if we impose instead $(\xi_1', \partial_u) = -1$, we get

$$\xi_1' = -du - \frac{4}{(\nabla\Psi_1)^2} dv - 4\frac{\partial_i\Psi_1}{(\nabla\Psi_1)^2} dx^i. \tag{3.17}$$

Then on disk 2, $(\xi_2, \partial_u) = -1$ implies

$$\xi_2 = -du - \frac{1}{4}(\nabla\Psi_2)^2 dv - \partial_i\Psi_2 dx^i. \tag{3.18}$$

These two surfaces intersect on C, thus the normal, ξ, has to be continuous across C. This means that for the A-S wave at $b = 0$, when $\Phi_1 = \Phi_2$ implying $\Psi_1 = \Psi_2$, we need to have

$$(\nabla\Psi_1)^2 = (\nabla\Psi_2)^2 = 4. \tag{3.19}$$

Then in $D = 4$, replacing the explicit form of ϕ we get

$$\Psi = \Phi = -8G\mu \ln\rho/\rho_c \Rightarrow \rho_c = 4G\mu = r_h, \tag{3.20}$$

whereas for $D > 4$

$$\Psi = \frac{16\pi G\mu}{\Omega_{D-3}(D-4)\rho^{D-4}} \Rightarrow \rho_c = \left(\frac{8\pi G\mu}{\Omega_{D-3}}\right)^{1/(D-3)}. \tag{3.21}$$

In the bar coordinates, both disks correspond as we said to $\bar{u} = \bar{v} = 0$ and $\bar{x}^i = x^i$. But this surface in the bar coordinates is just flat (on it, the metric is Minkowski), so the area (volume of balls) is just the area of two flat balls of radius $\bar{\rho}_c = \rho_c$. The area (volume) of a flat unit D dimensional ball is $V_{\text{ball},D} = \Omega_{D-1}/D$, so the total area of the trapped surface in D spacetime dimensions (two flat balls) is

$$A_{\min}(S) = 2V_{\text{ball},D-2}\rho_c^{D-2} = \frac{2}{D-2}\Omega_{D-3}\rho_c^{D-2}, \tag{3.22}$$

whereas, from the explicit form of the Schwarzschild solution in D dimensions the horizon radius of a black hole of mass $\sqrt{s} = 2\mu$ is

$$r_h = \left[\frac{32\pi G\mu}{(D-2)\Omega_{D-2}}\right]^{1/(D-3)}, \tag{3.23}$$

so that the horizon area of the mass $= \sqrt{s}$ black hole is

$$A_{\text{Sch}} = \Omega_{D-2} r_h^{D-2} \Rightarrow \frac{A_{\min}(S)}{A_{\text{Sch}}} = \frac{1}{2}\left[\frac{(D-2)\Omega_{D-2}}{4\Omega_{D-3}}\right]^{1/(D-3)} \equiv \frac{\epsilon}{2}. \tag{3.24}$$

The area of the trapped surface is smaller than the horizon area of the black hole to form (since the horizon is by definition outside the trapped surface), and we can express the area of the *disks (balls)* as the area of horizon *spheres* that will form, so $r \leq r_h$, where r is defined by $\text{Area}(S) = \Omega_{D-2} r^{D-2}$, implying that the mass of the formed black hole satisfies

$$\frac{16\pi G M_{\text{BH}}}{(D-2)\Omega_{D-2}} = r_h^{D-3} \geq r^{D-3} = \left[\frac{\text{Area}(S)}{\Omega_{D-2}}\right]^{(D-3)/(D-2)} \Rightarrow \frac{M_{\text{BH}}}{\sqrt{s}} \geq \frac{1}{2}\left[\frac{(D-2)\Omega_{D-2}}{2\Omega_{D-3}}\right]^{1/(D-2)} \tag{3.25}$$

(we have put in the explicit form of Area(S) and of ρ_c in terms of $\mu = \sqrt{s}/2$). Both $\frac{A_{\min}(S)}{A_{\text{Sch}}}$ and $\frac{M_{\text{BH}}}{\sqrt{s}}$ match the explicit numbers in [11].

B. Extension

In the previous discussion we have already set up the formalism so that it is valid for any function $\Phi(\vec{x})$ characterizing the shock wave. We will be applying this later for different Φ's.

Let us now try to extend this for the case of nonzero b in any dimension. For the $b = 0$, $D = 4$ A-S wave we had

$$\Psi = \Phi = -8G\mu \ln\rho/\rho_c, \tag{3.26}$$

meaning that $\Psi > 0$ for $\rho < \rho_c$. For $b = 0$, $D > 4$ we have

$$\Psi = \Phi - \Phi(\rho = \rho_c); \qquad \Phi = \frac{16\pi G\mu}{\Omega_{D-3}(D-4)\rho^{D-4}} \tag{3.27}$$

and again $\Psi > 0$ for $\rho < \rho_c$.

For $b > 0$, $D \geq 4$ now, we would need both ψ_1 and ψ_2 to be zero on the same surface (curve, for $D = 4$) C, not on two surfaces C_1 and C_2, since then the intersection of C_1 and C_2 would have $D - 4$ dimensions (points, for $D = 4$). So we cannot use in $D = 4$ for instance

$$\Psi_i = \Phi_i = -8G\mu \ln\frac{|\rho - \rho_{0i}|}{\rho_c}. \tag{3.28}$$

That would define two disks in $\bar{x} = x$ with the centers displaced by b, and while each of the disk boundaries would be a circle, the two circles will intersect in two points, so C would be composed of these two points.

The correct solution, which was explored in [11] using a self-consistent approach (which does not guarantee finding ALL solutions) is that $\Psi_i \neq \Phi_i$, just

$$\nabla^2\Psi_i = \nabla^2\Phi_i \propto \delta(\vec{x} - \vec{x}_{0i}), \tag{3.29}$$

which means that Ψ_1, Ψ_2 are Green's functions for sources at $\vec{x}_{01}, \vec{x}_{02}$ which *both* are zero on the same curve C enclosing $\vec{x}_{01}$ and $\vec{x}_{02}$. Then one imposes the condition for continuity of the null normal ξ which gives

$$\nabla\Psi_1 \cdot \nabla\Psi_2 = 4, \tag{3.30}$$

which fixes (together with the previous conditions) the form of C.

Clearly for very small b (much smaller than ρ_c), we have that C is well approximated by the boundary (envelope) of the union of the two disks. We will assume that in $D = 4$ the two points C_1 and C_2 (intersection of the two circles = boundaries of disks) are still part of the curve C even at b large, which seems like a reasonable assumption, though not well justified. Let us see what we can deduce out of it. Clearly the two sources will be inside C, so we will therefore assume that the curve C is outside the parallelogram made from C_1, C_2, x_{01}, x_{02}. We still call the distance between $C_{1,2}$ and x_{01}, x_{02} (radius of the circles) ρ_c, and if we then impose (3.30) on $\Psi_1 = \Phi_1$ and $\Psi_2 = \Phi_2$ (which are still good Green's functions for the circles that both pass through the two points C_1, C_2, even if they are not for the whole curve C), we get the equation ($\cos\theta/2 = \sqrt{\rho_c^2 - b^2/4}/\rho_c$)

$$\left|\nabla\frac{\Psi_1}{2}\right| \cdot \left|\nabla\frac{\Psi_2}{2}\right| \cos\theta = 1 \Rightarrow \frac{R_s^2}{\rho_c^2}\left(1 - \frac{b^2}{2\rho_c^2}\right) = 1 \quad (3.31)$$

(where $R_s = 4\mu G$) which gives the value of ρ_c as

$$\rho_c^2 = \frac{R_s^2}{2}\left(1 + \sqrt{1 - \frac{2b^2}{R_s^2}}\right). \quad (3.32)$$

We can check that if $b = 0$ we reproduce the known result of $\rho_c = R_s$. This formula means that the maximum impact parameter for which we can have a black hole forming within this approximate formalism is $b_{\max} = R_s/\sqrt{2} = 4G\mu/\sqrt{2}$ and the minimum radius is $\rho_{c,\min} = \rho_c(b_{\max}) = R_s/\sqrt{2} = b_{\max}$, and the area of the trapped surface satisfies

$$S \geq \sqrt{b^2\rho_c^2 - \frac{b^4}{4}} = \frac{b}{\sqrt{2}}\sqrt{R_s^2\left(1 + \sqrt{1 - \frac{2b^2}{R_s^2}}\right) - \frac{b^2}{2}} \equiv S_{\min} \quad (3.33)$$

so that $S_{\min}$ at the maximum b is $S_{\min} = R_s^2\sqrt{3}/4 = 4\sqrt{3}(\mu G)^2$.

Comparing now with the results of [11] we have that $b_{\max} = 4G\mu/\sqrt{2} \simeq 2.83G\mu$ is smaller than their result of $3.219G\mu$. Since $b_{\max} < R_s$ it is even physically acceptable (we would have a problem if it would be bigger). As for the estimate of the area of the trapped surface, $S_{\min} = 4\sqrt{3}(\mu G)^2$, it is sensibly smaller than the result of [11] which can be found to be (replacing the value of their parameter $a_{\max}$ in the formula for the area) $40.852(\mu G)^2$, so we have a much more conservative estimate.

But the advantage is that this procedure can be now easily extended in higher dimensions.

Indeed, in $D > 4$,

$$\vec{\nabla}\Phi = -\frac{16\pi\mu G}{\Omega_{D-3}}\frac{\vec{x}}{\rho^{D-2}}, \quad (3.34)$$

and so the condition $\vec{\nabla}\Phi_1 \cdot \vec{\nabla}\Phi_2 = 4$ implies

$$\left(\frac{\epsilon R_s}{\rho_c}\right)^{2D-6}\left(1 - \frac{b^2}{2\rho_c^2}\right) = 1, \quad (3.35)$$

where $R_s \equiv r_h$ is the horizon radius of the black hole, ϵ is defined in (3.24), and this equation can be rewritten as

$$f(x) = 4x^{D-2} - 4\alpha x + 2b^2\alpha = 0, \quad (3.36)$$

where $x = \rho_c^2$ and $\alpha = (\epsilon R_s)^{2D-6}$. We can easily find the maximum value of the impact parameter b from it. Since $\rho_c = \sqrt{x}$ is the biggest of the solutions to Eq. (3.36), we impose that $f(x_0) \leq 0$, where x_0 is the highest root of $f'(x_0) = 0$. This condition implies

$$b^2 \leq 2\left[\frac{\alpha}{D-2}\right]^{1/(D-3)}\frac{D-3}{D-2} = \frac{2(\epsilon R_s)^2}{[D-2]^{(D-2)/(D-3)}}(D-3). \quad (3.37)$$

We can check that indeed in $D = 4$ we recover the result $b_{\max} = R_s/\sqrt{2}$, since then $\epsilon = 1$. In $D = 5$, that means $b \leq 0.9523R_s < R_s$. We can also calculate the lower limit on the area of the trapped surface as before, except that now the area of the parallelogram C_1, C_2, x_{01}, x_{02} is replaced by the volume of a "surface of revolution" in $D - 4$ transverse directions around the axis x_{01}, x_{02}. The geometry in higher dimensions is more complicated, but for $D = 5$ this is just two cones glued on their bases, of height $h = b/2$ and base radius $\rho_c\cos\theta/2$, and so "$S_{\min}$" (volume of the cones) is

$$2\frac{S_0 h}{3} = \frac{\pi b}{3}\left(\rho_c^2 - \frac{b^2}{4}\right). \quad (3.38)$$

In conclusion, we have set up a formalism for shock waves in which we can calculate trapped surfaces at $b = 0$ and to some degree at nonzero b, for a general shock wave form.

C. Collision of sourceless waves

We have seen that for A-S−type waves colliding, in general we get a trapped surface in the future of the collision, which indicates a black hole horizon being formed. From this, we conclude that a black hole is formed in the high energy collision of two high energy photons (massless particles, with an energy-momentum source).

But what happens if two sourceless waves (gravitational solutions to the pure Einstein's equations) collide? We would expect to be able to associate this phenomenon with the collision of two gravitons, in which case we would expect to create a black hole in the collision. It is in fact true that there is a theorem stating that a singularity will form in the future of a collision of two sourceless waves [34,35]. It is also a theorem that, for Einstein gravity in flat space, a singularity cannot be naked, so we would expect to be able to find a trapped surface, indicating the formation of a horizon in a sourceless wave collision.

Unfortunately, we will see that this is not so, and we will speculate on why, after we see the problem.

Khan and Penrose found a solution [31] describing the (head-on, zero impact parameter) collision of two source-free gravitational pp waves of the type

$$ds^2 = -dU(2dV + (X^2 - Y^2)h(U)dU) + dX^2 + dY^2, \tag{3.39}$$

with $h(U) = \delta(U)$ (the function Φ defined before satisfies the source-free equation $\partial_i^2\Phi = 0$ solved by $\Phi = -X^2 + Y^2$). After the coordinate transformation

$$U = u, \qquad V = v + x^2/2FF' + y^2/2GG'; \qquad X = xF; \qquad Y = yG, \tag{3.40}$$

with $F'' = -Fh, G'' = Gh$, solved by $F = 1 - u\theta(u), G = 1 + u\theta(u)$ [$F' = \theta(u)$, as $u\delta(u) = 0$, and thus also $F\delta(u) = \delta(u)$], we get the wave in the form

$$ds^2 = -2dudv + F^2dx^2 + G^2dy^2. \tag{3.41}$$

The collision will involve two such waves, one in u and the other in v, at zero impact parameter (b). Thus the colliding wave solution of Khan and Penrose is

$$ds^2 = -\frac{2t^3dudv}{rw(pq + rw)^2} + t^2\left(\frac{r+q}{r-q}\right)\left(\frac{w+p}{w-p}\right)dx^2 + t^2\left(\frac{r-q}{r+q}\right)\left(\frac{w-p}{w+p}\right)dy^2, \tag{3.42}$$

where

$$p = u\theta(u); \qquad q = v\theta(v); \qquad r = \sqrt{1-p^2}; \qquad w = \sqrt{1-q^2}; \qquad t = \sqrt{1-p^2-q^2}. \tag{3.43}$$

In the region $u \geq 0, v < 0$ (before the coming of the second wave), we can check that the Khan-Penrose solution becomes

$$ds^2 = -2dudv + (1+p)^2dx^2 + (1-p)^2dy^2, \tag{3.44}$$

that is, of the sourceless wave form (3.41), and we see that there is a coordinate singularity at $u = 1$. Then in the collision region $u > 0, v > 0$ we have

$$ds^2 = -2\sqrt{\frac{1-u^2-v^2}{(1-u^2)(1-v^2)}}\frac{(\sqrt{(1-u^2)(1-v^2)} - uv)}{(\sqrt{(1-u^2)(1-v^2)} + uv)}dudv + \frac{(\sqrt{1-u^2}+v)^2(\sqrt{1-v^2}+u)^2dx^2 + (\sqrt{1-u^2}-v)^2(\sqrt{1-v^2}-u)^2dy^2}{1-u^2-v^2}. \tag{3.45}$$

Putting $v = 0$ we get back to the sourceless wave solution (3.41). Kahn and Penrose [31] found that, in the collision region, the line $u^2 + v^2 = 1$ has a scalar curvature singularity. We can calculate that for $u^2 + v^2 = 1 - \epsilon$ the metric is

$$ds^2 = \epsilon\left[\frac{\sqrt{\epsilon}du^2}{2u^2v^4} + \left(\frac{4uv}{\epsilon}\right)^2dx^2 + \left(\frac{\epsilon}{4uv}\right)^2dy^2\right], \tag{3.46}$$

so clearly the metric is singular, but there does not seem to be any good way to define a finite area of the singularity. Indeed, at $u =$ fixed, $v =$ fixed,

$$dS = ds_x \cdot ds_y = \epsilon dxdy. \tag{3.47}$$

So we cannot calculate this way a minimum on a horizon area of a black hole that would probably form.

We can still try to apply the formalism of Eardley and Giddings and calculate the area of a trapped surface that we assume will (should) form. Indeed, the individual gravitational waves that collide are still of the general form used in the previous subsection. The only difference is that instead of $\Phi = -8G\mu\ln\rho/\rho_c$ we have

$$\Phi = -\frac{X^2 - Y^2}{\rho_c}; \qquad (\rho_c = 4G\mu), \tag{3.48}$$

where we have rescaled U and V to introduce the dimensionful parameter ρ_c describing the strength of the wave. The problem is though that, in order to be able to choose $\Psi = \Phi$ like we did for the $b = 0$ A-S wave collision, Φ would have to be zero on the curve C at its boundary, so that $\rho = \rho_c$ as we shall see. But $\Phi = 0$ for $X = \pm Y$ (we could shift Φ by a constant, and then C would be a hyperbola), so that we cannot actually choose $\Psi = \Phi$.

Thus we could only use the above function for $X = \pm Y$, and these are four points ($X = \pm\rho_c/\sqrt{2}, X = \pm Y$) that would presumably lie on the curve C if there would be such a curve.

However, the correct treatment would involve solving the 2D Green's function (“electric potential”) for the Laplace equation $\nabla^2\Psi = 0$ with Dirichlet boundary conditions $\Psi = 0$ on a curve C where $\nabla\Psi$ (the “electric field”) has unit norm, and this is impossible.

Thus we seem to have proven that the assumption of a trapped surface is in fact wrong!

So there really seems to be no way of obtaining a trapped surface in the Khan-Penrose solution, even though we do obtain a singularity.

This is a most bizarre situation in itself, which could be perhaps saved by the fact that in such singular spacetimes the usual censorship theorems do not apply, but correlated with the expectation that the Khan-Penrose solution should describe graviton-graviton scattering, this is really puzzling.

One could perhaps think that the Khan-Penrose metric is not the correct sourceless wave to describe graviton-graviton scattering. After all, there is a plethora of sourceless wave scattering solutions, as reviewed to a certain extent in [33].

One can analyze their behavior though (we will not do it here explicitly) and convince oneself that these solutions do not describe graviton scattering. The simplest of them is the Szekeres solution in [32], which has the same singularity structure as the Khan-Penrose solution, but is described by a function Φ of the type $\Phi(\tilde{u}) = f(\tilde{u})\theta(\tilde{u})$ as opposed to the delta function profile $\Phi(\tilde{u}) = \delta(\tilde{u})$ of the incoming waves in the Khan-Penrose solution. The rest of the possible solutions are even more complicated, and they really describe the collision of realistic gravitational waves, as opposed to the idealized delta function waves of the Khan-Penrose solution. Therefore the Khan-Penrose solution is the only one that can claim to represent the collision of two (idealized) gravitons.

Of course, all these solutions were in 4D general relativity.

Gutperle and Pioline [37] set out to generalize these solutions to $2n + 2$ dimensions and to add p-form field strength to it, the ultimate goal being to scatter 2 maximally SUSY pp waves of IIB, or rather a shock wave generalization of it. They fall kind of short of the goal. The first try at the generalization gives exact solutions which however do not satisfy appropriate boundary conditions: the incoming waves are different from the Khan-Penrose and Szekeres profiles.

A perturbative attempt near the light cone (or for the strength of a wave much smaller than the other) produces a higher dimensional solution, as well as the p-form generalization.

Then, Chen *et al.* [38] also produce some generalizations of this type (see also [39,40]), with better singularity structure, but they do not analyze the incoming waves in Brinkman form, so it is not clear what they correspond to.

In conclusion, the Khan-Penrose solution is the only one that has a chance of describing the collision of two idealized gravitons, and we seem to obtain the existence of a naked singularity (no black hole) in the future of the collision. A good explanation of this paradox is still lacking.

IV. STRING CORRECTIONS

We will now try to apply the previously derived formalism to shock wave metrics that incorporate string corrections to the high energy scattering of two photons.

There are two such formalisms. One is due to Amati and Klimcik [24], and the other due to Amati, Ciafaloni, and Veneziano [23] (see also [41–43]).

The approach by [23] involves writing down an effective shock wave metric from which one can calculate an S matrix, which then is matched with a string-corrected S matrix (string calculation).

The S matrix is defined as $\exp(iS/\hbar)$ where the action is a function of the classical effective metric, with a source coupling to an external $T_{\mu\nu}$. Namely, the S matrix is

$$S_{\mathrm{eff}}(b, E) = \langle e^{iA(h_{\mu\nu})/\hbar}\rangle_{\mathrm{tree}} = e^{iA(h_{cl}^{\mu\nu})/\hbar}, \qquad (4.1)$$

where

$$A(h^{\mu\nu}) = \int d^4x[L_{\mathrm{eff}}(h^{\mu\nu}) + T_{\mu\nu}h^{\mu\nu}] \qquad (4.2)$$

is the action evaluated on its classical solution with sources given by two shock waves at $x = 0$ and $x = b$,

$$T_{--} = kE\delta(x^-)\delta^2(\vec{x}); \qquad T_{++} = kE\delta(x^+)\delta^2(\vec{x} - \vec{b}), \qquad (4.3)$$

and L_{eff} is an effective Lagrangian by Lipatov [44]. Amati *et al.* [23] showed as we mentioned that this calculation reproduces the result for the string correction to the scattering matrix S.

The string-corrected A-S−type metric obtained in [23] for $D = 4$ can be expressed in terms of a Φ of the form

$$\Phi = \Phi^{(0)}(A - S) + \Phi^{(1)} = kE\left[-\frac{1}{2\pi}\log\frac{|z|^2}{L^2} + R_s^2 a^{(1)}(z)\right], \qquad (4.4)$$

where

$$a^{(1)}(z) = \frac{1}{4\pi}\left[\frac{1}{|z|^2}\left|1 - \frac{z}{b}\right|^2 \log\left|1 - \frac{z}{b}\right|^2 + \frac{1}{bz} + \frac{1}{bz^*} - \frac{1}{b^2}\log\frac{L^2}{b^2}\right], \qquad (4.5)$$

and $kE = 8\pi G\mu$ ($k = 8\pi G$, $E \equiv \mu$), so the coefficient of the log term in $\Phi^{(0)}$ is $-R_s$. Here $z = x^1 + ix^2$ are complex transverse coordinates so that $|b - z|^2 = (b - x_1)^2 + x_2^2$, etc.

Then

$$\partial_1\Phi^{(1)} = \frac{R_s^3}{b(x_1^2 + x_2^2)^2}\left[(x_1^2 - x_2^2 - bx_1)\log\frac{(b - x_1)^2 + x_2^2}{b^2} + \frac{x_1}{b}(x_1^2 + x_2^2 - 2bx_1)\right]$$

$$\partial_2\Phi^{(1)} = \frac{R_s^3 x_2}{b(x_1^2 + x_2^2)^2}\left[(2x_1 - b)\log\frac{(b - x_1)^2 + x_2^2}{b^2} + \frac{1}{b}(x_1^2 + x_2^2 - 2bx_1)\right]. \qquad (4.6)$$

But the problem is that the string-corrected metric is only valid for $b > R_s$ (when no black hole forms yet), whereas we want to have a perturbation in b small.

We have tried to just plug in this metric in the continuity condition $\vec{\nabla}\Psi_1 \cdot \vec{\nabla}\Psi_2 = 4$, and treat it perturbatively in R_s/b as in [23], but one gets corrections of the order R_s^2/ρ_c^2, and then in a perturbative solution, when ρ_c is replaced by the first order solution which is of $o(R_s)$, the corrections are of order one. Thus this perturbation is useless.

But one can still do a small calculation, namely, to take the corrected metric (with b nonzero and moreover $>R_s$) and see what it does to the continuity condition for the collision at $b = 0$ of two uncorrected metrics, namely $(\nabla\Phi)^2 = 4$, now with

$$\partial_i\Phi = -2R_s\frac{x_i}{\rho^2} + \partial_i\Phi^{(1)} \tag{4.7}$$

and expand in R_s^2/b^2. This will not be very relevant (since the corrections disappear in the $b \to \infty$ limit, which is not what we want), but just to see what kind of effect string corrections have. We obtain

$$\begin{aligned}\rho_c^2 &= R_s^2\bigg[1 - \frac{R_s^2}{b\rho_c^2}A + \frac{R_s^4}{b^2\rho_c^4}\bigg[(\rho_c^2 - x_1^2)\\ &\qquad \times \log^2\bigg(\frac{\rho_c^2 - 2bx_1 + b^2}{b^2}\bigg) + A\bigg]\bigg]\\ A &= \bigg(\frac{\rho_c^2 - 2bx_1}{b} + (x_1 - b)\log\frac{\rho_c^2 - 2bx_1 + b^2}{b^2}\bigg),\end{aligned} \tag{4.8}$$

where we should have $x_1^2 + x_2^2 = \rho_c^2$, but obviously since we have used an asymmetric solution (where b is a distance on the x_1 axis), the solution we get for $\rho_c(R_s)$ also depends on our choice for x_1, x_2, so $\rho_c = \rho_c(x_1, x_2)$.

The above solution is exact, but we only need to expand in ρ_c/b. Expanding to the first two nontrivial orders we get (after some algebra)

$$\begin{aligned}\rho_c^2 \simeq R_s^2\bigg[&1 + \frac{x_1}{b^3}\bigg(\frac{x_1^2}{3} + x_2^2\bigg)\\ &+ \frac{1}{2b^4}\bigg(8x_1^2x_2^2 - R_s^4 + 4x_1^2R_s^2 - \frac{8}{3}x_1^4\bigg)\dots\bigg].\end{aligned} \tag{4.9}$$

The first correction is proportional to x_1 (times a positive quantity), so when we calculate the area of the curve $\rho_c(x_1, x_2)$, the positive contribution for $x_1 > 0$ will cancel against the negative one for $x_1 < 0$. So we need to turn to the next correction to see whether or not the area increases.

Defining

$$\begin{aligned}f(x_1^2) &= 8x_1^2x_2^2 - R_s^4 + 4x_1^2R_s^2 - \frac{8}{3}x_1^4\\ &\simeq 12x_1^2R_s^2 - \frac{32}{3}x_1^4 - R_s^4\end{aligned} \tag{4.10}$$

for $y = x_1^2/R_s^2$ between 0 and 1, we can check that the function is positive for $y > 0.09$ (most of the domain), so a simple estimate shows that the area of the trapped surface will indeed increase.

But after so many approximations it is not clear this still is relevant.

We turn instead to the approach of Amati and Klimcik [24].

Amati and Klimcik [24] first generalize the 't Hooft and Dray and 't Hooft calculation, as we explained in Sec. II. A shock wave metric

$$ds^2 = -dudv + \Phi(x)\delta(u)du^2 + dx^2 \tag{4.11}$$

would shift the geodesics at $u = 0$ by $\Delta v = \Phi$ and the S matrix was described by 't Hooft by the Fourier transform of the shifted wave function, giving essentially

$$S = e^{ip_v\Delta v}. \tag{4.12}$$

In string theory, the 't Hooft scattering in the shock wave background gives (for an open string $\to$ photon)

$$\Delta v = \frac{1}{\pi}\int_0^\pi \Phi(X(\sigma, 0))d\sigma \tag{4.13}$$

and the S matrix is defined as acting on creation/annihilation operators as $S^+a_{\text{in}}S = a_{\text{out}}$. Then

$$S = e^{[(ip_v)/\pi]\int_0^\pi d\sigma\Phi(X^u(\sigma,0))}. \tag{4.14}$$

This matches the resummed string calculation of [42] if

$$\Phi(y) = -q^v\int_0^\pi \frac{4}{s} : a_{\text{tree}}(s, y - X^d(\sigma_d, 0)) : \frac{d\sigma_d}{\pi}, \tag{4.15}$$

where $\frac{2p_vq^v}{s} = -1$, $b = x^u - x^d$, and $\hat{X}^u, \hat{X}^d$ are nonzero modes. Here the indices u, d refer to "up" and "down," necessary when we evaluate $\Phi(X(\sigma, 0))$.

We note that here b refers just to a parameter in the calculation of the shape of one modified A-S metric. We have not reached the scattering of two A-S–type waves yet. In that case, we will denote the impact parameter of the two waves by B, to avoid confusion.

Then we match with the S matrix obtained by resumming string diagrams,

$$\begin{aligned}S = \exp\bigg[2i\int_0^\pi &: a_{\text{tree}}(s, b + \hat{X}^u(\sigma_u, 0)\\ &- \hat{X}(\sigma_d, 0)) : \frac{d\sigma_u d\sigma_d}{\pi^2}\bigg],\end{aligned} \tag{4.16}$$

and the tree amplitude is

$$a_{\text{tree}}(s, b) = \frac{G_N s}{2\pi^{D/2-2}}b^{4-D}\int_0^{b^2/(Y-i\pi/2)/4} dte^{-t}t^{D/2-3} \tag{4.17}$$

($Y = \alpha' \log s$). Then $\Phi(y)$ becomes the function for the A-S wave at $y \gg \sqrt{Y} = \sqrt{\alpha' \log s}$. So S is dominated by graviton exchange at large b (Aichelburg-Sexl) and by absorption at small b.

As a first approximation, we can neglect all string oscillators in $\Phi(y)$ and obtain

$$\Phi(x) = -\frac{4q^v}{s} a_{\text{tree}}(s, x), \tag{4.18}$$

where $a_{\text{tree}}(s, x)$ is $a_{\text{tree}}(s, b)$ (impact parameter space), and becomes equal to the A-S result at large b. We can rewrite it also as (g = gauge coupling)

$$\begin{aligned} a_{\text{tree}}(b, s) &= \frac{\alpha' g^2 s}{16\pi} \frac{1}{(4\pi\bar{Y})^{D/2-2}} \int_0^1 d\rho \rho^{D/2-3} e^{-[b^2/(4\bar{Y})]\rho} \\ &= \frac{g^2 s \alpha'}{16\pi} \frac{1}{\pi^{D/2-2} b^{D-4}} \int_0^{b^2/(4\bar{Y})} dt e^{-t} t^{D/2-3}, \end{aligned} \tag{4.19}$$

where $\bar{Y} = \alpha' \log(-is) = Y - i\pi\alpha'/2$. We are only interested in the real part of a_{tree}, as it is the only one that we can use in the classical gravitational wave scattering calculation. It is obtained just by replacing $\bar{Y}$ with Y. For $b^2 \gg Y$, we obtain

$$\begin{aligned} \text{Re}\ a_{\text{tree}}(s, b) &\simeq \frac{g^2 s}{16\pi} \frac{\alpha'}{\pi^{D/2-2} b^{D-4}} \left(\Gamma(D/2-2) - e^{-[b^2/(4\bar{Y})]} \left(\frac{b^2}{4Y}\right)^{D/2-3} \left(1 + (D/2-3)\frac{4Y}{b^2}\right) + \cdots\right) \\ &= \frac{g^2 s\alpha'}{16\pi}\left(\frac{2}{\Omega_{D-5} b^{D-4}} + \cdots\right), \end{aligned} \tag{4.20}$$

whereas for $b^2 \ll Y$ we get

$$\text{Re}\ a_{\text{tree}}(s, b) \simeq \frac{g^2 s\alpha'}{16\pi} \frac{2}{(4Y)^{D/2-2}} \times \left(\frac{1}{D-4} - \frac{b^2}{4Y(D-2)} + \cdots\right). \tag{4.21}$$

One need just repeat the Eardley-Giddings–type calculation now, as we have set it up in the previous section.

The regime we are working in is small g, large $G_N s = g^2\alpha' s/(8\pi)$. Since $R_s^2 = 4G_N^2 s = g^4 \alpha'^2 s/(4\pi)^2$ and $Y = \alpha' \log(\alpha' s)$,

$$\frac{R_s^2}{Y} = \frac{g^2}{\log(\alpha' s)} \frac{g^2 \alpha' s}{(4\pi)^2} \tag{4.22}$$

can still be arbitrary, in particular, it can be very large. Since to first order $b_{\max} = \rho_c = R_s$ (for Aichelburg-Sexl), and at large b the metric is A-S plus corrections, in the regime $R_s^2/Y \gg 1$ we can use the large b ($b^2/Y \gg 1$) expansion of $\Phi(b)$.

Then in $D = 4$ we get, with $p^u q^v = s \Rightarrow q^v = \sqrt{s}$ (with the choice $p^u = q^v$ due to center of mass scattering, with equal strength shock waves scattering)

$$\begin{aligned} \Phi(b) &= -\frac{g^2\sqrt{s}}{4\pi}\alpha'\left(2\log\frac{b}{R_s} - e^{-[b^2/(4Y)]}\left(\frac{b^2}{4Y}\right)^{-1} + \cdots\right) \\ &= -R_s\left(2\log\frac{b}{R_s} - e^{-[b^2/(4Y)]}\left(\frac{b^2}{4Y}\right)^{-1} + \cdots\right). \end{aligned} \tag{4.23}$$

Then the condition for the trapped surface appearing in the scattering of two Amati-Klimcik waves at zero impact parameter, $(\nabla\Phi)^2 = 4$, gives

$$b_{\max} = \rho_c \simeq R_s\left(1 + \left(1 + \frac{4Y}{R_s^2}\right)e^{-[R_s^2/(4Y)]}\right) \tag{4.24}$$

(for $b^2/Y \gg 1$, so $R_s^2/Y \gg 1$) thus increases, so the area of the formed black hole also increases (since the black hole area is proportional to ρ_c^2). The area of the trapped surface giving the bound on the horizon area is $S_{\min} = 2\pi\rho_c^2 = 4\pi r_h^2$ and $r_h = 2M_{bh}G$, so

$$M_{bh} = \frac{\rho_c}{2\sqrt{2}G} \tag{4.25}$$

also increases.

At nonzero impact parameter of the two Amati-Klimcik waves, parameter denoted by B as we mentioned (to avoid confusion with the b that was used previously), applying the same approximation for finding ρ_c as was used in the flat space A-S case, the normal continuity condition is $\partial_i\Phi_1 \cdot \partial_i\Phi_2 = 4$, so $\partial_i\Phi(\vec{x} - \vec{x}_1)\cdot\partial_i\Phi(\vec{x} - \vec{x}_2) = 4$, so we only get an extra factor of

$$\cos^2\theta = 1 - \frac{B^2}{2\rho_c^2} \tag{4.26}$$

to the condition, which thus gets modified to

$$\frac{\rho_c}{R_s} = \sqrt{1 - \frac{B^2}{2\rho_c^2}}(1 + e^{-[\rho_c^2/(4Y)]} + \cdots) \tag{4.27}$$

solved perturbatively by

$$\begin{aligned} \frac{\rho_c^2(R_s, B)}{R_s^2} &= \frac{1}{2}\left(1 + \sqrt{1 - \frac{2B^2}{R_s^2} + 8\left(y_0 - \frac{B^2}{2R_s^2}\right)e^{-[R_s^2/(4Y)]y_0}}\right) + \cdots; \\ y_0 &= \frac{1 + \sqrt{1 - \frac{2B^2}{R_s^2}}}{2}, \end{aligned} \tag{4.28}$$

which means that $B_{\max} = R_s/\sqrt{2}(1 + e^{-R_s^2/(8Y)})$.

Finally, let us see what happens if $R_s^2/Y \ll 1$. At first, we would guess that we can use the small b expansion of the metric $b^2/Y \ll 1$, for which

$$\Phi(b) = -2R_s\left(\frac{1}{D-4} - \frac{b^2}{4Y(D-2)} + \cdots\right). \quad (4.29)$$

But if we plug it into the continuity equation for getting ρ_c, $(\nabla\Phi)^2 = 4$, we would get $\rho_c = 4Y/R_s$ to first order, meaning that $\rho_c^2/(4Y) = 4Y/R_s^2$, that is we would seem to be in the opposite regime, so the perturbation expansion used was invalid. The solution is of course that $R_s^2/Y \ll 1$ will correspond to $\rho_c^2/Y \sim 1$, so we would need to use the full solution, which however is difficult to handle.

But in any case we can say that, for $R_s^2/Y \ll 1$, classically (A-S wave) we have $\rho_c = R_s$, but in string theory we get $\rho_c \sim \sqrt{Y} \gg R_s$, so we have a great increase in the area of the black hole formed, thus it is natural to assume the cross section will also increase.

V. RANDALL-SUNDRUM–TYPE MODELS

The next application of the black hole creation formalism is to see what kind of corrections appear if we have the black hole being created in a physical setting, namely, for a Randall-Sundrum scenario for low Planck scale. Emparan [45] found an A-S–type wave in the background of the one-brane Randall-Sundrum (RS) scenario (shock wave on the UV brane), and analyzed the scattering *à la* 't Hooft in this wave.

We will not be interested in the 't Hooft analysis for the scattering and its phenomenological consequences, which was the main focus of [45]. Instead, we will try to see how the addition of the RS background affects the Eardley-Giddings calculation for the flat space black hole creation. We will keep the wave on the brane, as in the Emparan calculation.

A. A first attempt—applying the formalism

The solution for an A-S–type wave in the RS background is

$$ds^2 = dy^2 + e^{-2|y|/l}(-dudv + dx^i dx^i + h_{uu}(u, x^i, y)du^2), \quad (5.1)$$

where

$$h_{uu} = \frac{4G_{d+1}}{(2\pi)^{(d-4)/2}} p\delta(u) \frac{e^{d|y|/(2l)}}{r^{(d-4)/2}} \times \int_0^\infty dq q^{(d-4)/2} J_{(d-4)/2}(qr) \frac{K_{d/2}(e^{|y|/l}lq)}{K_{d/2-1}(lq)}, \quad (5.2)$$

which is a solution of Einstein's equation with $t_{uu} = 2\pi p\delta(q_0 + q_1)$. Yet another form for the metric is

$$e^{-2|y|/l}h_{uu}(u, r, y) = -4G_4 p\delta(u)\left[e^{-2|y|/l}\log\frac{r^2}{l^2} - \frac{2l}{\pi}\int_0^\infty dm K_0(mr)\frac{Y_1(ml)J_2(mle^{|y|/l}) - J_1(ml)Y_2(mle^{|y|/l})}{J_1^2(ml) + Y_1^2(ml)}\right], \quad (5.3)$$

which means that on the brane ($y = 0$)

$$h_{uu}(u, r, y = 0) = -4G_4 p\delta(u)\left[\log\frac{r^2}{l^2} - \frac{4}{\pi^2} \times \int_0^\infty \frac{dm}{m}\frac{K_0(mr)}{J_1^2(ml) + Y_1^2(ml)}\right]. \quad (5.4)$$

The Einstein tensor for this solution is linear in h_{uu}, and thus even though this is found as a solution to the linearized equations of motion, it is also an exact solution.

At large distances, $r \gg l$,

$$h_{uu}(u, r; y = 0) = -4G_4 p\delta(u)\left[\log\frac{r^2}{l^2} - \frac{l^2}{r^2} + \frac{2l^4}{r^4} \times \left(\log\frac{r^2}{l^2} - 1\right) + \cdots\right], \quad (5.5)$$

whereas at small distances $r \ll l$,

$$h_{uu}(u, r; y=0) = -4G_4 p\delta(u)\left[-\frac{l}{r} + \frac{3}{2}\log\frac{r}{l} + \frac{3r}{8l} + \cdots\right]. \quad (5.6)$$

We can use the formalism developed previously, since the solution can also be expressed as just a modification of the Φ function. Now we can at least calculate the zero impact parameter (b) values of $S_{\rm min}$ (the area of the trapped surface) and the mass of the corresponding black hole. We can also estimate the nonzero b parameter values of $\rho_c(R_s)$, $b_{\rm max}$, $S_{\rm min}$.

The new function Φ is now

$$\Phi(u, \rho, y = 0) = -R_s\left[\log\frac{\rho^2}{l^2} - \frac{4}{\pi^2}\int_0^\infty \frac{dm}{m}\frac{K_0(m\rho)}{J_1^2(ml) + Y_1^2(ml)}\right], \quad (5.7)$$

which means that

$$\partial_i\Phi = -R_s\frac{x_i}{\rho}\left[\frac{2}{\rho} - \frac{4}{\pi^2}\int_0^\infty dm\frac{K_0'(m\rho)}{J_1^2(ml) + Y_1^2(ml)}\right] \quad (5.8)$$

and thus imposing the continuity of the normal condition $(\partial_i\Phi)^2 = 4$ and rescaling the variables by R_s we get the integral equation for ρ_c:

$$\frac{\rho_c}{R_s} = 1 - \frac{2\rho_c/R_s}{\pi^2}\int_0^\infty dy\frac{K_0'(y\rho_c/R_s)}{J_1^2(yl/R_s) + Y_1^2(yl/R_s)}. \quad (5.9)$$

As before, the area of the trapped surface is the area of two disks, so it is

$$S_{\rm min} = 2\pi\rho_c^2 = 4\pi r_h^2, \quad (5.10)$$

where r_h is the horizon radius of the formed black hole, and

$r_h = 2GM_{bh}$, so

$$M_{bh} = \frac{\rho_c}{2\sqrt{2}G}. \tag{5.11}$$

We can use the expansion for $\rho \gg l$ and $\rho \ll l$ to calculate the form of ρ_c from Eq. (5.9), for $R_s \gg l$ and $R_s \ll l$. For $R_s \gg l$ we have

$$\Phi = \Phi^{(0)} + \Phi^{(1)}; \quad \Phi^{(1)} \simeq R_s\left[\frac{l^2}{\rho^2} - \frac{2l^4}{\rho^4}\left(ln\frac{\rho^2}{l^2} - 1\right) + \cdots\right], \tag{5.12}$$

and thus imposing $(\partial_i\Phi)^2 = 4$ we get

$$\rho_c^2 \simeq R_s^2\left[1 + \frac{2l^2}{R_s^2} - \frac{l^4}{R_s^4}\left(8\ln\frac{R_s^2}{l^2} - 13\right)\right]. \tag{5.13}$$

For $R_s \ll l$ we get

$$\Phi = -R_s\left[-\frac{l}{\rho} + \frac{3}{2}\ln\frac{\rho}{l} + \frac{3\rho}{8l} + \cdots\right] \tag{5.14}$$

and then

$$\rho_c \simeq \sqrt{\frac{lR_s}{2}\left(1 + \frac{3}{2}\sqrt{\frac{R_s}{2l}} + \frac{3}{2}\frac{R_s}{2l} + \cdots\right)}. \tag{5.15}$$

Note that $\rho_c = R_s$ is what one gets in flat 4 dimensions, whereas $\rho_c = \sqrt{2G_5\mu} = \sqrt{R_s l/2}$ is what one gets in flat 5 dimensions, so the formula is correct to zeroth order.

So the mass of the black hole is

$$M_{bh} \simeq \frac{\sqrt{s}}{\sqrt{2}}\left(1 + \frac{l^2}{R_s^2} + \cdots\right) l \ll R_s$$
$$M_{bh} \simeq \frac{\sqrt{s}}{2}\sqrt{\frac{l}{R_s}}\left(1 + \frac{3}{4}\sqrt{\frac{R_s}{2l}} + \cdots\right) l \gg R_s. \tag{5.16}$$

Notice that the limit of small l is the limit in which the space is very four-dimensional (large exponential warping in the extra dimension), so the four-dimensional result should hold, and we find that (just small corrections to the usual 4D result). The limit of large l is when the background space is approximately flat 5D space, so we have to modify the results to account for the creation of a 5D black hole. The condition $(\nabla\Phi)^2 = 4$ is independent of dimension, but it becomes $(\partial_i\Phi)^2 + (\partial_y\Phi)^2 = 4$ in a general dimension (with y being the transverse dimensions), and it will be modified for a general background.

Thus in the general case the trapped surface is something in between two disks and 2 balls, so 2 fat disks, or flattened balls. In the 2 limiting cases, the trapped surface can be approximated by 2 disks or 2 balls, respectively. One can still define the black hole projected onto 4 dimensions.

We will come back to the correct treatment in the next subsection, and we will see that, whereas the zeroth order formulas are correct, the first order corrections get modified.

At nonzero b, applying the same approximation for finding ρ_c as was used in the flat case, the normal continuity condition $\partial_i\Phi_1 \cdot \partial_i\Phi_2 = 4$ becomes $\partial_i\Phi(\vec{x} - \vec{x}_1) \cdot \partial_i\Phi(\vec{x} - \vec{x}_2) = 4$, so we only get an extra factor of

$$\cos\theta = 1 - \frac{b^2}{2\rho_c^2} \tag{5.17}$$

to the condition, so that now

$$\frac{\rho_c}{R_s} = \sqrt{1 - \frac{b^2}{2\rho_c^2}}\left[1 - \frac{2\rho_c/R_s}{\pi} \times \int_0^\infty dy \frac{K_0'(y\rho_c/R_s)}{J_a^2(yl/R_s) + Y_1^2(yl/R_s)}\right], \tag{5.18}$$

whereas the expression for the (very conservative) estimate of the trapped area, $S_{\min}$, remains the same as a function of ρ_c and b,

$$S_{\min} = \sqrt{b^2\rho_c^2 - \frac{b^4}{4}}. \tag{5.19}$$

Expanding in the $l \ll R_s$ regime we get

$$\frac{\rho_c^2}{R_s^2} = \left(1 - \frac{b^2}{2\rho_c^2}\right)\left[1 + \frac{2l^2}{\rho_c^2} - \frac{l^4}{\rho_c^4}\left(8\ln\frac{\rho_c^2}{l^2} - 13\right) + \cdots\right], \tag{5.20}$$

so that

$$\frac{\rho_c^2}{R_s^2} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{2b^2}{R_s^2} + \frac{8l^2}{R_s^2}\left(1 - \frac{b^2}{2R_s^2 y_0}\right) + o\left(\frac{l^4}{R_s^4}\right)}\right) \tag{5.21}$$

[so that $b_{\max}^2 \simeq R_s^2/2(1 + 4l^2/R_s^2)$].

In the $l \gg R_s$ regime we have

$$\rho_c^2 = \frac{lR_s}{2}\sqrt{1 - \frac{b^2}{4\rho_c^2}}\left[1 + \frac{3}{2}\frac{\rho_c}{l} + \frac{3}{8}\frac{\rho_c^2}{l^2} + \cdots\right]. \tag{5.22}$$

The first term gives the equation

$$x^3 = a^2\left(x - \frac{b^2}{4}\right); \qquad x = \rho_c^2; \qquad a = \frac{lR_s}{2}. \tag{5.23}$$

Solving this equation and selecting the solution that gives $x = a$ in the limit of $b = 0$, we get (also calculating the first two corrections)

$$\frac{2\rho_c^2}{lR_s} = \frac{x}{a} = \alpha\left(1 + \frac{3}{2}\sqrt{\frac{R_s\alpha}{2l}} + \frac{3}{8}\frac{R_s\alpha}{2l} + \cdots\right), \tag{5.24}$$

where

$$\alpha = \frac{1}{\sqrt{3}}\left(\Delta + \frac{1}{\Delta}\right); \qquad \Delta = \left(-\beta + \sqrt{-1 + \beta^2}\right)^{1/3}; \qquad \beta = \frac{9b^2}{4\sqrt{3}lR_s}. \tag{5.25}$$

If $\beta \le 1$, then α is real and is

$$\alpha = \frac{\cos\theta/3}{\sqrt{3}/2}; \qquad \text{where } \cos\theta = -\beta \Rightarrow \Delta = e^{i\theta/3}. \tag{5.26}$$

If $\beta > 1$, the solution is complex, thus

$$b^2_{\max} = \frac{4\sqrt{3}lR_s}{9}. \tag{5.27}$$

B. Correct treatment: generalizing the formalism to curved higher dimensional background

Let us try to understand what happens to the black hole area when we have a curved spacetime background of the RS type:

$$ds^2 = e^{-2|y|/l}[-dudv + dx_i^2] + dy^2. \tag{5.28}$$

Let us denote $e^{-2|y|/l} = A$ and $g_{ij} = A\bar{g}_{ij}$ represents the metric in both x and y coordinates (transverse). Then a straightforward calculation along the lines of the flat space case finds the vector normal to the surface is

$$\xi = -\frac{1}{4}\bar{g}^{ij}\partial_i\Psi\partial_j\Psi du - dv - \partial_i\Psi dx^i, \tag{5.29}$$

and so similarly to the flat case the continuity condition for the normal is

$$\bar{g}^{ij}\partial_i\Psi\partial_j\Psi = 4 \Rightarrow (\nabla\Psi)^2 + A(\partial_y\Psi)^2 = 4 \tag{5.30}$$

(the relation fixing the boundary of the trapped surface, or its radius)

For these and the next relations it is necessary to calculate the coordinate transformation from the coordinate system

$$ds^2 = e^{-2|\bar{y}|/l}(-d\bar{u}d\bar{v} + d\vec{x}^2 + h\delta(\bar{u})d\bar{u}^2) + d\bar{y}^2 \tag{5.31}$$

to the coordinate system without delta function discontinuities, up to order u (near $u = 0$). The calculation is a straightforward but tedious generalization of the flat space case, and one finds after the coordinate transformation

$$\begin{aligned} \bar{u} &= u \quad \bar{v} = v + h\theta(u) + \frac{u\theta(u)}{4}(\partial_i h\partial_j h\bar{g}^{ij} + A(\partial_y h)^2) \\ \bar{x}^i &= x^i + \frac{u\theta(u)}{2}\bar{g}^{ij}\partial_j h \quad \bar{y} = y + \frac{u\theta(u)}{2}A\partial_y h \end{aligned} \tag{5.32}$$

that

$$\begin{aligned} ds^2 = {}& A[-dudv + dx_i^2 + u\theta(u)\partial_i\partial_j h dx^i dx^j] \\ &+ dy^2[1 + u\theta(u)A\partial_y^2 h] + dydx^i u\theta(u)A\partial_i\partial_y h \\ &+ dydAu\theta(u)\partial_y h + o(u^2), \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} &A = e^{-2|\bar{y}|/l} + o(u)^2 \Rightarrow A|_{u=0} = e^{-2|y|/l}; \\ &dA|_{u=0} = -\frac{2}{l}A\left[dy + \frac{A}{2}\partial_y h du\right]. \end{aligned} \tag{5.34}$$

The convergence of the normals $\theta = g^{ij}D_i\xi_j$ is now again

$$\theta = -\nabla^2(\Psi - h), \tag{5.35}$$

where $h_{uu} = h\delta(u)(\equiv \Phi\delta(u))$ and

$$\nabla^2 = \frac{1}{A}\nabla_x^2 + \partial_y^2 - \frac{d}{l}\,\mathrm{sgn}(y)\partial_y. \tag{5.36}$$

Therefore we write

$$\Psi = \Phi + \zeta; \qquad \nabla^2\zeta = 0. \tag{5.37}$$

So now the trapped surface is a surface $f(\rho, y) = 0$ defined by both $\Psi = C$ (const) and by $\bar{g}^{ij}\partial_i\Psi\partial_j\Psi = 4$. In the flat case the first implied $\rho = \rho_0$ and the second $\rho_0 = R_s$. But we also saw that the nonzero b case had the same problem as we have now: find a surface C and a function ζ that satisfies both $\Psi = \text{const}$ and $\nabla^2\Psi = 4$ with $\Psi = \Phi + \zeta$.

In general it is a hard problem, but at least perturbatively, in the two limits $l \to 0$ and $l \to \infty$ we expect to find approximate disks and approximate balls, respectively (and fat disks in between). We would also expect that in the $l \to 0$ the surface is the same disk $\rho = R_s$ as for flat 4D space.

The formula for Φ (h) at nonzero y is (in [45], it is not Φ but $\Phi e^{-2|y|/l}$), so

$$\begin{aligned} \Phi = -R_s\bigg[&\log\frac{r^2}{l^2} - \frac{2l}{\pi}e^{2|y|/l}\int_0^\infty dm K_0(mr) \\ &\times \frac{Y_1(ml)J_2(mle^{|y|/l}) - J_1(ml)Y_2(mle^{|y|/l})}{J_1^2(ml) + Y_1^2(ml)}\bigg] \end{aligned} \tag{5.38}$$

(and actually, this is defined up to a constant, so the $\log r^2/l^2$ is conventional, we could have $\log r^2/r_0^2$).

Then we have

$$\begin{aligned} \partial_y\Phi|_{y=0} = \frac{2R_s}{\pi}\bigg[&-\frac{4}{l\pi}\int_0^\infty \frac{d(ml)}{ml}\frac{K_0(mr)}{J_1^2(ml) + Y_1^2(ml)} \\ &+ \frac{2}{l}\int_0^\infty d(ml)K_0(mr) \\ &\times \frac{Y_1(ml)J_2(ml) - J_1(ml)Y_2(ml)}{J_1^2(ml) + Y_1^2(ml)}\bigg] = 0! \end{aligned} \tag{5.39}$$

where we have used that $Y_\nu(x)J'_{\nu+1}(x) - J_\nu Y'_{\nu+1}(x) = -2(\nu + 1)/\pi x^2$, which we can easily deduce from the Bessel function properties.

Then we find

$$\partial_y^2\Phi|_{y=0} = \frac{2R_s}{\pi^2}\bigg[-\frac{8}{l^2}\int_0^\infty \frac{d(ml)}{ml}\frac{K_0(mr)}{J_1^2(ml)+Y_1^2(ml)} + \pi\partial_y^2\int_0^\infty d(ml)K_0(mr) \times \frac{Y_1(ml)J_2(mle^{|y|/l}) - J_1(ml)Y_2(mle^{|y|/l})}{J_1^2(ml)+Y_1^2(ml)}\bigg]\bigg|_{y=0}. \tag{5.40}$$

Let us now analyze the perturbation in l/r (the space is approximately flat 4D).

Using the relation

$$Y_\nu(x)J''_{\nu+1}(x) - J_\nu(x)Y''_{\nu+1}(x) = \frac{2}{\pi x}\left(\frac{6}{x^2}-1\right) \tag{5.41}$$

which can be easily derived, and also the expansion

$$J_1(x) \sim x/2; \qquad \pi Y_1(x) \sim -\frac{2}{x} + x\log\frac{x}{2} + \cdots, \tag{5.42}$$

we find

$$\partial_y^2\Phi|_{y=0} = -\frac{4R_s l^2}{r^4} + o(l^4/r^4). \tag{5.43}$$

We also have

$$\Phi|_{y=0} = -2R_s\log\frac{r}{l} + R_s\frac{l^2}{r^2} + \cdots. \tag{5.44}$$

Let us expand ζ near $y=0$ as

$$\zeta = \zeta_0(r) + \zeta_1(r)y + \frac{y^2}{2}\zeta_2(r). \tag{5.45}$$

Then at $y=0$ $\nabla^2\zeta = 0$ implies

$$\partial_x^2\zeta_0(r) + \zeta_2(r) - \frac{d}{l}\zeta_1(r) = 0, \tag{5.46}$$

and we do not want to upset the flat space solution, so we will take $\zeta_0 = 0$ [otherwise the continuity condition $(\nabla\Psi)^2 = 4$ implies a different radius for the trapped disks]. So $\zeta_2 = \frac{d}{l}\zeta_1$.

From

$$(\partial_i\Psi)^2 + e^{-2|y|/l}(\partial_y\Psi)^2 = 4, \tag{5.47}$$

we see that if $\partial_y\Psi$ has a y-independent piece, we will change the continuity equation at $y=0$, and we do not want that to happen to leading order in l. As $\partial\Phi|_{y=0} = 0$ already, we must put $\zeta_1 = 0$ to leading order, so at least $\zeta_1 \sim o(l)$, which implies $\zeta_2 = o(1)$ as well.

Then

$$\Psi = f + ay + \frac{y^2}{2}g + \cdots, \tag{5.48}$$

where

$$\begin{aligned} f &= \Phi_{|y=0} = -2R_s\log r/l + R_s l^2/r^2 + \cdots, \qquad a = \zeta_1 \\ g &= \partial_y^2\Phi|_{y=0} + \frac{d}{l}\zeta_1 = -\frac{4R_s l^2}{r^4} + o(l^4/r^4) + \frac{d}{l}\zeta_1 \equiv g_0 + \frac{d}{l}\zeta_1 + \cdots. \end{aligned} \tag{5.49}$$

We have to check now that the two surfaces in (r, y) defined by $\Psi = \text{const}$ and the normal continuity are the same to first nontrivial order in y and l.

$$\Psi = C = f + ay + \frac{y^2}{2}g + \cdots \tag{5.50}$$

and the other

$$\begin{aligned} C' = 4 &= \left(f' + ya' + \frac{y^2}{2}g' + \cdots\right)^2 + \left(1 - \frac{2y}{l} + 2\frac{y^2}{l^2} + \cdots\right)(a + yg + \cdots)^2 \\ &= f'^2 + a^2 + y\left(2a'f' - 2\frac{a^2}{l} + 2ag\right) + \cdots, \end{aligned} \tag{5.51}$$

if a is nonzero and

$$C' = 4 = f'^2 + y^2(f'g' + g^2) + \cdots, \tag{5.52}$$

if $a = 0$. If $a = 0$, we get to order y^2 (first nontrivial) for $\Psi = C$

$$2R_s\log\frac{r}{r_0} - R_s\frac{l^2}{r^2} + \cdots = y^2\left(-\frac{2R_s l^2}{r^4}\right) \tag{5.53}$$

(we have traded C for r_0) and for the continuity equation

$$\begin{aligned} (4 - f'^2 - a^2 =)4 - 4\frac{R_s^2}{r^2}\left(1 + 2\frac{l^2}{r^2}\right) - a^2 \\ = y^2(g^2 + f'g') = y^2\left(-4\frac{8R_s^2 l^2}{r^6}\right). \end{aligned} \tag{5.54}$$

Notice that at $l = 0$ the left-hand side of the two equations would be $2R_s\delta r/r_0$ and $8\delta r/R_s$, respectively, so with $r_0 = R_s$ (from $y = 0$) the two equations are not the same. So we have to put a nonzero ζ_1.

Also note that since the constant C (and hence r_0) is an arbitrary constant, at $y = 0$ but l nonzero we do not need to have the same l dependence in the two equations, we can absorb the unwanted l dependence in the redefinition of r_0. The l dependence of the radius $r_{\max}$ is deduced from the continuity equation (which does not have a free parameter).

Also note that *a priori* one could check the values for Φ and its y derivatives by using the alternative solution for Φ in [45]. We have tried to use perturbation theory on the alternate form (integral of ratio of K function) of Φ, but as Emparan noted, it is much harder to do so. In particular, one has to use the freedom to add an arbitrary constant to h (this is related to a rescaling of u and v).

If now we put $a = \zeta_1 \neq 0$ (and so $g = g_0 + \frac{4a}{l}$), the first order in y is linear, and by requiring that at $l = 0$ we get the same y dependence in both equations, we get the condition

$$a \simeq \frac{R_s}{4}\left(2ag_0 + \frac{6a^2}{l} - \frac{4R_s}{r}a'\right). \tag{5.55}$$

Thus if we put

$$a = \frac{\alpha R_s l}{r^2} \tag{5.56}$$

at $l = 0$ and $r = r_{\max} = R_s$ and since $g_0 \sim o(l^2)$ is negligible, we get $3/2\alpha = -1$, or $\alpha = -2/3$.

Then at $y = 0$, $l \neq 0$ the condition $\Psi = C$ is irrelevant as we said, since we can redefine the constant C. Then from the second (continuity) equation we get

$$\left(\frac{2R_s}{r_{\max}}\right)^2\left(1 + \frac{2l^2}{r^2} + \frac{\alpha^2 l^2}{4r^2} + \cdots\right) = 4 \Rightarrow r_{\max}^2 \equiv \rho_c^2 = R_s^2\left(1 + \frac{19}{9}\frac{l^2}{r^2} + \cdots\right) \Rightarrow M_{bh} \simeq \frac{\sqrt{s}}{2}\left(1 + \frac{19}{18}\frac{l^2}{r^2} + \cdots\right), \tag{5.57}$$

and in the treatment of the previous subsection we had thus neglected the α^2 term, the equation needed to be modified, but the sign of the correction is the same.

We can now also correct the calculation at nonzero b, by just putting the familiar $\cos\theta$ term

$$\frac{\rho_c^2}{R_s^2} = \left(1 - \frac{b^2}{2\rho_c^2}\right)\left(1 + \frac{l^2}{\rho_c^2}\left(1 + \frac{\alpha^2}{8}\right) + \cdots\right) \tag{5.58}$$

from which we get

$$\frac{\rho_c^2}{R_s^2} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{2b^2}{R_s^2} + \frac{8l^2}{R_s^2}\left(1 + \frac{\alpha^2}{8}\right)\left(1 - \frac{b^2}{2R_s^2 y_0}\right) + \cdots}\right). \tag{5.59}$$

The maximum impact parameter (and thus the scattering cross section $\sigma = \pi b_{\max}^2$) gets also modified:

$$b_{\max}^2 \simeq \frac{R_s^2}{2}\left[1 + \frac{4l^2}{R_s^2}\left(1 + \frac{\alpha^2}{8}\right)\right]. \tag{5.60}$$

The perturbation for $l \gg r$ (around flat 5D) will be left for future work.

VI. AICHELBURG-SEXL SOLUTION IN ADS BACKGROUND AND SCATTERING ANALYSIS

In this section we will analyze the case of an A-S wave in AdS (for future application to the gauge-gravity duality). First, we have to derive the solution for the A-S wave inside AdS.

A. Aichelburg-Sexl solution in AdS background

Let us notice that [29] analyzed putting A-S shock waves in more general backgrounds, of the type

$$ds^2 = 2A(u, v)dudv + g(u, v)h_{ij}(x^i)dx^i dx^j. \tag{6.1}$$

The calculation of the A-S solution in this background, with a source = massless photon at $u = 0, \rho = 0$ was done as in the flat background, just by gluing two regions at $u = 0$ with a shift $\Delta v = f = f(x^i)$. In [29], it was found that the Einstein equations are satisfied if

$$A_{,v} = 0 = g_{,v} \qquad \frac{A}{g}\Delta f - \frac{g_{,uv}}{g}f = 32\pi pGA^2\delta(\rho). \tag{6.2}$$

Indeed, in Minkowski background ($A = -1/2,\ g = 1$) one finds the Aichelburg-Sexl solution, $\Delta f = -16\pi pG\delta^{(2)}(\rho)$. Notice that, if the equations are not satisfied, it just means that one cannot find a solution for the ansatz taken. For example, spherical sourceless ($p = 0$) waves of this type in flat space are excluded [$A = -1/2, g = r^2 = (u - v)^2/4$ does not satisfy the conditions], but Penrose found another type of solution.

The authors of [29] were able to find such shock waves in the Schwarzschild solution in Kruskal-Szekeres coordinates,

$$ds^2 = -32\frac{m^3}{r}e^{-r/2m}dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$uv \equiv -(r/2m - 1)e^{r/2m}, \tag{6.3}$$

namely,

$$f(\theta, \phi) = k\int_0^\infty \frac{\sqrt{1/2}\cos(\sqrt{3}s/2)}{(\cosh s - \cos\theta)^{1/2}}ds. \tag{6.4}$$

Notice that if one would like to put AdS in the form in (6.1), one cannot: For a shock wave moving on the brane, the AdS background would be written as

$$ds^2 = \frac{1}{z^2}(dudv + d\vec{x}_2^2 + dz^2), \tag{6.5}$$

which is not of the desired form, whereas for a wave moving in the z direction

$$ds^2 = \frac{dudv + d\vec{x}_3^2}{(u - v)^2}, \tag{6.6}$$

which does not satisfy the conditions. But there could still be a solution of a different type.

Note that neither the previous metric nor the global AdS form,

$$ds^2 = l^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2), \quad (6.7)$$

nor the other forms,

$$ds^2 = \frac{l^2}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_3^2) \quad (6.8)$$

or (with $r/l = \sinh\rho = \tan\theta$)

$$ds^2 = -d\tau^2\left(1 + \frac{r^2}{l^2}\right) + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega_3^2, \quad (6.9)$$

help us in putting AdS into the form desired by [29], so we do need something else.

Indeed, we will see that instead we can follow closely the calculation of Emparan [45], so we will describe it, modifying it for our purposes.

Emparan [45] uses the metric of the one-brane RS model, perturbed with a general gravitational wave, in the form

$$ds^2 = e^{-2|y|/l}(-dudv + d\vec{x}^2 + h_{uu}(u, x^i, y)du^2) + dy^2. \quad (6.10)$$

But all we have to do in order to go to AdS is to replace $|y| \to y$. Then, under the coordinate transformation

$$y/l = \ln z/l, \quad (6.11)$$

we would get

$$ds^2 = \frac{l^2}{z^2}(-dudv + d\vec{x}^2 + h_{uu}(u, x^i, y)du^2 + dz^2), \quad (6.12)$$

which is the form that we wanted to obtain using the [29] formalism.

But [45] gives the Einstein tensor for the RS metric (6.10) as

$$\begin{aligned} G_{yy} &= \frac{d(d-1)}{2l^2} g_{yy} \\ G_{\mu\nu} &= \left(\frac{d(d-1)}{2l^2} - \frac{2(d-1)}{l}\delta(y)\right) g_{\mu\nu} \\ &\quad - \frac{1}{2}\partial_\mu u \partial_\nu u\left[e^{-2|y|/l}\left(\partial_y^2 - \mathrm{sgn}(y)\frac{d}{l}\partial_y\right) + \nabla_x^2\right] h_{uu}, \end{aligned} \quad (6.13)$$

where we have actually corrected the [45] result by putting the $\mathrm{sgn}(y)$ function. In the AdS case however, the $\mathrm{sgn}(y)$ is absent (since it came from $\partial_y|y|$).

The RS equations in the absence of h_{uu} are

$$G_{AB} = \Lambda g_{AB} + \lambda\delta(y) g_{\mu\nu}\delta_{AB}^{\mu\nu} \quad (6.14)$$

(cosmological constant Λ in the bulk and on the brane $\lambda =$ brane tension) and can be seen to be satisfied, we could read out what Λ and λ are. Then note that the equation for h_{uu} is linear.

In our case, adding the energy-momentum tensor of a photon of momentum p (which will generate the A-S metric), traveling at fixed x^i and fixed radial position in AdS, y_0,

$$t_{AB} = p\delta(u)\delta^{d-2}(x^i)\delta(y - y_0)\delta_{AB}^{uu}, \quad (6.15)$$

we get an equation, with $h_{uu} \equiv \Phi\delta(u)$,

$$\begin{aligned} &-\frac{1}{2}\left[e^{-2y/l}\left(\partial_y^2 - \frac{d}{l}\partial_y\right) + \nabla_x^2\right]\Phi \\ &\quad = 8\pi G_{d+1} p\delta^{d-2}(x^i)\delta(y - y_0). \end{aligned} \quad (6.16)$$

Note that the flat space limit $l \to \infty$ gives the correct result, $-1/2\partial_i^2 h = 8\pi G_{d+1} p\delta^{d-1}(x)$.

Going to 4D Fourier space

$$\Phi(q, y) = \int d^{d-2}x e^{-iq\cdot x}\Phi(x, y) \quad (6.17)$$

and similarly for t_{uu}, one obtains

$$\begin{aligned} &\Phi(q, y)'' - \frac{d}{l}\Phi(q, y)' - q^2 e^{2y/l}\Phi(q, y) \\ &\quad = -16\pi p G_{d+1}\delta(y - y_0). \end{aligned} \quad (6.18)$$

Going back to Emparan's case [45], the previous equation would have $d/l\,\mathrm{sgn}(y)$ and $e^{2|y|/l}$. The solution to that equation in Emparan's case is

$$A e^{(d|y|)/(2l)} K_{d/2}(e^{|y|/l} lq), \quad (6.19)$$

where the Bessel function K was chosen among the 2 solutions to the Bessel equation because of the boundary conditions: one wanted that at $y \to \infty$ the solution dies off, not blows up ($I_{d/2}$, the other solution, blows up exponentially at infinity). The $|y|$ in $e^{d|y|/2l}$ was because of the $\mathrm{sgn}(y)$ in the equation, and the $|y|$ in the $e^{|y|/l}$ argument was due to the $e^{2|y|/l}$ in the equation. Then both at $y = \infty$ and $-\infty$ we need the behavior of $K_\nu(x)$ for $x \to \infty$.

Finally, the constant is fixed by normalizing the coefficient of the delta function

$$A\left(\frac{d}{2l}K_{d/2}(lq) + qK'_{d/2}(lq)\right) = -8\pi G_{d+1} p, \quad (6.20)$$

and using an identity for Bessel functions A can be put to a simpler form. Also using a more general energy-momentum tensor for the momentum space wave, $t_{uu}(q)\delta(y)$ one has

$$h_{uu}(q, y) = 8\pi G t_{uu}(q) e^{(d|y|)/(2l)}\frac{K_{d/2}(e^{|y|/l} lq)}{qK_{d/2-1}(lq)}. \quad (6.21)$$

For the photon energy-momentum tensor, going back to x space and making the angular integrations, using

$$\int d\Omega_{d-3}e^{iqr\cos\theta} = \Omega_{d-4}\int_0^\pi d\theta \sin^{d-4}\theta d\theta e^{iqr\cos\theta} = (2\pi)^{(d-2)/2}\frac{J_{(d-4)/2}(qr)}{(qr)^{(d-4)/2}}, \quad (6.22)$$

one gets

$$h_{uu}(u,r,y) = \frac{4G_{d+1}}{(2\pi)^{(d-4)/2}}p\delta(u)\frac{e^{(d|y|)/(2l)}}{r^{(d-4)/2}} \times \int_0^\infty dq q^{(d-4)/2}J_{(d-4)/2}(qr)\frac{K_{d/2}(e^{|y|/l}lq)}{K_{d/2-1}(lq)}. \quad (6.23)$$

In our case, the generalization is very simple. There are no $|y|$ in the Eq. (6.18), so none in the solution. Again the solution at $y \to \infty$ has to decay, so we choose the Bessel function K for $y > y_0$. But now for $y_0 > y \to -\infty$ we get the exponent of the Bessel function becoming $K(x)$, $x \to 0$, for which $K_\nu(x)$ blows up as $x^{-\nu}$. Instead, the Bessel function $I_\nu(x)$ behaves smoothly, as x^ν. So the solution for $y < y_0$ is with $I_{d/2}$ instead of $K_{d/2}$. The normalization of the delta function is also different.

The solution is now of the type

$$\Phi = A_1 e^{(dy)/(2l)}K_{d/2}(e^{y/l}lq) \quad y > y_0$$
$$= A_2 e^{(dy)/(2l)}I_{d/2}(e^{y/l}lq) \quad y < y_0. \quad (6.24)$$

Continuity at y_0 gives

$$\frac{A_1}{A_2} = \frac{I_{d/2}(e^{y_0/l}lq)}{K_{d/2}(e^{y_0/l}lq)}, \quad (6.25)$$

and the jump in the derivative gives the delta function normalization $[(\Delta\Phi'(y_0)) = -16\pi G_{d+1}pe^{2y_0/l}]$. Using the Bessel function relations

$$zI'_\nu(z) + \nu I_\nu(z) = zI_{\nu-1}(z)$$
$$zK'_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$
$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z}, \quad (6.26)$$

we finally get

$$h_{uu}(q,y) = 8\pi G_{d+1}t_{uu}(q)e^{(dy)/(2l)}e^{[(4-d)/(2l)]y_0}K_{d/2}(e^{y/l}lq)2lI_{d/2}(e^{y_0/l}lq) \quad y > y_0$$
$$= 8\pi G_{d+1}t_{uu}(q)e^{(dy)/(2l)}e^{[(4-d)/(2l)]y_0}I_{d/2}(e^{y/l}lq)2lK_{d/2}(e^{y_0/l}lq) \quad y < y_0. \quad (6.27)$$

So now going in x space, taking the usual photon energy-momentum tensor, and making the angular integrations, we get

$$h_{uu}(u,r,y) = \frac{8G_{d+1}l}{(2\pi)^{(d-4)/2}}p\delta(u)\frac{e^{(dy)/(2l)}e^{[(4-d)/(2l)]y_0}}{r^{(d-4)/2}}\int_0^\infty dq q^{(d-2)/2}J_{(d-4)/2}(qr)K_{d/2}(e^{y/l}lq)I_{d/2}(e^{y_0/l}lq) \quad y > y_0$$
$$= \frac{8G_{d+1}l}{(2\pi)^{(d-4)/2}}p\delta(u)\frac{e^{(dy)/(2l)}e^{[(4-d)/(2l)]y_0}}{r^{(d-4)/2}}\int_0^\infty dq q^{(d-2)/2}J_{(d-4)/2}(qr)I_{d/2}(e^{y/l}lq)K_{d/2}(e^{y_0/l}lq) \quad y < y_0. \quad (6.28)$$

Again, the last integration cannot be done, except on a certain hypersurface. Indeed, we have the relation

$$\int_0^\infty dx x^{\nu+1}K_\mu(ax)I_\mu(bx)J_\nu(cx) = \frac{(ab)^{-\nu-1}c^\nu e^{-(\nu+1/2)\pi i}Q^{\nu+1/2}_{\mu-1/2}(\mu)}{\sqrt{2\pi}(\mu^2-1)^{\nu/2+1/4}} \quad (6.29)$$

[where $Q^\nu_\mu(z)$ is the associated Legendre function of the second kind], that is of the desired form, which is however valid only if $\mathrm{Re}(a) > |\mathrm{Re}(b)| + |\mathrm{Im}(c)|$, $\mathrm{Re}(\nu) > -1$, $\mathrm{Re}(\mu+\nu) > -1$ (all satisfied), and $2ab\mu = a^2 + b^2 + c^2$, which imposes a constraint.

Thus we obtain

$$h_{uu}(u,r,y) = C\frac{8G_{d+1}l}{(2\pi)^{(d-4)/2}}p\delta(u)e^{(y-y_0)/l}l^{2-d}e^{[(4-d)/l]y_0};$$
$$C = \frac{i^{(3-d)/2}Q^{(d-3)/2}_{(d-1)/2}(\frac{d}{2})}{\sqrt{2\pi}(\frac{d^2}{4}-1)^{(d-3)/4}} \quad (6.30)$$

(for both $y < y_0$ and $y > y_0$) on the hypersurface

$$r^2 = l^2e^{2y_0/l}(de^{(y-y_0)/l} - 1 - e^{[2(y-y_0)]/l}). \quad (6.31)$$

One could presumably check this by the Aichelburg-Sexl procedure, namely, of boosting the AdS black hole and then taking the limit where the mass of the black hole goes to zero as the boost goes to infinity. It is however quite difficult in practice.

B. Scattering analysis

Let us look now at the AdS scattering. Let us first obtain the limits of AdS–A-S wave. Defining as before $h_{uu} = \Phi\delta(u)$, we get

$$\Phi = \bar{C}\frac{e^{(dy)/(2l)}}{r^{d-2}}e^{[(4-d)/(2l)]y_0}\int_0^\infty dz z^{(d/2)-1}J_{(d-4)/2}(z)K_{d/2}\times\left(e^{y/l}\frac{lz}{r}\right)I_{d/2}\left(e^{y_0/l}\frac{lz}{r}\right), \quad (6.32)$$

with $\bar{C} = 8G_{d+1}lp/(2\pi)^{(D-4)/2}$. As we can see, for $r \gg l$ the integral is dominated by the region of small argument of I and K and we can use

$$I_\nu(x) \sim \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)};$$
$$K_\nu(x) \sim \frac{\pi}{2\sin\nu\pi}\frac{(x/2)^{-\nu}}{\Gamma(-\nu+1)} \Rightarrow K_n(x) \sim \frac{1}{2}(n-1)!\left(\frac{x}{2}\right)^{-n}. \tag{6.33}$$

But

$$\int_0^\infty dx x^{2n+1} J_0(x) = 0, \tag{6.34}$$

so we need to expand $I_2(bx)K_2(ax)$ up to the first term that is not of x^{2n} type. We find

$$I_2(bx)K_2(ax) = \frac{b^2}{4a^2} + \text{const}x^2 + \text{const}x^4 - \frac{1}{64}a^2b^2x^4\log(x) + o(x^5), \tag{6.35}$$

and using

$$\int_0^\infty dx x^5 \log(x) J_0(x) = -64, \tag{6.36}$$

we get

$$\Phi = \frac{\bar{C}l^4}{r^6}e^{(2/l)(2y+y_0)}. \tag{6.37}$$

Instead, when $r \ll l$ (actually, for $e^{y/l}l/r \gg 1$), we can use the large argument expansion of I and K,

$$I_\nu(x) \sim \frac{e^x}{\sqrt{\pi 2x}}; \qquad K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x} \tag{6.38}$$

and obtain (for $d = 4$)

$$\Phi \simeq \frac{\bar{C}e^{(3y-y_0)/(2l)}}{2l}\frac{1}{\sqrt{r^2 + l^2(e^{y/l} - e^{y_0/l})^2}} \tag{6.39}$$

and so if we also have y/l, $y_0/l \ll 1$ we obtain as expected the 5D result

$$\Phi \simeq \frac{C}{2l\sqrt{r^2 + (y-y_0)^2}}. \tag{6.40}$$

Note that the result in (6.39) can be obtained also if $r/l \sim 1$, $e^{y/l} \gg 1$, which means $y/l \sim$ a few (not too large).

Another particular case of interest is $y = y_0$. Then we can do the integral at all values of r and obtain ($\bar{C} = 2R_s l^2$, R_s is 4D the Schwarzschild radius)

$$\Phi = 2R_s\left[-1 + \frac{r^2}{2l^2}e^{-2y_0/l}\left(-1 + \sqrt{1 + \frac{4l^2}{r^2e^{-2y_0/l}}}\right) + \frac{l^2}{r^2e^{-2y_0/l}}\frac{1}{\sqrt{1 + \frac{4l^2}{r^2e^{-2y_0/l}}}}\right], \tag{6.41}$$

and we can check that for $r \gg l$ (and $y_0/l \sim 1$ or $\ll 1$) we get

$$\Phi \simeq \frac{2R_s l^6}{r^6 e^{-6y_0/l}}, \tag{6.42}$$

the same as the result that we obtain in this limit from the above answer for all y ($\neq y_0$).

We can also check that at $e^{y_0/l}l/r \gg 1$ we have

$$\Phi \simeq lR_s\frac{e^{y_0/l}}{r}, \tag{6.43}$$

as we obtained from the formula at arbitrary y.

Finally, let us now look at 't Hooft scattering in AdS_5 in the two limits. For $r \ll l$ or $l/r \sim 1$, $e^{y/l} \gg 1$ (so that $lq \gg 1$ or $lq \sim 1$, $e^{y/l} \gg 1$)

$$\Phi \simeq \frac{\bar{C}}{2l}\frac{e^{(3y-y_0)/(2l)}}{\sqrt{r^2 + l^2(e^{y/l} - e^{y_0/l})^2}}, \tag{6.44}$$

and hence (since $\delta = p_-^{(1)}\Phi$, and going to $z = qb \equiv qr$ variables and using $p_-^{(1)}p^{(2)} = s/4$)

$$\delta(b, s) = \frac{G_5 s e^{(3y-y_0)/(2l)} q}{\sqrt{z^2 + l^2q^2(e^{y/l} - e^{y_0/l})^2}}, \tag{6.45}$$

and thus if δ is small the amplitude is

$$\mathcal{A} \simeq \frac{AG_5 s e^{(3y-y_0)/(2l)}}{q} \times \int_0^\infty dz z \frac{1}{\sqrt{z^2 + l^2q^2(e^{y/l} - e^{y_0/l})^2}} J_0(z) = \frac{G_5}{2\pi}\frac{s}{\sqrt{t}}e^{(3y-y_0)/(2l)}\exp[-(\sqrt{t}l(e^{y/l} - e^{y_0/l}))], \tag{6.46}$$

where the exponent is therefore large.

However, $\delta \ll 1$ means either $y \neq y_0$ and $G_4 s e^{3(y-y_0)/(2l)} \ll 1$ or $y \simeq y_0$ and $G_4 s \frac{l}{r} e^{y_0/l} \ll 1$, so the only possibility is $y \neq y_0 < l$, $r \ll l$, $G_4 s \ll 1$, but we still want $G_4 s \sim 1$, but <1 for 't Hooft scattering, so it is not clear that there is a good regime in between.

For $r \gg l$ (or rather $lq \ll 1$), we obtain in $D = 4$

$$\delta(b, s) \simeq 2\frac{G_5 s l^5 e^{2(2y+y_0)/l} q^6}{z^6}, \tag{6.47}$$

and therefore now δ is always small, so

$$\mathcal{A} \simeq \frac{A}{q^{D-2}} \int_0^\infty dz z^{D/2-1} J_{D/2-2}(z)\delta(z). \tag{6.48}$$

Unfortunately, the result for $\mathcal{A}$ in $D = 4$ is infinite; hence, we need to regularize by removing an infinite constant piece and keeping only the t-dependent piece from the integral. To obtain the finite t-dependent piece we would need to get δ at general D, which seems to be quite difficult to do, but we can at least say that the result in $D = 4 + 2\epsilon$ will change as $(q/z)^6 \Rightarrow (q/z)^{6+m\epsilon}$ and so the result will be

$$\mathcal{A}(D = 4) = \frac{G_4 s l^6 t^2}{2\pi 2^7} e^{(2/l)(2y+y_0)} \frac{m-2}{3-m} \ln t \propto G_4 s l^6 t^2 \ln t e^{(2/l)(2y+y_0)}. \tag{6.49}$$

VII. CONCLUSIONS

In this paper we have reanalyzed the question of black hole formation in the high energy collision of two particles via the classical scattering of two shock waves.

We have found that string corrections increase the horizon area. For the effective shock wave metric in [23], we have found that if we scatter head-on (at $b = 0$) two such waves, each characterized by an impact parameter $b > R_s$, we obtain trapped surfaces which are deformed disks of area higher than the area obtained from A-S wave scattering. For the effective shock wave metric in [24], in the case of $R_S^2/Y \gg 1$ [$Y = \alpha' \log(\alpha' s)$], we get an increase of the area of the black hole formed, as well as of the classical scattering cross section, $\sigma = \pi b_{\max}^2$, while in the $R_s^2/Y \ll 1$ we get that the area of the formed black hole is of the order of Y (modified string scale), not R_s^2, so much larger.

For higher dimensions, we have found a conservative approximation scheme for the area of the horizon formed which gives us a maximum impact parameter (indicative of the scattering cross section, as we expect that $\sigma = \pi b_{\max}^2$). We have thus obtained that in $D = 4$, $b_{\max} = R_s/\sqrt{2}$, and in $D = 5$ for instance $b_{\max} \simeq 0.9523 R_s$, which is again a more conservative estimate as the one in [11].

What was more surprising was the fact that, although graviton-graviton scattering should be described by the collision of two ideal sourceless waves, given in the Khan-Penrose solution, there does not seem to be a horizon forming even at zero impact parameter. There is a theorem that a singularity will form in the future of any sourceless wave collision, yet we cannot find a trapped surface, namely, the usual trapped surface calculation does not have a solution. We have speculated that maybe the gravitons cannot be described by sourceless waves at all, or maybe trapped surfaces are inherently different from the [11] case, namely, that the surfaces form only in the interacting region $u > 0$, $v > 0$, not at the border ($u = 0$, $v = 0$) as in the photon scattering case.

We have extended the formalism to curved backgrounds. For more realistic scenarios, involving possible creation of black hole at accelerators for low fundamental scale, we have chosen the one-brane Randall-Sundrum case. In the case that the 5th direction is highly curved, we have obtained just corrections to the flat 4D case, whereas for a weakly curved 5th direction, we have corrections about the 5D flat space black hole creation.

Finally, we have found a solution for an Aichelburg-Sexl wave inside an AdS background, and we have calculated the scattering amplitude for 't Hooft scattering in such a wave, at small and large distances r. This was done for later use [17] for analysis of the gravity dual of QCD high energy scattering.

ACKNOWLEDGMENTS

We would like to acknowledge useful discussions with Matt Strassler, Radu Roiban, and Antal Jevicki. This research was supported in part by DOE Grant No. DE-FE0291ER40688-Task A.

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