

# Class of exact solutions of Einstein's field equations in higher dimensional spacetimes, $d \geq 4$ : Majumdar-Papapetrou solutions

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The Newtonian theory of gravitation and electrostatics admits equilibrium configurations of charged fluids where the charge density can be equal to the mass density, in appropriate units. The general relativistic analog for charged dust stars was discovered by Majumdar and by Papapetrou. In the present work we consider Einstein-Maxwell solutions in  $d$ -dimensional spacetimes and show that there are Majumdar-Papapetrou type solutions for all  $d \geq 4$ . It is verified that the equilibrium is independent of the shape of the distribution of the charged matter. It is also showed that for perfect fluid solutions satisfying the Majumdar-Papapetrou condition with a boundary where the pressure is zero, the pressure vanishes everywhere, and that the  $(d - 1)$ -dimensional spatial section of the spacetime is conformal to a Ricci-flat space. The Weyl  $d$ -dimensional axisymmetric solutions are generalized to include electric field and charged matter.

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## I. INTRODUCTION

Objects formed by elementary components with electric charge to mass ratio equal to one have been considered for a long time now in general relativity, and in other theories of gravity. It is clear that the Newtonian theory of gravitation and Coulomb electrostatics conjointly admit equilibrium configurations for charged fluids where the electrical charge density  $\rho_e$  is equal to the mass density  $\rho$ , in appropriate units. Such a neutral equilibrium is possible owing to the exact balancing of the gravitational and Coulombian electric forces on every fluid particle. Thus, a static distribution of charged dust of any shape can in principle be built. The general relativistic analog for such extremal charged dust configurations was discovered by Majumdar [1] and independently by Papapetrou [2]. First, Weyl [3], while studying the electrostatic field in vacuum Einstein-Maxwell theory in an axisymmetric static four-dimensional spacetime, found that if the metric component  $g_{tt} \equiv V(x^i)$  and the electric potential  $\phi(x^i)$  (where  $x^i$  represent the spatial coordinates,  $i = 1, 2, 3$ ) are related by a functional form  $V = V(\phi)$ , then this function is given by

$$V = A + B\phi + \phi^2, \quad (1)$$

where  $A$  and  $B$  are arbitrary constants, and we use geometrical units,  $G = 1$ ,  $c = 1$ . Majumdar [1] extended this result by showing that it holds for a large class of static spacetimes with no particular spatial symmetry, axial or otherwise, for which the metric can be written as

$$ds^2 = -Vdt^2 + h_{ij}dx^i dx^j, \quad i, j = 1, 2, 3, \quad (2)$$

where  $V$  and  $h_{ij}$  are functions of the spatial coordinates  $x^i$ . Moreover, by choosing  $B = \pm 2\sqrt{A}$ , in which case the

potential  $V$  assumes the form of a perfect square,

$$V = (\sqrt{A} \pm \phi)^2. \quad (3)$$

Majumdar was able to show that the Einstein-Maxwell equations in the presence of charged dust imply exactly the same relation of the Newtonian theory

$$\rho_e = \pm \rho, \quad (4)$$

with both the gravitational potential  $V$  and the electric potential  $\phi$  satisfying a Poisson-like equation. As in the Newtonian case, the relativistic solutions are static configurations of charged dust (a perfect fluid with zero pressure) and need not have any spatial symmetry. Majumdar [1] also showed that in the case  $V$  is a perfect square as in Eq. (3) the metric of the three-space is conformal to a flat metric whose conformal factor is given by  $1/V$ , and in such a case all the stresses in the charged matter vanish. Similar results were found by Papapetrou [2], who assumed as starting point a perfect square relation among  $V$  and  $\phi$ . Condition (3) is called the Majumdar-Papapetrou condition. We shall see that the condition (3) implies (4), whereas the converse is not generally true. Solutions in which conditions (3) and (4) hold are called Majumdar-Papapetrou solutions.

After the works of Majumdar [1] and Papapetrou [2], several authors have studied different aspects of static spacetimes satisfying the Majumdar-Papapetrou conditions. In vacuum the extreme Reissner-Nordström spacetime and the corresponding multi-black-hole solutions were analyzed first by Hartle and Hawking [4] (see also [5]). Other solutions in vacuum are the Israel-Wilson-Perjés rotating solutions [6,7]. In matter, several authors

have dealt with these solutions [8–27], showing new analysis and results. An interesting result that concerns us here was given by Das [9], where it was shown that in the case of static charged incoherent matter distributions the condition balance (4),  $\rho_e = \rho$ , implies the Majumdar-Papapetrou condition (3),  $V = (\sqrt{A} \pm \phi)^2$ . This analysis was further developed in [20]. The other works dealt with a large number of models of different charged bodies with several types of shapes and mass density profiles which obey the Majumdar-Papapetrou relation displaying new analysis and results.

Now, the multi-black-hole Reissner-Nordström solutions have the interesting property of being supersymmetric [28], a result that can be extended to all solutions belonging to the Majumdar-Papapetrou system [29]. These solutions saturate supersymmetric Bogomol’nyi bounds, are stable, and may be considered as ground states of the theory [30]. These Majumdar-Papapetrou solutions can also be embedded in supergravities and superstring theories with dimensions higher than four. This has been done for vacuum solutions where generalized extreme Reissner-Nordström single [31] and extreme multi-black-holes [32] have been studied, as well as brane extensions to higher dimensions [33,34]. The study of  $d$ -dimensional solutions, with  $d \geq 4$ , in several theories is a hot topic. The higher dimensional Kerr solutions were discussed in [35]. Higher dimensional Weyl metrics were analyzed in [36]. There is now a five-dimensional rotating black ring solution [37] and its charged generalization [38] which extreme case belongs to the Majumdar-Papapetrou class in 5 dimensions (see also [39,40]). All the  $d$ -dimensional solutions, with  $d \geq 4$ , mentioned above are vacuum solutions. It is then, of course, important to analyze  $d$ -dimensional,  $d \geq 4$ , Majumdar-Papapetrou type solutions in matter, a subject which we pursue here. One can ask whether lower dimensional theories,  $d < 4$ , admit Majumdar-Papapetrou solutions or not. As will be seen,  $d = 2$  and  $d = 3$  yield singular expressions (either give zero or infinity) in our formulas for generic  $d$  Majumdar-Papapetrou systems. This is no surprise. In  $d = 3$  general relativity the analogues of Majumdar-Papapetrou solutions are pure point sources, since gravity does not propagate [41]. General relativity in  $d = 2$  does not exist and so there is no analogue. On the other hand, for lower dimensional theories of gravity, other than general relativity, with extra fields such as a dilaton field, analogues of the Majumdar-Papapetrou solutions might be found, see [42] for a  $d = 2$  system. Because of their singular behavior we avoid treating the lower dimensional Majumdar-Papapetrou systems, and work for the  $d \geq 4$  systems only.

In line with the beautiful paper of Majumdar [1] we will follow closely his analysis and render his results in four dimensions into higher  $d$  dimensions. We will also incorporate the interesting developments of Guilfoyle [20]. The plan of the present work is as follows: Starting with a

charged perfect fluid in  $d$  dimensions, we impose a Weyl type relation among the gravitational potential  $V$  and the electrostatic potential  $\phi$ ,  $V = V(\phi)$ , obtain a relation among the pressure and the charge and mass densities of the fluid, and show that, for matter distributions that have a boundary where the pressure is zero, only the case of charged dust matter (zero pressure everywhere) has non-singular solutions. This is done in Secs. II, III, IV, V, and VI. In Sec. III solutions involving a relation between  $g_{tt}$  and  $\phi$  are analyzed. In Sec. IV we show that the  $(d - 1)$ -dimensional spacelike submanifold is conformal to a Ricci-flat space only if the pressure vanishes. In Sec. V the nature of the solutions is analyzed. In Sec. VI the vanishing of the material stresses is analyzed, showing also that if it is assumed that the  $(d - 1)$ -dimensional submanifold is conformal to a Ricci-flat space then the pressure vanishes. In Sec. VII an examination of generalized Weyl’s axially symmetric solutions in  $d$ -dimensional spacetimes is done for electrovacuum and in presence of charged matter. In Sec. VIII a brief analysis of boundary value problems is made. Finally, in Sec. IX we present final comments and conclusions.

## II. THE FUNDAMENTAL EQUATIONS

We write the Einstein-Maxwell equations as ( $c = 1$ )

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + E_{\mu\nu}), \quad (5)$$

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \quad (6)$$

where Greek indices  $\mu, \nu$ , etc., run from 0 to  $d - 1$ . We have put  $G_d$ , the  $d$ -dimensional gravitational constant equal to one,  $G_d = 1$ , as well as  $c = 1$  throughout. Also,  $g_{\mu\nu}$  is the metric,  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  is the Einstein tensor, with  $R_{\mu\nu}$  being the Ricci tensor, and  $R$  the Ricci scalar.  $E_{\mu\nu}$  is the electromagnetic energy-momentum tensor, given by

$$4\pi E_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \quad (7)$$

where the Maxwell tensor is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (8)$$

$\nabla_\mu$  being the covariant derivative, and  $A_\mu$  the electromagnetic gauge field. In addition,

$$J_\mu = \rho_e U_\mu \quad (9)$$

is the current density,  $\rho_e$  is the electric charge density in the  $d$ -dimensional spacetime, and  $U_\mu$  is the fluid four-velocity.  $T_{\mu\nu}$  is the material energy-momentum tensor given by

$$T_{\mu\nu} = \rho_m U_\mu U_\nu + M_{\mu\nu}, \quad (10)$$

where  $\rho_m$  is the fluid matter energy density in the  $d$ -dimensional spacetime, and  $M_{\mu\nu}$  is the stress tensor. Following Guilfoyle [20] in our analysis we will use mainly a perfect fluid, in which case  $M_{\mu\nu}$  is given by

$$M_{\mu\nu}^{\text{perfect fluid}} = p(U_\mu U_\nu + g_{\mu\nu}). \quad (11)$$

Note that Majumdar [1] uses mainly electrovacuum. Since the gravitational constant  $G_d$  has dimensions of  $(\text{length})^{(d-3)} \times (\text{mass})^{-1}$ , and we have set it to one,  $G_d = 1$ , it implies that mass has dimensions of  $(\text{length})^{(d-3)}$ , while  $\rho_m$  has dimensions of  $(\text{length})^{-2}$ .

We assume the spacetime is static and that the metric can be written in the form

$$ds^2 = -V dt^2 + h_{ij} dx^i dx^j, \quad i, j = 1, \dots, d-1, \quad (12)$$

a direct extension of Eq. (2) to extra dimensions. The gauge field and four-velocity are then given by

$$A_\mu = \phi \delta_\mu^0, \quad (13)$$

$$U_\mu = -\sqrt{V} \delta_\mu^0. \quad (14)$$

The metric spatial tensor  $h_{ij}$ , the metric potential  $V$  and the electrostatic potential  $\phi$  are functions of the spatial coordinates  $x^i$  alone.

Initially, we are interested in the equations determining the metric potential  $V$  and the electric potential  $\phi$ . These are obtained, respectively, from the  $tt$  component of Einstein equations (5) and from the  $t$  component of Maxwell equations. These equations give

$$\begin{aligned} \partial_i(\sqrt{h} h^{ij} \partial_j V) &= \frac{\sqrt{h}}{2V} h^{ij} \partial_i V \partial_j V + 4 \frac{d-3}{d-2} \sqrt{h} \left[ h^{ij} \partial_i \phi \partial_j \phi \right. \\ &\quad \left. + 4\pi V \left( \rho_m + \frac{d-1}{d-3} p \right) \right], \end{aligned} \quad (15)$$

$$\partial_i(\sqrt{h} h^{ij} \partial_j \phi) = \frac{1}{2V} \sqrt{h} h^{ij} \partial_i V \partial_j \phi + 4\pi \sqrt{h} V \rho_e, \quad (16)$$

where  $h$  stands for the determinant of the metric  $h_{ij}$ , and  $\partial_i$  denotes the partial derivative with respect to the coordinate  $x^i$ . Notice that Maxwell equations (6) imply just one equation, the  $t$  component,  $\nabla_i F^{ti} = 4\pi J^t$ , showed in (16).

Equations (15) and (16) determine the potentials  $V$  and  $\phi$  in terms of a set of unknown quantities. Namely, the  $(d-1)(d-2)/2$  spatial metric coefficients  $h_{ij}$ , the fluid variables, energy density  $\rho_m$  and pressure  $p$ , and the electric charge density  $\rho_e$ . There are exactly  $(d-1)(d-2)/2$  additional equations that come from the Einstein equations, which in principle determine the  $h_{ij}$  metric components in terms of  $\rho_m$ ,  $p$  and  $\rho_e$ . Hence, to complete the system of equations it is necessary to provide the energy and charge density functions,  $\rho_m$  and  $\rho_e$ , and also to specify the pressure  $p$  or an equation of state for the perfect fluid. In the present analysis, we will not need the explicit form of the space metric  $h_{ij}$  and so the corresponding Einstein equations will not be written here. Additional equations that can be used are the conservation equations,  $\nabla_\nu T^{\mu\nu} = 0$ , which are sometimes useful in replacing a

subset of Einstein's equations. In the present case the conservation equations yield

$$\partial_i p + \frac{1}{2V} (\rho_m + p) \partial_i V - \frac{1}{\sqrt{V}} \rho_e \partial_i \phi = 0. \quad (17)$$

This is the relativistic analogous to the Euler equation, and carries the information of how the pressure gradients balance the equilibrium of the system. In what follows we investigate some particular cases of the above set of equations including electrovacuum, dust fluid, and a perfect fluid.

### III. SOLUTIONS INVOLVING A FUNCTIONAL RELATION BETWEEN $g_{tt} \equiv V$ AND $\phi$

#### A. The equations

We now assume the solutions of the  $d$ -dimensional spacetime to be of Weyl type where the metric potential  $g_{tt} \equiv V$  is a functional of the gauge potential  $\phi$ ,  $V = V(\phi)$ . Hence, Eqs. (15) and (16) read, respectively,

$$\begin{aligned} \partial_i(\sqrt{h} h^{ij} \partial_j \phi) &= \sqrt{h} \left( \frac{V'}{2V} + \frac{4}{V'} \frac{d-3}{d-2} - \frac{V''}{V'} \right) h^{ij} \partial_i \phi \partial_j \phi \\ &\quad + 16\pi \sqrt{h} \frac{V}{V'} \left( \frac{d-3}{d-2} \rho_m + \frac{d-1}{d-2} p \right), \end{aligned} \quad (18)$$

$$\partial_i(\sqrt{h} h^{ij} \partial_j \phi) = \frac{1}{2} \frac{V'}{V} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + 4\pi \sqrt{h} V \rho_e, \quad (19)$$

where we have defined  $V' = dV/d\phi$  and  $V'' = d^2V/d\phi^2$ . Using the last two equations we get

$$\begin{aligned} \left( 4 \frac{d-3}{d-2} - V'' \right) h^{ij} \partial_i \phi \partial_j \phi - 4\pi V' \sqrt{V} \rho_e \\ + 16\pi V \left( \frac{d-3}{d-2} \rho_m + \frac{d-1}{d-2} p \right) = 0. \end{aligned} \quad (20)$$

Note the singular behavior of the lower dimensional cases,  $d=2$  and  $d=3$ , which will not be treated here. Substituting the functional relation  $V = V(\phi)$  into the conservation Eqs. (17) it follows

$$\partial_i p + \left[ (\rho_m + p) \frac{V'}{2V} - \frac{\rho_e}{\sqrt{V}} \right] \partial_i \phi = 0. \quad (21)$$

From this equation it is possible to show that the pressure  $p$  turns out to be a function of  $\phi$  alone, and depends only indirectly on the matter and charge densities, see the appendix. Hence, there is a relation of the form

$$p = p(\phi), \quad (22)$$

among the pressure  $p$  and the electric potential and  $\phi$ , or equivalently among  $p$  and the metric potential  $V$ ,  $p = p(V)$ . A particular case of this relation appears in Sec. III C. The basic system of equations to be solved is composed by Eqs. (19)–(21).

## B. Electrovacuum solutions

Here we generalize to  $d$  dimensions, first the Weyl form and then the Majumdar-Papapetrou form of electrovacuum solutions to  $d$ -dimensional spacetimes.

### 1. Weyl form

In vacuum one has  $\rho_m = 0$ ,  $\rho_e = 0$ , and  $M_{\mu\nu} = 0$ . Using these in Eq. (20) and assuming  $h^{ij}\partial_i\phi\partial_j\phi \neq 0$ , it follows that  $4(d-3)/(d-2) - V'' = 0$ , whose solution is

$$V(\phi) = A + B\phi + 2\frac{d-3}{d-2}\phi^2, \quad (23)$$

where  $A$  and  $B$  are arbitrary constants. These are Weyl type solutions generalized from four to higher dimensions. This result holds for any spatial symmetry. Thus, it generalizes Majumdar's result, where he noticed that the function form (23) with  $d = 4$  would hold not only for the axial symmetry imposed by Weyl [3], but also for any spatial symmetry.

### 2. Majumdar-Papapetrou form

The problem is further simplified by choosing  $V(\phi)$  in the form of a perfect square, as done by Majumdar [1]. In the higher dimensional case, this is accomplished by choosing  $B = \pm 2\sqrt{2A(d-3)/(d-2)}$ . Therefore, the metric potential  $V$  reads

$$V(\phi) = \left(\sqrt{A} \pm \sqrt{2\frac{d-3}{d-2}}\phi\right)^2. \quad (24)$$

Equation (24) is the Majumdar-Papapetrou condition in  $d$  dimensions for  $d \geq 4$ .

A word on the choice of the relation between the potentials  $V$  and  $\phi$  is in order. The constant  $A$  can be normalized by some particular asymptotic condition on the metric, or can be set to zero by a redefinition of the electric potential  $\phi$ . This can be done since the Einstein-Maxwell equations, Eqs. (5) and (6), depend upon  $\phi$  only through its derivatives, so we may remove the additive constant  $A$  in Eq. (24) by performing the transformation  $\phi \rightarrow \phi \mp [(d-2)/2(d-3)]A$ , and writing  $V = 2[(d-3)/(d-2)]\phi^2$ .

## C. Charged matter solutions

### 1. Weyl form

As mentioned above, the condition imposed on  $V(\phi)$  to produce Weyl type solutions in vacuum is  $V'' = 4(d-3)/(d-2)$ , see Eq. (23). If the system has matter, we see that, when the condition  $V'' = 4(d-3)/(d-2)$  is satisfied, Eq. (20) holds only if

$$\rho_e = 4\frac{\sqrt{V}}{V'}\left(\frac{d-3}{d-2}\rho_m + \frac{d-1}{d-2}p\right). \quad (25)$$

This relation holds whether or not the potential  $V$  is a perfect square in  $\phi$ . This together with Eq. (23) are the

Weyl type conditions valid for matter systems. Usually this condition is not discussed in the literature, see however [20] for the four-dimensional analysis.

As in the electrovacuum case, lower dimensional spacetimes deserve special attention. The Weyl condition in three dimensions follows by substituting  $d = 3$  into Eqs. (23) and (25), which gives

$$\rho_e = 8\frac{\sqrt{A+B\phi}}{B}p.$$

This equation, together with Eqs. (17) and (19) indicates that there can be found interesting Weyl type charged fluid solutions in three-dimensional spacetimes.

### 2. Majumdar-Papapetrou form

Now, we want to specialize from Weyl type solutions in matter to Majumdar-Papapetrou solutions in matter. We then look for particular solutions of Eq. (20) by choosing the metric potential  $V$  in the Majumdar-Papapetrou form, i.e., in the form of a perfect square of the form (24). Substituting such a potential  $V(\phi)$  into (25) it follows

$$\rho_e = \pm\sqrt{2\frac{d-3}{d-2}}\left(\rho_m + \frac{d-1}{d-3}p\right), \quad (26)$$

which is the generalized Majumdar-Papapetrou condition for matter in spacetimes whose number of dimensions is  $d \geq 4$ . The plus sign goes with positive electric charge and the minus sign is chosen for negative electric charge. This analysis is analogous to the four-dimensional case, see [9,14,16,20]. The Majumdar-Papapetrou solution for  $V$  in terms of  $\phi$  is then complete, while the solution for  $\rho_e$  is given, as a function of  $\rho_m$  and  $p$ , by Eq. (26).

We now, following the interesting result of [20] in four dimensions, show that the Majumdar-Papapetrou conditions rule out  $d$ -dimensional perfect fluid solutions with boundary surfaces on which  $p = 0$ , but  $p \neq 0$  in the bulk. In order to see that, we substitute Eqs. (24) and (26) into (21) and obtain  $(d-3)V\partial_i p = p\partial_i V$ . This equation can easily be integrated giving

$$p^{d-3} = kV, \quad (27)$$

where  $k$  is an integration constant. It follows from the last equation that the surface of zero pressure is also a surface where the  $g_{tt}$  coefficient of the metric vanishes, implying a metric singularity. In the static spacetimes we are considering here, the vanishing of  $g_{tt}$  means infinite redshift, and such a kind of surface is not allowed in the solution for a self-gravitating perfect fluid. Suppose a localized object such as a star, for instance. The surface of the star is usually defined by imposing  $p = 0$  as a boundary condition. Equation (27) implies that the surface of the star would be singular (an infinite redshift surface), and the solution cannot represent a star. In cases like this, it is then necessary to take  $k = 0$ , implying that pure Majumdar-

Papapetrou type stars do exist only for charged dust, i.e., for solutions in which  $p = 0$  everywhere, which is the case most treated in the literature [8–27]. Of course, Majumdar-Papapetrou solutions without boundaries can have  $p \neq 0$  throughout matter. Another possibility is to consider Majumdar-Papapetrou solutions with some thin shell at the surface.

#### IV. A CLASS OF EXACT SOLUTIONS

Here we show that the  $(d - 1)$ -dimensional spatial section of the spacetime satisfying the Majumdar-Papapetrou condition has Ricci tensor proportional to the pressure of the fluid. If the pressure vanishes the subspace is conformal to a Ricci-flat Riemannian space.

Let us then factor out a conformal factor in the  $(d - 1)$ -space metric as follows,

$$h_{ij}dx^i dx^j = \frac{1}{W} \hat{h}_{ij} dx^i dx^j, \quad (28)$$

where  $W$  and  $\hat{h}_{ij}$  are functions of the space coordinates  $x^i$ ,  $i = 1, 2, \dots, d - 1$ . There is no loss of generality in the above choice, since  $W$  and  $\hat{h}_{ij}$  are arbitrary functions.

The next step is writing Einstein-Maxwell equations in terms of the potentials  $V$  and  $W$ , and in terms of the tensors in the conformal space. The convention adopted in the following is that all quantities wearing hats belong to the conformal space and are associated to the conformal metric  $\hat{h}_{ij}$ . For instance, the connection coefficients may be written as  $\Gamma_{jk}^i = \hat{\Gamma}_{jk}^i - [\delta_k^i \hat{\nabla}_j W + \delta_j^i \hat{\nabla}_k W - \hat{h}_{jk} \hat{\nabla}^i W]/(2W)$ , where the connection coefficients  $\hat{\Gamma}_{jk}^i$ , and the covariant derivative  $\hat{\nabla}_i$ , are built from the metric  $\hat{h}_{ij}$ .

Now, the explicit form of the Ricci tensor in terms of  $V$ ,  $W$  and  $\hat{\Gamma}_{jk}^i$  is needed. After some algebra we obtain

$$\begin{aligned} R_{tt} &= -\frac{W}{2} \hat{\nabla}^2 V + \frac{W}{4V} \hat{\nabla} V \cdot \hat{\nabla} V + \frac{d-3}{4} \hat{\nabla} V \cdot \hat{\nabla} W, \quad (29) \\ R_{ij} &= \hat{R}_{ij} - \frac{1}{4V^2} (2V \hat{\nabla}_i \hat{\nabla}_j V - \hat{\nabla}_i V \hat{\nabla}_j V) \\ &\quad - \frac{1}{4VW} (\hat{\nabla}_i V \hat{\nabla}_j W + \hat{\nabla}_j V \hat{\nabla}_i W - \hat{h}_{ij} \hat{\nabla} V \cdot \hat{\nabla} W) \\ &\quad + \frac{d-3}{4W^2} \left( 2W \hat{\nabla}_i \hat{\nabla}_j W - \hat{\nabla}_i W \hat{\nabla}_j W \right. \\ &\quad \left. - \frac{d-1}{d-3} \hat{h}_{ij} \hat{\nabla} W \cdot \hat{\nabla} W + \frac{2W}{d-3} \hat{h}_{ij} \hat{\nabla}^2 W \right), \quad (30) \end{aligned}$$

where the dot stands for the scalar product with respect to the metric  $\hat{h}_{ij}$ . It is then seen that the expressions for the components of the Ricci tensor are greatly simplified by choosing  $V(d-3)\hat{\nabla} W = W\hat{\nabla} V$ , which means

$$W = V^{1/(d-3)}. \quad (31)$$

As argued previously, this choice can be done without loss of generality since the metric  $\hat{h}_{ij}$  is arbitrary. Substituting (31) into (29) and (30) it follows

$$R_{tt} = -\frac{1}{2} V^{1/(d-3)} \left( \hat{\nabla}^2 V - \frac{1}{V} \hat{\nabla} V \cdot \hat{\nabla} V \right), \quad (32)$$

$$\begin{aligned} R_{ij} &= \hat{R}_{ij} - \frac{1}{4} \frac{d-2}{d-3} \frac{\hat{\nabla}_i V}{V} \frac{\hat{\nabla}_j V}{V} \\ &\quad + \frac{1}{d-3} \frac{\hat{h}_{ij}}{2V} \left( \hat{\nabla}^2 V - \frac{1}{V} \hat{\nabla} V \cdot \hat{\nabla} V \right). \quad (33) \end{aligned}$$

Calculating the energy-momentum tensor it is found

$$\begin{aligned} G_{tt} - \frac{g_{tt}}{d-2} G &= -2 \frac{d-3}{d-2} V^{1/(d-3)} \hat{\nabla} \phi \cdot \hat{\nabla} \phi \\ &\quad + 8\pi V \frac{d-3}{d-2} \left( \rho_m + \frac{d-1}{d-3} p \right), \quad (34) \end{aligned}$$

$$\begin{aligned} G_{ij} - \frac{g_{ij}}{d-2} G &= -2 \hat{\nabla}_i \phi \hat{\nabla}_j \phi + \frac{2}{d-2} \hat{h}_{ij} \nabla \phi \cdot \nabla \phi \\ &\quad + \frac{8\pi}{d-2} \frac{\hat{h}_{ij}}{V^{1/(d-3)}} (\rho_m - p). \quad (35) \end{aligned}$$

From the  $tt$  component of Einstein equations, and Eqs. (32) and (34), we obtain the following equation for  $V$ :

$$\begin{aligned} \hat{\nabla}^2 V - \frac{1}{V} \hat{\nabla} V \cdot \hat{\nabla} V &= 4 \frac{d-3}{d-2} \frac{1}{V} \hat{\nabla} \phi \cdot \hat{\nabla} \phi \\ &\quad + 16\pi \frac{d-3}{d-2} \frac{1}{V^{1/(d-3)}} \\ &\quad \times \left( \rho_m + \frac{d-1}{d-3} p \right). \quad (36) \end{aligned}$$

Additionally, using Eqs. (33) and (35), and the Einstein equations, it follows

$$\begin{aligned} \hat{R}_{ij} &= \frac{1}{4} \frac{d-2}{d-3} \frac{\hat{\nabla}_i V}{V} \frac{\hat{\nabla}_j V}{V} \\ &\quad - \frac{1}{2} \frac{1}{d-3} \frac{\hat{h}_{ij}}{V} \left( \hat{\nabla}^2 V - \frac{1}{V} \hat{\nabla} V \cdot \hat{\nabla} V \right) \\ &\quad - 2 \hat{\nabla}_i \phi \hat{\nabla}_j \phi + \frac{h_{ij}}{d-2} \left[ 2 \hat{\nabla} \phi \cdot \hat{\nabla} \phi \right. \\ &\quad \left. + \frac{8\pi}{V^{1/(d-3)}} (\rho_m - p) \right]. \quad (37) \end{aligned}$$

Comparing the last two equations we get

$$\begin{aligned} \hat{R}_{ij} &= \frac{1}{4} \frac{d-2}{d-3} \frac{\hat{\nabla}_i V}{V} \frac{\hat{\nabla}_j V}{V} - 2 \hat{\nabla}_i \phi \hat{\nabla}_j \phi \\ &\quad - \frac{16\pi}{d-3} \frac{p}{V^{1/(d-3)}} \hat{h}_{ij}. \quad (38) \end{aligned}$$

We also need to rewrite the Maxwell equation (16) under the choices given by Eqs. (28) and (31). The resulting

equation for the electric potential  $\phi$  is then

$$\hat{\nabla}^2 \phi = \frac{\hat{\nabla} V}{2V} \cdot \hat{\nabla} \phi + 4\pi \frac{1}{V^{1/(d-2)}} \rho_e. \quad (39)$$

Once the energy density, pressure and charge density are specified, the system of equations formed by (36), (38), and (39) supply all the equations needed to determine the variables  $V$ ,  $\hat{h}_{ij}$  and  $\phi$ . Equation (36) can be thought of as the equation that furnishes  $V$ , and Eqs. (38) determine the inner metric  $\hat{h}_{ij}$ , while the Maxwell equation (39) determines the electric potential  $\phi$ .

If we assume the functional relation among  $V$  and  $\phi$  is given by the Weyl type potential (23), then the number of unknown variables is reduced by one. The same happens to the number of equations because, in such a case, Eqs. (36) and (39) become identical, and the fluid variables  $\rho_m$ ,  $p$  and  $\rho_e$  are connected by relation (25). If we further assume that the relation between  $V$  and  $\phi$  is the Majumdar-Papapetrou condition (24), Eqs. (38) read

$$\hat{R}_{ij} = -16\pi \frac{p}{V^{1/(d-3)}} \hat{h}_{ij}. \quad (40)$$

A further conformal transformation such that  $\hat{h}_{ij} = V^{1/(d-3)} \bar{h}_{ij}/p$  leads to a new Ricci tensor  $\bar{R}_{ij} = -16\pi \bar{h}_{ij}$ , which is characteristic of a  $(d-1)$ -space of constant scalar curvature,  $\bar{R} = \text{constant}$ . The static spacetime fulfilled by a charged fluid whose fields satisfy the Majumdar-Papapetrou condition has a  $(d-1)$ -dimensional spacelike submanifold which is conformal to a space of constant curvature. When the pressure vanishes, the curvature is zero and the submanifold is conformal to a Ricci-flat space. These results are independent of the spatial symmetry of the matter distribution.

Defining now

$$U = \frac{1}{\sqrt{V}}, \quad (41)$$

the metric of the spacetime in the presence of a generic charged fluid can be put into the form

$$ds^2 = -U^{-2} dt^2 + U^{2/(d-3)} \hat{h}_{ij} dx^i dx^j. \quad (42)$$

Using this definition of  $U$  and Eq. (36), one obtains a Poisson-like equation for  $U$ ,

$$\hat{\nabla}^2 U = -8\pi \frac{d-3}{d-2} U^{(d-1)/(d-3)} \left( \rho_m + \frac{d-1}{d-3} p \right). \quad (43)$$

In the particular case of vacuum, the last equation reduces to the Laplace equation, just as in four dimensions [1]. Moreover, using Eqs. (24), the relation between  $U$  and  $\phi$  results

$$U = \frac{1}{\sqrt{A} \pm \sqrt{2 \frac{d-3}{d-2}} \phi}. \quad (44)$$

A spacetime metric in the form (42) was used in Ref. [32], as an ansatz, to study  $d$ -dimensional Majumdar-Papapetrou vacuum solutions.

## V. NATURE OF THE SOLUTIONS

The class of exact solutions discussed in the previous section corresponds to static spacetimes whose metric coefficient  $g_{tt}$  is a special function of the electric potential  $\phi$ ,  $g_{tt} = V = 1/U^2$  with  $U$  and  $\phi$  related by (44). Note, however, that there is some arbitrariness on the choice of the relation between the potentials  $U$  and  $\phi$ . Two interesting possibilities are worth mentioning here. First, the additive constant  $A$  can be made equal to zero by performing the transformation  $\phi \rightarrow \phi \mp A\sqrt{(1/2)[(d-2)/(d-3)]}$ , and writing  $\sqrt{V} = 1/U = 2[(d-3)/(d-2)]\phi^2$  (see also the discussion at the end of Sec. IV). Second, if  $A \neq 0$ , a reparametrization of the time coordinate of the form  $t \rightarrow t/\sqrt{A}$  transforms the  $g_{tt}$  coefficient into the form  $g_{tt} = 1/U^2 = (1 \pm \sqrt{2A[(d-3)/(d-2)]}\phi)^2$ . Furthermore, since  $A$  is an arbitrary integration constant, it can then be chosen appropriately according to the choice of the units of electric charge. That is to say, one may choose  $A$  such that  $2A(d-3)/(d-2) = 1$ , which implies  $1/U = 1 \pm \phi$  in appropriate units.

In the Newtonian limit one has  $U \simeq 1 + \varphi$ , and Eq. (43) reduces to the Poisson equation for the gravitational potential  $\varphi$ ,  $\hat{\nabla}^2 \varphi = -8\pi[(d-3)(d-2)]\rho_{\text{eff}}$ , with the effective Newtonian matter density given by  $\rho_{\text{eff}} = \rho_m + [(d-1)/(d-3)]p$ . It also follows that  $\varphi = \mp \phi$ . As in four dimensions, the  $d$ -dimensional spacetime fulfilled with a charged fluid satisfying the Majumdar-Papapetrou condition is analogous to a  $d$ -dimensional Newtonian system of charged self-gravitating fluid in static equilibrium. In fact, the quantity  $\rho_{\text{eff}} = \rho_m + [(d-1)/(d-3)]p$  is the effective energy density acting as source of the gravitational field, and it can be thought of as the Newtonian matter density. The effect of the dimensionality of the spacetime on the effective energy density is in the sense of diminishing the weight of the pressure, as  $\rho_{\text{eff}}$  varies from  $\rho_m + 3p$  for  $d = 4$  to  $\rho_m + p$  in the limit  $d \rightarrow \infty$ . Additionally, the factor  $\sqrt{2[(d-3)/(d-2)]}$  also depends on the dimension of the spacetime. However, this factor can be made equal to unity by choosing appropriate units. In fact, by putting back the gravitational constant of gravitation ( $G_d$ ) into Eq. (26), the factor  $\sqrt{2[(d-3)/(d-2)]}$  is replaced by  $\sqrt{2[(d-3)/(d-2)]G_d}$ . Then, we may choose units such that  $2[(d-3)/(d-2)]G_d = 1$ , yielding  $\rho_e/\rho_{\text{eff}} = \pm 1$  for all  $d > 3$ . This is in accordance to the fact that what really matters for the balancing of the gravitational and electromagnetic forces is the relation between  $g_{tt}$  and  $\phi$  being the same at every point of spacetime, as Eq. (24) shows. In other words, if the charge and the effective energy densities bear the same constant of proportionality, the system will be in static equilibrium

owing to the balancing of electric and gravitational forces. The value of such a proportionality parameter depends on the system of units one chooses, and this choice is, of course, dependent upon the dimensionality of the spacetime.

## VI. VANISHING OF THE MATERIAL STRESSES FOR A PROPER CHOICE OF THE INTERNAL FIELD

In three dimensions the Ricci tensor is proportional to the Riemann tensor, so the three-space Ricci tensor  $\hat{R}_{ij}$  is proportional to the three-space Riemann tensor  $\hat{R}_{ijkl}$ . Now, when  $p = 0$ , Eq. (40) shows that the spatial Ricci tensor is zero,  $\hat{R}_{ij} = 0$ , so that the Riemann tensor is also zero, and the inner spatial three-space is flat. The converse is also true, i.e., if the inner three-space is flat then  $p = 0$ . This was shown by Majumdar [1].

In higher dimensions, however, the proportionality between Ricci and Riemann tensors is not valid, so the vanishing of the pressure does not imply a conformally flat  $(d - 1)$ -space. On the other hand, if we assume that the metric is of the form (42) and assume further that the internal metric is Euclidean,  $\hat{h}_{ij} = \delta_{ij}$ , then  $\hat{R}_{ij}$  are identically zero and Eqs. (40) imply that the pressure vanishes. This can also be done for more general material stresses other than perfect fluid pressures, partially generalizing thus Majumdar's result for four dimensions.

## VII. EXAMINATION OF GENERALIZED WEYL'S AXIALLY SYMMETRIC SOLUTIONS IN HIGHER DIMENSIONS

A further study appearing in Majumdar's paper [1] is a discussion and analysis of Weyl's results [3] on axisymmetric solutions. Majumdar showed that when the relation between the metric potential and the electric potential is of the form of Eq. (1), then there is no need of requiring axial symmetry. The original works by Weyl [3] and Majumdar [1] analyzed axisymmetric solutions mainly in electrovacuum case. The problem of pure gravitational field inside uncharged matter (with no electromagnetic field) was considered by Majumdar without assuming any particular kind of matter. The case inside charged matter was only touched upon by Majumdar in Sec. VII B of his paper [1]. In the spirit of the present work, we extend the analysis of axisymmetric Weyl solutions to the case of higher dimensions both in vacuum and with charged matter. As a particular case we discuss the four-dimensional case in charged matter, somehow completing Majumdar's discussion. The main aim of this section is then to find the explicit form of the equations for the metric potentials and for the electrostatic potential in the axisymmetric Weyl form. The special cases where the metric potential  $V$  assumes, as function of the electric potential  $\phi$ , the Weyl and the Majumdar-Papapetrou forms are depicted separately. This is, in cer-

tain sense, an example of what was found in the previous sections, particularized to the Weyl axisymmetric form of the metric in a  $d$ -dimensional spacetime.

### A. The equations

In four dimensions for the Weyl axisymmetric metric one usually uses, instead of  $V$  in Eq. (2), the potential  $\mu_0$  such that the potential  $V$  appearing in Eq. (2) is  $V = e^{2\mu_0}$  where  $\mu_0$  is a function of  $r$  and  $z$  (see also Sec. IV). There are two other Weyl potentials, one is  $\mu_1$  which is usually related to the Weyl radial coordinate  $r$ , as  $r = e^{2\mu_1}$ , the other is  $\nu$  which is generally a function of  $r$  and  $z$ . Then, the four-dimensional metric in Weyl axisymmetric form is written in terms of the functions  $\mu_0$ ,  $\mu_1$ , and  $\nu$ , as  $ds^2 = e^{2\mu_0}(-dt^2 + e^{2\mu_1}d\varphi^2) + e^{2\nu}(dr^2 + dz^2)$ , or putting  $e^{2\mu_1} = r$ , as  $ds^2 = e^{2\mu_0}(-dt^2 + r^2d\varphi^2) + e^{2\nu}(dr^2 + dz^2)$ , where  $(t, \varphi, r, z)$  are spacetime cylindrical type coordinates. The Einstein field equations can then be obtained and analyzed in terms of the Weyl coordinates. Interestingly enough this can also be done in higher dimensions. The axisymmetric metric for higher dimensions is given by Emparan and Reall [37] (see also [36] and references therein for different higher dimensional generalizations of Weyl axisymmetric solutions). They assumed that the spacetime has  $d - 2$  non-null Killing vectors in which case the metric can be put into the form

$$ds^2 = -e^{2\mu_0}dt^2 + e^{-[2\mu_0/(d-3)]} \sum_{i=1}^{d-3} e^{2\mu_i}(dx^i)^2 + e^{2\nu}(dr^2 + dz^2), \quad (45)$$

where the functions  $\mu_0$ ,  $\mu_i$  and  $\nu$  depend upon the coordinates  $r$  and  $z$  only. The function  $\mu_0$  again plays the role of the gravitational potential. The Latin index  $i$  runs from 1 to  $d - 3$  and the coordinates  $x^i$  label  $d - 3$  spatial dimensions of the spacetime, while the two remaining spatial dimensions are labeled as  $x^{d-2} = r$  and  $x^{d-1} = z$ . The metric (45) satisfies the constraint (see e.g. Ref. [40]),

$$\exp\left(\sum_{i=1}^{d-3} \mu_i\right) = r. \quad (46)$$

Such a constraint implies that the function  $\Phi$ , defined by  $\Phi = \sum_i^{d-3} \mu_i$ , is harmonic

$$\nabla^2 \sum_{i=1}^{d-3} \mu_i = 0, \quad (47)$$

i.e., it satisfies a Laplace equation. It was found that, in vacuum, Einstein equations imply that the functions  $\mu_0$  and  $\mu_i$  satisfy a Laplace equation in a flat metric,  $\nabla^2 \mu_\alpha = 0$ , where  $\nabla^2$  is the Laplacian operator in  $d$  dimensions, and  $\alpha$  runs from 0 to  $d - 3$ , while  $\nu$  is given as a function of  $\mu_0$  and  $\mu_i$ . Some particular solutions have been found in the vacuum case for  $d = 5$ , and also in higher dimensions (see [40] and references therein). Moreover, the black ring

solution of Emparan and Reall [37] was generalized to include electric charge [38]. We then consider the generalized  $d$ -dimensional static axisymmetric Weyl spacetimes, whose metric is written as in Eq. (45), and investigate the general properties of the solutions in the presence of charged matter and in electrovacuum. We assume the  $d$ -dimensional spacetime inside matter satisfies the following conditions: (i) there exists a line element of the form (45), where the functions  $\mu_i$  satisfy the constraint (46), (ii) the matter content is given by an energy-momentum tensor of the form (10), with  $M_{\mu\nu}$  given by (11), (iii) the metric potential  $g_{tt} = e^{2\mu_0}$  is connected to the electric potential  $\phi$  by (23). Then, using the metric in the form (45) and the constraint (46), the Einstein equations yield

$$\nabla^2 \mu_0 = 16\pi \frac{d-3}{d-2} e^{2\nu} \left[ \rho_m + \left( \frac{d-1}{d-3} \right) p \right] + 4 \frac{d-3}{d-2} e^{-2\mu_0} [(\partial_r \phi)^2 + (\partial_z \phi)^2], \quad (48)$$

$$\nabla^2 \mu_i = 32\pi \frac{e^{2\nu}}{d-3} p, \quad (49)$$

$$\partial_r \nu = -\frac{1}{2r} - \frac{1}{d-3} \partial_r \mu_0 + \frac{r}{2} \sum_{i=1}^{d-3} [(\partial_r \mu_i)^2 - (\partial_z \mu_i)^2] + \frac{r}{2} \left[ \frac{d-2}{d-3} [(\partial_r \mu_0)^2 - (\partial_z \mu_0)^2] - 2e^{-2\mu_0} [(\partial_r \phi)^2 - (\partial_z \phi)^2] \right], \quad (50)$$

$$\partial_z \nu = -\frac{1}{d-3} \partial_z \mu_0 + r \sum_{i=1}^{d-3} \partial_r \mu_i \partial_z \mu_i + r \left[ \frac{d-2}{d-3} \partial_r \mu_0 \partial_z \mu_0 - 2e^{-2\mu_0} \partial_r \phi \partial_z \phi \right], \quad (51)$$

where the Latin index  $i$  runs from 1 to  $d-3$ . The above equations can also be obtained from Eqs. (38) and (43) with the obvious identifications  $V = e^{2\mu_0}$ ,  $\hat{h}_{ij} = e^{2\mu_i}$  for  $i = 1$  to  $d-3$ , and for the other two spacelike coordinates,  $x^{(d-2)} = r$  and  $x^{(d-1)}$ ,  $\hat{h}_{rr} = \hat{h}_{zz} = e^{2[\nu - \mu_0/(d-3)]}$ . The solely Maxwell equation is

$$\nabla^2 \phi = 4\pi e^{2\nu + \mu_0} \rho_e + 2\partial_r \phi \partial_r \mu_0 + 2\partial_z \phi \partial_z \mu_0, \quad (52)$$

which has exactly the same form as in four-dimensional spacetime.

Note that the special form of the metric (45) implies that the pressure vanishes. For adding Eq. (49) over  $i$ , from  $i = 1$  to  $i = d-3$ , one finds  $\nabla^2 \sum_{i=1}^{d-3} \mu_i = 32\pi e^{2\nu} p$ , and comparing to (47) gives  $p = 0$ . This result is consistent with what was shown in Sec. IV. The choice of the metric of the  $d$ -dimensional spacetime in the axisymmetric form (45) together with the Weyl condition (23) implies the

vanishing of the pressure of the perfect fluid. It is straightforward to show that this is true for any kind of matter, not only for a perfect fluid, i.e., in the present conditions, all the material stresses  $M_{ij}$  vanish. In the following we investigate some general properties of the solutions to the above system of equations.

## B. Electrovacuum solutions

Consider first the electrovacuum case. The equations governing the metric and electric potentials are obtained from the system (48)–(52) with  $\rho_m = 0$ ,  $p = 0$ , and  $\rho_e = 0$ . Let us consider the Weyl and Majumdar-Papapetrou cases separately.

### 1. Weyl form

In the  $d$ -dimensional case, the Weyl form of the relation between the metric coefficient  $g_{tt}$  and the electric potential  $\phi$  is [see also Eq. (23)]  $g_{tt} = e^{2\mu_0} = A + B\phi + 2[(d-3)/(d-2)]\phi^2$ , with  $B \neq \pm 2\sqrt{2}[(d-3)/(d-2)]A$ . Following Weyl [3], and defining an auxiliary function  $\chi$  by

$$\chi = \int \frac{d\phi}{A + B\phi + 2\frac{d-3}{d-2}\phi^2} = \frac{2}{\sqrt{B^2 - 8\frac{d-3}{d-2}A}} \tanh^{-1} \left( \frac{4\frac{d-3}{d-2}\phi + B}{\sqrt{B^2 - 8\frac{d-3}{d-2}A}} \right), \quad (53)$$

it is possible to show that Eqs. (48) and (52) become identical, and assume the Laplacian form i.e.,

$$\nabla^2 \chi = 0. \quad (54)$$

For  $d = 4$ , Eq. (53) reduces to the result presented by Majumdar [3],  $\chi = -\int \{d\phi/(A + B\phi + \phi^2)\} = 2 \tanh^{-1}[(2\phi + B)/\sqrt{B^2 - 4A^2}]/\sqrt{B^2 - 4A^2}$ . The properties of the other metric potential  $\nu$  are analyzed in detail by Majumdar [1] and will not be considered here.

### 2. Majumdar-Papapetrou form

The particular Majumdar-Papapetrou form happens when  $B = \pm 2\sqrt{2}[(d-3)/(d-2)]A$ , so that  $e^{2\mu_0} = (\sqrt{A} \pm \sqrt{2}[(d-3)/(d-2)]\phi)^2$ . With this, the relation among  $\chi$  and the electric potential  $\phi$  results very simple,

$$\chi = \sqrt{\frac{1}{2} \frac{d-2}{d-3}} \frac{1}{\sqrt{A} \pm \sqrt{2\frac{d-3}{d-2}}\phi}, \quad (55)$$

with  $\chi$  satisfying the Laplace equation (54), which is obtained from (48), or from (52). Notice that  $\chi$  is proportional to the function  $U$  introduced in Sec. IV [see Eq. (44)].

It is interesting to study here the four-dimensional case. Substituting  $d = 4$  into Eq. (55) gives  $\chi = 1/(\sqrt{A} \pm \phi)$ .



Hence, in the four-dimensional Majumdar-Papapetrou case, the potential  $\chi$  is connected to the metric potential by the well known relation  $\chi = 1/\sqrt{V} = e^{-\mu_0}$ , as found by Majumdar [1] and Papapetrou [2]. Moreover, we find, for instance,  $\partial_r \mu_0 \partial_z \mu_0 = e^{2\mu_0} (\sqrt{A} \pm \phi)^2 \partial_r \phi \partial_z \phi$ , etc. Therefore, imposing the Majumdar-Papapetrou condition we get  $\partial_r \nu = -\partial_r \mu_0$  and  $\partial_z \nu = -\partial_z \mu_0$ , which yields

$$\nu = -\mu_0 + \text{constant}. \tag{56}$$

The four-dimensional metric is then of the form  $ds^2 = e^{2\mu_0} dt^2 + e^{-2\mu_0} (dr^2 + r^2 d\theta^2 + dz^2)$ , where the spatial section of the spacetime is conformal to a flat space. The inner (three-dimensional) metric is Euclidean, in accordance to the results of Sec. IV for spacetimes with matter without stresses and satisfying the Majumdar-Papapetrou condition. The stresses are zero here due to the *a priori* chosen axial symmetry of the metric.

On the other hand, differently from the four-dimensional case, in  $d$  dimensions there is no relation between  $\nu$  and  $\mu_0$  such as Eq. (56). This can be seen by substituting the Majumdar-Papapetrou condition into Eqs. (50) and (51), which gives

$$\partial_r \nu = -\frac{1}{2r} - \frac{1}{d-3} \partial_r \mu_0 + \frac{r}{2} \sum_{i=1}^{d-3} [(\partial_r \mu_i)^2 - (\partial_z \mu_i)^2], \tag{57}$$

$$\partial_z \nu = -\frac{1}{d-3} \partial_z \mu_0 + r \sum_{i=1}^{d-3} \partial_r \mu_i \partial_z \mu_i. \tag{58}$$

Therefore, in the general  $d$ -dimensional Weyl axisymmetric spacetime, the Majumdar-Papapetrou condition (24) does not imply any further special restriction to the space metric other than the Weyl condition (23) does. This result is related to the fact that in  $d$ -dimensional spacetimes the Majumdar-Papapetrou condition does not imply that the metric of the inner space,  $\hat{h}_{ij}$  (see Sec. IV), for vanishing matter stresses, is necessarily flat, as it does in the case of four-dimensional spacetimes.

### C. Charged matter solutions

The main properties of the solutions inside charged matter are analyzed in this section.

#### 1. Weyl form

As shown in Sec. III, if we impose the Weyl relation  $e^{2\mu_0} = A + B\phi + 2[(d-3)/(d-2)]\phi^2$ , then Eq. (48) becomes identical to (52) if the following equation holds [see Eq. (25)]:

$$\begin{aligned} \rho_e &= 2 \frac{d-3}{d-2} \frac{e^{-\mu_0}}{\mu_0'} \rho_m \\ &= 2 \frac{d-3}{d-2} \frac{\sqrt{A + B\phi + 2 \frac{(d-3)}{d-2} \phi^2}}{B + 4 \frac{(d-3)}{d-2} \phi} \rho_m, \end{aligned} \tag{59}$$

where the prime stands for the total derivative with respect to  $\phi$ . In  $d=4$  dimensions one obtains  $\rho_e = \pm 2[\sqrt{A + B\phi + \phi^2}/(B + 2\phi)]\rho_m$ . From Eq. (59) it is obtained the resulting equation for the potential  $\chi$  defined by (53). It is

$$\nabla^2 \chi = -8\pi \frac{d-3}{d-2} e^{2\nu} \frac{\sqrt{\frac{B^2}{4} - 2 \frac{d-3}{d-2} A}}{\tanh(\sqrt{\frac{B^2}{4} - 2 \frac{d-3}{d-2} A} \chi)} \rho_m. \tag{60}$$

Thus, the general properties of the axisymmetric solution satisfying the Weyl condition (23) are verified as expected. The particular four-dimensional case considered by Weyl [3] and Majumdar [1] follows from the above analysis by choosing  $d=4$ , and noting that the energy density function  $\sigma$  used by Majumdar is related to the scalar invariant energy density  $\rho_m$  by

$$\sigma = \frac{\sqrt{\frac{B^2}{4} - Ae^{2\nu}}}{\tanh(\sqrt{\frac{B^2}{4} - A} \chi)} \rho_m.$$

#### 2. Majumdar-Papapetrou form

The Majumdar-Papapetrou relation is now  $e^{\mu_0} = (\sqrt{A} \pm \sqrt{2[(d-3)/(d-2)]}\phi)$ , and it is straightforward to show that Eq. (59) reduces to (26) but here with  $p=0$ , i.e.,  $\rho_e = \sqrt{2[(d-3)/(d-2)]}\rho_m$ . As in the electrovacuum case, the potential  $\chi$  which satisfies the Majumdar-Papapetrou condition given by (55) is proportional to  $U$ , where  $U = e^{-\mu_0}$  is the potential defined in Eq. (44). In addition, the resulting equation for  $V = e^{2\mu_0}$  (or for  $\phi$ ), obtained from (48) [or (52)], can be put into the form (43), namely,

$$\nabla^2 \chi = -8\pi \frac{d-3}{d-2} e^{2\nu} \chi \rho_m. \tag{61}$$

Hence, the spacetime metric can be written in Majumdar-Papapetrou form (42), in which the metric of the inner space,  $\hat{h}_{ij}$ , is the flat metric. In four dimensions we have  $e^{2\nu} = e^{-2\mu_0} = \chi^2$ , and the potential  $\chi$  satisfies the equation  $\nabla^2 \chi = -4\pi \chi^3 \rho_m$ , as found in Refs. [1,2]. The same comments with respect to the function  $\nu$  made at the end of the section on the  $d$ -dimensional electrovacuum case hold also here.

### VIII. BOUNDARY VALUE PROBLEMS

We have shown above that if  $g_{tt}$  and  $\phi$  are functionally related, then every level surface of  $g_{tt}$  is also a level surface of  $\phi$ . In the four-dimensional case, Majumdar also proved the proposition that if there exists a surface  $S$  on which  $g_{tt}$  and  $\phi$  are both constant, and if one of the two domains into which such a surface divides the entire spacetime is free from matter, then, in this domain, every level surface of  $g_{tt}$  will also be a level surface of  $\phi$ , and therefore  $g_{tt}$  and  $\phi$  will be functionally related by an equation like (1). Here we prove that this theorem holds also in higher dimensions. For we write Eq. (16) in the absence of electric charge

$$\partial_i \left( \frac{\sqrt{-g}}{V} h^{ij} \partial_j \phi \right) = 0, \quad (62)$$

where  $g = -Vh$  is the determinant of the metric. Moreover, Eq. (15) in the absence of matter can be cast into the form

$$\partial_i \left( \frac{\sqrt{-g}}{V} h^{ij} \partial_j V \right) = 4 \frac{d-3}{d-2} \frac{\sqrt{-g}}{V} h^{ij} \partial_i \phi \partial_j \phi. \quad (63)$$

Multiplying Eq. (62) by  $4[(d-3)/(d-2)]\phi$ , subtracting from (63) and rearranging, we have

$$\partial_i \left[ \frac{\sqrt{-g}}{V} h^{ij} \partial_j \left( V - 2 \frac{d-3}{d-2} \phi^2 \right) \right] = 0, \quad (64)$$

which has the same form of Eq. (58) of Ref. [1]. Now we notice that an arbitrary constant can be added to the potential  $\phi$  in Eq. (62) [and/or also to the function  $V - 2[(d-3)/(d-2)]\phi^2$  in Eq. (63)]. Hence, adding such a constant and multiplying (62) by another constant  $B$  and subtracting from (64) we find

$$\partial_i \left[ \frac{\sqrt{-g}}{V} h^{ij} \partial_j \left( V - 2 \frac{d-3}{d-2} \phi^2 - B\phi - A \right) \right] = 0, \quad (65)$$

where  $A$  is a constant. The proof of the theorem is then completed following exactly the same steps as done by Majumdar. In particular, it follows that  $V$  and  $\phi$  are functionally related by  $V - 2[(d-3)/(d-2)]\phi^2 - B\phi - A = 0$ .

### IX. CONCLUSIONS

We have investigated here the main properties of charged fluid distributions in higher dimensional Einstein-Maxwell gravity, imposing Weyl type and Majumdar-Papapetrou type conditions. Several properties of such solutions in four-dimensional spacetime are shown to hold also in  $d$ -dimensional spacetimes. At any dimension higher than three, a distribution of charged dust with constant matter to charge densities ratio can be stable, independently of the shape of the body. We also showed here that for  $d > 4$  the  $(d-1)$ -space is conformal to a Ricci-flat space (which can be nonflat). In  $d = 4$ , every solution satisfying Majumdar-Papapetrou conditions has a

three-dimensional spatial subspace conformal to a flat space, because the three-dimensional Ricci tensor is proportional to the Riemann tensor. There exists a vast literature in particular of Majumdar-Papapetrou type solutions for charged dust in four-dimensional general relativity [22], and also including dilaton or other scalar fields [5]. The study of analogous problems in  $d$ -dimensional spacetimes is of course of interest.

### APPENDIX: FUNCTIONAL RELATION $p = p(\phi)$

Equation (21) is of the form

$$\frac{\partial p}{\partial x^i} = f(x^j) \frac{\partial \phi}{\partial x^i}, \quad (A1)$$

where  $x^i$ ,  $i = 1, 2, \dots, d-1$ , are spacelike coordinates in the  $(d-1)$ -dimensional space, and  $f(x^j) \equiv (\rho_m + p)V'/2V - \rho_e/\sqrt{V}$  is a function of the coordinates. Multiplying both sides of Eq. (A1) by  $dx^i$  and adding over  $i$  results in  $(\partial p/\partial x^i)dx^i = f(x^j)(\partial \phi/\partial x^i)dx^i$ , which is equivalent to

$$dp = f(x^i)d\phi. \quad (A2)$$

The potential  $\phi$  is also a function of the coordinates,  $\phi = \phi(x^i)$ . Assuming that  $\phi$  is an invertible function of (at least) one of the coordinates,  $x^1$ , say, then we can write  $x^1$  in terms of  $\phi$  and of the other coordinates  $x^a$ ,  $a = 2, 3, \dots, d-1$ . The pressure  $p$  is then a function of  $\phi$  and of the remaining spacelike coordinates  $x^a$ ,  $p = p(\phi, x^a)$ . Therefore, it follows

$$dp = \left( \frac{\partial p}{\partial \phi} \right)_{x^a} d\phi + \left( \frac{\partial p}{\partial x^a} \right)_{\phi} dx^a, \quad (A3)$$

where  $(\partial p/\partial \phi)_{x^a}$  means the derivative is done with  $x^a$  held constant, and  $(\partial p/\partial x^a)_{\phi}$  means the derivative is done with  $\phi$  held constant. Comparing this last equation to Eq. (A2) it follows

$$\left( \frac{\partial p}{\partial \phi} \right)_{x^a} = f(\phi, x^a), \quad (A4)$$

$$\left( \frac{\partial p}{\partial x^a} \right)_{\phi} = 0. \quad (A5)$$

Equation (A4) gives the dependence of  $p$  on  $\phi$ , while Eq. (A5) says that for constant  $\phi$ , the pressure  $p$  does not depend on any one of the coordinates  $x^a$ ,  $a = 2, 3, \dots, d-1$ . Finally, by repeating the above procedure with  $x^1$  replaced, e.g., by  $x^2$ , it is also shown that  $(\partial p/\partial x^1)_{\phi} = 0$ . So, the partial derivatives of  $p$  with respect to all the coordinates  $x^i$ , with  $\phi$  held constant, are zero,

$$\left( \frac{\partial p}{\partial x^i} \right)_{\phi} = 0, \quad i = 1, 2, \dots, d-1,$$

completing the proof.

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