Anisotropic cosmological models with nonminimally coupled magnetic field

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Motivated by the structure of one-loop vacuum polarization effects in curved space-time we discuss a nonminimal extension of the Einstein-Maxwell equations. This formalism is applied to Bianchi I models with magnetic field. We obtain several exact solutions of the nonminimal system including those which describe an isotropization process. We show that there are inflationary solutions in which the cosmological constant is determined by the nonminimal coupling parameters. Furthermore, we find an isotropic de Sitter solution characterized by a "screening" of the magnetic field as a consequence of the nonminimal coupling.

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I. INTRODUCTION

The Einstein-Maxwell theory has been a subject of investigations for a long time[1]. As far as cosmological implications are concerned, the possible role of a primordial magnetic field has attracted particular interest. Studying the impact of a magnetic field on the dynamical evolution requires anisotropic cosmological models. A general discussion of this type of models with magnetic field and references to early activities along this line may be found in [2]. By now there are strong limits on the current magnitude of such a field [3] which seems to render magnetic fields on cosmological scales unimportant at the present stage of the cosmic evolution. This does not imply, however, that a magnetic field did not influence the dynamics of the early Universe. Moreover, quantum effects are expected to become relevant at early cosmic stages. Quantum-electrodynamical consideration shows that vacuum polarization effects in curved space-time give rise to nonminimal modifications of the (minimal) Einstein-Maxwell Lagrangian [4]. The investigation of a nonminimal coupling of gravity with electromagnetic fields was initiated by Prasanna [5]. Prasanna introduced the additional invariant $R^{ikmn}F_{ik}F_{mn}$ (R^{ikmn} is the Riemann tensor, F_{ik} is the Maxwell tensor) into the Lagrangian for the gravito-electromagnetic system and obtained a nonminimal one-parameter modification of the Einstein-Maxwell equations [6] . Novello and Salim [7] included the (gauge-dependent) terms RA^kA_k and $R^{ik}A_iA_k$ in the Lagrangian (A_k is the electromagnetic potential four-vector, R^{ik} is the Ricci tensor and R is the curvature scalar). A qualitatively new step has then been made by Drummond and Hathrell [4] by calculating quantum-electrodynamical one-loop corrections in curved space-time. The Lagrangian of such a theory contains the three U(1) gauge-invariant scalars $R^{ikmn}F_{ik}F_{mn}$, $R^{ik}g^{mn}F_{im}F_{kn}$ and $RF_{mn}F^{mn}$ with coefficients proportional to the square of the Compton wavelength of the electron. Subsequently, a nonminimal coupling of gravity and electromagnetism has been discussed by a number of authors in different settings [8–18]. Nonminimally extended theories were used as a framework to discuss potential limitations of the equivalence principle [19–21]). A further quantum-electrodynamical motivation of the use of the generalized Maxwell equations can be found in [22–24]. The effect of birefringence induced by curvature, first discussed in [4], and some of its consequences for electrodynamic systems have been investigated for the pp-wave background in [25–29]. A curvature force has been introduced to describe the accelerated expansion of the universe [30,31]. Nonminimal interactions in which torsion is coupled to the electromagnetic field were studied in [32,33] (see also [34] for a review). Finally, we mention a mediated nonminimal coupling in which the scalar Higgs field ϕ is coupled to gravity via a $\xi\phi^2R$ term and to a Yang-Mills potential A_k by $\phi^2A_kA^k$ [35].

Most of the investigations so far were devoted to the analysis of the nonminimally modified Maxwell equations on a given background. With the exceptions [8,15] the impact of the nonminimal coupling on the gravitational field has been outside the focus of interest. However, one may expect that the rich structure of such type of theories gives rise to novel features in the gravitational dynamics as well. Our purpose here is to demonstrate this aspect for a specific class of nonminimal interactions in Bianchi I cosmological models. These interactions are modeled according to the already mentioned general structure obtained in [4] as the result of quantum-electrodynamical calculations. While the coupling constants are fixed in [4], they will be considered as arbitrary, constant parameters in our analysis (cf. [13]). On this basis we shall first obtain the general set of equations for a nonminimally extended Einstein-Maxwell system with linear electrodynamics (For a nonlinear, nonminimal extension of the

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Einstein-Maxwell theory see [36]). Then we specify this set to the homogenous but anisotropic case of Bianchi I cosmological models with magnetic field and a matter component with generally anisotropic equations of state. We obtain simple exact solutions for several choices of the nonminimal coupling parameters. Among them are solutions with axial symmetry which isotropize in the long time limit. As a specific feature of inflationary solutions of the nonminimal theory we find direct relations between a cosmological constant and the nonminimal coupling parameters. There exists an isotropic de Sitter solution with this property as well for which the corresponding set of parameters makes the gravitational dynamics independent of the (nonvanishing) magnetic field.

The paper is organized as follows. In Sec. II we present the general formalism of the nonminimally extended set of gravito-electromagnetic field equations, based on [4]. The (linear) electrodynamical field equations are obtained an discussed in Sec. III. Section IV is devoted to the gravitational field equations in the general case. In Sec. V the material of the previous sections is applied to the Bianchi I geometry. Several particular models are then studied in Sec. VI. Section VII provides a summary of the paper.

II. NON-MINIMAL COUPLING OF GRAVITY AND ELECTROMAGNETISM

A. General formalism

A nonminimal extension of the Einstein-Maxwell theory can be derived from the action functional

$$S[g, A] = \int d^4x\sqrt{-g}L \tag{1}$$

with the Lagrangian

$$L = \frac{R + 2\Lambda}{\kappa} + \mathcal{L}_{\mathrm{matter}} + \frac{1}{2}F_{mn}F^{mn} + \frac{1}{2}\mathcal{R}^{ikmn}F_{ik}F_{mn}. \tag{2}$$

Here, Λ is a cosmological constant, g is the determinant of the metric tensor g_{ik}, the constant κ is equal to $\kappa = \frac{8\pi G}{c^4}$ where G is Newton's gravitational constant. The quantity $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian of neutral matter and F_{ik} is the Maxwell tensor $F_{ik} = \nabla_i A_k - \nabla_k A_i$, where ∇_i denotes the covariant derivative and A_i is the four potential. The last term describes a [U(1) gauge-invariant] nonminimal coupling between gravity and electromagnetism, mediated through the tensor

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2}(g^{im}g^{kn} - g^{in}g^{km})R + \frac{q_2}{2}(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) + q_3 R^{ikmn}, \tag{3}$$

where q_1, q_2, and q_3 are phenomenological coupling constants with the dimension $[\mathrm{length}]^2$. This structure of the coupling is motivated by quantum-electrodynamical calculations of vacuum polarization effects in curved spacetime by Drummond and Hathrell [4]. While q_1, q_2, and q_3 have definite values in [4], we assume them to be arbitrary constant parameters in our analysis. The choice (2), (3) is a generalization of previous nonminimal modifications of Maxwell's theory. The case $q_1 = q_2 = 0$ was investigated in [5,6]. For $q_2 = q_3 = 0$ one obtains a model considered in [15].

The tensor $\mathcal{R}^{ikmn}$ has the same symmetry properties as the Riemann tensor R^{ikmn}. Contraction yields

$$\begin{aligned} g_{kn}\mathcal{R}^{ikmn} &= R^{im}(q_2 + q_3) + \frac{1}{2}Rg^{im}(3q_1 + q_2), \\ g_{kn}g_{im}\mathcal{R}^{ikmn} &= R(6q_1 + 3q_2 + q_3). \end{aligned} \tag{4}$$

The case of a vanishing trace which will be of interest in later applications is characterized by

$$g_{kn}g_{im}\mathcal{R}^{ikmn} = 0 \quad \Rightarrow \quad 6q_1 + 3q_2 + q_3 = 0. \tag{5}$$

III. ELECTRODYNAMIC EQUATIONS

The equations of nonminimal electrodynamics are obtained by varying the action functional with the Lagrangian (2) with respect to the four-potential A_i of the electromagnetic field. They are of the standard form

$$\nabla_k H^{ik} = 0, \qquad \nabla_k F^{*ik} = 0, \tag{6}$$

where the induction tensor H^{ik} is given by

$$H^{ik} \equiv F^{ik} + \mathcal{R}^{ikmn}F_{mn}. \tag{7}$$

This relation has the structure of a constitutive law in which $\mathcal{R}^{ikmn}$ plays the role of a susceptibility tensor.

A. Nonminimal constitutive equations

The linear constitutive Eq. (7) has the standard form $H^{ik} = C^{ikmn}F_{mn}$ [37–39] with a "material" tensor

$$C^{ikmn} \equiv \frac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}) + \mathcal{R}^{ikmn}. \tag{8}$$

This tensor describes the linear electromagnetic response of the system, which may also be characterized by the dielectric and magnetic permeabilities, as well as by possible magnetoelectric effects [37–39]. C^{ikmn} can uniquely be decomposed with respect to the four velocity U^i (normalized by $U^iU_i = 1$) of the medium:

$$\begin{aligned} C^{ikmn} = {}& \frac{1}{2}[\varepsilon^{im}U^kU^n - \varepsilon^{in}U^kU^m + \varepsilon^{kn}U^iU^m - \varepsilon^{km}U^iU^n] \\ & - \frac{1}{2}\eta^{ikl}(\mu^{-1})_{ls}\eta^{mns} - \frac{1}{2}[\eta^{ikl}(U^m\nu_{l\cdot}^{\ n} - U^n\nu_{l\cdot}^{\ m}) \\ & + \eta^{lmn}(U^i\nu_{l\cdot}^{\ k} - U^k\nu_{l\cdot}^{\ i})]. \end{aligned} \tag{9}$$

Here ε^{im} and $(\mu^{-1})_{pq}$ are the dielectric and magnetic permeability tensors, respectively, and $\nu_{p\cdot}^{\ m}$ is a tensor of magnetoelectric coefficients. These quantities are defined as

$$\varepsilon^{im} = 2C^{ikmn}U_k U_n, \qquad (\mu^{-1})_{pq} = -\frac{1}{2}\eta_{pik}C^{ikmn}\eta_{mnq}, \tag{10}$$

$$\nu_{p\cdot}^{\ m} = \eta_{pik}C^{ikmn}U_n = U_k C^{mkln}\eta_{lnp}. \tag{11}$$

The dot denotes the position of the second index when lowered. The tensors η_{mnl} and η^{ikl} are antisymmetric and orthogonal to U^i,

$$\eta_{mnl} \equiv \epsilon_{mnls}U^s, \qquad \eta^{ikl} \equiv \epsilon^{ikls}U_s. \tag{12}$$

They satisfy the identity

$$-\eta^{ikp}\eta_{mnp} = \delta^{ikl}_{mns}U_l U^s = \Delta^i_m\Delta^k_n - \Delta^i_n\Delta^k_m, \tag{13}$$

where δ^{ikl}_{mns} is the generalized 6-indices $\delta-$ Kronecker tensor. The spatial projection tensor Δ^{ik} is defined by

$$\Delta^{ik} = g^{ik} - U^i U^k. \tag{14}$$

Contracting Eq. (13) yields

$$\frac{1}{2}\eta^{ikl}\eta_{klm} = -\delta^{il}_{ms}U_l U^s = -\Delta^i_m. \tag{15}$$

The tensors ε_{ik} and $(\mu^{-1})_{ik}$ are symmetric, but ν_{lk} is generally nonsymmetric. All these tensors are orthogonal to U^i,

$$\varepsilon_{ik}U^k = 0, \qquad (\mu^{-1})_{ik}U^k = 0, \qquad \nu_{l\cdot}^{\ k}U^l = 0 = \nu_{l\cdot}^{\ k}U_k. \tag{16}$$

Use of (8) in (10) and (11) provides us with

$$\varepsilon^{im} = \Delta^{im} + 2\mathcal{R}^{ikmn}U_k U_n, \tag{17}$$

$$(\mu^{-1})_{pq} = \Delta_{pq} - \frac{1}{2}\eta_{pik}\mathcal{R}^{ikmn}\eta_{mnq}, \tag{18}$$

$$\nu_{p\cdot}^{\ m} = \eta_{pik}\mathcal{R}^{ikmn}U_n. \tag{19}$$

The nonminimal coupling of gravitational and electromagnetic fields may be interpreted as a change of the dielectric and magnetic properties of the vacuum, including a specific magnetoelectric interaction. The vacuum acquires properties of a quasimedium under the influence of a nonvanishing tensor $\mathcal{R}^{ikmn}$. The analogy between nonminimally extended electrodynamics and macroscopic media was pointed out, e.g., in [22]. This analogy may be completed by introducing the electric induction D^i, the magnetic field H^i, the electric field E^i and the magnetic induction B^i [37]:

$$D^i = \varepsilon^{im}E_m - B^l\nu_{l\cdot}^{\ i}, \qquad H_i = \nu_{i\cdot}^{\ m}E_m + (\mu^{-1})_{im}B^m. \tag{20}$$

The vectors D^i, H^i, E^i, and B^i are defined by [40]:

$$D^i = H^{ik}U_k, \qquad H^i = H^{*ik}U_k, \qquad E^i = F^{ik}U_k, \qquad B^i = F^{*ik}U_k. \tag{21}$$

They are orthogonal to the velocity four-vector U^i:

$$D^iU_i = 0 = E^iU_i, \qquad H^iU_i = 0 = B^iU_i, \tag{22}$$

and form the basis for the decomposition of the tensors F_{mn} and H_{mn}:

$$F_{mn} = E_mU_n - E_nU_m - \eta_{mnl}B^l, \qquad H_{mn} = D_mU_n - D_nU_m - \eta_{mnl}H^l. \tag{23}$$

IV. GRAVITATIONAL FIELD EQUATIONS

The gravitational field equations are obtained by varying the action (1) with the Lagrangian (2) with respect to the metric tensor. They can be written in the standard form

$$R_{ik} - \frac{1}{2}Rg_{ik} = \Lambda g_{ik} + \kappa T^{(\mathrm{eff})}_{ik}, \tag{24}$$

where

$$T^{(\mathrm{eff})}_{ik} = T^{(\mathrm{matter})}_{ik} + T^{(0)}_{ik} + q_1T^{(1)}_{ik} + q_2T^{(2)}_{ik} + q_3T^{(3)}_{ik}. \tag{25}$$

The stress-energy tensor of the matter $T^{(\mathrm{matter})}_{ik}$ may be decomposed according to

$$T^{(\mathrm{matter})}_{ik} = WU_iU_k + I^{(q)}_iU_k + I^{(q)}_kU_i + P_{ik}, \tag{26}$$

where W is the matter energy density scalar, P_{ik} is the symmetric (generally anisotropic) pressure tensor, orthogonal to the velocity four-vector ($P_{ik}U^k = 0$), and $I^{(q)}_i$ is the energy-flux four-vector, orthogonal to the four velocity ($I^{(q)}_iU^i = 0$). By $T^{(0)}_{ik}$ we denote the usual stress-energy tensor of the electromagnetic field,

$$T^{(0)}_{ik} \equiv \frac{1}{4}g_{ik}F_{mn}F^{mn} - F_{in}F_{k}^{\cdot n}. \tag{27}$$

The contributions from the nonminimal interaction are

$$T^{(1)}_{ik} = RT^{(0)}_{ik} - \frac{1}{2}R_{ik}F_{mn}F^{mn} - \frac{1}{2}g_{ik}\nabla^l\nabla_l(F_{mn}F^{mn}) + \frac{1}{2}\nabla_i\nabla_k(F_{mn}F^{mn}), \tag{28}$$

$$\begin{aligned}T^{(2)}_{ik} = &-\frac{1}{2}g_{ik}[\nabla_m\nabla_l(F^{mn}F^{l}_{\cdot n}) - R_{lm}F^{mn}F^{l}_{\cdot n}] \\ &- F^{ln}(R_{il}F_{kn} + R_{kl}F_{in}) - R^{mn}F_{im}F_{kn} \\ &- \frac{1}{2}\nabla^l\nabla_l(F_{in}F_{k}^{\cdot n}) + \frac{1}{2}\nabla_l[\nabla_i(F_{kn}F^{ln}) \\ &+ \nabla_k(F_{in}F^{ln})],\end{aligned} \tag{29}$$

and

$$T^{(3)}_{ik} = \frac{1}{4} g_{ik} R^{mnls} F_{mn} F_{ls} - \frac{3}{4} F^{ls} (F_i^{\cdot n} R_{knls} + F_k^{\cdot n} R_{inls}) - \frac{1}{2} \nabla_m \nabla_n (F_i^{\cdot n} F_k^{\cdot m} + F_k^{\cdot n} F_i^{\cdot m}). \tag{30}$$

The tensor $T^{(1)}_{ik}$ is proportional to the corresponding term in [15], the part $T^{(3)}_{ik}$ reproduces the stress-energy tensor of [16]. The tensor $T^{(2)}_{ik}$ is new. All three terms are supposed to contribute to the total stress-energy tensor in the following. In contrast to the traceless electromagnetic stress-energy tensor $T^{(0)}_{ik}$ the tensors $T^{(1)}_{ik}$, $T^{(2)}_{ik}$, and $T^{(3)}_{ik}$ have nonvanishing traces:

$$g^{ik} T^{(1)}_{ik} = -q_1 \left[\frac{1}{2} R F_{mn} F^{mn} + \frac{3}{2} \nabla^k \nabla_k (F_{mn} F^{mn}) \right], \tag{31}$$

$$g^{ik} T^{(2)}_{ik} = -q_2 \left[R^{mn} F^k_{\cdot m} F_{kn} + \frac{1}{2} \nabla^k \nabla_k (F_{mn} F^{mn}) + \nabla^m \nabla_n (F^{kn} F_{km}) \right], \tag{32}$$

$$g^{ik} T^{(3)}_{ik} = -q_3 \left[\frac{1}{2} R^{mnls} F_{mn} F_{ls} + \nabla^m \nabla_n (F^{kn} F_{km}) \right]. \tag{33}$$

Nonvanishing traces of effective stress-energy tensors are also features of nonlinear electrodynamic models (see, e.g., [41]).

The effective stress-energy tensor (25) in Eq. (24) has to be divergence-free, i.e.

$$\nabla^k T^{(\mathrm{eff})}_{ik} = 0. \tag{34}$$

We assume the stress-energy tensor of the matter $T^{(\mathrm{matter})}_{ik}$ to be conserved separately, i.e., $\nabla^k T^{(\mathrm{matter})}_{ik} = 0$. The remaining part of the effective stress-energy tensor is then automatically conserved if F_{ik} is a solution of Maxwell's equations. In order to check this fact directly, one has to use the Maxwell Equations (6) with (7), the Bianchi identities, the symmetry properties of the Riemann tensor and the commutation rules for the covariant derivatives. This procedure is analogous to the one described in [16] and we omit it.

V. ANISOTROPIC COSMOLOGICAL MODELS

A. Metric structure

We consider now the Bianchi I cosmological model with the line element [1,2]

$$ds^2 = dt^2 - a^2(t)(dx^1)^2 - b^2(t)(dx^2)^2 - c^2(t)(dx^3)^2. \tag{35}$$

Because of the symmetry of the metric only six components of the Riemann tensor are different from zero:

$$R^{01}_{\cdot\cdot 01} = -\frac{\ddot{a}}{a}, \quad R^{02}_{\cdot\cdot 02} = -\frac{\ddot{b}}{b}, \quad R^{03}_{\cdot\cdot 03} = -\frac{\ddot{c}}{c},$$
$$R^{12}_{\cdot\cdot 12} = -\frac{\dot{a}}{a}\frac{\dot{b}}{b}, \quad R^{13}_{\cdot\cdot 13} = -\frac{\dot{a}}{a}\frac{\dot{c}}{c}, \quad R^{23}_{\cdot\cdot 23} = -\frac{\dot{b}}{b}\frac{\dot{c}}{c}. \tag{36}$$

B. Exact solution of Maxwell's equations

Since the susceptibility tensor $\mathcal{R}^{ikmn}$ has no nonvanishing components with only one index zero ($\mathcal{R}^{\alpha\beta\gamma 0} = 0$, Greek indices denote spatial coordinates), it follows from relation (19) that all magnetoelectric coefficients vanish. The gravitational field (35) does not mix pure electric and pure magnetic fields. From relations (17) and (18) we find the dielectric and magnetic permeability tensors

$$\varepsilon^\alpha_\beta = \delta^\alpha_\beta + 2\mathcal{R}^\alpha_{0\beta 0}, \qquad (\mu^{-1})^\alpha_\beta = \delta^\alpha_\beta - \frac{1}{2} \eta^{\alpha\gamma\sigma} \mathcal{R}_{\gamma\sigma}^{\cdot\cdot\mu\nu} \eta_{\mu\nu\beta}. \tag{37}$$

Let us consider now a magnetic field directed along the $0z$ axis (which is also the direction of the shear eigenvector [42]). The symmetry of the problem then fixes F_{ik} and H^{ik} to be of the following structure:

$$F_{ik} = (\delta^1_i \delta^2_k - \delta^2_i \delta^1_k) F_{12}, \tag{38}$$

$$H^{ik} = (g^{i1} g^{k2} - g^{i2} g^{k1})[1 + q_1 R + q_2 (R^1_1 + R^2_2) + q_3 R^{12}_{\cdot\cdot 12}] F_{12}. \tag{39}$$

The second set of the Eqs. (6) yields $F_{12} = \mathrm{const}$. The first set of Eqs. (6) is identically satisfied since all the components H^{i0} are equal to zero. Introducing the scalar value of the magnetic field $B(t)$ by

$$B^2(t) = \frac{1}{2} F_{ik} F^{ik} = F_{12} F^{12}, \tag{40}$$

we reproduce the result [cf. [2])]

$$B^2(t) = \frac{\mathrm{const}^2}{a^2(t) b^2(t)}, \tag{41}$$

i.e., $B(t)a(t)b(t) = \mathrm{const}$.

C. Einstein's equations

In a comoving frame with $U^i = \delta^i_0$, the field Eqs. (24) reduce to the following system:

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = \Lambda + \kappa W + \frac{1}{2}\kappa B^2(t) + \kappa B^2(t) \left\{ q_1 \left[2\left(\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 \right) + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) \right] + q_2 \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)^2 + q_3 \frac{\dot{a}}{a}\frac{\dot{b}}{b} \right\}, \tag{42}$$

$$\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}=\Lambda-\kappa P_{(1)}-\frac{1}{2}\kappa B^2(t)+\kappa B^2(t)\Big\{q_1\Big[4\frac{\ddot{a}}{a}+3\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}-6\Big(\frac{\dot{a}}{a}\Big)^2-4\Big(\frac{\dot{b}}{b}\Big)^2-4\frac{\dot{a}}{a}\frac{\dot{b}}{b}+4\frac{\dot{a}}{a}\frac{\dot{c}}{c}+3\frac{\dot{b}}{b}\frac{\dot{c}}{c}\Big]$$
$$+q_2\Big[2\Big(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}\Big)-3\Big(\Big(\frac{\dot{a}}{a}\Big)^2+\Big(\frac{\dot{b}}{b}\Big)^2\Big)-2\frac{\dot{a}}{a}\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\Big)\Big]+q_3\Big[\frac{\ddot{b}}{b}-2\Big(\frac{\dot{b}}{b}\Big)^2+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\Big]\Big\}, \tag{43}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}=\Lambda-\kappa P_{(2)}-\frac{1}{2}\kappa B^2(t)+\kappa B^2(t)\Big\{q_1\Big[4\frac{\ddot{b}}{b}+3\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}-6\Big(\frac{\dot{b}}{b}\Big)^2-4\Big(\frac{\dot{a}}{a}\Big)^2-4\frac{\dot{a}}{a}\frac{\dot{b}}{b}+4\frac{\dot{b}}{b}\frac{\dot{c}}{c}+3\frac{\dot{a}}{a}\frac{\dot{c}}{c}\Big]$$
$$+q_2\Big[2\Big(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}\Big)-3\Big(\Big(\frac{\dot{a}}{a}\Big)^2+\Big(\frac{\dot{b}}{b}\Big)^2\Big)-2\frac{\dot{a}}{a}\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\Big)\Big]+q_3\Big[\frac{\ddot{a}}{a}-2\Big(\frac{\dot{a}}{a}\Big)^2+\frac{\dot{a}}{a}\frac{\dot{c}}{c}\Big]\Big\}, \tag{44}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}=\Lambda-\kappa P_{(3)}+\frac{1}{2}\kappa B^2(t)+\kappa B^2(t)\Big\{q_1\Big[\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}-4\Big(\Big(\frac{\dot{a}}{a}\Big)^2+\Big(\frac{\dot{b}}{b}\Big)^2\Big)-5\frac{\dot{a}}{a}\frac{\dot{b}}{b}\Big]-q_2\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\Big)^2-q_3\frac{\dot{a}}{a}\frac{\dot{b}}{b}\Big\}. \tag{45}$$

Here, $P_{(1)}$, $P_{(2)}$, and $P_{(3)}$ are the eigenvalues of the anisotropic pressure tensor P_{ik}. Summing up Eqs. (42)–(45) we obtain the trace equation:

$$2\Big(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\Big)=4\Lambda+\kappa(W-P_{(1)}-P_{(2)}-P_{(3)})+\kappa B^2(t)\Big\{(8q_1+4q_2+q_3)\Big(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}\Big)$$
$$+2q_1\frac{\ddot{c}}{c}-2(6q_1+3q_2+q_3)\Big(\Big(\frac{\dot{a}}{a}\Big)^2+\Big(\frac{\dot{b}}{b}\Big)^2\Big)$$
$$-2(5q_1+2q_2)\frac{\dot{a}}{a}\frac{\dot{b}}{b}+(8q_1+4q_2+q_3)\frac{\dot{c}}{c}\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\Big)\Big\}. \tag{46}$$

Differentiating (42) and using (43)–(45) leads to the conservation law for the matter:

$$\dot{W}+\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\Big)W+\frac{\dot{a}}{a}P_{(1)}+\frac{\dot{b}}{b}P_{(2)}+\frac{\dot{c}}{c}P_{(3)}=0. \tag{47}$$

It follows that $T^{(0)}_{ik}+q_1T^{(1)}_{ik}+q_2T^{(2)}_{ik}+q_3T^{(3)}_{ik}$ is conserved as well. The Eqs. (42)–(45) represent a modified (compared to minimal coupling) dynamical system, since the nonminimal terms contribute to the coefficients before the second order derivatives. Its complete analysis should be the subject of future investigations along the lines described in [43].

VI. PARTICULAR MODELS

A. Quasi-one-dimensional solutions with pure magnetic field

As the first application we consider a model without matter ($W=0$, $P_{ik}=0$) and with constant values of two of the metric functions: $a(t)=a_0$ and $b(t)=b_0$. Only $c(t)$ is assumed to vary. Then $B^2(t)=\text{const}$ [see (41)] and the Eqs. (42)–(45) reduce to

$$0=\Lambda+\frac{1}{2}\kappa B^2(t_0),$$
$$\frac{\ddot{c}}{c}[1-q_1\kappa B^2(t_0)]=\Lambda-\frac{1}{2}\kappa B^2(t_0). \tag{48}$$

Only one of the nonminimal parameters enters the dynamics. According to the first equation this model requires a negative cosmological constant which exactly compensates the magnetic field term, i.e., $\kappa B^2=-2\Lambda$. The equation for $c(t)$ is

$$\ddot{c}(t)+c(t)\Big[\frac{\kappa B^2(t_0)}{1-q_1\kappa B^2(t_0)}\Big]=0. \tag{49}$$

For $B^2=\Lambda=0$ we obtain the vacuum solution $c\propto t$ which is the degenerate case of a Kasner solution, equivalent to the flat space-time Milne universe (cf. [2]). For nonvanishing $\kappa B^2=-2\Lambda$ we have three cases.

First case: $q_1\kappa B^2(t_0)<1$.

This condition includes the case $q_1=0$. The solution of Eq. (49) is oscillatory:

$$c(t)=c(t_0)\cos\nu(t-t_0)+\frac{\dot{c}(t_0)}{\nu}\sin\nu(t-t_0), \tag{50}$$

where

$$\nu^2\equiv\Big[\frac{\kappa B^2(t_0)}{1-q_1\kappa B^2(t_0)}\Big]. \tag{51}$$

The zeros of $c(t)$ denote singularities of the model.

Second case: $q_1\kappa B^2(t_0)=1$.

The Eqs. (48) contradict each other. This case is incompatible with $B(t_0)\neq 0$.

Third case: $q_1\kappa B^2(t_0)>1$.

This condition requires q_1 to be positive. The solution of (49) is

$$c(t) = c(t_0)\cosh\mu(t - t_0) + \frac{\dot{c}(t_0)}{\mu}\sinh\mu(t - t_0), \quad (52)$$

where

$$\mu^2 \equiv \left[\frac{\kappa B^2(t_0)}{q_1\kappa B^2(t_0) - 1}\right]. \quad (53)$$

For $t \gg t_0$ the function $c(t)$ behaves as $c(t) \propto e^{\mu t}$. The model is nonsingular when $|\frac{c(t_0)\mu}{\dot{c}(t_0)}| \geq 1$. If $|\frac{c(t_0)\mu}{\dot{c}(t_0)}| < 1$, then there exists a singularity at t^*, given by $\tanh\mu(t^* - t_0) = -\frac{c(t_0)\mu}{\dot{c}(t_0)}$ and $c(t^*) = 0$.

B. Quasi-two-dimensional solutions with magnetic field and matter

A second model with $B(t) =$ const is obtained for $a(t)b(t) =$ const. We may write $a(t) = a(t_0)E(t)$ and $b(t) = b(t_0)E^{-1}(t)$ with $E(t_0) = 1$ where t_0 is some reference time. If we additionally assume $c(t) =$ const, the dynamics is restricted to the x^1Ox^2 plane. The Eqs. (42)–(45) reduce to the system:

$$-L\left(\frac{\dot{E}}{E}\right)^2 = \Lambda + \kappa W + \frac{1}{2}\kappa B^2,$$
$$-L\frac{\ddot{E}}{E} + 2\left(\frac{\dot{E}}{E}\right)^2 = \Lambda - \kappa P_{(1)} - \frac{1}{2}\kappa B^2, \quad (54)$$

$$L\left(\frac{\dot{E}}{E}\right)^2 = \Lambda - \kappa P_{(3)} + \frac{1}{2}\kappa B^2,$$
$$L\frac{\ddot{E}}{E} + 2(1 - L)\left(\frac{\dot{E}}{E}\right)^2 = \Lambda - \kappa P_{(2)} - \frac{1}{2}\kappa B^2, \quad (55)$$

where

$$L \equiv 1 + (q_1 - q_3)\kappa B^2 = \text{const}. \quad (56)$$

For $L \neq 0$ the system (54) and (55) is equivalent to

$$2L\left(\frac{\dot{E}}{E}\right)^2 = -\kappa(W + P_{(3)}), \qquad 2L\left(\frac{\dot{E}}{E}\right)^{\cdot} = \kappa(P_{(1)} - P_{(2)}), \quad (57)$$

$$P_{(3)} = W + B^2 + \frac{2\Lambda}{\kappa}, \qquad P_{(1)} + P_{(2)} = 2W + \frac{4\Lambda}{\kappa}. \quad (58)$$

The minimally coupled case $q_1 = q_2 = q_3 = 0$ corresponds to $L = 1$. It is dynamically indistinguishable from a nonminimal configuration with $q_1 = q_3$ and arbitrary q_2. The case $L = 1$ requires $W + P_{(3)} < 0$. This can only be achieved by a sufficiently negative cosmological constant, $2W + B^2 + \frac{2\Lambda}{\kappa} < 0$ which also implies that $P_{(1)} + P_{(2)} < 0$.

1. Case $L = 0$

For $q_3 - q_1 = \frac{1}{\kappa B^2}$ the quantity $E(t)$ is constant, i.e., the universe is static. The matter distribution is characterized by constant quantities as well:

$$P_{(1)} = P_{(2)} = \frac{\Lambda}{\kappa} - \frac{1}{2}B^2, \qquad W = -P_{(3)} = -\frac{\Lambda}{\kappa} - \frac{1}{2}B^2. \quad (59)$$

The energy density W is positive if $\Lambda < -\frac{1}{2}\kappa B^2$. All the pressure eigenvalues become negative in this case.

2. Case $L \neq 0$, $P_{(1)} = P_{(2)}$

Here we obtain

$$E(t) = e^{H_0(t-t_0)}, \qquad W = -LH_0^2 - \frac{\Lambda}{\kappa} - \frac{1}{2}B^2, \quad (60)$$

$$P_{(3)} = -LH_0^2 + \frac{\Lambda}{\kappa} + \frac{1}{2}B^2,$$
$$P_{(1)} = P_{(2)} = -LH_0^2 + \frac{\Lambda}{\kappa} - \frac{1}{2}B^2, \quad (61)$$

where H_0 is an arbitrary integration constant. For any $L > 0$ a sufficiently negative cosmological constant is required for the energy density W to be positive. After reparametrization of the coordinates and the time the metric takes a form

$$ds^2 = dt^2 - (e^{H_0 t}dx^2 + e^{-H_0 t}dy^2) - dz^2. \quad (62)$$

3. Ultrarelativistic matter with $L \neq 0$, $P_{(1)} = P_{(2)}$

If the matter is ultrarelativistic, i.e., $T^k_{k(\text{matter})} = 0$ and, consequently, $W = P_{(1)} + P_{(2)} + P_{(3)}$, one obtains

$$H_0^2 = \frac{2\Lambda}{\kappa L}, \qquad W = -\frac{3}{2}LH_0^2 - \frac{1}{2}B^2, \quad (63)$$

$$P_{(3)} = -\frac{1}{2}LH_0^2 + \frac{1}{2}B^2,$$
$$P_{(1)} = P_{(2)} = -\frac{1}{2}LH_0^2 - \frac{1}{2}B^2. \quad (64)$$

The energy density W is positive for $LH_0^2 < -\frac{1}{3}B^2$, which requires the constant L to be negative, i.e., $q_3 - q_1 > \frac{1}{\kappa B^2}$. In this case the longitudinal pressure $P_{(3)}$ is positive as well. The transversal pressure $P_{(1)} = P_{(2)}$ is positive for $LH_0^2 < -B^2$. The condition $L < 0$ necessarily implies a negative cosmological constant again.

C. Axial Symmetry: $a(t) = b(t)$, $P_{(1)} = P_{(2)} \equiv P_{(\text{tr})}$

All the metric functions are assumed to be time dependent now. Einstein's equations reduce to the following system of three equations for two unknown functions:

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} = \Lambda + \kappa W + \kappa B^2(t)\left\{\frac{1}{2} + (7q_1 + 4q_2 + q_3) \times \left(\frac{\dot{a}}{a}\right)^2 + 2q_1\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right\}, \quad (65)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} = \Lambda - \kappa P_{(\mathrm{tr})} + \kappa B^2(t)\left\{-\frac{1}{2} + q_1\frac{\ddot{c}}{c} + (7q_1 + 4q_2 + q_3)\left[\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{\dot{c}}{c}\right]\right\}, \quad (66)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \kappa P_{(3)} + \kappa B^2(t)\left\{\frac{1}{2} + 2q_1\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \times (13q_1 + 4q_2 + q_3)\right\}. \quad (67)$$

The matter conservation law takes the form

$$\dot{W} + 2\frac{\dot{a}}{a}(W + P_{(\mathrm{tr})}) + \frac{\dot{c}}{c}(W + P_{(3)}) = 0. \quad (68)$$

Formally, the function $c(t)$ does not appear in Eq. (67). However, in general there is a coupling to $c(t)$ via the pressure $P_{(3)}$ through the conservation law (68). There are two simple cases for which Eq. (67) decouples from Eqs. (65) and (66). The first one is $P_{(3)} = 0$, i.e., a vanishing longitudinal pressure, the second one is $W + P_{(3)} = 0$, i.e., a vacuum type behavior in longitudinal direction together with a transversal equation of state $P_{(\mathrm{tr})} = P_{(\mathrm{tr})}(W)$. In the following we consider both cases separately.

1. *"Longitudinal dust":* $P_{(3)} = 0$

With the substitution

$$\dot{a}(t) = a^\sigma\sqrt{Z(a)}, \qquad \sigma = \frac{13q_1 + 4q_2 + q_3}{2q_1}, \qquad B^2(t) = \frac{M^2}{a^4}, \quad (69)$$

where $M^2 = \mathrm{const}$ and $q_1 \neq 0$, Eq. (67) is transformed into the first order equation

$$\left(1 - \frac{\kappa q_1 M^2}{a^4}\right)\frac{dZ(a)}{da} + (2\sigma + 1)\frac{Z}{a} = \left(\Lambda + \frac{\kappa M^2}{2a^4}\right)a^{1-2\sigma}. \quad (70)$$

The solution of Eq. (70) can be represented as quadrature:

$$Z(a) = H_a^2(a_0)a_0^{2-2\sigma}\left(\frac{a_0^4 - \kappa q_1 M^2}{a^4 - \kappa q_1 M^2}\right)^{(2\sigma+1)/4} + (a^4 - \kappa q_1 M^2)^{-(2\sigma+1)/4} \times \int_{a_0}^{a} dx x^{1-2\sigma}(x^4 - \kappa q_1 M^2)^{(2\sigma-3)/4}\left(\Lambda x^4 + \frac{1}{2}\kappa M^2\right), \quad (71)$$

where $H_a(a_0) \equiv a^{\sigma-1}\sqrt{Z(a_0)}$ with $a_0 \equiv a(t_0)$ and $H_a(t) \equiv \frac{\dot{a}}{a} = H_b(t) \equiv \frac{\dot{b}}{b}$ is the expansion rate in the x^1 and x^2 directions. Because of the expressions $(x^4 - \kappa q_1 M^2)^{(2\sigma-3)/4}$ and $(a^4 - \kappa q_1 M^2)^{(2\sigma-3)/4}$ in Eq. (71), the result of the integration is sensitive to the sign of q_1. For $q_1 > 0$ the term $(x^4 - \kappa q_1 M^2)$ has two real zeros $x_{1,2} = \pm(\kappa q_1 M^2)^{1/4}$. If at least one zero belongs to the interval $(a_0, a(t))$, the integral in Eq. (71) diverges for $\frac{2\sigma-3}{4} \leq -1$, i.e., for $2\sigma + 1 \leq 0$. If $q_1 < 0$, such a singularity does not appear. The asymptotic behavior of the function $Z(a)$ for $a \to \infty$ is

$$Z(a \to \infty) = \frac{\Lambda}{3}a^{2-2\sigma}. \quad (72)$$

Thus, asymptotically all the models yield

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} = H_a = \mathrm{const}, \qquad a(t) = a(t_0)e^{H_a(t-t_0)}, \quad (73)$$

i.e., a de Sitter type expansion for $a(t)$, independent of the parameter σ. Moreover, one can check directly that the solution (73) is an exact solution of the Eq. (67) with

$$H_a^{-2} = 2(11q_1 + 4q_2 + q_3), \quad (74)$$

where $11q_1 + 4q_2 + q_3 \neq 0$. It is remarkable, that the constant expansion rate H_a is determined both by Λ via Eq. (73) and by the nonminimal coupling parameters via Eq. (74). The relation (74) has no counterpart in the minimal theory. Combining (73) and (74) allows us to establish the following relation between Λ and the coupling parameters:

$$\Lambda = \frac{3}{2(11q_1 + 4q_2 + q_3)}. \quad (75)$$

The cosmological constant is expressed in terms of quantities which are supposed to be the result of quantum field theoretical calculations. In quantum electrodynamics the parameters q_1, q_2, and q_3 are [4] $q_1 = -\frac{\alpha\lambda_e^2}{180\pi}$, $q_2 = -13q_1$, $q_3 = 2q_1$, where α is the fine structure constant, i.e., they are proportional to the square of the Compton wavelength λ_e of the electron. This would give rise to a value $\Lambda_{\mathrm{QED}} = \frac{90}{13}\frac{\pi}{\alpha\lambda_e^2}$ of the cosmological constant. While one does not expect a quantum-electrodynamical length to set the scale for an early de Sitter stage (recall that we assume q_1, q_2, and q_3 to be free parameters) this result may nevertheless indicate a potential relevance of nonminimal interactions for an early inflationary dynamics.

With a transversal equation of state $P_{(\mathrm{tr})}(t) = (\gamma - 1)W(t)$ and with (73) the conservation law (68) yields

$$W(t) = W(t_0)\frac{c(t_0)}{c(t)}e^{-2H_a\gamma(t-t_0)}. \qquad (76)$$

The Eq. (65) for $c(t)$ can be rewritten in the form

$$\frac{d}{dz}\{[1-\alpha z]^{-\xi}Y(z)\} = -[1-\alpha z]^{-(\xi+1)}\frac{\kappa W(t_0)}{8H_a^2}z^{(2\gamma-3)/4}, \qquad (77)$$

where

$$z \equiv \left(\frac{a_0}{a(t)}\right)^4, \qquad \alpha \equiv \frac{q_1\kappa M^2}{a_0^4},$$
$$\xi \equiv \frac{1-2q_1H_a^2}{8q_1H_a^2}, \qquad Y \equiv \frac{c(t)a(t_0)}{a(t)c(t_0)}. \qquad (78)$$

For $\alpha \neq 1$ the solution for $c(t)$ with the initial value $c(t_0)$ has the following explicit form

$$c(t) = c(t_0)e^{H_a(t-t_0)}[1-\alpha e^{-4H_a(t-t_0)}]^{\xi}\cdot\left\{(1-\alpha)^{-\xi} - \frac{\kappa W(t_0)}{8H_a^2}\int_1^{e^{-4H_a(t-t_0)}}dx x^{(2\gamma-3)/4}[1-\alpha x]^{-(\xi+1)}\right\}. \qquad (79)$$

At $\alpha x = 1$ the integral in Eq. (79) is nonsingular for $\xi < 0$. In the asymptotic regime $t \to \infty$ one obtains from (79)

$$c(t) \approx c(t_0)e^{H_a(t-t_0)}\Gamma, \qquad (80)$$

where the constant value Γ is equal to

$$\Gamma \equiv (1-\alpha)^{-\xi} - \frac{\kappa W(t_0)}{8H_a^2}\int_1^0 dx x^{(2\gamma-3)/4}[1-\alpha x]^{-(\xi+1)}. \qquad (81)$$

For $x \to 0$ the constant Γ remains finite for $\frac{2\gamma-3}{4} > -1$, i.e., $2\gamma + 1 > 0$.

The expression (80) shows that the expansion rate in $0x^3$ direction tends asymptotically to the expansion rate in the orthogonal direction, i.e., the universe becomes isotropic. The isotropization rate is characterized by the function $K(t)$,

$$K(t) \equiv \log Y, \qquad \dot{K}(t) = \frac{\dot{c}(t)}{c(t)} - \frac{\dot{a}(t)}{a(t)},$$
$$\dot{K}(t \to \infty) \to 0. \qquad (82)$$

Since $11q_1 + 4q_2 + q_3 \neq 0$ was assumed in Eq. (74), the minimal limit $q_1 = q_2 = q_3 = 0$ cannot be taken here, since it implies $H_a \to \infty$. To check whether or not a corresponding isotropization takes place in the minimally coupled theory as well one has to solve the system (65)–(67) with $q_1 = q_2 = q_3 = 0$. It is straightforward to realize that a solution $3H_a^2 = \Lambda$ of the minimally coupled system requires $P_{(3)} = \frac{1}{2}B^2$. Consequently, a dust equation of state $P_{(3)} = 0$ means the absence of the magnetic field. While the nonminimal theory in our example admits an isotropization in the presence of a magnetic field, the minimally coupled theory does not.

For $\alpha = 1$ the initial value problem for $c(t)$ with the initial value $c(t_0)$ degenerates. This requires a special treatment and we do not consider this case here.

2. *"Longitudinal quasivacuum"*: $W + P_{(3)} = 0$

When $P_{(tr)}(t) = (\gamma - 1)W(t)$ and $W + P_{(3)} = 0$, the conservation law yields

$$W(t) = W(t_0)\left(\frac{a(t_0)}{a(t)}\right)^{2\gamma}, \qquad (83)$$

and $a(t)$ can be found in quadratures from the Eq. (67). The solution $a(t) = a(t_0)e^{H_a(t-t_0)}$ holds in this case as well if the magnetic field term with $B^2(t) \propto a^{-4}(t)$ is compensated by the longitudinal pressure $P_{(3)}(t) = -W(t) \propto a^{-2\gamma}(t)$. This requires a transversal stiff matter equation of state, i.e., $\gamma = 2$. As in the previous subsection [see the discussion following Eq. (75)] we obtain two expressions for H_a,

$$H_a = \sqrt{\frac{\Lambda}{3}} \quad \text{and} \quad H_a^2 = \frac{2W(t_0)a_0^4 + M^2}{2M^2(11q_1 + 4q_2 + q_3)}, \qquad (84)$$

which again imply a relation between Λ and the parameters of the nonminimal interaction. From Eq. (65) we find

$$c(t) = c(t_0)e^{H_a(t-t_0)}[1-\alpha e^{-4H_a(t-t_0)}]^{\zeta}[1-\alpha]^{-\zeta}, \qquad (85)$$

where

$$\zeta \equiv \frac{2W(t_0)a_0^4 + M^2(1-2q_1H_a^2)}{8q_1M^2H_a^2}. \qquad (86)$$

Again, the solution for $c(t)$ has the same asymptotical behavior as $a(t)$, i.e., the universe becomes isotropic. This solution is nonsingular if q_1 is negative. $c(t)$ increases monotonically for $4\zeta < 1$. If $4\zeta > 1$ the function $c(t)$ increases after it passed a minimum value. If q_1 is positive, there exists a time t^* with $c(t^*) = 0$, where t^* is given by

$$t^* = t_0 + \frac{1}{4H_a}\log\alpha, \qquad \alpha = \frac{\kappa q_1 M^2}{a_0^4}. \qquad (87)$$

The model is then applicable for $t > t^*$.

Also in this case the isotropization process is a property of the nonminimal theory only. The corresponding solution $H_a = \text{const}$ of the minimal theory is necessarily isotropic for $W + P_{(3)} = 0$.

D. Isotropic universe model with "hidden" magnetic field

Let us consider now the conditions, under which an isotropic model with $a(t) = b(t) = c(t)$ is compatible with the existence of a magnetic field. The Einstein equations reduce to

$$3\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \kappa W + \kappa B^2(t)\left\{\frac{1}{2} + (9q_1 + 4q_2 + q_3)\left(\frac{\dot{a}}{a}\right)^2\right\}, \tag{88}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \kappa P_{(\mathrm{tr})} + \kappa B^2(t)\left\{-\frac{1}{2} + (8q_1 + 4q_2 + q_3)\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2(7q_1 + 4q_2 + q_3)\right\} \tag{89}$$

and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \kappa P_{(3)} + \kappa B^2(t)\left\{\frac{1}{2} + 2q_1\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2(13q_1 + 4q_2 + q_3)\right\}. \tag{90}$$

Furthermore, the trace equation becomes

$$6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 4\Lambda + \kappa(W - 2P_{(\mathrm{tr})} - P_{(3)}) + 2\kappa B^2(t) \times (9q_1 + 4q_2 + q_3)\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right], \tag{91}$$

and the conservation law simplifies to

$$\dot{W} + \frac{\dot{a}}{a}(3W + 2P_{(\mathrm{tr})} + P_{(3)}) = 0. \tag{92}$$

Equations (89) and (90) are compatible when

$$\frac{\ddot{a}}{a}(6q_1 + 4q_2 + q_3) + 6q_1\left(\frac{\dot{a}}{a}\right)^2 = 1 - \frac{P_{(3)} - P_{(\mathrm{tr})}}{B^2(t)}. \tag{93}$$

In the minimally coupled case compatibility requires $P_{(3)} = P_{(\mathrm{tr})} + B^2$. This excludes an isotropic matter configuration $P_{(3)} = P_{(\mathrm{tr})}$ together with a nonvanishing magnetic field. For the present nonminimal coupling the situation is different. Here, the compatibility condition (93) takes the form

$$\dot{H}(t)Q_1 + H^2(t)Q_2 = 1, \qquad H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{94}$$

where

$$Q_1 = 6q_1 + 4q_2 + q_3, \qquad Q_2 = 12q_1 + 4q_2 + q_3. \tag{95}$$

(In the present isotropic case we have $H_a = \frac{\dot{a}}{a} = H_b = \frac{\dot{b}}{b} = H_c = \frac{\dot{c}}{c} = H$). The solutions of this compatibility condition can be classified as follows.

Case $Q_1 = 0$

The Hubble parameter is constant and satisfies the condition $6H^2q_1 = 1$.

Case $Q_1 \neq 0$

For positive Q_2 the function $H(t)$ satisfies the relation

$$\frac{1 - \sqrt{Q_2}H(t)}{1 + \sqrt{Q_2}H(t)} = \frac{1 - \sqrt{Q_2}H(t_0)}{1 + \sqrt{Q_2}H(t_0)}e^{-[2\sqrt{Q_2}(t-t_0)]/Q_1}. \tag{96}$$

For the subcase $H(t_0) = \frac{1}{\sqrt{Q_2}}$, there exists a special constant solution

$$H^2(t) = H^2(t_0) = \frac{1}{(12q_1 + 4q_2 + q_3)}. \tag{97}$$

For negative Q_2 the Hubble function is

$$H(t)\frac{1}{\sqrt{|Q_2|}}\tan\left\{\arctan\sqrt{|Q_2|}H(t_0) + \frac{\sqrt{|Q_2|}(t - t_0)}{Q_1}\right\}. \tag{98}$$

For $Q_2 = 0$ the Hubble parameter is linear in time

$$H(t) = H(t_0) - \frac{t - t_0}{6q_1}. \tag{99}$$

Furthermore, for nonminimal coupling the richer structure of the field equations admits a configuration for which a nonzero magnetic field does not appear in Einstein's equations. This happens if all the multipliers of the terms $\kappa B^2(t)$ in (88)–(90) vanish simultaneously. The corresponding conditions are

$$H(t) \equiv \frac{\dot{a}}{a} = \mathrm{const} \equiv H_0, \qquad q_1 = \frac{1}{2H_0^2}, \qquad 4q_2 + q_3 = -\frac{5}{H_0^2}. \tag{100}$$

It is interesting to realize that this solution coincides with (97). The third relation admits the particular case

$$q_2 = -\frac{2}{H_0^2}, \qquad q_3 = \frac{3}{H_0^2}, \tag{101}$$

for which $6q_1 + 3q_2 + q_3 = 0$ and the trace of the susceptibility tensor vanishes [see, Eq. (5)]. As a consequence, there exists a stationary cosmological solution with

$$P_{(3)} = P_{(\mathrm{tr})} \equiv P, \quad W(t) = \mathrm{const} \equiv W_0, \quad P = \mathrm{const} = -W_0, \quad 3H_0^2 = \kappa W_0 + \Lambda, \quad H(t) = H_0, \quad a(t) = a(t_0)e^{H_0(t-t_0)}. \tag{102}$$

The nonvanishing magnetic field is *hidden* as far as the space-time evolution is concerned. The additional coupling terms give rise to a *nonminimal screening* of the magnetic field.

Also the solution (102) with (100) and (101) is characterized by a Hubble expansion that is directly determined by the nonminimal coupling strength. While this solution does not constitute a real inflationary model since there is no exit from the de Sitter phase, we hope that it may provide the starting point for a more general approach in which the parameters q_1, q_2, and q_3 are no longer constants but dynamical degrees of freedom, e.g., a multiplet of scalar fields. In such a context more ''realistic'' solutions might well occur. The circumstance that the impact of the nonminimal coupling weakens in the long time limit becomes obvious if we try to solve, say Eq. (90), with a power law ansatz $a \propto t^{\nu}$. All the terms in the braces on the right hand side, except for the first one which is independent of the nonminimal coupling, decay as t^{-2}. Consequently, these terms play a role at early times but they become irrelevant in the long time limit.

VII. DISCUSSION

Inspired by a well motivated nonminimal coupling between gravity and electromagnetism we have explicitly demonstrated that the richer structure of the corresponding theory gives rise to novel features of the cosmological dynamics. We have obtained a number of simple exact solutions for Bianchi I models with magnetic field. For axially symmetric configurations we found inflationary type solutions with magnetic field which describe an isotropization process as a result of the nonminimal coupling, i.e., without a counterpart in the minimally coupled theory. Furthermore, some solutions of the nonminimal theory establish a direct relation between a cosmological constant and the coupling parameters of the nonminimal interaction. Finally, we have shown that there exists an isotropic de Sitter solution for which the magnetic field is screened by the nonminimal coupling.

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[1] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[2] Ya. B. Zel'dovich and I. D. Novikov, *The Structure and Evolution of the Universe* (University of Chicago, Chicago, 1983).
[3] J. D. Barrow, P. Ferreira, and J. Silk, Phys. Rev. Lett. **78**, 3610 (1997).
[4] I. T. Drummond and S. J. Hathrell, Phys. Rev. D **22**, 343 (1980).
[5] A. R. Prasanna, Phys. Lett. A **37**, 331 (1971).
[6] A. R. Prasanna, Lett. Nuovo Cimento Soc. Ital. Fis. **6**, 420 (1973).
[7] M. Novello and J. M. Salim, Phys. Rev. D **20**, 377 (1979).
[8] A. J. Accioly, A. N. Vaidya, and M. M. Som, Phys. Rev. D **28**, 1853 (1983).
[9] M. Novello and H. Heintzmann Gen. Relativ. Gravit. **16**, 527 (1984).
[10] J. G. Souza, M. L. Bedran, and B. Lesche, Revista Brasileira de Fisica **14**, 488 (1984).
[11] H. Goenner Found. Phys. **14**, 865 (1984).
[12] A. J. Accioly and N. L. P. Pereira da Silva Prog. Theor. Phys. **76**, 1179 (1986) Progress Letters.
[13] M. S. Turner and L. M. Widrow Phys. Rev. D **37**, 2743 (1988).
[14] F. Müller-Hoissen, Phys. Lett. B **201**, 325 (1988).
[15] M. Novello, L. A. R. Oliveira, and J. M. Salim Classical Quantum Gravity **7**, 51 (1990).
[16] A. Accioly, A. D. Azeredo, C. M. L. De Aragão, and H.Mukai Classical Quantum Gravity **14**, 1163 (1997).
[17] S. Mohanty and A. R. Prasanna Nucl. Phys. B **526**, 501 (1998).
[18] P. Teyssandier , gr-qc/0303081.
[19] R. Lafrance and R. C. Myers Phys. Rev. D **51**, 2584 (1995).
[20] A. R. Prasanna and S. Mohanty, Classical Quantum Gravity **20**, 3023 (2003).
[21] S. K. Solanki, O. Preuss, M. P. Haugan, A. Gandorfer, H. P. Povel, P. Steiner, K. Stucki, P. N. Bernasconi, and D. Soltau, Phys. Rev. D **69**, 062001 (2004).
[22] D. Colladay and V. A. Kostelecký Phys. Rev. D **58**, 116002 (1998).
[23] V. A. Kostelecký and M. Mewes Phys. Rev. Lett. **87**, 251304 (2001).
[24] V. A. Kostelecký and M. Mewes, Phys. Rev. D **66**, 056005 (2002).
[25] A. B. Balakin Classical Quantum Gravity **14**, 2881 (1997).
[26] A. B. Balakin, Ann. Phys. (Leipzig) **9**, SI-21 (2000).
[27] A. B. Balakin and J. P. S. Lemos Classical Quantum Gravity **18**, 941 (2001).
[28] A. B. Balakin , R. Kerner, and J. P. S. Lemos, Classical Quantum Gravity **18**, 2217 (2001).
[29] A. B. Balakin and J. P. S. Lemos Classical Quantum Gravity **19**, 4897 (2002).
[30] W. Zimdahl, D. J. Schwarz, A. B. Balakin, and D. Pavón, Phys. Rev. D **64**, 063501 (2001).
[31] A. B. Balakin, D. Pavón, D. J. Schwarz, and W. Zimdahl, New J. Phys. **5**, 85 (2003).
[32] G. F. Rubilar, Yu. N. Obukhov, and F. W. Hehl, Classical Quantum Gravity **20**, L185 (2003).
[33] Ya. Itin and F. W. Hehl Phys. Rev. D **68**, 127701 (2003).
[34] F. W. Hehl and Yu. N. Obukhov, Lect. Notes Phys. **562**, 479 (2001).
[35] J. J. van der Bij and E. Radu, Nucl. Phys. B **585**, 637 (2000).
[36] A. B. Balakin and J. P. S. Lemos Classical Quantum Gravity **22**, 1867 (2005).
[37] A. C. Eringen and G. A. Maugin, *Electrodynamics of Continua* (Springer-Verlag, New York, 1989).
[38] L. D. Landau, E. M. Lifchitz, and L. P. Pitaevskii *Electrodynamics of Continuous Media* (Butterworth Heinemann, Oxford, 1996).
[39] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser, Boston, 2003).

[40] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967) p. 84.
[41] J. P. S. Lemos and R. Kerner, Gravitation Cosmol. **6**, 49 (2000).
[42] C. G. Tsagas and R. Maartens, Classical Quantum Gravity **17**, 2215 (2000).
[43] *Dynamical systems in cosmology*, edited by J. Wainwright and G. F. R. Ellis (Cambridge University Press, Cambridge, England, 1997).