PHYSICAL REVIEW D **71,** 124012 (2005)

Issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity

J. M. Pons*

Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona, and Institut de Física d'Altes Energies, Avenida Diagonal 647, 08028 Barcelona, Catalonia, Spain

D. C. Salisbury†

Department of Physics, Austin College, Sherman, Texas 75090-4440, USA

(Received 29 March 2005; published 6 June 2005)

Diffeomorphism-induced symmetry transformations and time evolution are distinct operations in generally covariant theories formulated in phase space. Time is not frozen. Diffeomorphism invariants are consequently not necessarily constants of the motion. Time-dependent invariants arise through the choice of an intrinsic time, or equivalently through the imposition of time-dependent gauge fixation conditions. One example of such a time-dependent gauge fixing is the Komar-Bergmann use of Weyl curvature scalars in general relativity. An analogous gauge fixing is also imposed for the relativistic free particle and the resulting complete set time-dependent invariants for this exactly solvable model are displayed. In contrast with the free particle case, we show that gauge invariants that are simultaneously constants of motion cannot exist in general relativity. They vary with intrinsic time.

DOI: 10.1103/PhysRevD.71.124012 PACS numbers: 04.60.Ds, 04.20.Fy

I. INTRODUCTION

Generally covariant theories in phase space have in common that the Hamiltonian is a linear combination of first class constraints. This means that the Hamiltonian vanishes "on shell," i.e., when the equations of motion are satisfied [1]. Certain combinations of first class constraints generate gauge symmetries. And since rigid translation in time coordinate is a spacetime diffeomorphism which does engender corresponding gauge symmetries of dynamical variables in configuration-velocity space, some authors have concluded that the Hamiltonian is itself a symmetry generator. This interpretation has led to the claim that since time evolution is just a gauge symmetry transformation there is no real physical evolution of states in the classical canonical formulation of generally covariant theories [2]. So it would appear that the canonical phase space approach encounters a disturbing conceptual problem: if there is no physical time evolution (a) the theory seems to no longer coincide with the formulation in configuration-velocity space and (b) the very concept of time as an evolutionary parameter seems to lose any meaning. This assertion, that time evolution equals gauge symmetry, can be viewed from other perspectives. For instance, it is encountered again when one applies a gauge fixing (GF) and finds that the final evolution generator vanishes; one then speaks of the frozen time problem. Finally, a third view of the problem comes from the definition of observables since the claim that time evolution is gauge leads to the statement that the only possible observables are constants of motion. Of course, this unsettling state of affairs deserves careful scrutiny.

In this paper we will show that there is no conceptual problem whatsoever for the canonical formulation of generally covariant theories because the mathematical identification of the Hamiltonian as a gauge generator is erroneous. Briefly, the Hamiltonian evolves solutions from their initial data; the gauge generator, as a symmetry of the equations of motion, maps entire solution trajectories into new solution trajectories [3].

The distinction between time evolution and gauge symmetry can be made in configuration-velocity space. But it is perhaps most interesting in phase space since this is the arena in which one hopes to make the canonical transition to quantum theory. We will apply our remarks frequently to phase space; this perspective is made possible by recent work in which it was shown how the four-dimensional diffeomorphism-induced gauge symmetry is realized as a canonical transformation group on the full set of canonical variables, including the lapse and shift [4–7]. Consequently we can demonstrate in detail the diffeomorphism invariance of the phase space functions proposed originally by Komar and Bergmann.

The point of view we advocate in this paper is consistent with statements made by Marolf [8] and Rovelli [9] regarding the nature of diffeomorphism invariants. One of the merits of the present work is that we describe precisely in what sense observables are and are not time dependent. We provide explicit examples, and we stress the difference between arbitrary gauge fixing and the construction of observables which are indeed amenable to measurement.

We begin in Sec. II with a technical presentation of the classical "problem of time" and with two familiar ex-

*Email address: pons@ecm.ub.es
†Email address: dsalisbury@austincollege.edu

amples that exhibit it, the relativistic free particle and conventional general relativity. The reparametrization symmetry of the free particle is nontrivial, and therefore this toy model offers edifying illustrations of many ideas and techniques related to time evolution, gauge fixing, and reparametrization invariants. A resolution of the time puzzle is given in Sec. III where we address and dismiss, from a conceptual point of view, the supposed equivalence of time evolution and gauge transformation. In Sec. IV the problems associated with gauge fixing procedures are analyzed and resolved. In Sec. V we introduce Komar-Bergmann intrinsic coordinates which make use of Weyl curvature scalars. In Sec. VI we show that the Komar-Bergmann approach can be interpreted as a gauge fixing procedure that fulfills the requirements discussed in Sec. IV, and we show, in particular, that time dependence is necessarily retained through the compulsory use of at least one explicitly time-dependent gauge condition. Section VII is devoted to the issue of observables. We present with full detail the well known result that scalar functions of intrinsic coordinates which are themselves defined as scalar functions of dynamical variables are diffeomorphism invariants. The construction is carried out in complete detail for the free particle where we confirm that the most general class of invariants are *not* constants of the motion. (Henceforth, ''invariant'' and ''gauge invariant'' will mean the same.) Then a somewhat different perspective (with equivalent results) is given for general relativity. We present our conclusions in Sec. VIII, including possible implications of this work regarding an eventual quantum theory of gravity.

II. GENERALLY COVARIANT THEORIES IN PHASE SPACE

Here we review the formulation of generally covariant theories in phase space with its diffeomorphism-induced gauge group. We also consider the possibility that, besides diffeomorphism invariance, internal gauge symmetries may be present, thus including cases like Einstein-Yang-Mills, tetrad, and connection formulations of general relativity [4–7]. Our starting point is always a variational principle formulated with a Lagrangian density, which is a function in configuration-velocity space. Its corresponding phase space formulation is given by the Dirac-Bergmann theory of constrained dynamical systems.

The Dirac Hamiltonian takes the form

$$H_\lambda = n^A \mathcal{H}_A + \lambda^A P_A, \tag{1}$$

where λ^A are arbitrary functions of spacetime coordinates. The canonical variables n^A are fixed by these arbitrary functions under time evolution and are often called the gauge functions, although the redundancy of variables that is caused by the gauge symmetry is not exhausted by them. Their canonical momenta P_A are primary constraints. The physical phase space is further constrained by secondary constraints $\mathcal{H}_A$. These constraints do not depend on n^A or P_A. If there is no symmetry in configuration-velocity space beyond general covariance, the range of the index A is simply the dimensionality of the underlying coordinate manifold and $n^0 = n$ and n^a are the lapse and shift variables. If additional internal symmetries are present A will also range over the dimension of the group Lie algebra. This is the case, for example, with Yang-Mills gauge fields in general relativity and also for tetrad connection approaches to gravity.

The complete generator of infinitesimal gauge symmetries which are projectable onto phase space under the Legendre map takes the general form

$$G_\xi(t) = P_A \dot{\xi}^A + (\mathcal{H}_A + P_{C''} n^{B'} C^{C''}_{AB'})\xi^A, \tag{2}$$

where the structure functions are obtained from the closed Poisson bracket algebra

$$\{\mathcal{H}_A, \mathcal{H}_{B'}\} = C^{C''}_{AB'} \mathcal{H}_{C''}, \tag{3}$$

and where spatial integrations at time t over corresponding repeated capital indices are assumed hereafter. The generators $G_\xi(t)$ act on phase space through the equal-time Poisson brackets, and map solution trajectories into other solutions. In this sense, it is assumed that all phase space variables appearing in (2) are solution trajectories $y_\lambda(t)$. Poisson brackets at time t are evaluated with respect to the canonical set $y_\lambda(t)$. The ''descriptors'' ξ^A are arbitrary spacetime functions and $\dot{\xi}^A$ stands for the time derivative of ξ^A. When internal symmetries are present, the previously projectable diffeomorphisms which alter spacetime foliations are no longer projectable to phase space; they must be accompanied by internal gauge ''rotations'' fixed by the spacetime descriptors ξ^μ [5–7].

Notice that since dynamically $\dot{n}^A_\lambda(t)$ equals $\lambda^A(t)$, when the functions ξ^A take the values n^A, it appears that the gauge generator G_ξ coincides with the Dirac Hamiltonian (1). This is the technical setting of the problem; H_λ appears naively to be included within the family of G_ξ, leading to the (spurious) conclusion that the motion generated by H_λ is gauge.

Now we present two examples of generally covariant theories that exhibit the phenomenon just described.

A. The relativistic free particle

We employ the Lagrangian

$$L = \frac{1}{2n}\dot{q}^2 - \frac{1}{2}n,$$

where q^μ are the Cartesian spacetime coordinates for the trajectory of a unit mass particle in Minkowski space and the auxiliary variable n plays the role of a lapse function on the one-dimensional parameter space. The resulting Dirac Hamiltonian is

$$H_\lambda(t) = \frac{1}{2} n(p^2 + 1) + \lambda(t)\pi, \tag{4}$$

where p_μ and π are the variables conjugate to q^μ and n. λ is an arbitrary function that reflects the reparametrization gauge freedom of the model. Notice that the equations of motion yield $\dot{n} = \lambda$. n is therefore fixed, up to an integration constant, by the arbitrarily chosen function λ. The gauge generator $G_\xi(t)$ is constructed with the first class constraints $(p^2 + 1) \approx 0$ and $\pi \approx 0$:

$$G_\xi(t) = \frac{1}{2} \xi(p^2 + 1) + \dot{\xi}\pi, \tag{5}$$

and ξ is an arbitrary function. $G_\xi(t)$ generates variations of dynamical variables resulting from infinitesimal reparametrizations of the form $t' = t - n^{-1}(t)\xi(t)$. Note that since the dynamics fixes $\dot{n} = \lambda$, when ξ happens to be equal to n times an obvious infinitesimal factor, H_λ is a particular case of G_ξ. Hence the claim, that we will prove spurious, that dynamical evolution is a gauge transformation.

B. Conventional canonical general relativity

In canonical general relativity the Dirac Hamiltonian takes the form

$$H_\lambda = (n^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu), \tag{6}$$

where [we use in the following the standard index notation $\mu = (0, a)$] $n^0 := n$ is the lapse and n^a are the components of the shift 3 vector. P_μ are the variables conjugate to the lapse and shift and are primary constraints. $\mathcal{H}_\mu \approx 0$ are the so-called Hamiltonian and momentum secondary constraints. The gauge generator is, at a given time x^0,

$$G_\xi(x^0) = P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + n^\rho C^\nu_{\mu\rho} P_\nu)\xi^\mu. \tag{7}$$

(From now on, repeated index summation includes an integration over 3 space.) The ξ^μ are arbitrary descriptor functions of the spacetime coordinates and $C^\nu_{\mu\rho}$ are the structure functions for the Poisson bracket algebra of the Hamiltonian and momentum constraints. The functions ξ^μ are related to the functions ϵ^μ that define an infinitesimal diffeomorphism (in the passive view: $x^\mu \rightarrow x^\mu - \epsilon^\mu$) by the following construction [4,10]. Construct the vectors $\mathcal{N}^\mu$ orthonormal to the constant-time surfaces,

$$\mathcal{N}^\mu = \delta^\mu_0 n^{-1} - \delta^\mu_a n^{-1} n^a. \tag{8}$$

Then

$$\epsilon^\mu = \delta^\mu_a \xi^a + \mathcal{N}^\mu \xi^0. \tag{9}$$

Note that the Hamiltonian coincides with the gauge generator when the arbitrary functions ξ^μ are chosen to be $\xi^\mu = n^\mu$. Thus it appears at first sight that in this case the gauge generator and Hamiltonian are identical. In the next section we will present in depth arguments that there is no problem of time if the roles of gauge operator versus the Hamiltonian are properly understood.

III. THE RESOLUTION OF THE TIME EVOLUTION VERSUS GAUGE PROBLEM

A. The space of field configurations

The first answer we give to the question as to whether in generally covariant theories the dynamical evolution is just gauge follows from this consideration: gauge transformations, as a special case of symmetries, map solutions of the equations of motion into solutions. Therefore the natural arena for the action of gauge transformations is just the space of solution field configurations, i.e., the space of histories. In reparametrization covariant particle models, for example, these are just the particle world lines. An element in this space is a specific spacetime description—a history—of the fields and particles that are present in the physical setting. The action of the gauge group on this space defines orbits. An orbit is the set of all field configurations connected by diffeomorphisms. In the passive view of diffeomorphisms, an orbit is understood as the set of all field configurations that correspond to a unique physical situation but expressed in different coordinates. Infinitesimal variations of histories in the active point of view are simply Lie derivatives along the direction of the vector field ϵ^μ, associated with infinitesimal coordinate transformations of the passive view. In general relativity some of these coordinate choices may have physical content in the sense that each may correspond to a set of observers with a scheme for physically achieving time simultaneity and readjustment of proper time clocks. But the theory must also carry with it instructions on how to move from one coordinate fixing to another; this is the action of the gauge group. On the other hand, the role of the Hamiltonian could not be more distinct: it defines, through the Poisson brackets, the differential equations that enable us to build the whole configuration of the fields out of initial data at a given equal-time surface. It is obvious then that the Hamiltonian has no action on the space of field configurations for it simply defines how to build the elements of this space.

B. Finite evolution and gauge operators

The equations of motion fix the evolution of the gauge variables after the arbitrary functions λ^A have been selected. We may then write down a formal solution of the dynamical equations, given initial conditions at time $t = 0$, in terms of the finite evolution operator

$$U_\lambda(t) := T \exp\left(\int_0^t dt' \{-, H_\lambda(t')\}_{y_0}\right), \tag{10}$$

where T stands for the t-ordering operator that places the highest t on the right. All Poisson brackets in the expansion defined on the right hand side are evaluated in terms of

$y_0 := y(t=0)$. (We must only be careful to take account of the explicit time dependence of the functions λ^A.) Thus it is possible to express all dynamical variables, including the arbitrary gauge variables n^A in terms of initial values y_0.

The finite form of the gauge transformation looks quite different:

$$V_\xi(s,t) = \exp(s\{-, G_\xi(t)\}_{y_\lambda(t)}). \tag{11}$$

(The functions ξ, being arbitrary, may develop a dependence on the parameter s; in this case the finite operator for gauge transformations will contain an s-ordering operator as well [11].)

C. Application to the relativistic free particle

We demonstrate the action of (10) with the free particle using (4). We find

$$n(t) = T\exp\left(\int_0^t dt'\{-, H_\lambda(t')\}_y\right)n_0$$
$$= n_0 + \int_0^t dt_1\{n_0, H_\lambda(t_1)\} \tag{12}$$

$$= n_0 + \int_0^t dt_1\{n_0, \lambda(t_1)\pi\} = n_0 + \int_0^t dt_1 \lambda(t_1). \tag{13}$$

(In this expression and henceforth we will let the variable name with the zero subscript represent the initial value of the variable.) Notice that all the remaining nested Poisson brackets vanish since the first yields a numerical function. Similarly, $p_\mu(t) = p_\mu$, and

$$\begin{aligned} q^\mu(t) &= q_0^\mu + \int_0^t dt_1\left\{q_0^\mu, \frac{n_0}{2}p^2\right\} \\ &\quad + \int_0^t dt_2\int_0^{t_2} dt_1\left\{\left\{q_0^\mu, \frac{n_0}{2}p^2\right\}, \lambda(t_1)\pi_0\right\} \\ &= q^\mu + \int_0^t dt_1 n_0 p^\mu + \int_0^t dt_2\int_0^{t_2} dt_1 p^\mu\lambda(t_1) \\ &= q_0^\mu + p^\mu\int_0^t dt_1 n_0(t_1). \end{aligned} \tag{14}$$

Since we shall require this result below when we compute the action of the finite gauge generator, let us also calculate the evolution of the constraint $\pi(t)$,

$$\pi(t) = \pi_0 + t\left\{\pi_0, \frac{1}{2}N(p^2+1)\right\} = \pi_0 - \frac{1}{2}(p^2+1)t. \tag{15}$$

[Note that the additional constraint term is required to preserve the canonical Poisson Bracket $\{\pi(t), q^\mu(t)\} = 0$.]

Let us return to the generator of gauge transformations $G_\xi(t)$. For each time t this object generates an infinitesimal variation of solution trajectories to produce new solution trajectories.

In an effort to minimize misunderstandings concerning the action of the finite gauge generator (11) we will calculate its action in two equivalent ways, first calculating Poisson brackets with respect to the canonical variables $y(t)$, and then alternatively in terms of the initial variables y_0.

First, using the gauge generator (5), written as $G_\xi = \frac{1}{2}\xi(t)(p^2(t)+1) + \dot\xi(t)\pi(t))$, we find

$$\begin{aligned} q_s^\mu(t) &= q^\mu(t) + s\frac{\partial G_\xi(t)}{\partial p_\mu(t)} = q^\mu(t) + s\xi(t)p^\mu(t) \\ &= q^\mu(t) + s\xi(t)p^\mu. \end{aligned} \tag{16}$$

Of course, since the $y(t)$ are obtained from y through a canonical transformation, we can equivalently calculate Poisson brackets with respect to the initial variables y_0:

$$\begin{aligned} q_s^\mu(t) &= (\exp(s\{-, G_\xi(t)\}_{y_0}))q^\mu(t) \\ &= q^\mu(t) + s\left\{q_0^\mu + p^\mu n_0 t, \frac{1}{2}\xi(t)(p^2+1)\right. \\ &\quad \left.+ \dot\xi(t)\left(\pi_0 - \frac{1}{2}t(p^2+1)\right)\right\}_{y_0} \\ &= q^\mu(t) + s\xi(t)p^\mu, \end{aligned} \tag{17}$$

The corresponding expression for $n_s(t)$ is

$$n_s(t) = (\exp(s\{-, G_\xi(t)\}_{y_\lambda(t)})n(t) = n(t) + s\dot\xi(t). \tag{18}$$

IV. THE GAUGE FIXING RESOLUTION OF THE EVOLUTION VERSUS GAUGE PUZZLE

Another way to rephrase the claim "dynamical evolution equals gauge transformation" makes use of the gauge fixing methods. For, it is argued, suppose we consider a complete set of GF constraints, say $\chi^A = 0$, complete in the sense that they eliminate all of the gauge freedom:

$$\{\chi^A, G_\xi(t)\} = 0, \qquad \forall t \Rightarrow \xi^A = 0, \tag{19}$$

that is, the arbitrary functions ξ^A in $G_\xi(t)$ [see Eq. (2)] become zero and no gauge freedom is left. Equation (19) expresses the fact that, after implementing the GF constraints, the gauge evolution is frozen because any gauge motion will take our field configurations (or trajectories) out of the GF constraints surface. Then one makes the assertion: since H is a particular case of $G(t)$, the dynamics must therefore also be frozen, for the dynamics will take the field configurations out of the GF constraints surface. But what might seem an insurmountable problem is easily overcome when we recognize that the GF constraints (or at least one of them) may depend on the time variable. The dynamical evolution for an explicitly time-dependent constraint is

$$\frac{d\chi}{dt} = \frac{\partial\chi}{\partial t} + \{\chi, H\},$$

and to require this to vanish no longer freezes the dynamics. In fact what seemed to be a problem is a theorem: *in generally covariant theories, at least one of the GF constraints must exhibit an explicit dependence on time* [4,12]. This time-dependent constraint plays the role of defining the time in terms of the dynamical variables. This argument will be worked out in full detail for the free relativistic particle and for general relativity.

A. The Dirac bracket puzzle

The Dirac bracket argument for frozen dynamics is the previous GF argument in disguise. It proceeds as follows. The GF procedure makes the first class constraints of the theory second class through the addition of the appropriate GF constraints. Since the Dirac Hamiltonian H_λ in a generally covariant theory is made up of first class constraints, when the Dirac bracket $\{-,-\}^*$ is introduced, all constraints (which are now second class) can be taken to be zero inside the bracket, that is

$$\{-, H_\lambda\}^* = \{-, 0\}^* = 0.$$

Then it appears that no dynamical evolution remains, independently of whether the GF constraints have an explicit time dependence or not. As before the flaw in this argument can be traced back to a failure to appropriately take into account the presence of time-dependent gauge fixing constraints. Our starting point is a first class Dirac Hamiltonian

$$H_\lambda = H_c + \lambda^i \phi_i,$$

where H_c is the canonical Hamiltonian and the ϕ_i are first class constraints. We implement a set χ_i of GF constraint. The on-shell dynamics does not change if we substitute all constraints, the original and the gauge fixing constraints, into the Dirac Hamiltonian, each multiplied by a different arbitrary function that plays the role of a Lagrange multiplier. The stabilization of the constraints, i.e., their conservation in time, then determines all these Lagrange multipliers. Let us use the notation ψ_n for the complete set of now second class constraints. Using the extended Hamiltonian $H_{e\lambda} = H_c + \lambda^n \psi_n$ the result on shell is

$$\frac{d}{dt}\psi_n = \frac{\partial}{\partial t}\psi_n + \{\psi_n, H_c\} + \lambda^m \{\psi_n, \psi_m\} = 0, \tag{20}$$

that determines

$$\lambda^m = -\left(\frac{\partial}{\partial t}\psi_n + \{\psi_n, H_c\}\right) M^{mn}, \tag{21}$$

with M^{mn} being the inverse matrix of $\{\psi_m, \psi_n\}$. Substituting these values for λ^m into $H_{e\lambda}$ we obtain (always on the constraint surface)

$$\{-, H_{e\lambda}\} = \{-, H_c\}^* - \{-, \psi_m\} M^{mn} \frac{\partial \psi_n}{\partial t}, \tag{22}$$

where we have used the standard notation for the Dirac brackets,

$$\{-,-\}^* = \{-,-\} - \{-, \psi_m\} M^{mn} \{\psi_n, -\}.$$

Therefore, even when the first term on the right hand side in (22) vanishes (as is the case in all generally covariant theories), a nontrivial dynamical evolution still obtains as long as at least one GF constraint has an explicit time dependence, where the gauge-fixed Hamiltonian is

$$H_{GF} = -\psi_m M^{mn} \frac{\partial \psi_n}{\partial t}. \tag{23}$$

B. Dirac brackets for the free relativistic particle

We shall illustrate the nontrivial evolution which results from an explicitly time-dependent gauge fixing constraint with the free relativistic particle. Let us impose the gauge condition $\psi_3 := q^0(t) - f(t) \approx 0$, where f is an arbitrary monotonically increasing function. Letting $\psi_1 := \frac{1}{2}(p^2 + 1) \approx 0$ and $\psi_2 := \pi \approx 0$ represent the original first class constraints, preservation of the constraint ψ_3 under time evolution leads to a fourth constraint $\psi_4 := p^0 n(t) - \dot{f}(t) \approx 0$. The Poisson brackets among the constraints displayed as a matrix then takes the form

$$\{\psi_m, \psi_n\} = \begin{pmatrix} 0 & 0 & -p^0 & 0 \\ 0 & 0 & 0 & -p^0 \\ p^0 & 0 & 0 & -n(t) \\ 0 & p^0 & n(t) & 0 \end{pmatrix}, \tag{24}$$

with inverse

$$M^{mn} = \begin{pmatrix} 0 & -\frac{n(t)}{(p^0)^2} & \frac{1}{p^0} & 0 \\ \frac{n(t)}{(p^0)^2} & 0 & 0 & \frac{1}{p^0} \\ -\frac{1}{p^0} & 0 & 0 & 0 \\ 0 & -\frac{1}{p^0} & 0 & 0 \end{pmatrix}. \tag{25}$$

Since the nonvanishing explicit time derivatives of the constraints are $(\partial\psi_3/\partial t) = \dot{f}$ and $(\partial\psi_4/\partial t) = \ddot{f}$, the extended Dirac Hamiltonian, once we have set the canonical Hamiltonian to zero inside the Dirac bracket, see (23), becomes

$$H_{GF} = \psi_1 M^{13}\dot{f} + \psi_2 M^{24}\ddot{f} = \frac{1}{2p^0}(p^2 + 1)\dot{f} + \frac{1}{p^0}\pi\ddot{f}. \tag{26}$$

This yields the equations of motion

$$\dot{q}^\mu(t) = \frac{p^\mu}{p^0}\dot{f}(t), \tag{27}$$

and

$$\dot{n}(t) = \frac{\ddot{f}(t)}{p^0}. \tag{28}$$

V. INTRINSIC COORDINATES IN GENERALLY COVARIANT THEORIES

In this and the following section we introduce the gauge fixing method of Komar and Bergmann that has a direct application to the preceding discussion. The method employs Weyl scalars to fix intrinsic coordinates. We first present their definition, and show that they do not depend on the lapse and shift. Komar and Bergmann proposed their use in vacuum spacetimes. We shall show that they can equally well be used in spacetimes with other fields present, as we will see in the Einstein-Maxwell case. In the following subsection we show that regardless of the arbitrary coordinate system in which one may be working initially, transformation to the intrinsic coordinate system yields identical metric functions. The explanation of the use of Weyl scalars as a gauge fixing is given in Sec. VI.

A. Weyl curvature scalars

We begin with the general expression for the conformal tensor in terms of the Riemann tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\sigma}R_{\rho]\mu} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma[\nu}. \tag{29}$$

We will be concerned only with the conformal tensor evaluated on solutions of the equations of motion,

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{30}$$

where $T_{\mu\nu}$ is the stress-energy tensor and T is its trace.

In the vacuum case which we consider initially where $R_{\mu\nu} := R_{\rho\mu\nu\sigma}g^{\rho\sigma} = 0$ the conformal and Riemann tensors coincide on shell. Bergmann and Komar discovered that spatial components of the Riemann tensor, and also contractions with the normal $\mathcal{N}^\mu$ to the fixed time hypersurfaces could be expressed in terms of canonical variables [13,14]. The construction uses the projection tensor

$$e^{\mu\nu} := g^{\mu\nu} + \mathcal{N}^\mu\mathcal{N}^\nu, \tag{31}$$

and the Gauss-Codazzi relations,

$$R_{abcd} = {}^3R_{abcd} - K_{bc}K_{ad} + K_{bd}K_{ac}, \tag{32}$$

and

$$R_{\mu abc}\mathcal{N}^\mu = K_{ab|c} - K_{ac|b}, \tag{33}$$

with the observation that the canonical momentum written in terms of the extrinsic curvature K_{ab} is

$$\pi_{ab} = \sqrt{g}(K_{ab} - K^c_c g_{ab}). \tag{34}$$

Thus we may invert to find the extrinsic curvature in terms of the momentum

$$K_{ab} = \frac{1}{\sqrt{g}}\left(\pi_{ab} - \frac{1}{2}\pi^c_c g_{ab}\right). \tag{35}$$

In all of these expressions indices are raised with e^{ab} and "$|$" signifies covariant derivative with respect to the spatial metric.

Referring to (32) and (35) we see that on shell the spatial components of the conformal tensor may be written in terms of canonical variables as promised,

$$C_{abcd} = {}^3R_{abcd} - K_{bc}K_{ad} + K_{bd}K_{ac}. \tag{36}$$

From (33) we have on shell

$$C_{abc} := C_{\mu abc}\mathcal{N}^\mu = K_{ab|c} - K_{ac|b}. \tag{37}$$

Finally we find on shell, using (31), that

$$C_{ab} := C_{\mu ab\nu}\mathcal{N}^\mu\mathcal{N}^\nu = C_{cabd}e^{cd}. \tag{38}$$

All three expressions can therefore independent of the canonical variables $\mathcal{N}$ and $\mathcal{N}^a$.

We are finally prepared to construct the Weyl scalars which are most conveniently written for our purposes as [14]

$$W^1 = C_{\alpha\alpha'\beta\beta'}g^{\beta\beta'\gamma\gamma'}C_{\gamma\gamma'\delta\delta'}g^{\delta\delta'\alpha\alpha'}, \tag{39}$$

$$W^2 = C_{\alpha\alpha'\beta\beta'}g^{\beta\beta'\gamma\gamma'}C_{\gamma\gamma'\delta\delta'}\epsilon^{\delta\delta'\alpha\alpha'}, \tag{40}$$

$$W^3 = C_{\alpha\alpha'\beta\beta'}g^{\beta\beta'\gamma\gamma'}C_{\gamma\gamma'\delta\delta'}g^{\delta\delta'\rho\rho'}C_{\rho\rho'\sigma\sigma'}g^{\sigma\sigma'\alpha\alpha'}, \tag{41}$$

$$W^4 = C_{\alpha\alpha'\beta\beta'}g^{\beta\beta'\gamma\gamma'}C_{\gamma\gamma'\delta\delta'}g^{\delta\delta'\rho\rho'}C_{\rho\rho'\sigma\sigma'}\epsilon^{\sigma\sigma'\alpha\alpha'}, \tag{42}$$

where

$$\begin{aligned} g^{\beta\beta'\gamma\gamma'} &:= 2g^{\beta[\gamma}g^{\gamma']\beta'} \\ &= 2e^{\beta[\gamma}e^{\gamma']\beta'} - 2\mathcal{N}^\beta\mathcal{N}^{[\gamma}e^{\gamma']\beta'} \\ &\quad - 2e^{\beta[\gamma}\mathcal{N}^{\gamma']}\mathcal{N}^{\beta'}. \end{aligned} \tag{43}$$

Substitution of (43) into (39) yields, for example [13],

$$W^1 = C_{abcd}C^{abcd} + 4C_{abc}C^{abc} + 4C_{ab}C^{ab}. \tag{44}$$

One might suspect from (29) that this construction could be generalized to include other fields. We shall now show that after substitution of the Einstein equations into (29) the right hand side is indeed independent of the lapse and shift, and depends only on the remaining material and metric phase space variables. We carry out the construction explicitly for Einstein-Maxwell theory.

The stress-energy tensor is up to a constant

$$T_{\mu\nu} = F_{\alpha\mu}F_{\beta\nu}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

where $F_{\alpha\beta}$ is the Maxwell tensor. We need the spatial components, and we want to write them in terms of the canonical momentum

$$P^a = \sqrt{|{}^4g|}F_{\mu\nu}g^{a\mu}g^{0\nu}.$$

Substituting for the metric it turns out that

$$P^a = \sqrt{|{}^4g|}n^{-2}(F_{0b}e^{ab} + F_{bc}e^{ab}n^c),$$

so

$$F_{0a} = \frac{1}{\sqrt{|{}^4g|}}n^2 P_a - F_{ab}n^b,$$

where the index is lowered with the three metric. Let us first substitute into the $F_{\alpha\beta}F^{\alpha\beta}$ term:

$$F_{\alpha\beta}F^{\alpha\beta} = 2F_{0a}F_{0b}g^{00}g^{ab} + 2F_{0a}F_{b0}g^{0b}g^{a0} + 4F_{0a}F_{bc}g^{0b}g^{ac} + F_{ab}F_{cd}g^{ac}g^{bd}.$$

Some remarkable cancellations ultimately yield

$$F_{\alpha\beta}F^{\alpha\beta} = -2\frac{1}{|{}^4g|}n^2P^aP^bg_{ab} + F_{ab}F_{cd}e^{ac}e^{bd}.$$

A similar calculation yields

$$F_{\alpha a}F_{\beta b}g^{\alpha\beta} = -\frac{1}{|{}^4g|}n^2P_aP_b + F_{ca}F_{db}e^{cd}.$$

Noting that $|{}^4g| = ng$, where g represents the determinant of the three metric, we find

$$T_{ab} = -\frac{1}{g}P_aP_b + \frac{1}{2g}g_{ab}P^cP^dg_{cd} - \frac{1}{4}F_{ab}F_{cd}e^{ac}e^{bd} + F_{ca}F_{db}e^{cd}.$$

Now we can calculate the required components and projections of the conformal tensor. Start with (29), noting that for Einstein-Maxwell theory the stress-energy is traceless implying that the curvature scalar R vanishes,

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a}.$$

So we merely have to replace the Ricci tensor R_{ab} by $8\pi T_{ab}$, and we have our first result, the independence of C_{abcd} of the lapse and shift as promised.

The independence of $C_{abc} := C_{\mu abc}\mathcal{N}^\mu$ and $C_{ab} := C_{\mu ab\nu}\mathcal{N}^\mu\mathcal{N}^\nu$ of the lapse and shift is simpler to see. Consider first the contraction $(-g_{\mu[a}R_{b]\nu} + g_{\nu[a}R_{b]\mu})\mathcal{N}^\mu$. The first terms gives the covariant spatial component of the normal, which is zero. The second term is a projection of the Einstein equations (since $R = 0$), and is therefore a constraint. So both contributions vanish on shell. A similar argument holds for the second projection. So we have the final result: The Weyl scalars are independent of the lapse and shift canonical variables.

B. Komar-Bergmann intrinsic coordinates in general relativity

We now show how the Weyl scalars can be used to construct quantities which are invariant under diffeomorphisms. Bergmann and Komar were pioneers in this approach [13,15–17].

We will consider only the generic asymmetric case when the four Weyl scalars $W^I; I = 0, \cdots 3$ are independent.

If the metric g is locally described in a given system of coordinates $\{x^\mu\}$ as $g^{\mu\nu}(x)$ then four independent functions $A^I(W)$ of the four scalars become four scalar functions of the coordinates

$$a^I_g(x) := A^I(W[g(x)]). \tag{45}$$

Independence of the four Weyl scalars and the functions A^I implies

$$\det\left(\frac{\partial a^I_g(x)}{\partial x^\mu}\right) \neq 0. \tag{46}$$

Consider a metric g', infinitesimally close to g, and related to it by an active diffeomorphism generated by the infinitesimal vector field $\epsilon^\mu\partial_\mu$, with $\epsilon^\mu(x)$ arbitrary functions (so g and g' belong to the same gauge orbit). If $g'^{\mu\nu}(x)$ is the local description of g in the coordinates $\{x^\mu\}$, we can write

$$g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \mathcal{L}_{\epsilon^\mu\partial_\mu}(g^{\mu\nu}(x)),$$

where $\mathcal{L}_{\epsilon^\mu\partial_\mu}$ represents the Lie derivative with respect to the vector $\epsilon^\mu\partial_\mu$.

Since A^I are scalars,

$$A^I(W[g'(x')]) = A^I(W[g(x)]),$$

that is,

$$a^I_{g'}(x') = a^I_g(x). \tag{47}$$

The Bergmann and Komar procedure consists in implementing a metric-dependent change of coordinates dictated by the functions a^I_g. The new coordinates will be written as X^I, and are related to the old ones by

$$X^I(x) := a^I_g(x). \tag{48}$$

(In Sec. VI we will find conditions which must be satisfied by the functions A^I in order that X^0 actually labels spacelike foliations of spacetime.) We will call these new coordinates "intrinsic." As is clear from the notation, the change of coordinates (48) is g dependent.

Consider now the passive coordinate transformation that results from first transforming from x to x', and then to intrinsic coordinates, $X'(x'(x))$

$$X'^I(x'(x)) := a^I_{g'}(x'(x)).$$

Recalling (47), we find

$$X'^I(x') = X^I(x).$$

Now we can express the metric G, used to define the intrinsic coordinates X^I, in terms of these coordinates. It will take the form $G^{IJ}(X)$, with

$$G^{IJ}(X) = g^{\mu\nu}(x)\frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu}.$$

Notice that indices I, J, used to enumerate the four scalar A^I, now play a role indistinguishable from spacetime indices. Had we started the whole procedure from g' instead of g we would have ended up with $G'^{IJ}(X')$ instead of $G^{IJ}(X)$. But the fact that the new coordinates have been constructed out of scalars guarantees that the functions G'^{IJ} and G^{IJ} coincide, as we now demonstrate. Since $X'(x') = X(x)$,

$$\begin{aligned} G'^{IJ}(X(x)) = G'^{IJ}(X'(x')) &= g'^{\mu\nu}(x')\frac{\partial X'^I(x')}{\partial x'^\mu}\frac{\partial X'^J(x')}{\partial x'^\nu} \\ &= g'^{\mu\nu}(x')\frac{\partial X^I(x)}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial X^J(x)}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\nu} \\ &= g^{\rho\sigma}(x)\frac{\partial X^I(x)}{\partial x^\rho}\frac{\partial X^J(x)}{\partial x^\sigma} = G^{IJ}(X(x)). \end{aligned} \tag{49}$$

Let us recapitulate. We assume an observer has made an initial arbitrary choice of coordinates x and is working with a fixed solution trajectory $g(x)$ in these coordinates. This first observer is instructed how to select a new coordinate choice $X(x)$ resulting in a new functional form of solution $G(X)$ in terms of the new coordinates. The metric at x is mapped to a metric at X. We assume that a second observer is working with the same physical solution trajectory, which means that the second trajectory must be obtainable from the first through a passive coordinate transformation $x'(x)$ with the functional form $g'(x')$. This second observer then follows the same instructions to transform to intrinsic coordinates. We discover that the composite coordinate transformation $X'(x'(x))$ is the same as the original one step transformation $X(x)$. Thus the second observer agrees not only with the functional form of the solution in intrinsic coordinates, but also with the values of the individual metric components assigned to the same intrinsic coordinate location.

This procedure only works provided the intrinsic coordinates are determined by scalar functions of the dynamical fields. Indeed, as we shall discuss further in Sec. VII, one can view these scalars as identifying points in the spacetime manifold. In fact, this procedure can be used to "test" [15] the physical equivalency of a pair of solutions. This will be the case if the metric tensors are connected by a diffeomorphism. The test is performed by going to the intrinsic coordinates (48), and then checking whether the new functions G^{IJ} that describe the metric in the new coordinates coincide. Notice that the choice of the intrinsic coordinates is subject to the choice of the scalars A^I. Obviously, any "coordinate transformation," subject to the conditions to be determined in Sec. VI,

$$A^I \rightarrow \hat{A}^I = f^I(A), \tag{50}$$

defined by a set of functions f^I (defining an invertible coordinate transformation) gives a new set of scalars $\hat{A}^I$ as good as the former one as regards the application of our procedure. Therefore, in order to correctly perform the test, the form of the scalars A^I must be given and agreed upon by all the observers who wish to check whether their respective physics agrees.

C. Intrinsic coordinates for the relativistic free particle

We now refer to the relativistic free particle for a simple implementation of intrinsic coordinates. There are several ways to consider analogs of the Weyl scalars for the relativistic free particle. One could use, for instance, functions of the temporal coordinate q^0 of the trajectory in Minkowski space. The particle proper time is also a useful analogue of the Weyl scalars. We will explore both cases.

Let us first use Minkowski time. In this case we select the analogue of the Weyl scalar to be $A[q^\mu] = f^{-1}(q^0)$, where f^{-1} is an arbitrary monotonically increasing function of its argument. (Recall that each component q^μ transforms as a scalar under reparametrizations.) We represent the parameter time by t and we set the intrinsic coordinate time T equal to the appropriate scalar function of the dynamical variables

$$T := a_q(t) := A[q(t)] = f^{-1}(q^0(t)). \tag{51}$$

Referring to the general solution (14) our observer is instructed to set

$$f(T) = q^0(t) = q^0 + p^0\int_0^t dt_1 n(t_1), \tag{52}$$

so

$$\int_0^t dt_1 n(t_1) = \frac{1}{p^0}(f(T) - q^0). \tag{53}$$

Also, differentiating (51) we find

$$\frac{dt}{dT} = \frac{\frac{df}{dT}}{p^0 n(t)}. \tag{54}$$

Substituting into the solution (14) we find

$$Q^\mu(T) := q^\mu(t(T)) = q_0^\mu + \frac{p^\mu}{p^0}(f(T) - q_0^0), \tag{55}$$

and

$$N(T) = n(t)\frac{dt}{dT} = \frac{\frac{df(T)}{dT}}{p^0}. \tag{56}$$

We want to compare this solution in terms of the intrinsic coordinate T to that obtained by a second observer working with a different parametrization. Let us use our canonically implemented reparametrization transformed solution (17) since we wish to analyze this construction later on from the perspective of gauge fixing by transforming along gauge orbits. So our second observer is instructed to set his $q_s^0(t_s) = f(T_s)$, resulting in the following determination of $f(T_s)$ in terms of the gauge descriptor and lapse,

$$s\xi(t_s) + \int_0^{t_s} dt_1 n(t_1) = \frac{1}{p^0}(f(T_s) - q^0). \qquad (57)$$

Substituting into the gauge transformed trajectory we find again that

$$Q_s^\mu(T_s) := q_s^\mu(t_s(T_s)) = q_0^\mu + \frac{p^\mu}{p^0}(f(T_s) - q_0^0), \qquad (58)$$

and

$$N_s(T_s) = \frac{\frac{df(T_s)}{dT_s}}{p^0} \qquad (59)$$

Thus $Q_s^\mu(T) = Q^\mu(T)$ and $N_s(T) = N(T)$ consistently with our general claim that all observers find the same solution.

Let us also examine what happens to the transformation to intrinsic coordinates when we first go to coordinate t' before passing to intrinsic coordinates. For this purpose we will assume that we undertake an infinitesimal coordinate transformation

$$t' = t - n^{-1}(t)\xi(t),$$

for infinitesimal $\xi(t)$, so

$$t = t' + n^{-1}(t')\xi(t').$$

Under this transformation we find

$$q'^0(t') = q^0(t(t')) = q^0(t') + \dot{q}^0(t')n^{-1}(t')\xi(t'),$$

and therefore in passing to intrinsic coordinates

$$\begin{aligned} T_\xi(t') &= q^0(t(t')) = q^0(t') + \dot{q}^0(t')n^{-1}(t')\xi(t') \\ &= q^0(t) = T(t). \end{aligned}$$

Of course we could have avoided writing this out in detail by using the fact that q^0 transforms as a scalar under $t'(t)$. Nevertheless it is instructive to see that even though the functional dependence of q^0 does change we still find that $T_\xi(t'(t)) = T(t)$, i.e., it is the same transformation from t to T.

Proper time (and functions thereof) may also be used as intrinsic coordinates. If we wished to use proper time we would set the intrinsic coordinate T equal to $q^0(t)/p^0$. It is straightforward to show that in this case the resulting unique trajectories in terms of this new intrinsic coordinate are

$$Q^\mu(T) = q^\mu - \frac{p^\mu}{p^0}q^0 + p^\mu T.$$

We will discuss on the physical significance of these results in Sec. VII.

VI. THE KOMAR-BERGMANN PROCEDURE AS A GAUGE FIXING.

Once two observers agree on the set of scalars A^I to use, we claim that they will describe the same physics if their descriptions in their respective intrinsic coordinates coincide.

The results of the previous section can be given a different perspective from the point of view of gauge fixing. Indeed, given a metric g, the functions G^{IJ} of Sec. V B are the solution of the four gauge fixing constraints

$$\Phi^I := x^I - A^I(W[g(x)]) = 0. \qquad (60)$$

We employ here the usual definition of gauge fixing. Given a solution of the Einstein equations in some given coordinate system we consider all solutions obtainable from this solution through the action of all finite gauge transformations generated by (7) for arbitrary finite ξ. These solutions lie on a gauge orbit. Among these functionally different solutions we demand that there exist only one for which (60) is identically satisfied.

Indeed, consider that g is a metric solution of the constraints $\Phi^I = 0$. Then an infinitesimally close metric in the same gauge orbit,

$$g' = g + \mathcal{L}_{\epsilon^\mu \partial_\mu}(g)$$

cannot be a solution of the constraints. In fact, using (46),

$$\begin{aligned} A^I(W[g'(x)]) &= A^I(W[g(x+\epsilon)]) = a_g^I(x+\epsilon) \\ &= a_g^I(x) + \epsilon^\mu \partial_\mu a_g^I(x) \neq a_g^I(x) = x^I, \end{aligned}$$

that is,

$$A^I(W[g'(x)]) \neq x^I$$

for at least one value of I. Notice that the solution of the constraints in each gauge orbit obviously changes if we adopt a different set of scalars $\hat{A}^I$ as defined in (50).

A complementary interpretation is available for fixing the gauge. Since we have at our disposal finite gauge transformations corresponding to finite changes of coordinates, we can find the in general dynamical-solution-dependent gauge transformation which will transform any given solution to one satisfying the gauge conditions. The resulting gauge transformed solutions are by construction invariants—i.e., observables.

In addition, the formalism provides instructions on transforming from one set of observables to another set. In the context of Komar-Bergmann gauge fixing, these instructions amount to implementing diffeomorphisms on the Weyl scalar coordinates, as expressed in (50). Different choices of the functional form of the scalars A^I will correspond to different sets of observables, and the formalism tells us how to convert from one set to another. We will exhibit this procedure in detail for the free relativistic particle.

Up to now we have verified that the constraints ϕ^I produce a good gauge fixing in the space of metric configurations, for they select, at least locally, a single representative for each gauge orbit. For the remainder of this subsection we will consider that the dynamics of the metrics is given by general relativity (GR) and will study the role of the Komar-Bergmann constraints in fixing the dynamics, that is, in the building of a solution of Einstein equations starting from a well posed set of initial data. We will work in a local system of coordinates x^μ such that x^0 has a "time" interpretation, that is, the surfaces defined by the constancy of x^0 are spacelike. Also, we will formulate the dynamics in phase space. It is worth remembering that the scalars A^I are indeed functionals of a reduced set of the phase space variables, namely g_{ab} and π^{ab}, assuming, of course, that we are working with solutions of the Einstein equations [13].

The constraint structure of canonical (ADM) GR is given by four primary constraints, which are the canonical conjugate P_μ to the lapse $n^0 := n$ and shift n^a variables, and four secondary constraints, which are the so-called Hamiltonians $\mathcal{H}_\mu(g_{ab}, \pi^{ab})$ [18]. All constraints are first class and the Poisson brackets between the Hamiltonians define a set of structure functions $C^\sigma_{\mu\nu}$. The gauge generator in phase space associated with infinitesimal diffeomorphisms was given in [4] (here we keep explicit the 3-space integration)

$$G_\xi(x^0) = \int d^3x (P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + n^\rho C^\nu_{\mu\rho} P_\nu)\xi^\mu), \quad (61)$$

for ξ^μ arbitrary descriptor functions of the spacetime coordinates. A good gauge fixing is one for which the vanishing at all times of the equal-time Poisson brackets of the gauge fixing constraints with the gauge generator eliminates all possible gauge transformation freedom, that is

$$\{\Phi^I, G_\xi(x^0)\} = 0, \qquad \forall x^0 \longrightarrow \xi^\mu = 0.$$

This means that

$$\det(\{A^I, \mathcal{H}_\mu\}) \neq 0. \quad (62)$$

The dynamical generator of time evolution, the Dirac Hamiltonian, is [19]

$$H_\lambda = \int d^3x (n^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu) =: H_c + \int d^3x \lambda^\mu P_\mu, \quad (63)$$

where λ^μ are Lagrange multipliers that must become determined when the gauge fixing constraints are implemented. In fact, the time stabilization of the gauge fixing constraints gives new constraints

$$\dot{\Phi}^I = \frac{\partial \Phi^I}{\partial t} + \{\Phi^I, H_\lambda\} = \delta^I_0 - \int d^3x' \{A^I, \mathcal{H}'_\mu\} n'^\mu = 0 \quad (64)$$

(where primes indicate dependences with respect to the coordinates x') which, upon a new time stabilization,

$$\ddot{\Phi}^I = -\{\{A^I, H_c\}, H_c\} - \int d^3x' \{A^I, \mathcal{H}'_\nu\} \lambda'^\nu = 0, \quad (65)$$

determines the Lagrange multipliers, taking into account (62).

Notice that since by assumption the full set of constraints and gauge conditions is second class, the Poisson bracket $\{A^I, \mathcal{H}_\mu\}$ possesses a matrix inverse, and (64) may be solved for the lapse and shift as functions of the nongauge variables. It is noteworthy that had the gauge fixing conditions not possessed an explicit time dependence (in this case in Φ^0), the lapse and shift would have been zero. Our constraints Φ^I thus conform with the general result [4,12], already cited in Sec. IV, that the gauge fixing constraints for generally covariant theories must always include an explicit time dependence. In fact, the Komar-Bergmann constraints exhibit explicit dependence, not only on the time coordinate, but on all spacetime coordinates since each constraint solves for one coordinate.

We can obtain an equivalent explicit expression for the lapse and shift taking into account that the A^I are scalars, and that we are working in a coordinate system in which (we switch to greek indices for convenience) $x^\mu = A^\mu$, so $\partial_\nu A^\sigma = \delta^\sigma_\nu$. Consequently

$$\begin{aligned}\delta A^\sigma = \epsilon^\mu \partial_\mu A^\sigma &= \xi^0 \mathcal{N}^\mu \partial_\mu A^\sigma + \xi^a \partial_a A^\sigma \\ &= \xi^0 n^\sigma + \xi^a \delta^\sigma_a. \end{aligned} \quad (66)$$

On the other hand this infinitesimal transformation is generated by (7). Comparison gives the results

$$\begin{aligned}\{A^\sigma(x^0, \mathbf{x}), \mathcal{H}_\mu(x^0, \mathbf{x}')\} = (&\delta^0_\mu \mathcal{N}^\sigma(x^0, \mathbf{x}') + \delta^a_\mu \delta^\sigma_a) \\ &\times \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (67)$$

These relations impose conditions on the functional forms of $A^\sigma(W)$. Perhaps the most significant is that A^0 must be chosen so that $\mathcal{N}^0$ is positive definite, but the requirement that

$$\{A^\mu, \mathcal{H}_a\} = \delta^\mu_a \delta(\mathbf{x} - \mathbf{x}') \quad (68)$$

is also nontrivial. It follows from (67) that

$$\mathcal{N}^\mu(x^0, \vec{x}) = \left\{A^\mu(x^0, \vec{x}), \int d^3x' \mathcal{H}_0(x^0, \vec{x}')\right\}. \quad (69)$$

Care must be exercised in interpreting (69). It is actually a constraint which expresses the canonical lapse and shift variables, see (8), in terms of the remaining variables. It is equivalent to (64), whereas (68) are just the spatial derivatives of the Komar-Bergmann constraints.

Also, we can substitute (67) directly into the λ dependent term in (65) to determine the Lagrange multipliers. We obtain

$$\lambda^0 = -n\{\{A^0, H_c\}, H_c\},$$
$$\lambda^a = -\{\{nA^a + A^a, H_c\}, H_c\}, \tag{70}$$

and therefore the Dirac Hamiltonian can be written as

$$H_D = \int d^3x (n^\mu(\mathcal{H}_\mu - \{\{A^0, H_c\}, H_c\} P_\mu) - \{\{A^a, H_c\}, H_c\} P_a).$$

A. Gauge fixing for the free relativistic particle

We conclude this section with an illustration of gauge fixing and the associated determination of lapse and Lagrange multiplier for the free relativistic particle. The preceding discussion is applicable since the only assumption used explicitly was that the coordinate time be set equal to a scalar function of the nongauge dynamical variables.

So let us investigate the implications of the gauge fixing

$$\Phi := t - f^{-1}(q^0) \approx 0, \tag{71}$$

where f^{-1} is a monotonically increasing but otherwise arbitrary function. The function f^{-1} plays the role of A^0. Now according to (69) the lapse must be given by

$$n^{-1}(t) = \left\{ f^{-1}(q^0(t)), \frac{1}{2}(p^2 + 1) \right\} = p^0 \frac{df^{-1}(q^0(t))}{dq^0(t)}. \tag{72}$$

But notice that since $f(f^{-1}(q^0)) = q^0$, $1 = \dot{f}[(df^{-1}(q^0))/dq^0]$, and differentiating once more we find that $0 = (\ddot{f}/\dot{f}^2) + \dot{f}[(d^2 f^{-1}(q^0))/d(q^0)^2]$. Therefore

$$n(t) = \frac{\dot{f}(t)}{p^0}, \tag{73}$$

which agrees with (28). Finally, according to (70)

$$\lambda = -n(t)\left\{ n(t) p^0 \frac{df^{-1}(q^0)}{dq^0}, n(t)\frac{1}{2}(p^2 + 1) \right\} = -n^3(t) p^0 \frac{\frac{d^2 f^{-1}(q^0)}{d(q^0)^2}}{\frac{df^{-1}(q^0)}{dq^0}} = \frac{\ddot{f}(t)}{p^0}. \tag{74}$$

B. Degrees of freedom through the Komar-Bergmann method

We have studied the role of the Komar-Bergmann gauge fixing constraints (60) in two different frameworks. The first, in Sec. V B, was the space of spacetime metric histories with no dynamical content; no dynamical stabilization algorithm was invoked. Once the scalars A^I have been chosen, the gauge fixing (60) selects, at least locally, a single metric in each gauge orbit. The global question is left unanswered because we are not able to rule out the possible appearance of Gribov-type ambiguities. The second framework, analyzed in this section, was the space of solutions of Einstein equations. We showed that the constraints (60) fix completely the Einstein dynamics because the stabilization algorithm fixed uniquely in (65) the Lagrangian multipliers in the Dirac Hamiltonian (63). Now let us count the number of independent variables. The lapse and shift variables are determined through (64) in terms of the other variables. Also, since the primary constraints P_μ are the canonical conjugate variables to the lapse and shift variables, they are determined as well—to be zero—and so we have $2 \times 4 = 8$ variables already determined. At this point we are left with g_{ab} and π^{ab} as independent variables, adding up to a total of 12. To these variables we must apply the four restrictions coming from the secondary constraints $\mathcal{H}_\mu = 0$ (the Hamiltonian and momentum constraints) and also the original Komar-Bergmann gauge fixing constraints (60). We are left with $12 - 8 = 4$ independent variables, corresponding to the two standard degrees of freedom of general relativity. Since the Einstein dynamics has been completely fixed, that is, it has become a deterministic dynamics, we could study the degrees of freedom as the freedom of setting the initial data at, say, $x^0 = 0$. Giving values at that time to the four independent variables will determine a solution with a unique physical content. Giving other values to the four independent variables will determine a physically distinct solution. So changing the initial values of the four independent variables amounts to changing the gauge orbit; all the metrics in the same gauge orbit define the same physics. The freedom to give arbitrary values to the four independent variables is consistent with the fact that, in the space of metrics, the Bergmann-Komar gauge fixing constraints (60) select, at least locally, a single metric in each gauge orbit.

VII. OBSERVABLES

We interpret observables in any generally covariant theory to be those functions of dynamical variables which are invariant under diffeomorphisms. In phase space formulations of generally covariant theories this characterization must be altered to read "invariance under diffeomorphism-induced transformations." We shall first present a concise general argument for a Komar-Bergmann type construction. Then in the following subsection we demonstrate explicitly the invariant nature of objects constructed using Komar-Bergmann gauge fixing and we inquire into their physical observability. In the next subsection we write down invariants for the free particle and we show explicitly that they remain invariant under the action of the gauge group. In this context we also show that, contrary to initial expectations, there is no necessary relation between invariants and additional symmetries of the equations of motion, and we will explain why.

A. Komar-Bergmann type observables

The primary ingredients in the Komar-Bergmann construction are an intrinsic coordinate fixation using a scalar function of dynamical variables, and a scalar function of variables expressed in these coordinates. The idea that the specification of four independent scalars could bring observables for GR is an old one. Besides the work of Komar and Bergmann which is an elaboration of a suggestion by Géhéniau and Debever that Weyl scalars be used for this purpose [20,21], DeWitt [22], Isham and Kuchar [23], Hartle [24], and Marolf [8] have also advocated the use of scalars. Let us explore again, this time in a formally precise way, why this procedure delivers invariants.

We consider a generic generally covariant theory in which we have dynamical variables, or functions of variables, which transform as scalars under diffeomorphisms. Let $s(x)$ represent an independent set of scalars equal in number to the dimension of spacetime. We suppose they are of second differential order in the dynamical fields, after imposition of the equations of motion, and can therefore be expressed in terms of phase space variables. We let $q(x)$ represent the full set of dynamical variables. The first step in the construction is to set intrinsic coordinates X equal to $s(x) =: a(x)$, where we suppose that a is an invertible coordinate transformation. We interpret $X = s(x)$ as a dynamical variable dependent coordinate transformation which depends only implicitly on x through the x dependence of s. The geometric variables obtained under this map are $Q(X) = a^* q(x)$. But suppose we first undertake an arbitrary finite coordinate transformation $x_f = f(x, q)$, where we permit this transformation to even depend on the dynamical variables. Under this transformation $s_f(x_f) = s(x)$. Follow this transformation with the transformation to intrinsic coordinates, then because the s are scalars:

$$X_f = s_f(x_f) = s(x) = X. \tag{75}$$

This key result, $X_f(x) = X(x)$ shows that the numerical values of the intrinsic coordinates are the same, and the resulting coordinate transformation from x to X is identical, in spite of the indirect route. It follows therefore that the map of geometric objects a^* is identical, and that the resulting geometrical objects expressed in terms of intrinsic coordinates are identical, i.e., they are invariant under the arbitrary coordinate map f.

The invariance we are discussing here is the usual notion of invariance in any theory which possesses a local gauge symmetry. We imagine we have a solution of the equations of motion which we have expressed in an arbitrary coordinate system. Thus we want to consider the objects we construct by going to intrinsic coordinates as phase space functions of these original variables. They undergo nontrivial variations engendered by our symmetry generator $G[\xi]$. On the other hand, if we were to take a solution which already satisfies the gauge fixing condition (rather than our invariant function of solutions) and perform a gauge transformation on it, we would of course obtain a solution which no longer satisfies the gauge condition. We will illustrate these ideas in detail in the next section using the relativistic free particle.

Let us now consider the physical measurement of the full four-dimensional metric tensor in an intrinsic coordinate system. There is in principal a well-defined procedure at our disposal. It relies on a device first envisioned by Peter Szekeres which he has called a "gravitational compass" [25]. It consists of a tetrahedral arrangement of springs. By measuring the stresses in the springs one can determine components of the curvature tensor. In the vacuum case three compasses will suffice to determine all of the local components of the Weyl tensor. Four compasses are required to determine the full local Riemann curvature tensor in the presence of matter sources. These measurements can in principal be used to establish the intrinsic coordinate system fixed by the Weyl scalars. Supplemental measurements of distances using light ranging will then determine components of the metric in this coordinate system.

B. Observables for the free relativistic particle

We now consider the construction of invariants for the free relativistic particle in the manner just described. Actually the job was already completed in Sec. V C. The idea is that we choose a scalar function of the dynamical variables, and then use this scalar to define a parameter transformation $T(t)$. Then we can construct invariants out of all components of the spacetime position and the lapse by setting $Q^\mu(T) = q^\mu(t(T))$ and $N(T) = N(t(T))(dt/dT)$. We will explicitly construct classes of gauge invariants for the free particle corresponding to a wide class of gauge choices.

We first consider invariants using the intrinsic coordinate

$$T = f^{-1}(q^0(t)), \tag{76}$$

where f^{-1} is a monotonically increasing but otherwise arbitrary function. We found in obtaining (55) and (13) that we did not need to solve explicitly for t in terms of T. We merely substituted the t dependent term in the general solution for $q^0(t)$ into the expression for $q^a(t)$, obtaining

$$Q^a(T) = q^a(t(T)) = q_0^a + \frac{p^a}{p^0}(f(T) - q_0^0). \tag{77}$$

Similarly we solved for (dt/dT) to find

$$N(T) = \frac{\frac{df(T)}{dT}}{p^0}. \tag{78}$$

These are our putative invariants.

It will be useful to rewrite these invariants in terms of the solution trajectories $n(t)$ and $q^\mu(t)$ given by (13) and (14).

Substituting for the initial values q_0^μ we find

$$Q^a(t) = q^a(t) + \frac{p^a}{p^0}(f(t) - q^0(t)), \tag{79}$$

while $N(t) = \frac{1}{p^0}[(df(t))/dt]$ is unchanged.

Now let us examine the variations of these objects under an arbitrary infinitesimal canonical gauge transformation. We shall demonstrate invariance in two equivalent ways. In the first procedure we express all phase space variables in terms of initial values, and Poisson brackets will be computed in terms of these initial value phase space coordinates. We note that the only relevant nonvanishing variations engendered by

$$G_\xi(t) = \frac{1}{2}\xi(t)(p^2 + 1) + \dot{\xi}(t)\pi(t), \tag{80}$$

are (since none of the invariants depend on n)

$$\delta q_0^\mu = (\xi(t) - t\dot{\xi}(t))p^\mu. \tag{81}$$

Therefore $Q^0(t)$ and $N(t)$ are trivially invariant, while $Q^a(t)$ is invariant since the q_0^μ coordinates appear in the combination $q_0^a - \frac{p^a}{p^0}q_0^0$.

A second equivalent procedure available to us is to compute Poisson brackets at the time t with respect to the canonically evolved phase space variables at time t. Thus the relevant nonvanishing variations generated at time t are

$$\begin{aligned}\delta q^a(t) &= \left\{q^a(t), \frac{1}{2}\xi(t)p^2(t) + \dot{\xi}(t)\pi(t)\right\}_{y(t)} \\ &= \xi(t)p^a(t) = \xi(t)p^a.\end{aligned} \tag{82}$$

Therefore, referring to (79),

$$\delta Q^a(t) = \delta q^a(t) - \frac{p^a}{p^0}\delta q^0(t) = \xi(t)(p^a - p^a) = 0. \tag{83}$$

Notice that these invariants are in general dependent on t, and *not* constants of the motion. The independence of gauge and time evolution is made strikingly evident in this example. Notice also that our observables are also invariant under the action of the gauge-fixed Hamiltonian (26). At first sight this may appear to be a contradiction since we have simply expressed an arbitrary solution of the equations of motion in terms of intrinsic coordinates. We might well ask: should this solution satisfy the equations of motion? If it did satisfy the equations of motion then its Poisson bracket with the Hamiltonian would *not* vanish. But the apparent contradiction is resolved when one realizes that $Q^a(t)$ given by (79) exhibits explicit time dependence. Thus we have written the invariants as the sum of a part constructed with solution trajectories plus a part which contains an explicit t dependence. Therefore

$$\frac{dQ^\mu(t)}{dt} = \frac{\partial Q^\mu(t)}{\partial t} + \{Q^\mu(t), H_D\} = \frac{p^\mu}{p^0}\dot{f} + 0 = \frac{p^a}{p^0}\dot{f}, \tag{84}$$

which agrees with (27).

There is a widespread mistaken notion in the literature that gauge invariants in generally covariant theories must be constants of the motion. (See the conclusions in Sec. VIII for further discussion and references.) Our gauge invariants for the free particle are a counterexample. And since they are not constants of motion, they should not be expected to generate symmetries of the equations of motion.

There do exist invariants for the free particle which *are* constants of the motion, and it will be instructive to examine some of them. One such class can be obtained even before adopting intrinsic time. Consider the solutions (14). They satisfy

$$q^\mu(t) - \frac{p^\mu}{p^0}q^0(t) = q^\mu(0) - \frac{p^\mu}{p^0}q^0(0) \tag{85}$$

that is,

$$q^\mu(t) - \frac{p^\mu}{p^0}q^0(t) \tag{86}$$

are constants of motion (the time component vanishing) with no explicit time dependence. One can check that they are also gauge invariant quantities.

Notice that these very same gauge invariant quantities (86) can be presented, when described with the intrinsic coordinates, with explicit time dependence. Indeed, this can be achieved by isolating the new initial conditions on the trajectory (77),

$$Q^\mu(0) = q_0^\mu + \frac{p^\mu}{p^0}(f(0) - q_0^0), \tag{87}$$

which are evidently gauge invariant quantities. Then the trajectory can be expressed as

$$Q^\mu(t) = Q^\mu(0) + \frac{p^\mu}{p^0}(f(t) - f(0)), \tag{88}$$

which identifies the time-dependent constants of the motion

$$L^a := Q^a(t) - \frac{p^a}{p^0}f(t). \tag{89}$$

These constants of the motion are just the constant initial values,

$$L^a = q^a - \frac{p^a}{p^0}q^0. \tag{90}$$

Constants of the motion are generators of symmetries of the equations of motion, and map solutions into solutions. It is not unusual for a constant of motion to be time dependent, as are, for example, the Noether constants of motion associated with Galilean boosts. In fact, the L^a are generators of Lorentz boosts as can be straightforwardly shown. We notice that nothing analogous to these boost generators in a gauge-fixed theory can exists in vacuum general relativity because there exists no dynamical symmetries beyond general covariance.

We have pursued this example in some detail to make a significant point. Torre has asserted that in general relativity there can exist no observables built as spatial integrals of local functions [26]. In fact the Komar-Bergmann construction in the case of the free particle provides local observables. It is true that observables commute with the Hamiltonian constraint. But whereas constants of the motion generate symmetries and map solutions onto solutions, nonconstant invariants do *not* map solutions onto solutions. What Torre has actually proven, and it is in our view no less significant, is that in vacuum general relativity there exists no *constant* in time observables built as spatial integrals of local phase space functions. The Komar-Bergmann observables are indeed local in both space and time. This follows from the fact that the intrinsic coordinates are local functions of the spatial metric and conjugate momenta and spatial derivatives thereof. These are in turn algebraic functions of spatial and time coordinates. Thus the mapping from arbitrary spacetime coordinates to intrinsic coordinates is local, as is the inversion map. In addition, the metric components in the original coordinate patch are local functions of the coordinates, and they therefore remain local functions when expressed in terms of the intrinsic coordinates.

It is also clear from this free particle example that given *any* parametrization of the particle world line there is a corresponding set of invariants, corresponding to the choice of the function f. Are these invariants measurable, and therefore observable? Indeed they are in the context of flat spacetime where we assume we have coordinate clocks distributed throughout space. These coordinate clocks are usually set to run with the gauge fixing condition $f(t) = t$, so $q^0 = t$. The reading of the clock constitutes a partial observable in the sense of Rovelli [9]. Complete observables are correlations between partial observables, and the correlations are fixed by the theory. The observables cited above admirably fit this description when we take into account that choices of the gauge fixing function f merely correspond to differing instructions on adjusting the rate of rotation of the clock hands with respect to the flow of Minkowski time.

C. Invariants, gauge fixing and Dirac brackets

Since we are now able to implement finite diffeomorphism-induced gauge transformations we have at our disposal a standard procedure for producing gauge invariants through the imposition of gauge conditions. After describing the general method we will apply it to the relativistic particle, and then comment on the general relationship between invariants, gauge fixing, and Dirac brackets.

As before we let $y(t)$ represent the set of canonical solution trajectories corresponding to the Dirac Hamiltonian H_λ. But we shall alter our previous notation somewhat and represent a finite gauge transformed trajectory with descriptor ξ and group parameter $s = 1$ as $y_\xi(t)$. Let us impose gauge conditions $\chi(y(t)) = 0$. We achieve this condition by performing the appropriate $y(t)$ dependent gauge transformation on $y(t)$, described by a $y(t)$ dependent descriptor ξ. Thus the objects $y_{\xi(y(t))}(t)$ are manifest gauge invariants.

In the case of the free relativistic particle the descriptor is fixed by the gauge condition

$$f(t) = q^0_\xi(t) = q^0(t) + p^0\xi(t), \tag{91}$$

resulting in $\xi(y(t), t) = \frac{1}{p^0}(f(t) - q^0(t))$. Thus we recover the gauge invariants displayed above:

$$\begin{aligned} q^a_{\xi(q(t),p(t),t)} &= q^a(t) + \frac{p^a}{p^0}(f(t) - q^0(t)) \\ &= q^a + \frac{p^a}{p^0}(f(t) - q^0). \end{aligned} \tag{92}$$

We complete this section noting that Dirac brackets are simply the ordinary Poisson brackets of our invariants. For example, employing the inverse matrix M^{mn} in (25) we find

$$\begin{aligned} \{n(t), q^a(t)\}^* &= -\{n(t), \psi_2(t)\}M^{21}(t)\{\psi_1(t), q^a(t)\} \\ &= \frac{n(t)}{(p^0)^2}p^a = \{N(t), Q^a(t)\}, \end{aligned} \tag{93}$$

where in the last equality (77) and (78) have been used.

VIII. CONCLUSIONS

Our focus throughout this paper has been the distinction between time evolution and diffeomorphism gauge symmetries in generally covariant theories, and the startling physical consequences of this distinction. Time evolution is of course the mapping of initial data to produce solution trajectories. Diffeomorphism gauge transformations map entire solution trajectories into solution trajectories. The distinction is obvious in the usual configuration-velocity space formulation. Clearly if, for example, $g_{\mu\nu}(x^0, \vec{x})$ represents a solution of Einstein's equations, then under an infinitesimal coordinate transformation $x'^\mu = x^\mu - \epsilon^\mu(x^0, \vec{x})$, the corresponding active variation of $g_{\mu\nu}(x^0, \vec{x})$ is just the Lie derivative

$$\mathcal{L}_\epsilon g_{\mu\nu} = g_{\mu\nu,\alpha}\epsilon^\alpha + \epsilon^\alpha_{,\mu}g_{\alpha\nu} + \epsilon^\alpha_{,\nu}g_{\mu\alpha}. \tag{94}$$

Clearly there is a different variation at each time x^0. It is in the transition to phase space that one can easily lose one's way.

Fortunately we now have at our disposal a concrete realization of the full diffeomorphism-induced gauge transformation group in phase space. And the distinction between time evolution and gauge is made even more transparent when lapse and shift functions are retained as canonical phase space variables. There is an essential distinction between the Hamiltonian and the generator of gauge transformations. They are similar in appearance, but in the Hamiltonian we have the arbitrary coordinate functions λ^A, whereas in the gauge generator these coordinate functions are replaced by the canonical variables $\dot{n}^A$.

The misidentification of evolution and gauge has led to the often repeated assertion that gauge invariants in generally covariant theories must be constants of the motion. It is true that the Poisson brackets of invariants with all of the first class constraints in a generally covariant theory must vanish [since these constraints all appear multiplying arbitrary functions in the generator $G(t)$]. But this assertion fails to take into account any explicit time dependence (as opposed to implicit time dependence, that is, time dependence appearing in canonical variables) in the invariants of the theory. We have shown that such an explicit time dependence arises in any acceptable gauge fixing. It is compulsory in order to uniquely fix a solution on the gauge orbit. We have shown in detail how the Hamiltonian dynamics in generally covariant theories accommodates time-dependent gauge fixing. In particular, the dynamics is not frozen; time evolution is nontrivial after the imposition of time-dependent gauge conditions.

We have explored a special class of gauge conditions in which scalar functions of dynamical variables are selected to define intrinsic coordinates. Komar and Bergmann made the original concrete proposal of this type in general relativity with the suggestion that Weyl curvature scalars serve as intrinsic coordinates in vacuum general relativity. (We have shown that Weyl scalars may be used for this purpose also when material sources are present.) We demonstrated why this procedure produces invariants.

Throughout this paper we have illustrated our discussion with applications to the free relativistic particle. This is a dynamical model for which we possess the general solution but which nevertheless possesses a highly nontrivial symmetry structure in phase space. We imposed an analogue of the Komar-Bergmann gauge fixing, recovered a nontrivial dynamics due to the explicit time dependence of this gauge choice, and ultimately displayed diffeomorphism invariants which were not constants of the motion. We also displayed constant invariants which were found to generate Lorentz boosts of the particle world lines. We pointed out that a theorem due to Torre can be reinterpreted as a demonstration that in vacuum general relativity, as opposed to the relativistic free particle, there can exist no constant (of the motion) diffeomorphism invariants constructed with spatial integrals of local canonical variables since in vacuum general relativity there exists no dynamical symmetries beyond general covariance.

We interpret invariants as observables. That the invariants displayed for the free particle are indeed observable we think is beyond dispute since they represent correlations observed in daily practice in modern laboratories. In the language proposed by Rovelli, the time t for the free particle is a partial observable, recorded by a suitably regulated clock, and our observables are complete in the sense that for a choice of the regulating function f the theory gives a univocal prediction for particle spacetime position.

The invariants produced in the Komar-Bergmann gauge fixing in general relativity enjoy a similar status. Intrinsic spacetime coordinates are in principle measurable and constitute partial observables. Given these coordinates for a specific spacetime the theory univocally predicts spacetime distances along arbitrary routes. And schemes exist, using light ranging, for example, for measuring these spacetime distances. The invariants resulting from Komar-Bergmann gauge are a univocal correlation for these measurements, and therefore constitute complete observables.

The assertion that diffeomorphism invariants must be constant in time has a long and distinguished history, and is traceable at least as far as Komar and Bergmann [27]. Yet these authors explicitly note situations, namely, in regard to the use of intrinsic coordinates, in which invariants display time dependence [17]. Rovelli has explicitly addressed comparable apparent contradictions when on the one hand he states that "physical observables must be invariant under evolution in t" but points out that such a statement is "ill posed, because it confuses evolution with respect to coordinate time t and physical evolution" [9]. In all of the works cited the paradox is resolved through the method of coincidences, or equivalently, intrinsic coordinates. The view apparently espoused by these authors is that we can and should distinguish between intrinsic time, on which variables might depend, and our initially arbitrarily chosen coordinate time. Invariants must be independent of this latter choice. We fully agree. We have given a formal elaboration of this distinction in our enlarged phase space in which the full four-dimensional diffeomorphism-induced symmetry group is realized as a canonical transformation group.

We close with some comments about the implications of this work for an eventual quantum theory of gravity. The implications are profound. A nonquantum evolutionary parameter which we should interpret as the time will appear naturally in a Heisenberg picture formalism in which states are functionals of the three metric—and perhaps also of lapse and shift. The time-dependent invariants which appear in the Komar-Bergmann gauge fixing may be promoted to operators (recognizing as always that factor ordering ambiguities may arise). These operators represent the full four-dimensional metric in intrinsic coordinates, and the full metric will therefore be subject to time-dependent fluctuations. Invariants have recently been constructed for a class of classical Bianchi type I cosmological models [28], and work is underway investigated the significance of these invariants in quantum cosmology.

ACKNOWLEDGMENTS

We thank the Department of Physics at the University of Florence, and particularly Giorgio Longhi for hospitality during the first stages of this work. We also thank Luca Lusanna for many stimulating discussions on the subject. D. C. S. thanks the Department of Physics at Syracuse University for its hospitality. He thanks Josh Goldberg for his critical reading of an earlier draft of this work, and many incisive comments. He also thanks A. P. Balachandran and Don Marolf for stimulating and enlightening discussions, as well as James Friedrichson, Josh Helpert, Larry Shepley, and Allison Schmitz for spirited debate. D. C. S. acknowledges recent financial support from the Priddy Foundation. J. M. P. acknowledges financial support by CICYT, AEN98-0431, and CIRIT, GC 1998SGR.

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[1] This is no longer strictly true when boundary terms are present as is the case in asymptotically flat spacetimes in general relativity; but these terms do not affect time evolution and will not be relevant to our discussion.
[2] In this paper we consider only the classical ''problem of time.'' Quantum aspects, such as the relation of time to the Wheeler–de Witt equation, will be dealt with in future publications.
[3] Note that in the case of diffeomorphisms as gauge transformations we will use throughout the active view for which the spacetime coordinates are preserved whereas the objects (metric and other fields) defined on them are transformed under the diffeomorphisms.
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