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Quasilocal energy-momentum and energy flux at null infinity

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The null infinity limit of the gravitational energy-momentum and energy flux determined by the covariant Hamiltonian quasilocal expressions is evaluated using the Newman-Penrose spin coefficients. The reference contribution is considered by three different embedding approaches. All of them give the expected Bondi energy and energy flux.

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I. INTRODUCTION

It is well known that, as a consequence of the equivalence principle, gravitational energy cannot be localized (see e.g. [1], Sec. 20.4). An alternative idea is quasilocal, namely, quantities associated with a closed two-surface [2]. During recent decades, there have been numerous intensive efforts made in the search for a better definition of quasilocal energy (as well as momentum and angular momentum) for gravitating systems, with the goal of obtaining quasilocal quantities which can provide a description of the gravitational field more elaborate than that given by the total quantities. A very nice review on the development and applications of quasilocal quantities in general relativity can be found in Ref. [3]. We are interested in testing certain quasilocal expressions for energy-momentum and energy flux obtained from the Hamiltonian boundary term using the covariant Hamiltonian formalism applied to gravity [4–7].

Some basic criteria are usually presumed for a physically reasonable definition of quasilocal gravitational energy [3,8]. One of the most important is to consider the asymptotic behavior of the quasilocal energy when the two-surface approaches null and spatial infinity. The first aim of this work is to check, using the Newman-Penrose (NP) spin coefficient formalism [9], the null infinity limit of the quasilocal energy-momentum associated with the boundary of a finite region for a certain covariant Hamiltonian quasilocal energy-momentum expression.

We are interested in gravitational quantities such as energy in the quasilocal sense, i.e., within a finite region of the space-time. There is an important issue concerning how much energy flux flows into and out of the considered region. There continues to be considerable interest in this topic [10–13]. In this work, we also investigate a natural expression for the energy flux associated with the aforementioned quasilocal energy [7]. In order to test whether the definition is suitable, we again look to the null infinity limit using the spin coefficient techniques. In this case, the gravitational energy flux is expected to be given by the well-known Bondi energy loss formula.

In order to have a reasonable definition of gravitational energy, the choice of reference plays an essential role. It is well known that the technique for choosing the reference is an important unsolved problem for the quasilocal energy issue. The ambiguity comes from how to embed the reference configuration into the physical space-time. Here we consider three different embeddings.

This paper is organized as follows: In the next section, we start from a basic review on the asymptotic behavior of space-time near null infinity including a discussion of the expansion of the Newman-Penrose coefficients which is more complete than the well-known ones in Refs. [9,14–17]. Our main result is contained in three parts: In Sec. III, we review the covariant Hamiltonian formalism and the associated boundary term approach to quasilocal energy-momentum and energy flux. We then rewrite the energy-momentum expression in terms of the Newman-Penrose formalism. In Sec. IV, we find the asymptotic behavior at null infinity under three different embedding methods. From the results, we find that the Brown-Lau-York (BLY) embedding [10] directly gives the standard Bondi mass aspect, while two other embeddings include an additional term which, however, vanishes upon integration. We identify the source of this difference. In Sec. V, we look at the direct definition of the energy flux and consider its null infinity limit to test whether this definition is reasonable. The detailed calculation is divided into several subsections. In the first subsection, we calculate the purely physical part of the energy flux and, in later subsections, we will consider the three different embedding methods which have been mentioned. All three types of embedding give the standard Bondi energy flux. In Sec. VI we test, using the spin coefficient formalism in the null infinity limit, the new Hamiltonian identity based expression for energy flux. It directly yields the expected Bondi energy-flux value. Section VII includes our concluding discussion.

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II. ASYMPTOTIC BEHAVIOR OF SPACE-TIME NEAR NULL INFINITY

We first review the asymptotic behavior of space-time near null infinity. Here we are interested in the cases in which the space-time is asymptotically flat. There have been many intensive investigations of this subject in the past decades. We will follow the method initiated by Newman, Penrose, and Tod [15,16]. Many additional useful results have been discussed in Refs. [16–20].

In accord with Penrose's conformal compactification method, i.e., the Penrose diagram, we assume that $(\tilde{M}, \tilde{g}_{\mu\nu}, \Omega)$ is the conformal compactified manifold of a physical space-time $(M, g_{\mu\nu})$ via a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. Suppose S is a section of future null infinity $\mathcal{I}^+$, we choose *Bondi coordinates* near $\mathcal{I}^+$ in the following way: (θ, ϕ) are spherical coordinates on S and u is the affine parameter generating $\mathcal{I}^+$ such that $u = 0$ on S. If k^μ is a null vector on $\mathcal{I}^+$ which is not tangential, namely, $k^\mu \notin T(\mathcal{I}^+)$, then the null geodesics generated by k^μ will take the coordinates (u, θ, ϕ) into the physical space-time $(M, g_{\mu\nu})$. Moreover, on each null geodesic $\psi(r) = \{u = \text{const}, \theta = \text{const}, \phi = \text{const}\}$, its affine parameter r, in the sense of the physical metric $g_{\mu\nu}$, can serve as the fourth coordinate (in the neighborhood of $\mathcal{I}^+$, we assume the null geodesics are complete). Thus, we have constructed coordinates (u, r, θ, ϕ) for the physical space-time (they are like Bondi's coordinates except that r is an affine parameter rather than a luminosity distance, a small difference asymptotically).

In addition to this coordinate construction, we impose, as has been introduced in Refs. [19,20], certain null frame gauges choices; stated in terms of the spin coefficients, they are

$$\begin{aligned}\rho - \bar{\rho} = \mu - \bar{\mu} = \kappa = \varepsilon = \tau - \bar{\alpha} - \beta \\ = \bar{\pi} - \bar{\alpha} - \beta = 0.\end{aligned} \tag{1}$$

The fact that ρ and μ are real functions is insured by the null frame m and $\bar{m}$ being always tangent to the two sphere $r = \text{const}$ and $u = \text{const}$. The meaning of the other gauge choices is that l is a null geodesic and the Lie transports of both n and m along l are tangent to the sphere.

We use the standard notation for derivatives, $D := l^a \nabla_a$, $D' := n^a \nabla_a$, $\delta := m^a \nabla_a$, $\bar{\delta} := \bar{m}^a \nabla_a$, and the usual complex coordinate on the two sphere, $\zeta := \cot\frac{\theta}{2} e^{i\phi}$. The null tetrad can be chosen as

$$l = \frac{\partial}{\partial r}, \tag{2}$$

$$n = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X\frac{\partial}{\partial \zeta} + \bar{X}\frac{\partial}{\partial \bar{\zeta}}, \tag{3}$$

$$m = \xi\frac{\partial}{\partial \zeta} + \bar{\eta}\frac{\partial}{\partial \bar{\zeta}}, \tag{4}$$

where U (real) and X, ξ, η (complex) are undetermined functions.

For simplicity, in this paper we will focus only on the vacuum case. Due to the asymptotic flatness, the behavior of the Weyl curvature satisfies the "peeling off theorem" [9,14], i.e.,

$$\Psi_i \sim O(r^{i-5}), \qquad i = 0, 1, 2, 3, 4. \tag{5}$$

Near null infinity, we expand all quantities in Taylor series with respect to $1/r$. Using the NP equations, the asymptotic behavior of the spin coefficients is [16,20]

$$\rho = -\frac{1}{r} - \frac{|\sigma^0|^2}{r^3} + O(r^{-5}), \tag{6}$$

$$\sigma = \frac{\sigma^0}{r^2} + \frac{|\sigma^0|^2\sigma^0 - \frac{1}{2}\Psi_0^0}{r^4} + O(r^{-5}), \tag{7}$$

$$\alpha = \frac{\alpha^0}{r} + \frac{\bar{\sigma}^0\bar{\alpha}^0 + \eth_0\bar{\sigma}^0}{r^2} + O(r^{-3}), \tag{8}$$

$$\beta = -\frac{\bar{\alpha}^0}{r} - \frac{\alpha^0\sigma^0}{r^2} + O(r^{-3}), \tag{9}$$

$$\tau = \bar{\pi} = \frac{\bar{\eth}_0\sigma^0}{r^2} + O(r^{-3}), \tag{10}$$

where $\sigma^0(u, \theta, \phi)$ and $\Psi_0^0(u, \theta, \phi)$, the leading order terms of σ (of order r^{-2}) and Ψ_0 (of order r^{-5}), are free functions. Moreover, the variable α^0 is an abbreviation for $\alpha^0 := (1/2\sqrt{2})\zeta$, the spin-weight operator $\eth_0$ is defined by $\eth_0 := (P/\sqrt{2})\frac{\partial}{\partial\bar{\zeta}} + 2s\bar{\alpha}^0$ acting on a variable with spin weight s [17], and $P := 1 + \zeta\bar{\zeta} = \sin^{-2}\frac{\theta}{2}$. For calculating the energy flux, we also need the asymptotic expansion for the other NP coefficients. From the vacuum NP equations, after imposing the gauge conditions (1), we have

$$D\gamma = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta + \tau\pi + \Psi_2. \tag{11}$$

Using the results in Eqs. (6)–(10) and the asymptotic condition (5), it is straightforward to get

$$\begin{aligned}\gamma &= \frac{\gamma_1}{r^2} + O(r^{-3}) \\ &= \frac{-\alpha^0\bar{\eth}_0\sigma^0 + \bar{\alpha}^0\eth_0\bar{\sigma}^0 - \frac{1}{2}\Psi_2^0}{r^2} + O(r^{-3}).\end{aligned} \tag{12}$$

In order to get the asymptotic expansion of the remaining NP coefficients, we need to have some more control on the null tetrad. From the commutation relations, one can easily derive the null tetrad control equations

$$DU = -(\gamma + \bar{\gamma}), \tag{13}$$

$$DX = (\bar{\tau} + \pi)\xi + (\tau + \bar{\pi})\eta, \tag{14}$$

$$D\xi = \bar{\rho}\xi + \sigma\eta, \tag{15}$$

$$D\eta = \bar{\sigma}\xi + \rho\eta, \tag{16}$$

which lead to the following asymptotic behavior of the null tetrad:

$$U = -\frac{1}{2} + \frac{\gamma_1 + \bar{\gamma}_1}{r} + O(r^{-2}), \tag{17}$$

$$X = -\frac{\xi^0 \eth_0 \bar{\sigma}^0}{r^2} + O(r^{-3}), \tag{18}$$

$$\xi = \frac{\xi^0}{r} + \frac{|\sigma^0|^2 \xi^0}{r^3} + O(r^{-5}), \tag{19}$$

$$\eta = -\frac{\bar{\sigma}^0 \xi^0}{r^2} + O(r^{-4}), \tag{20}$$

where the numerical factor $-\frac{1}{2}$ in (17) and the value of ξ^0 ($\xi^0 := P/\sqrt{2}$) are specified by the result from Minkowski space-time. The asymptotic behavior of μ and λ can now be retrieved from the following NP equations:

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + |\pi|^2 - (\bar{\alpha} - \beta)\pi + \Psi_2, \tag{21}$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi, \tag{22}$$

$$\delta\tau - D'\sigma = (\mu\sigma + \bar{\lambda}\rho) + \tau^2 - (\bar{\alpha} - \beta)\tau - (3\gamma - \bar{\gamma})\sigma. \tag{23}$$

The last equation is needed to determine the value of the leading order term of λ. Using the obtained results (17)–(20) and the property that the spin weight of π is -1, it is straightforward to get

$$\mu = -\frac{1}{2r} - \frac{\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0 + \eth_0^2 \bar{\sigma}^0}{r^2} + O(r^{-3}), \tag{24}$$

$$\lambda = \frac{\dot{\bar{\sigma}}^0}{r} + \frac{\frac{1}{2}\bar{\sigma}^0 - \bar{\eth}_0 \eth_0 \bar{\sigma}^0}{r^2} + O(r^{-3}), \tag{25}$$

where the dot means derivative with respect to u and the numerical factor $-\frac{1}{2}$ in the leading term of μ is specified by the Minkowski space-time result. (It will turn out that μ and λ will play the key roles in our results.)

Finally, the expansion for the coefficient ν can be obtained from the NP equation

$$D\nu - D'\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi + \Psi_3, \tag{26}$$

which gives

$$\nu = -\frac{\eth_0 \dot{\bar{\sigma}}^0 + \Psi_3^0}{r} + O(r^{-2}). \tag{27}$$

However, the leading term of the NP equation

$$\delta\lambda - \bar{\delta}\mu = (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 \tag{28}$$

shows that $\Psi_3^0 = -\eth_0 \dot{\bar{\sigma}}^0$ (the spin weight of $\bar{\sigma}^0$ is -2). Therefore, the leading order of ν indeed is r^{-2}, which does not make a contribution in the later calculation.

Moreover, for the later calculation, we still need the asymptotic properties of the induced volume element (2-dimensional) on the considered sphere S^2. The induced metric ${}^{(2)}ds^2$ on the sphere $S = \{u = \text{const}, t = \text{const}\}$ asymptotically should be

$${}^{(2)}ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) + O(r). \tag{29}$$

The induced volume element on S^2 can be obtained from the volume element $\boldsymbol{\epsilon} := i\boldsymbol{l} \wedge \mathbf{n} \wedge \mathbf{m} \wedge \bar{\mathbf{m}}$, in our case $\boldsymbol{\epsilon} = \sqrt{-g}du \wedge dr \wedge d\theta \wedge d\phi$, by

$$\begin{aligned} {}^2\epsilon_{cd} &= \epsilon_{abcd}\left(\frac{\partial}{\partial u}\right)^a \left(\frac{\partial}{\partial r}\right)^b = i\mathbf{m} \wedge \bar{\mathbf{m}} + O(r) \\ &= r^2\left(1 - \frac{|\sigma^0|^2}{r^2}\right)\sin\theta d\theta \wedge d\phi + O(r^{-2}). \end{aligned} \tag{30}$$

In the derivation we have used the relation $\frac{\partial}{\partial\zeta} = -\sin^2\frac{\theta}{2} e^{-i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)$.

III. QUASILOCAL ENERGY-MOMENTUM AND ITS NULL INFINITY LIMIT

There have been many proposals regarding quasilocal quantities. It should be noted that there is as yet no consensus regarding what approach should be used or even what are the proper criteria [3]. However, it has been argued that the Hamiltonian approach, which has been used by many researchers and which we adopt here, has certain merits (see e.g. [4–6,10,11,21–23]). For a general region Σ (finite or infinite) the Hamiltonian,

$$H(N, \Sigma) = \int_\Sigma N^a \mathcal{H}_a + \oint_S \mathcal{B}(N), \tag{31}$$

which displaces the region along a vector field N, includes not only an integral of a density over the 3-dimensional region but also an integral over its closed 2-surface boundary $S = \partial\Sigma$. For Einstein's general relativity (GR) (as well as any other geometric gravity theory), the Hamiltonian densities $\mathcal{H}_a$ are proportional to certain field equations—the initial value constraints—and so the *value* of the Hamiltonian, $E(N, S)$, is determined purely by the integral of the boundary term. For appropriate choices of the displacement N on the boundary, this Hamiltonian boundary term, for any gravitating system, determines the quasilocal values: in particular, from a suitable timelike translation, the quasilocal energy, and from a suitable spacelike translation, the quasilocal linear momentum. The approach is quite general; it can incorporate, in particular, not only all the Noether charge expressions but also all the traditional pseudotensor expressions (and thereby it rehabilitates this often discredited approach) while taming the notorious ambiguities: The choice of boundary expression is linked, via the boundary term in the variation of the Hamiltonian, with the choice of boundary condition, while the reference frame ambiguity can be associated with a choice of boundary reference values, which determine the choice of vacuum or ground state for the system. In this way the traditional ambiguities can be given a clear physical and geometric significance [6,21,22]. Within the covariant Hamiltonian formalism, certain *covariant symplectic* expressions for the conserved gravitational quantities have been proposed [4–6]. When we look at GR from this perspective, one of these expressions stands out as being suitable for most applications (among other virtues, it has an associated positive energy proof [24]). Here we consider only this particular expression, specifically, in geometric units

$$E(N,S) = \frac{1}{16\pi}\oint_S (\Delta\omega^{ab} \wedge i_N \eta_{ab} + \overset{\circ}{\nabla}{}^b \overset{\circ}{N}{}^a \Delta\eta_{ab}), \quad (32)$$

where $\Delta\omega^{ab} := \omega^{ab} - \omega^{\circ ab}$ is the difference between the orthonormal frame connection one-forms (i.e., the Ricci rotation one-forms) and their reference values, ∇° and $N^{\circ a}$ are the connection and the displacement vector in the reference space-time, $\eta_{ab} := (1/2)\epsilon_{abcd}\vartheta^c \wedge \vartheta^d$, $\Delta\eta_{ab} := (1/2)\epsilon_{abcd}(\vartheta^c \wedge \vartheta^d - \vartheta^{\circ c} \wedge \vartheta^{\circ d})$, with ϑ^a and $\vartheta^{\circ a}$ being, respectively, the dynamic and reference orthonormal coframes.

The physical and geometric significance of this particular choice of Hamiltonian boundary term expression is revealed by the resultant boundary term in the variation of the Hamiltonian:

$$\delta H(N,\Sigma) = \int_\Sigma (\text{field equation terms}) - \frac{1}{16\pi}\oint_S i_N(\Delta\omega^{ab} \wedge \delta\eta_{ab}). \quad (33)$$

This indicates that we should hold fixed on the boundary S the pullback of η_{ab}, i.e., certain projected components of the coframe—thus, effectively, certain projected components of the metric (arguably the most natural choice).

In the works already cited, this quasilocal expression has been tested in various ways. Here we are concerned with the requirement that the value of the quasilocal energy-momentum and the energy flux have the correct limit at null infinity. To this end, we take $S = \partial\Sigma$, the closed boundary of the 3-dimensional spacelike region Σ, to be a two sphere which approaches in the limit null infinity. The Hamiltonian formalism with a boundary approaching null infinity has been considered from several perspectives; see e.g. [10,25]. A nice detailed discussion of the topic addressing all of the important issues has been given recently [11]. Here we want to consider this problem in the general case.

By choosing the vector N, one can derive the 10 conserved quasilocal quantities for gravity based on the Poincaré symmetry. In particular, the covariant quasilocal energy and momentum associated with the time and space translation asymptotic symmetries are

$$p_\nu = \frac{1}{16\pi}\int_S (\Delta\omega^{ab} \wedge i_{N_\nu}\eta_{ab} + \overset{\circ}{\nabla}{}^b \overset{\circ}{N}{}^a_\nu \Delta\eta_{ab}). \quad (34)$$

Here the value of ν labels what quantities are evaluated, 0 for energy and 1, 2, 3 for the three components of momentum. In the asymptotically flat case, N_ν should be the translation part of the asymptotic Killing vectors. The translation part of the Bondi-Metzner-Sachs (BMS) group is well defined; its expansion, in the leading order, is of the form $N_\nu = N_\nu^{(0)} + O(r^{-1})$, with $\nu = 0$, $k = 1, 2, 3$:

$$\begin{aligned} N_0^{(0)} &= \frac{\partial}{\partial u} = f_0\Big(n + \frac{1}{2}l\Big) + O(r^{-1}),\\ N_k^{(0)} &= f_k\Big(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\Big) = f_k\Big(n - \frac{1}{2}l\Big) + O(r^{-1}), \end{aligned} \quad (35)$$

where $f_\nu = (1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$.

The energy-momentum expression (34) includes two parts which will be considered separately: the purely physical part and the part including the reference, i.e.,

$$p_\nu = p_\nu^{\text{phy}} + p_\nu^{\text{ref}}, \quad (36)$$

where

$$\begin{aligned} p_\nu^{\text{phy}} &:= \frac{1}{16\pi}\int_S (\omega^{ab} \wedge i_{N_\nu}\eta_{ab}),\\ p_\nu^{\text{ref}} &:= \frac{1}{16\pi}\int_S (-\overset{\circ}{\omega}{}^{ab} \wedge i_{N_\nu}\eta_{ab} + \overset{\circ}{\nabla}{}^a \overset{\circ}{N}{}^b_\nu \Delta\eta_{ab}). \end{aligned} \quad (37)$$

We first evaluate the physical part and leave the reference part and the final results to the next section. It is important to keep in mind that the integral is evaluated on a two sphere with constant u and r; therefore, the only contributing term is $\mathbf{m} \wedge \bar{\mathbf{m}}$. The gauge (1) guarantees that the vector m is always tangent to the two sphere. Hereafter, we will only present the coefficient of the 2-form $\mathbf{m} \wedge \bar{\mathbf{m}}$, denoting this as "$\cong$".

From the expansion of the vector N, (35), we realize that the significant contribution for $N = f_\nu(n \pm \frac{1}{2}l)$ is

$$\begin{aligned} \omega^{ab} \wedge i_{N_\nu}\eta_{ab} = f_\nu\{&-i2(\gamma - \bar{\gamma})\boldsymbol{l} \wedge \mathbf{n}\\ &\pm i(\alpha - \bar{\beta} + \bar{\tau} \pm 2\nu)\boldsymbol{l} \wedge \mathbf{m}\\ &\mp i(\bar{\alpha} - \beta + \tau \pm 2\bar{\nu})\boldsymbol{l} \wedge \bar{\mathbf{m}}\\ &- i2(\alpha - \bar{\beta} - \pi)\mathbf{n} \wedge \mathbf{m}\\ &+ i2(\bar{\alpha} - \beta - \bar{\pi})\mathbf{n} \wedge \bar{\mathbf{m}}\\ &+ i[2(\mu + \bar{\mu}) \pm (\rho + \bar{\rho})]\mathbf{m} \wedge \bar{\mathbf{m}}\}. \end{aligned} \quad (38)$$

The imaginary unit i comes from the volume element, i.e., $\epsilon = i\boldsymbol{l} \wedge \mathbf{n} \wedge \mathbf{m} \wedge \bar{\mathbf{m}}$ [26]. However, if we focus only on the value on the two sphere boundary, the result is

$$\omega^{ab} \wedge i_{N_\nu}\eta_{ab} \cong i f_\nu[2(\mu + \bar{\mu}) \pm (\rho + \bar{\rho})]\mathbf{m} \wedge \bar{\mathbf{m}}. \quad (39)$$

IV. GRAVITATIONAL ENERGY-MOMENTUM IN DIFFERENT EMBEDDINGS

Reference configurations play a crucial role in the expressions for gravitational energy and its energy flux (indeed, for all the quasilocal quantities). There are two essential related issues: (i) a "suitable" reference configuration choice and (ii) a proper embedding into the physical space-time.

There are two terms in the reference part. The NP formulation for the first one, $\omega^{\circ ab} \wedge i_N \eta_{ab}$, can be easily read out from (38) by replacing all NP coefficients and the frames within the connection with their reference values.

For an asymptotically flat space-time, the choice of reference configuration is more or less unambiguous—the Minkowski space-time. In Eddington-Finkelstein coordinates (u, r, θ, ϕ), which are related to the standard coordinates by $u = t - r$, the first fundamental form of the Minkowski space-time is

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (40)$$

It is easy to verify that the coordinates are just the Bondi coordinates.

However, the issue of embedding the two-surface S of physical space-time into the Minkowski space-time is not fully transparent yet. Various proposals have been made in earlier works. In this subsection, we consider several types of embedding used for calculating the reference part of the energy; later we will also use these embeddings for energy flux. We will see that the embeddings we consider actually give the same result for the gravitational energy flux at future null infinity, but not quite the same expression for energy itself.

A. Holonomic embedding

The simplest embedding technique is to identify the Bondi coordinates with the Minkowski coordinates; i.e., the embedded surface S_0 is just the standard coordinate round sphere. In this approach, the embedded null frame is

$$\mathring{l} = \frac{\partial}{\partial r}, \quad (41)$$

$$\mathring{n} = \frac{\partial}{\partial u} - \frac{1}{2}\frac{\partial}{\partial r}, \quad (42)$$

$$\mathring{m} = \frac{e^{-i\phi}}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right) = \frac{P}{\sqrt{2}r}\frac{\partial}{\partial\zeta}. \quad (43)$$

Therefore, the nonvanishing NP reference coefficients are

$$\mathring{\rho} = -\frac{1}{r}, \qquad \mathring{\mu} = -\frac{1}{2r}, \qquad \mathring{\alpha} = -\mathring{\bar{\beta}} = \frac{\zeta}{2\sqrt{2}r}. \quad (44)$$

The embedding of the time direction N° is $N^\circ{}_\nu = f_\nu(n^\circ + \frac{1}{2}l^\circ)$, which is a Killing vector of the Minkowski space-time. Properly, we should calculate $\omega^{\circ ab} = \Gamma^{\circ}{}_c{}^{ab}\vartheta^{\circ c}$. However, since the difference of the coframes ϑ^c and $\vartheta^{\circ c}$ is $o(r^{-1})$, we can make an approximation and take for the reference part

$$\mathring{\omega}^{ab} \wedge i_N\eta_{ab} \cong if_\nu[2(\mathring{\mu} + \mathring{\bar{\mu}}) \pm (\mathring{\rho} + \mathring{\bar{\rho}})]\mathbf{m} \wedge \bar{\mathbf{m}}. \quad (45)$$

Moreover, it is straightforward to check, for $N^\circ{}_\nu = f_\nu(n^\circ \pm \frac{1}{2}l^\circ) + O(r^{-1})$, that

$$\begin{aligned}\mathring{\nabla}^a \mathring{N}^b_\nu \Delta\eta_{ab} &= (\mathring{\partial}^a f_\nu)(\mathring{n} \pm \tfrac{1}{2}\mathring{l})^b \Delta\eta_{ab} \\ &\quad + f_\nu \mathring{\nabla}^a(\mathring{n} \pm \tfrac{1}{2}\mathring{l})^b \Delta\eta_{ab} + \mathring{\nabla} O(r^{-1})\Delta\eta \\ &= (\mathring{\partial}^a f_\nu)(\mathring{n} \pm \tfrac{1}{2}\mathring{l})^b \Delta\eta_{ab} - if_\nu[(\mathring{\gamma} + \mathring{\bar{\gamma}}) \\ &\quad \times \Delta(\mathbf{m}\wedge\bar{\mathbf{m}}) + \mathring{\nu}\Delta(\boldsymbol{l}\wedge\mathbf{m}) - \mathring{\bar{\nu}}\Delta(\boldsymbol{l}\wedge\bar{\mathbf{m}})] \\ &\quad + O(r^{-1}) \\ &\cong O(r^{-1}). \end{aligned} \quad (46)$$

Finally, from the results (39), (45), and (46), the energy-momentum in the holonomic embedding is

$$\begin{aligned}\lim_{I^+} p_\nu &= \lim_{I^+}\frac{1}{16\pi}\int_S f_\nu[2(\mu + \bar{\mu} - \mathring{\mu} - \mathring{\bar{\mu}}) \\ &\quad \pm (\rho + \bar{\rho} - \mathring{\rho} - \mathring{\bar{\rho}})]i\mathbf{m}\wedge\bar{\mathbf{m}} \\ &= -\frac{1}{4\pi}\int_S \mathrm{Re}(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0)f_\nu d\Omega^2, \end{aligned} \quad (47)$$

where $d\Omega^2 = \sin\theta d\theta d\phi$. The integrand differs slightly from the usual formula for the Bondi energy-momentum; however, the integral of the extra term vanishes at least for the energy—which is our real interest here—simply because $\eth\bar{\sigma}^0$ is of spin weight -1. Note that entirely analogous terms show up in equivalent calculations done directly in the Bondi-Sachs metric [7,25].

B. Ó Murchadha-Szabados-Tod embedding

In Ref. [27], Ó Murchadha, Szabados, and Tod (OST) introduced another kind of embedding method. Let us consider the spacelike region Σ_0 in the physical space-time, $\partial\Sigma_0 = S$. We suppose that S is a topological two sphere and that isothermal coordinates globally exist on S. In these coordinates, the induced metric ${}^{(2)}ds^2$ on S is

$${}^{(2)}ds^2 = \omega^2(\theta', \phi')(d\theta'^2 + \sin^2\theta' d\phi'^2), \quad (48)$$

where $\omega(\theta', \phi')$ is a positive function on S. This two-surface is embedded isometrically into the physical space-time (in Bondi coordinates) with

$$u = \text{const}, \qquad \theta = \theta', \qquad \phi = \phi', \qquad r = \omega(\theta, \phi). \quad (49)$$

Based on this embedding, we define a coordinate transformation in Minkowski space-time,

$$u \to U, \qquad r \to R + \omega, \qquad \theta \to \theta, \qquad \phi \to \phi. \quad (50)$$

Then the Minkowski metric becomes

$$\mathring{g} = \begin{pmatrix} -1 & -1 & -\partial_\theta\omega & -\partial_\phi\omega \\ -1 & 0 & 0 & 0 \\ -\partial_\theta\omega & 0 & (R+\omega)^2 & 0 \\ -\partial_\phi\omega & 0 & 0 & (R+\omega)^2\sin^2\theta \end{pmatrix}. \tag{51}$$

The NP reference tetrad is chosen to be

$$\begin{aligned} \mathring{l} &= \frac{\partial}{\partial R}, \\ \mathring{n} &= \frac{\partial}{\partial U} - \frac{1}{2}\Big(1 + \frac{|\delta_0\omega|^2}{(R+\omega)^2}\Big)\frac{\partial}{\partial R} + \frac{\partial_\theta\omega}{(R+\omega)^2}\frac{\partial}{\partial\theta} \\ &\quad + \frac{\partial_\phi\omega}{(R+\omega)^2\sin^2\theta}\frac{\partial}{\partial\phi}, \\ \mathring{m} &= \frac{e^{-i\phi}}{\sqrt{2}(R+\omega)}\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big), \end{aligned} \tag{52}$$

where $\delta_0 = \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}$. The associated dual reference tetrad is

$$\begin{aligned} \mathring{\mathbf{l}} &= -dU, \qquad \mathring{\mathbf{n}} = -\frac{1}{2}\Big(1 + \frac{|\delta_0\omega|^2}{(R+\omega)^2}\Big)dU - dR, \\ \mathring{\mathbf{m}} &= -\frac{e^{-i\phi}\delta_0\omega}{\sqrt{2}(R+\omega)}dU + \frac{1}{\sqrt{2}}e^{-i\phi}(R+\omega) \\ &\quad \times (d\theta + i\sin\theta d\phi). \end{aligned} \tag{53}$$

From the above metric, a direct calculation gives the nonvanishing NP reference coefficients as

$$\mathring{\rho} = -\frac{1}{R+\omega}, \tag{54}$$

$$\mathring{\mu} = -\frac{1}{2}\Big[\frac{1}{R+\omega} + \frac{|\delta_0\omega|^2}{(R+\omega)^3} - \frac{\partial_\theta^2\omega + \cot\theta\partial_\theta\omega + \frac{1}{\sin^2\theta}\partial_\phi^2\omega}{(R+\omega)^2}\Big], \tag{55}$$

$$\mathring{\alpha} = -\frac{e^{i\phi}}{2\sqrt{2}}\Big[\frac{\cot\frac{\theta}{2}}{R+\omega} + \frac{2\bar{\delta}_0\omega}{(R+\omega)^2}\Big], \tag{56}$$

$$\mathring{\bar{\beta}} = \frac{e^{i\phi}}{2\sqrt{2}}\frac{\cot\frac{\theta}{2}}{R+\omega}, \tag{57}$$

$$\mathring{\bar{\tau}} = \mathring{\pi} = -\frac{e^{i\phi}\bar{\delta}_0\omega}{\sqrt{2}(R+\omega)^2}, \tag{58}$$

$$\mathring{\gamma} = -\frac{1}{2}\Big[\frac{|\delta_0\omega|^2}{(R+\omega)^3} + \frac{i\cot\frac{\theta}{2}\partial_\phi\omega}{(R+\omega)^2\sin\theta}\Big], \tag{59}$$

$$\begin{aligned} \mathring{\lambda} &= \frac{e^{2i\phi}}{2(R+\omega)^2}\Big[\partial_\theta^2\omega - \frac{\partial_\phi^2\omega}{\sin^2\theta} - \cot\theta\partial_\theta\omega + \frac{2i\cot\theta\partial_\phi\omega}{\sin\theta} \\ &\quad - \frac{2i}{\sin\theta}\partial_\theta\partial_\phi\omega - \frac{2}{R+\omega}\Big(\partial_\theta\omega - \frac{i\partial_\phi\omega}{\sin\theta}\Big)^2\Big] \\ &= \frac{e^{2i\phi}}{2(R+\omega)^2}\Big[\bar{\delta}_0^2\omega - \cot\theta\bar{\delta}_0\omega - \frac{2}{R+\omega}(\bar{\delta}_0\omega)^2\Big], \end{aligned} \tag{60}$$

$$\begin{aligned} \mathring{\nu} &= -\frac{e^{i\phi}}{\sqrt{2}(R+\omega)^3}\Big[\frac{|\delta_0\omega|^2\bar{\delta}_0\omega}{R+\omega} - \delta_0\omega\Big(\bar{\delta}_0\partial_\theta\omega \\ &\quad + \frac{i\cot\theta\partial_\phi\omega}{\sin\theta}\Big) + \frac{i\partial_\phi\omega}{\sin\theta}\Big(\partial_\theta^2\omega + \cot\theta\partial_\theta\omega \\ &\quad + \frac{1}{\sin^2\theta}\partial_\phi^2\omega\Big)\Big]. \end{aligned} \tag{61}$$

The intrinsic geometry of the two sphere is preserved in the OST embedding; therefore, $\Delta\eta_{ab} \cong 0$. Moreover, on the considered two sphere S, $R = 0$ by the definition of the embedding and, from Eqs. (29) and (30), we have

$$\omega(\theta, \phi) = r - \frac{|\sigma^0|^2}{2r} + O(r^{-2}). \tag{62}$$

Therefore, this reference contribution differs from the result of the holonomic embedding only by higher orders of $1/r$. Hence, the energy-momentum at null infinity in the OST embedding is the same as in the holonomic embedding, i.e.,

$$\lim_{I^+} p_\nu = -\frac{1}{4\pi}\int_S \mathrm{Re}(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0) f_\nu d\Omega^2. \tag{63}$$

Again, the result differs from the usual Bondi integrand by the same extra term—which makes a vanishing contribution to the energy when integrated over the two sphere.

C. Brown-Lau-York embedding

In the above two subsections, the two methods used both embedded the surface S into a standard light cone in Minkowski space-time; i.e., the light cone N is the light cone from one point. Brown, Lau, and York [10] gave another way to do the embedding near null infinity. This method considers a more general light cone.

Suppose we choose a Bondi coordinate system (u, R, θ, ϕ) in Minkowski space-time. The asymptotic shear is $\sigma^{\circ 0}$. We can also do the formal Taylor extension near null infinity as in Eqs. (6)–(10). The only difference is that we have $\Psi^\circ{}_i = 0, n = 0, 1, 2, 3, 4$. From the NP equations we have

$$\Psi_3^{(0)} = -\eth_0\partial_u\mathring{\bar{\sigma}}^0, \qquad \Psi_4^{(0)} = -\partial_u^2\mathring{\bar{\sigma}}^0. \tag{64}$$

Because the spin weight of $\partial_u\bar{\sigma}^{\circ 0}$ is nonzero, the above results tell us that $\partial_u\bar{\sigma}^{\circ 0} = 0$ [14]. Inserting these results into Eqs. (6)–(10), the NP quantities in Minkowski space-time are then

$$\begin{aligned}
&\mathring{\rho} = -\frac{1}{R} - \frac{|\sigma^{\circ 0}|^2}{R^3} + O(R^{-5}), \qquad \mathring{\sigma} = \frac{\sigma^{\circ 0}}{R^2} + \frac{|\sigma^{\circ 0}|^2 \sigma^{\circ 0}}{R^4} + O(R^{-5}), \qquad \mathring{\alpha} = \frac{\alpha^0}{R} + \frac{\bar{\sigma}^{\circ 0}\bar{\alpha}^0 + \eth_0 \bar{\sigma}^{\circ 0}}{R^2} + O(R^{-3}),\\
&\mathring{\beta} = -\frac{\bar{\alpha}^0}{R} - \frac{\alpha^0 \sigma^{\circ 0}}{R^2} + O(R^{-3}), \qquad \mathring{\tau} = \mathring{\bar{\pi}} = \frac{\bar{\eth}_0 \sigma^{\circ 0}}{R^2} + O(R^{-3}),\\
&\mathring{\gamma} = \frac{\gamma^{\circ}{}_1}{R^2} + O(R^{-3}) = \frac{-\alpha^0 \bar{\eth}_0 \sigma^{\circ 0} + \bar{\alpha}^0 \eth_0 \bar{\sigma}^{\circ 0}}{R^2} + O(R^{-3}), \qquad \mathring{\mu} = -\frac{1}{2R} - \frac{\eth_0^2 \bar{\sigma}^{\circ 0}}{R^2} + O(R^{-3}),\\
&\mathring{\lambda} = \frac{\frac{1}{2}\bar{\sigma}^{\circ 0} - \bar{\eth}_0 \eth_0 \bar{\sigma}^{\circ 0}}{R^2} + O(R^{-3}), \qquad \mathring{\nu} = O(R^{-3}).
\end{aligned} \tag{65}$$

The tetrad part is

$$\begin{aligned}
\mathring{U} &= -\frac{1}{2} - \frac{\gamma^{\circ}{}_1 + \bar{\gamma}^{\circ}{}_1}{R} + O(R^{-2}),\\
\mathring{X} &= -\frac{\bar{\eth}_0 \sigma^{\circ 0} \xi^0}{R^2} + O(R^{-3}),\\
\mathring{\xi} &= \frac{\xi^0}{R} + \frac{|\sigma^{\circ 0}|^2 \xi^0}{R^3} + O(R^{-4}),\\
\mathring{\eta} &= -\frac{\bar{\sigma}^{\circ 0} \xi^0}{R^2} + O(R^{-4}).
\end{aligned} \tag{66}$$

The intrinsic geometry of the two sphere is preserved in the embedding, i.e.,

$$\mathcal{R} = \mathring{\mathcal{R}}, \tag{67}$$

where the 2-dimensional Ricci scalar $\mathcal{R}$ is given by $\mathcal{R} = -2\rho\mu - 2\bar{\rho}\,\bar{\mu} + 2\sigma\lambda + 2\bar{\sigma}\,\bar{\lambda} + 2\Psi_2 + 2\bar{\Psi}_2$. Consequently, this leads to

$$\begin{aligned}
&-\mu\rho + \lambda\sigma + \Psi_2 + \bar{\Psi}_2 + \bar{\lambda}\,\bar{\sigma} - \bar{\mu}\,\bar{\rho}\\
&= -\mathring{\mu}\,\mathring{\rho} - \mathring{\bar{\mu}}\,\mathring{\bar{\rho}} + \mathring{\lambda}\,\mathring{\sigma} + \mathring{\bar{\lambda}}\,\mathring{\bar{\sigma}}.
\end{aligned} \tag{68}$$

Using the Taylor expansion in Eqs. (6)–(10) and Eqs. (65), we find that the relation between the parameter r and R is

$$\begin{aligned}
&R = r + k + O(r^{-1})\\
&\quad \text{with } k = \eth_0^2 \mathring{\bar{\sigma}}^0 + \bar{\eth}_0^2 \mathring{\sigma}^0 - \eth_0^2 \bar{\sigma}^0 - \bar{\eth}_0^2 \sigma^0.
\end{aligned} \tag{69}$$

We choose $\sigma^{\circ 0}|_{S_0} = \sigma^0|_S$, where S is the section on $\mathcal{I}^+$ and S_0 is its image under the embedding. We have

$$R = r + O(r^{-1}) = r[1 + O(r^{-2})]. \tag{70}$$

As for the OST embedding, the two sphere geometry is preserved; therefore, $\Delta\eta_{ab} \cong 0$. Finally, the gravitational energy-momentum in the BLY embedding is

$$\lim_{\mathcal{I}^+} p_\nu = -\frac{1}{4\pi}\int_S \mathrm{Re}(\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0) f_\nu d\Omega^2. \tag{71}$$

It is worth noting that the BLY embedding is a little neater, in that it directly gives the standard Bondi energy-momentum integrand, whereas there is an additional term in the other two embeddings (which vanishes upon integration). That term is associated with the embedding methods in which the section S is not embedded into a standard light cone, generated by the null geodesics starting from a single point. This difference again shows us that keeping the inner geometry unchanged under the embedding is not enough to ensure physically reasonable quasilocal quantities; generally, as is especially clear from [3], we need more restrictions.

V. THE ENERGY FLUX AT NULL INFINITY VIA THE DIRECT METHOD

In this section, we directly calculate the energy flux through a two sphere. For simplicity, we choose the timelike translation to be $N = n + \frac{1}{2}l$. Asymptotically, this agrees with the natural choice, the time translation of the BMS group at null infinity ∂_u.

We consider the energy $E = H(N, S)$. Suppose Σ_0 is the spacelike region that we want to consider, $\partial\Sigma_0 = S$, and $\Sigma_{\Delta t}$ is the time evolution of Σ_0. The energy within the region during the time interval changes by the amount $E(\Sigma_{\Delta t}) - E(\Sigma_0)$; hence, a natural direct definition of the rate of energy change is

$$\dot{E} := \lim_{\Delta t \to 0} \frac{E(\Sigma_{\Delta t}) - E(\Sigma_0)}{\Delta t}. \tag{72}$$

Looking to the value of the Hamiltonian, taking into account the vanishing of the initial value constraints, from (32) we straightforwardly get the quasilocal energy-flux relation

$$\begin{aligned}
\dot{E} = \dot{H}(N, \Sigma) &:= \frac{1}{16\pi}\oint_S \mathcal{L}_N(\Delta\omega^{ab} \wedge i_N \eta_{ab} - \mathring{\nabla}^a \mathring{N}^b \Delta\eta_{ab})\\
&= \frac{1}{16\pi}\oint_S [\mathcal{L}_N(\omega^{ab} \wedge i_N \eta_{ab})\\
&\quad - \mathcal{L}_N(\mathring{\omega}^{ab} \wedge i_N \eta_{ab}) - \mathcal{L}_N(\mathring{\nabla}^a \mathring{N}^b \Delta\eta_{ab})].
\end{aligned} \tag{73}$$

The right-hand side defines an energy-flux expression F, which includes two parts that will be considered separately: the purely physical part and the other part, which includes the reference, i.e., $F = F_{\text{phy}} + F_{\text{ref}}$. The reason for making such a separation is that the part including the reference, as we have already seen, depends on the embedding of the reference configuration into physical spacetime.

We first evaluate the physical part of the flux,

$$\mathcal{L}_N(\omega^{ab}\wedge i_N\eta_{ab}) = \tfrac{1}{2}\mathcal{L}_l(\omega^{ab}\wedge i_N\eta_{ab}) + \mathcal{L}_n(\omega^{ab}\wedge i_N\eta_{ab}). \qquad (74)$$

We see that it is necessary to know the Lie derivative of all 2-form elements. However, we are interested only in the final results that can contribute to the integral: the terms proportional to $\mathbf{m}\wedge\bar{\mathbf{m}}$. After a straightforward verification, the contributing terms are

$$\begin{aligned}\mathcal{L}_l(\mathbf{m}\wedge\bar{\mathbf{m}}) &\cong -(\rho+\bar\rho)\mathbf{m}\wedge\bar{\mathbf{m}},\\ \mathcal{L}_n(\mathbf{m}\wedge\bar{\mathbf{m}}) &\cong (\mu+\bar\mu)\mathbf{m}\wedge\bar{\mathbf{m}},\\ \mathcal{L}_n(\mathbf{n}\wedge\mathbf{m}) &\cong -\bar\nu\mathbf{m}\wedge\bar{\mathbf{m}}.\end{aligned} \qquad (75)$$

Therefore, following the result of $\omega^{ab}\wedge i_N\eta_{ab}$ in Eq. (38), the first term is

$$\begin{aligned}\mathcal{L}_l(\omega^{ab}\wedge i_N\eta_{ab}) &\cong iD[(\rho+\bar\rho)+2(\mu+\bar\mu)]\mathbf{m}\wedge\bar{\mathbf{m}} + i[(\rho+\bar\rho)+2(\mu+\bar\mu)]\mathcal{L}_l(\mathbf{m}\wedge\bar{\mathbf{m}})\\ &\cong i\{D[(\rho+\bar\rho)+2(\mu+\bar\mu)] - (\rho+\bar\rho)[(\rho+\bar\rho)+2(\mu+\bar\mu)]\}\mathbf{m}\wedge\bar{\mathbf{m}}\\ &= i[2\delta\pi + 2\bar\delta\bar\pi + \mathcal{R} - 2\rho^2 + 2|\sigma|^2 + 4|\pi|^2 - 2(\bar\alpha-\beta)\pi - 2(\alpha-\bar\beta)\bar\pi]\mathbf{m}\wedge\bar{\mathbf{m}},\end{aligned} \qquad (76)$$

where $\mathcal{R}$ is the 2-dimensional Ricci scalar on the enclosed surface. Similarly, the second term is

$$\begin{aligned}\mathcal{L}_n(\omega^{ab}\wedge i_N\eta_{ab}) &\cong iD'[(\rho+\bar\rho)+2(\mu+\bar\mu)]\mathbf{m}\wedge\bar{\mathbf{m}} + i[(\rho+\bar\rho)+2(\mu+\bar\mu)]\mathcal{L}_n(\mathbf{m}\wedge\bar{\mathbf{m}})\\ &\quad - i2(\alpha-\bar\beta-\pi)\mathcal{L}_n(\mathbf{n}\wedge\mathbf{m}) + i2(\bar\alpha-\beta-\bar\pi)\mathcal{L}_n(\mathbf{n}\wedge\bar{\mathbf{m}})\\ &\cong i\{D'[(\rho+\bar\rho)+2(\mu+\bar\mu)] + (\mu+\bar\mu)[(\rho+\bar\rho)+2(\mu+\bar\mu)] + 2(\alpha-\bar\beta-\pi)\bar\nu - 2(\bar\alpha-\beta-\bar\pi)\nu\}\mathbf{m}\wedge\bar{\mathbf{m}}\\ &= i[\delta\bar\tau + \bar\delta\tau + 2\delta\nu + 2\bar\delta\bar\nu - \frac{1}{2}\mathcal{R} + 4\mu^2 - 4|\lambda|^2 + 2(\gamma+\bar\gamma)(\rho-2\mu) + 2|\pi|^2 + (\bar\alpha-\beta)\pi + (\alpha-\bar\beta)\bar\pi\\ &\quad + 2\nu(\bar\alpha+3\beta) + 2\bar\nu(\alpha+3\bar\beta) + 2(\alpha-\bar\beta-\pi)\bar\nu - 2(\bar\alpha-\beta-\bar\pi)\nu]\mathbf{m}\wedge\bar{\mathbf{m}}.\end{aligned} \qquad (77)$$

Finally, the purely physical contribution to the energy flux is

$$\begin{aligned}F_{\rm phy} &= \frac{1}{16\pi}\int_S \mathcal{L}_N(\omega^{ab}\wedge i_N\eta_{ab})\\ &= \frac{i}{8\pi}\int_S\bigg[\delta\pi + \bar\delta\bar\pi + \delta\nu + \bar\delta\bar\nu - \frac{1}{2}\rho^2 + \frac{1}{2}|\sigma|^2 + 2\mu^2 + 2|\pi|^2 - 2|\lambda|^2 + (\gamma+\bar\gamma)(\rho-2\mu) + \nu(\bar\alpha+3\beta)\\ &\quad + \bar\nu(\alpha+3\bar\beta) + (\alpha-\bar\beta-\pi)\bar\nu - (\bar\alpha-\beta-\bar\pi)\nu\bigg]\mathbf{m}\wedge\bar{\mathbf{m}}.\end{aligned} \qquad (78)$$

From the asymptotic expansion of all the NP coefficients (6)–(10), (12), (24), (25), and (27), we find that all except one of the terms fall off as $O(1/r^3)$ or faster; only the $-2|\lambda|^2$ term contributes asymptotically. Hence, the null infinity limit of the purely physical flux is

$$\lim_{I^+}F_{\rm phy} = \lim_{I^+}\frac{1}{8\pi}\int_S\bigg[-2\frac{|\dot{\bar\sigma}^0|^2}{r^2} + O(r^{-3})\bigg]i\mathbf{m}\wedge\bar{\mathbf{m}} = -\frac{1}{4\pi}\int_S|\dot{\bar\sigma}^0|^2 d\Omega^2. \qquad (79)$$

A. Holonomic embedding

A straightforward calculation gives

$$\begin{aligned}\mathcal{L}_N(\mathring{\omega}^{ab}\wedge i_N\eta_{ab}) &= \mathcal{L}_N[(\mathring\rho + \mathring{\bar\rho} + 2\mathring\mu + 2\mathring{\bar\mu})i\mathbf{m}\wedge\bar{\mathbf{m}} + (\mathring{\bar\alpha} - \mathring\beta)i\mathbf{n}\wedge\bar{\mathbf{m}} - (\mathring\alpha - \mathring{\bar\beta})i\mathbf{n}\wedge\mathbf{m} + O(r^{-3})]\\ &\cong [(\mathring\rho + \mathring{\bar\rho} + 2\mathring\mu + 2\mathring{\bar\mu})(-\tfrac{1}{2}\rho - \tfrac{1}{2}\bar\rho + \mu + \bar\mu) + O(r^{-3})]i\mathbf{m}\wedge\bar{\mathbf{m}},\\ \mathcal{L}_N(\mathring\nabla^a\mathring N^b\Delta\eta_{ab}) &= \mathcal{L}_N[-\tfrac{1}{2}(\mathring\gamma + \mathring{\bar\gamma})i(\mathbf{m}\wedge\bar{\mathbf{m}} - \mathring{\mathbf{m}}\wedge\mathring{\bar{\mathbf{m}}}) + \tfrac{1}{2}\mathring{\bar\nu}i(\boldsymbol{l}\wedge\bar{\mathbf{m}} - \mathring{\boldsymbol{l}}\wedge\mathring{\bar{\mathbf{m}}}) - \tfrac{1}{2}\mathring\nu i(\boldsymbol{l}\wedge\mathbf{m} - \mathring{\boldsymbol{l}}\wedge\mathring{\mathbf{m}})]\\ &\cong 0.\end{aligned} \qquad (80)$$

Therefore, the reference part in the holonomic embedding is

$$F_{\rm ref} = -\frac{1}{16\pi}\int_S\Big[(\mathring{\rho} + \mathring{\bar{\rho}} + 2\mathring{\mu} + 2\mathring{\bar{\mu}}) \times\Big(-\tfrac{1}{2}\rho - \tfrac{1}{2}\bar{\rho} + \mu + \bar{\mu}\Big) + O(r^{-3})\Big] i\mathbf{m}\wedge\bar{\mathbf{m}} = \frac{1}{16\pi}\int_S O(r^{-3}) i\mathbf{m}\wedge\bar{\mathbf{m}}. \quad (81)$$

Finally, we have

$$\lim_{I^+} F = -\frac{1}{4\pi}\int_S |\dot{\sigma}^0|^2 d\Omega^2. \quad (82)$$

This is just the standard expression of the Bondi energy flux at $\mathcal{I}^+$ [9,14,16,28].

B. Ó Murchadha-Szabados-Tod embedding

The reference part in the OST embedding is

$$\mathcal{L}_N(\mathring{\omega}^{ab}\wedge i_N\eta_{ab}) \cong \Big[(-\mathring{\bar{\pi}} + \mathring{\bar{\alpha}} - \mathring{\beta})\nu + (-\mathring{\pi} + \mathring{\alpha} - \mathring{\bar{\beta}})\bar{\nu} + (\mathring{\rho} + \mathring{\bar{\rho}} + \mathring{\mu} + \mathring{\bar{\mu}})\Big(-\frac{\rho+\bar{\rho}}{2} + \mu + \bar{\mu}\Big)\Big] i\mathbf{m}\wedge\bar{\mathbf{m}},$$
$$\mathcal{L}_N(\mathring{\nabla}^a \mathring{N}^b\Delta\eta_{ab}) \cong \mathring{\nabla}^a \mathring{N}^b\mathcal{L}_N(\Delta\eta_{ab}) \cong -[(\mathring{\gamma} + \mathring{\bar{\gamma}})\mathcal{L}_N\Delta(\mathbf{m}\wedge\bar{\mathbf{m}}) + \mathring{\nu}\mathcal{L}_N\Delta(\boldsymbol{l}\wedge\mathbf{m}) - \mathring{\bar{\nu}}\mathcal{L}_N\Delta(\boldsymbol{l}\wedge\bar{\mathbf{m}})]. \quad (83)$$

The image of S is the two sphere in Minkowski spacetime such that $U = \text{const}$ and $R = 0$. From the above calculation, we get the same result as before:

$$F_{\rm ref} = \int_S O(r^{-3}) i\mathbf{m}\wedge\bar{\mathbf{m}}. \quad (84)$$

Consequently, for the energy flux we get the same result as Eq. (82).

C. Brown-Lau-York embedding

Following up on the result in the case of the energy-momentum calculation, the reference part of the energy flux is

$$F_{\rm ref} = \int_S O(r^{-3}) i\mathbf{m}\wedge\bar{\mathbf{m}}. \quad (85)$$

Hence, for the total energy flux we again we get the same result as Eq. (82).

VI. THE ENERGY FLUX AT NULL INFINITY VIA AN IDENTITY

In this section, we calculate the energy flux through a two sphere using an interesting formal Hamiltonian identity [29] derived in detail in Ref. [7]. The identity is simply the analogue of the classical mechanics identity $\dot{H} \equiv 0$, which follows from $\delta H = \dot{q}^k\delta p_k - \dot{p}_k\delta q_k$ by simply replacing $\delta \to d/dt$. We can obtain it directly in essentially the same way from (33) simply by substituting the time derivative operator $\mathcal{L}_N$ for δ; then the field equation terms cancel identically (just as they did in the classical mechanics case), leaving

$$\dot{E} := \dot{H}(N,\Sigma) \equiv -\oint_S i_N(\Delta\omega^{ab}\wedge\mathcal{L}_N\eta_{ab}) \equiv \oint_S(-i_N\Delta\omega^{ab}\mathcal{L}_N\eta_{ab} + \Delta\omega^{ab}\wedge\mathcal{L}_N i_N\eta_{ab}). \quad (86)$$

This is a general quasilocal formula for energy flux, applicable to the boundary of any region. In Ref. [7] it was tested in the null infinity limit using the Bondi-Sachs metric.

Here we wish to transcribe this expression into the NP spin coefficient form and confirm that it gives the desired asymptotic results using that well developed technique. For our calculation here, we take $N = \partial_u = n^\circ + \frac{1}{2}l^\circ = n + \frac{1}{2}l + O(1/r)$, the reference geometry Killing field.

Let us consider Eq. (86) term by term. The first term is

$$\begin{aligned} i_N\Delta\omega^{ab}\mathcal{L}_N\eta_{ab} &= \Delta\omega^{ab}_n\mathcal{L}_n\eta_{ab} + \tfrac{1}{2}\Delta\omega^{ab}_n\mathcal{L}_l\eta_{ab} + \tfrac{1}{2}\Delta\omega^{ab}_l\mathcal{L}_n\eta_{ab} + \tfrac{1}{4}\Delta\omega^{ab}_l\mathcal{L}_l\eta_{ab} \\ &= 2\Delta(\gamma+\bar{\gamma})\mathcal{L}_n(i\mathbf{m}\wedge\bar{\mathbf{m}}) - 2\Delta\nu\mathcal{L}_n(i\mathbf{m}\wedge\boldsymbol{l}) - 2\Delta\bar{\nu}\mathcal{L}_n(i\boldsymbol{l}\wedge\bar{\mathbf{m}}) + 2\Delta\bar{\tau}\mathcal{L}_n(i\mathbf{n}\wedge\mathbf{m}) + 2\Delta\tau\mathcal{L}_n(i\bar{\mathbf{m}}\wedge\mathbf{n}) \\ &\quad + 2\Delta(\gamma-\bar{\gamma})\mathcal{L}_n(i\mathbf{n}\wedge\boldsymbol{l}) + \Delta(\gamma+\bar{\gamma})\mathcal{L}_l(i\mathbf{m}\wedge\bar{\mathbf{m}}) - \Delta\nu\mathcal{L}_l(i\mathbf{m}\wedge\boldsymbol{l}) - \Delta\bar{\nu}\mathcal{L}_l(i\boldsymbol{l}\wedge\bar{\mathbf{m}}) + \Delta\bar{\tau}\mathcal{L}_l(i\mathbf{n}\wedge\mathbf{m}) \\ &\quad + \Delta\tau\mathcal{L}_l(i\bar{\mathbf{m}}\wedge\mathbf{n}) + \Delta(\gamma-\bar{\gamma})\mathcal{L}_l(i\mathbf{n}\wedge\boldsymbol{l}) - \Delta(\varepsilon+\bar{\varepsilon})\mathcal{L}_n(i\mathbf{m}\wedge\bar{\mathbf{m}}) - \Delta\pi\mathcal{L}_n(i\mathbf{m}\wedge\boldsymbol{l}) - \Delta\bar{\pi}\mathcal{L}_n(i\boldsymbol{l}\wedge\bar{\mathbf{m}}) \\ &\quad - \Delta\bar{\kappa}\mathcal{L}_n(i\mathbf{n}\wedge\mathbf{m}) - \Delta\kappa\mathcal{L}_n(i\bar{\mathbf{m}}\wedge\mathbf{n}) + \Delta(\varepsilon-\bar{\varepsilon})\mathcal{L}_n(i\mathbf{n}\wedge\boldsymbol{l}) + \tfrac{1}{2}[-\Delta(\varepsilon+\bar{\varepsilon})\mathcal{L}_l(i\mathbf{m}\wedge\bar{\mathbf{m}}) \\ &\quad - \Delta\pi\mathcal{L}_l(i\mathbf{m}\wedge\mathbf{l}) - \Delta\bar{\pi}\mathcal{L}_l(i\boldsymbol{l}\wedge\bar{\mathbf{m}}) - \Delta\bar{\kappa}\mathcal{L}_l(i\mathbf{n}\wedge\mathbf{m}) - \Delta\kappa\mathcal{L}_l(i\bar{\mathbf{m}}\wedge\mathbf{n}) + \Delta(\varepsilon-\bar{\varepsilon})\mathcal{L}_l(i\mathbf{n}\wedge\boldsymbol{l})]. \quad (87) \end{aligned}$$

Based on the asymptotic estimations given in the previous sections and the three considered embedding methods, all of these terms are of higher order than $O(1/r)$; hence, they make no contribution to the flux asymptotically. Basically, this comes about because (i) in our gauge $\epsilon = \kappa = 0$, (ii) the Lie derivative introduces a factor of $1/r$, (iii) the spin coefficients $\sigma, \pi, \tau, \gamma, \nu$ are $O(1/r^2)$, and (iv) for ρ, α, β, μ the Δ operation removes their $O(1/r)$ part.

Similarly, for the second term we find

$$\begin{aligned}\Delta\omega^{ab}\wedge\mathcal{L}_N i_N\eta_{ab} &\cong (-\Delta\omega^{ab}_m i_{\bar{m}}\mathcal{L}_N i_N\eta_{ab} + \Delta\omega^{ab}_{\bar{m}} i_m\mathcal{L}_N i_N\eta_{ab})\mathbf{m}\wedge\bar{\mathbf{m}} \\ &= 2[\Delta(\bar{\alpha}+\beta)i_{\bar{m}}\mathcal{L}_N i_N\eta_{01} - (\Delta\mu)i_{\bar{m}}\mathcal{L}_N i_N\eta_{02} - (\Delta\bar{\lambda})i_{\bar{m}}\mathcal{L}_N i_N\eta_{03} + (\Delta\bar{\rho})i_{\bar{m}}\mathcal{L}_N i_N\eta_{12} \\ &\quad + (\Delta\sigma)i_{\bar{m}}\mathcal{L}_N i_N\eta_{13} + \Delta(\bar{\alpha}-\beta)i_{\bar{m}}\mathcal{L}_N i_N\eta_{23} - \Delta(\alpha+\bar{\beta})i_m\mathcal{L}_N i_N\eta_{01} + (\Delta\lambda)i_m\mathcal{L}_N i_N\eta_{02} \\ &\quad + (\Delta\bar{\mu})i_m\mathcal{L}_N i_N\eta_{03} - (\Delta\bar{\sigma})i_m\mathcal{L}_N i_N\eta_{12} - (\Delta\rho)i_m\mathcal{L}_N i_N\eta_{13} + \Delta(\alpha-\bar{\beta})i_m\mathcal{L}_N i_N\eta_{23}]\mathbf{m}\wedge\bar{\mathbf{m}} \\ &= -4|\lambda|^2 i\mathbf{m}\wedge\bar{\mathbf{m}} + O(\tfrac{1}{r}). \end{aligned} \tag{88}$$

Again we used the asymptotic estimations and the three considered embedding methods. In this case we get a nonvanishing asymptotic contribution from $\lambda = O(1/r)$ (25). Submitting these results into Eq. (86), we find that the null infinity limit of the energy flux calculated from that Hamiltonian identity relation is

$$\lim_{I^+} F = -\frac{1}{4\pi}\int_S |\dot{\sigma}^0|^2 d\Omega^2, \tag{89}$$

which is, just as was found from the direct calculation (82), the standard flux loss due to the Bondi news.

VII. DISCUSSION

We have tested certain expressions for the quasilocal energy-momentum and energy flux of gravitating systems. The expressions were obtained from the covariant Hamiltonian formalism. In this formalism, the quasilocal quantities are determined by the value of the boundary term in the Hamiltonian. The variation of the Hamiltonian associates the choice of boundary term with specific boundary conditions. Thus, the definition of the quasilocal energy-momentum of a gravitating system is linked to the choice of boundary conditions. The boundary term that corresponds to holding certain projected components of the orthonormal frame fixed seems to be the best choice for most purposes. We have considered only that choice here (the values for certain other choices are given in Ref. [7]).

Energy flux can be computed in more than one way. On the one hand, it can be obtained directly from the change in the energy expression. On the other hand, one can use an interesting identity associated with the specific role of the Hamiltonian and its variation. Here we have evaluated the energy flux by both techniques.

In strong field regions we do not have any sharp test as to what values we should find for energy-momentum and energy flux. Proposed expressions necessarily are first tested in the weak field linearized theory limits. Getting good values at spatial infinity is not the strongest test. The Bondi limit at future null infinity is more delicate. Here we tested our selected expression for energy-momentum and its associate energy flux in this limit.

Technically, we used a well-known and well-developed technique: the Newman-Penrose spin coefficients. We selected a suitable gauge and found that we needed certain quantities expanded in more detail than is usual [30]. In the quasilocal expressions, it is necessary to select reference values which determine the "vacuum" or "ground state." The natural choice is, of course, Minkowski space, but it is not so obvious how to embed the Minkowski space into the asymptotic part of the dynamic space. We considered three types of embeddings which have been used: holonomic, one due to Ó Murchadha, Szabados, and Tod [27], and one due to Brown, York, and Lau [10]. We found some interesting technical differences between the embeddings but in the end they all gave the same answer: namely, the expected Bondi energy and the Bondi energy flux determined by the Bondi news.

In the detailed calculation we noted that, at least in the selected gauge, the quasilocal energy was asymptotically determined by the deviation of the spin coefficient μ from its asymptotic Minkowski value and that the energy flux was determined by the spin coefficient λ (in the notation of Ref. [17], these coefficients are $-\rho'$ and $-\sigma'$, respectively).

We have shown that the values of these expressions can be practically calculated in terms of the NP spin coefficient technique; the expressions were found to have the desired asymptotic values. Thus, they satisfy an important criterion for quasilocal energy and energy-flux expressions.

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