

Cosmic string evolution in higher dimensions

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We obtain the equations of motion for cosmic strings in extensions of the $3 + 1$ Friedmann-Robertson-Walker (FRW) model with extra dimensions. From these we derive a generalization of the velocity-dependent one-scale model for cosmic string network evolution which we apply, first, to a higher-dimensional isotropic $D + 1$ FRW model and, second, to a $3 + 1$ FRW model with static flat extra dimensions. In the former case the string network does not achieve a scaling regime because of the diminishing rate of string intersections ($D > 3$), but this can be avoided in the latter case by considering compact, small extra dimensions, for which there is a reduced but still appreciable string intercommuting probability. We note that the velocity components lying in the three expanding dimensions are Hubble damped, whereas those in the static extra dimensions are only very weakly damped. This leads to the pathological possibility, in principle, that string motion in the three infinite dimensions can come to a halt preventing the strings from intersecting, with the result that scaling is not achieved and the strings irreversibly dominate the early universe. We note criteria by which this can be avoided, notably if the spatial structure of the network becomes essentially three-dimensional, as is expected for string networks produced in brane inflation. Applying our model to a brane inflation setting, we find scaling solutions in which the effective 3D string motion does not necessarily stop, but it is slowed down because of the excitations trapped in the extra dimensions. These effects are likely to influence cosmic string network evolution for a long period after formation and we discuss their more general implications.

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I. INTRODUCTION

There has been a recent resurgence of interest in cosmic strings for both theoretical and observational reasons. Of particular interest is the generic possibility that cosmic strings can be produced at the end of an inflationary phase in models of brane inflation [1,2]. The evolution of the network of the strings created in these models can be very different from the standard field theory case, thus providing a potential observational window on superstring physics [3–5]. On the observational side, there are perennial cases of astrophysical phenomena for which cosmic strings have been invoked as an explanation in the absence of some more orthodox mechanism; recent examples are two peculiar gravitational lensing events [6–9] but which need further independent follow-up. However, expected improvements in observational data, particularly from high resolution cosmic microwave background (CMB) experiments and gravitational wave detectors, present us with the very real prospect of detecting or constraining cosmic strings over a wide range of predicted energy scales (see, for example, [2,10]). Further theoretical motivation has come from a recent phenomenological study of supersymmetric grand unified theory (SUSY GUT) models [11], which again found generic cosmic string production (for all cases which solve the monopole problem).

It is the recent work on cosmic strings appearing in higher-dimensional theories, such as brane inflation, which

primarily motivates our present study. For spacetime dimension greater than four, strings no longer generically collide, so that loop production will be highly suppressed. Loops radiate away energy from the long string network, so this suppression will result in a much higher density of cosmic strings than in the usual $3 + 1$ -dimensional case. Jones, Stoica and Tye [3] have estimated this enhancement by using a three-dimensional one-scale model and introducing an intercommuting probability $P < 1$ to account for the fact that strings generically miss each other due to the presence of extra dimensions. They suggest that the enhancement on the energy density of the string network is of order P^{-2} , which can be orders of magnitude different than the usual case.

This approach however does not take into account string velocities in the extra dimensions. In general, cosmic strings are subject to the constraint that the average velocity squared of string segments must be less than $1/2$. Thus the fact that strings are moving in the extra dimensions will slow down their apparent three-dimensional motion. One might naively expect that velocities in the infinite dimensions will be redshifted by the expansion, while velocities in the compact dimensions will not if these dimensions are static. Hence there is the cosmologically dangerous possibility that velocities in the extra dimensions will accumulate and dominate while string motion in the three infinite dimensions will come to a halt. This can occur before SUSY breaking, when the fields corresponding to the string position in the extra dimensions are expected to become massive. However, if 3D string motion stops for long enough in the early universe, then the strings would irre-

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versibly dominate the energy density of the universe making a subsequent hot big bang model impossible. Thus we need to go beyond the simple analysis presented to date in order to gain a better quantitative understanding of cosmic string evolution in higher-dimensional spacetimes, notably taking into account the important role of velocities in the extra dimensions.

String evolution in three spatial dimensions has been studied by various authors. Kibble [12] described string networks by a single length scale, the ‘‘correlation length,’’ and showed that it evolves towards a scaling solution in which it stays constant with respect to the horizon size. Bennett [13,14] later modified this ‘‘one-scale’’ model with similar conclusions, subject to a condition on the efficiency of small loop production. The existence and stability of this scaling solution was verified by numerical simulations [15,16]. These studies also revealed new physics at smaller scales, in particular, the accumulation of significant small-scale structure on strings, which results in loop production at much smaller scales than initially thought.

To try to incorporate small-scale structure in analytic models, a number of different approaches have been attempted. These include a ‘‘kink-counting’’ model [17,18], a functional approach [19], a ‘‘three-scale’’ model [20] and a ‘‘wiggly’’ model [21]. Including small-scale structure in analytical models comes with the cost of introducing several extra parameters, which need to be fixed by simulations.

However, the large-scale properties of string networks can be quantitatively described by the velocity-dependent one-scale (VOS) model [22–24], which does not suffer from this problem. By introducing a variable rms string velocity, the VOS model extends its validity from the friction dominated regime at early times, through the matter-radiation transition to Λ domination at late times, thus describing the complete cosmological history of string networks. Though it does not directly model small-scale structure, it provides a ‘‘thermodynamic’’ large-scale description of cosmic string evolution, which agrees remarkably well with high resolution numerical simulations.

The purpose of this paper is to extend the VOS model to spacetimes of higher dimension. Although strong motivation is provided by brane inflation, where the extra dimensions are small and stabilized, we intend to keep the discussion as general as possible to include time-varying extra dimensions, as, for example, in a $(D + 1)$ -dimensional Friedmann-Robertson-Walker (FRW) universe.

The paper is organized as follows. In Sec. II we discuss cosmic string dynamics in a FRW spacetime with isotropic and flat (but possibly expanding) extra dimensions. Starting from the Nambu-Goto action in this spacetime we derive the equations of motion as well as an expression for the energy of a cosmic string network. In Sec. III we

derive the averaged equations describing the evolution of cosmic strings in that $(D + 1)$ -dimensional spacetime. After we briefly review the $3 + 1$ VOS model, we extend it to $D + 1$ dimensions and comment on some qualitative features of solutions. In Sec. IV we consider possible application of this extradimensional VOS (EDVOS) model to the case of brane inflation and discuss the dependence of the results on various parameters as, for example, the intercommuting probability of strings. We conclude in Sec. V. Finally, there is an appendix where we derive approximate formulas for the momentum parameters of the $(D + 1)$ -dimensional VOS model.

II. COSMIC STRING DYNAMICS IN HIGHER DIMENSIONS

We consider a cosmic string propagating in a $D + 1$ -dimensional spacetime with metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, \dots, D$). In the limit that its thickness is much smaller than its radius of curvature, the string can be regarded as a one-dimensional object with a world history described by a two-dimensional spacetime surface, the string world sheet

$$x^\mu = x^\mu(\zeta^\alpha), \quad \alpha = 0, 1. \quad (1)$$

The dynamics is given by the Nambu-Goto action

$$S = -\mu \int \sqrt{-\gamma} d^2 \zeta \quad (2)$$

where μ is the string tension and γ is the determinant of $\gamma_{\alpha\beta} = g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$, the pullback metric on the world sheet.

The equations of motion for the fields x^μ obtained from this action are given by

$$\nabla^2 x^\mu + \Gamma_{\nu\lambda}^\mu \gamma^{\alpha\beta} \partial_\alpha x^\nu \partial_\beta x^\lambda = 0 \quad (3)$$

where $\Gamma_{\nu\lambda}^\mu$ is the $(D + 1)$ -dimensional Christoffel symbol

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\kappa} (\partial_\lambda g_{\kappa\nu} + \partial_\nu g_{\kappa\lambda} - \partial_\kappa g_{\nu\lambda}) \quad (4)$$

and $\nabla^2 x^\mu$ the covariant Laplacian of the world sheet fields x^μ given by

$$\nabla^2 x^\mu = \frac{1}{\sqrt{-\gamma}} \partial_\alpha (\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\beta x^\mu). \quad (5)$$

By varying the action with respect to the background metric $g_{\mu\nu}$ we obtain a spacetime energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \mu \int d^2 \zeta \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \delta^{(4)}(x^\lambda - x^\lambda(\zeta^\alpha)). \quad (6)$$

We wish to study the evolution of cosmic strings in a FRW universe with $D - 3$ extra dimensions. For simplicity we choose the following metric, allowing toroidal compactification of the extra dimensions:

$$ds^2 = N(t)^2 dt^2 - a(t)^2 d\mathbf{x}^2 - b(t)^2 d\mathbf{l}^2 \quad (7)$$

where $t \equiv x^0$, $\mathbf{x} \equiv x^i$ with $i = 1, 2, 3$ and $\mathbf{l} \equiv x^\ell$ with $\ell = 4, 5, \dots, D$. The lapse function $N(t)$ allows us to switch from cosmic [$N(t) = 1$] to conformal time by simply setting $N(t) = a(t)$. For stabilized compact extra dimensions, $b(t)$ is set to a constant and the coordinates \mathbf{l} are periodically identified. Alternatively, the scale factor $b(t)$ of the “ ℓ space” can be set to $a(t)$ for a generalized $(D + 1)$ -dimensional FRW universe or even more generally be allowed to evolve independently of $a(t)$.

The action (2) is invariant under world sheet reparametrizations, which we can use to choose a gauge. In flat space either the “light-cone” ($\zeta^0 = t$, $\zeta^1 = z - t$) or the “conformal” gauge ($\dot{x} \cdot x' = 0$, $\dot{x}^2 + x'^2 = 0$) is usually chosen but for cosmological backgrounds it is convenient to work in the gauge

$$\zeta^0 = t, \quad \dot{x} \cdot x' = 0 \quad (8)$$

where dots and primes denote derivatives with respect to the timelike and spacelike world sheet coordinates ζ^0 and ζ^1 , respectively. By choosing this gauge we identify space-time and world sheet times, while we impose that the vector \dot{x} is perpendicular to the string tangent, thus representing the physically observable velocity.

In this gauge the equations of motion (3) in the space-time (7) are

$$\dot{\epsilon} = -N^{-2} \epsilon \left\{ N\dot{N} + a\dot{a} \left[\dot{\mathbf{x}}^2 - \left(\frac{\mathbf{x}'}{\epsilon} \right)^2 \right] + b\dot{b} \left[\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon} \right)^2 \right] \right\} \quad (9)$$

$$\ddot{\mathbf{x}} + \left\{ \frac{2\dot{a}}{a} - N^{-2} \left\{ N\dot{N} + a\dot{a} \left[\dot{\mathbf{x}}^2 - \left(\frac{\mathbf{x}'}{\epsilon} \right)^2 \right] + b\dot{b} \left[\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon} \right)^2 \right] \right\} \right\} \dot{\mathbf{x}} = \left(\frac{\mathbf{x}'}{\epsilon} \right)' \epsilon^{-1} \quad (10)$$

$$\ddot{\mathbf{l}} + \left\{ \frac{2\dot{b}}{b} - N^{-2} \left\{ N\dot{N} + a\dot{a} \left[\dot{\mathbf{x}}^2 - \left(\frac{\mathbf{x}'}{\epsilon} \right)^2 \right] + b\dot{b} \left[\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon} \right)^2 \right] \right\} \right\} \dot{\mathbf{l}} = \left(\frac{\mathbf{l}'}{\epsilon} \right)' \epsilon^{-1} \quad (11)$$

where ϵ is a scalar, the energy per unit coordinate length, defined by

$$\epsilon = \frac{-x'^2}{\sqrt{-\gamma}} = \left(\frac{a^2 \mathbf{x}'^2 + b^2 \mathbf{l}'^2}{N^2 - a^2 \dot{\mathbf{x}}^2 - b^2 \dot{\mathbf{l}}^2} \right)^{1/2}. \quad (12)$$

The energy-momentum tensor (6) for the metric (7) in the same gauge becomes

$$T^{\mu\nu} = \frac{1}{Na^3 b^{D-3}} \mu \int d\zeta (\epsilon \dot{x}^\mu \dot{x}^\nu - \epsilon^{-1} x'^\mu x'^\nu) \times \delta^{(D)}(\mathbf{x} - \mathbf{x}(\zeta, t), \mathbf{l} - \mathbf{l}(\zeta, t)) \quad (13)$$

where we have defined $\zeta \equiv \zeta^1$.

The energy of the cosmic string can be defined from $T^{\mu\nu}$ as

$$E = \int_{t=\text{const}} \sqrt{h} n_\mu n_\nu T^{\mu\nu} d^3 \mathbf{x} d^{D-3} \mathbf{l} \quad (14)$$

where $n_\mu = (N, \mathbf{0})$ is the normal covector to the spacelike D -dimensional surface $t = \text{const}$ and h is the determinant of the metric on that surface related to the volume element by

$$\begin{aligned} \frac{1}{D!} \sqrt{|g|} \epsilon_{\mu\mu_1\mu_2\dots\mu_D} n^\mu dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_D} \\ = \sqrt{h} d^3 \mathbf{x} d^{D-3} \mathbf{l}. \end{aligned} \quad (15)$$

With Eq. (13) the energy becomes

$$E(t) = N(t) \mu \int \epsilon d\zeta. \quad (16)$$

Note that when $\mathbf{l} = \mathbf{0}$ and $N(t) = a(t)$, Eqs. (9)–(11) and (16) reduce to the usual equations of motion and energy definition for a string in an FRW universe written in conformal time [25]. These have been used to study cosmic string evolution in an expanding $(3 + 1)$ -dimensional universe. In the next section we discuss cosmic string evolution in higher dimensions, based on Eqs. (9)–(11) and (16).

III. COSMIC STRING EVOLUTION

In this section we discuss the cosmological evolution of strings. After reviewing some basic methods and results in three dimensions, we derive the equations describing the evolution of cosmic strings in a higher-dimensional space-time with metric (7), giving particular attention to the case of an isotropic, $(D + 1)$ -dimensional FRW universe [$b(t) = a(t)$], and that of stabilized extra dimensions $b(t) = \text{const}$. Starting from Eqs. (9)–(11) and (16) we write down a higher-dimensional extension of the VOS model [24], which has been successfully used to provide an analytic “thermodynamical” description of the basic properties of evolving cosmic string networks. Application to cosmologically interesting cases is also discussed.

A. Basics: Evolution in 3 + 1 dimensions

Monte Carlo simulations of cosmic string formation after symmetry breaking phase transitions suggest that to a good approximation the strings have the shapes of random walks at the time of formation. Such “Brownian” strings can be described by a characteristic length L , which determines both the typical radius of curvature of strings and the typical distance between nearby string segments in the network. On average there is a string segment of length L in each volume L^3 and thus the density of the cosmic string network at formation is

$$\rho = \frac{\mu L}{L^3} = \frac{\mu}{L^2}. \quad (17)$$

A heuristic picture of the evolution of the density of the string network can be obtained as follows. Assuming that the strings are simply stretched by the cosmological expansion we have $\rho \sim a(t)^{-2}$. This decays slower than both the matter and radiation energy densities and so such non-interacting strings will soon dominate the universe.

This picture changes when the effect of interactions is taken into account. As the network evolves, the strings collide or curl back on themselves creating small loops, which oscillate and radiatively decay. Via these interactions energy is lost from the network so that string domination can be avoided. Each string segment travels on average a distance L before encountering another nearby segment in a volume L^3 . Assuming relativistic motion and that the produced loops have an average size L , the corresponding energy loss is given by $\dot{\rho}_{\text{loops}} \approx L^{-4} \mu L$. The energy loss rate equation becomes

$$\dot{\rho} \approx -2 \frac{\dot{a}}{a} \rho - \frac{\rho}{L}. \quad (18)$$

Cosmic string networks are known to evolve towards a ‘‘scaling’’ regime in which the characteristic length L stays constant relative to the horizon $d_H \sim t$ [12]. To see this we set $L = \gamma(t)t$ and substitute (17) into (18) to obtain

$$\frac{\dot{\gamma}}{\gamma} = \frac{1}{2t} \left(2(\beta - 1) + \frac{1}{\gamma} \right). \quad (19)$$

The parameter β is related to the scale factor $a(t)$ by $a(t) \propto t^\beta$ and is equal to $1/2$ and $2/3$ in the radiation and matter eras, respectively. Equation (19) has a scaling solution

$$\gamma = [2(1 - \beta)]^{-1} \quad (20)$$

(which depends on cosmology through the expansion exponent β) demonstrating that the characteristic length scales at a value $L \sim t$. If we start with a high density of strings, intercommuting will produce loops reducing the energy of the network, whereas if the initial density is low then there will not be enough intercommuting and γ will decrease. Given enough time, the two competing effects of stretching and fragmentation will always reach a steady state and the scaling regime will be approached.

Equation (19) was derived on physical grounds and it only captures the basic processes involved in string evolution, namely, the stretching and intercommuting of strings. It does not take into account other effects like the redshifting of string velocities due to Hubble expansion. However we can derive a more accurate evolution equation for the string energy density based on the Nambu-Goto action. By differentiating the energy (16) with respect to time for the case of a $3 + 1$ FRW universe and setting $E \propto \rho a^3$ we find

$$\dot{\rho} = - \left(2 \frac{\dot{a}}{a} + 2N^{-2} a \dot{a} \langle \dot{\mathbf{x}}^2 \rangle \right) \rho \quad (21)$$

where we have defined

$$\langle \dot{\mathbf{x}}^2 \rangle \equiv \frac{\int \dot{\mathbf{x}}^2 \epsilon d\zeta}{\int \epsilon d\zeta}. \quad (22)$$

We also introduce a phenomenological term to account for energy loss through loop production

$$\dot{\rho}_{\text{loops}} = - \frac{\tilde{c} \nu \rho}{L} \quad (23)$$

where \tilde{c} is the loop production parameter, related to the integral of an appropriate loop production function over all relevant loop sizes [see Eq. (37) and the discussion above it], and ν the average velocity of intercommuting string segments.

Using (17) and setting $L = \gamma(t)t$ as before, we obtain the following equation, expressed in physical time t :

$$\gamma^{-1} \frac{d\gamma}{dt} = \frac{1}{2t} \left(2\beta(1 + \nu^2) - 2 + \frac{\tilde{c}\nu}{\gamma} \right). \quad (24)$$

Here we defined ν , the average velocity of string segments, by

$$\nu^2 = \left\langle \frac{d\mathbf{x}^2}{d\tau} \right\rangle \equiv \frac{\int \frac{d\mathbf{x}^2}{d\tau} \epsilon d\zeta}{\int \epsilon d\zeta} \quad (25)$$

with τ the conformal time [$N(\tau) = a(\tau)$].

Equation (24) is of the same form as (19) but has an extra correction term $\beta\nu^2$ accounting for redshifting of velocities due to cosmological expansion. It also includes the parameter \tilde{c} , the value of which can be extracted from numerical simulations and it is of order unity [23,25].

To solve (24) we also need an evolution equation for ν . This can be obtained by differentiating (25) with respect to τ and using the three-dimensional version of the equation of motion for \mathbf{x} (10). The result (expressed in physical time t) is [22]

$$\frac{d\nu}{dt} = (1 - \nu^2) \left(\frac{k}{R} - 2H\nu \right) \quad (26)$$

where H is the Hubble parameter $\dot{a}/a = \beta t^{-1}$, R the average radius of curvature of strings in the network and k the momentum parameter (first introduced in [22]) defined by

$$\frac{k\nu(1 - \nu^2)}{R} = \langle \dot{\mathbf{x}} \cdot \mathbf{u}(1 - \dot{\mathbf{x}}^2) \rangle \quad (27)$$

\mathbf{u} being the curvature vector defined by

$$\frac{d^2\mathbf{x}}{ds^2} = \mathbf{u} = \frac{1}{R} \hat{\mathbf{u}} \quad (28)$$

with $ds = \sqrt{\dot{\mathbf{x}}^2} d\zeta$. For a Brownian network (and within the VOS assumptions) the average radius of curvature R is equal to the correlation length $L \equiv \gamma t$.

Equation (26) has a clear physical meaning: velocities of string segments are produced by string curvature and damped by the cosmological expansion. Together with (24) it constitutes the VOS model, which has been dem-

onstrated to be in very good agreement with numerical simulations [26]. It has the scaling solution

$$\gamma^2 = \frac{k(k + \tilde{c})}{4\beta(1 - \beta)} \quad v^2 = \frac{k(1 - \beta)}{\beta(k + \tilde{c})} \quad (29)$$

in terms of the expansion exponent β , the loop production parameter \tilde{c} and the momentum parameter k .

The momentum parameter is a measure of the angle between the curvature vector and the velocity of string segments and thus is related to the smoothness of the strings. Slowly moving strings are smooth so the velocity is more or less parallel to the curvature vector, resulting in a value of k of order unity. As v increases towards relativistic values the accumulation of small-scale structure renders the strings wiggly. Velocities become uncorrelated to curvature and k decreases. In particular it can be shown analytically that for flat space, where $v^2 = 1/2$, the momentum parameter vanishes for a wide range of known solutions [21,23].

An accurate ansatz for the momentum parameter k has been proposed in [24]

$$k = k(v) = \frac{2\sqrt{2}}{\pi} \frac{1 - 8v^6}{1 + 8v^6} \quad (30)$$

satisfying $k(1/\sqrt{2}) = 0$.

Note the fact that $v = 1/\sqrt{2}$ in flat spacetime can be shown analytically for closed loops only, but for long strings it is observed in numerical simulations [25]. For expanding or contracting spacetimes, v is less or greater than $1/\sqrt{2}$, respectively. Hence for an expanding universe, string velocities are subject to the constraint

$$v^2 \leq \frac{1}{2}. \quad (31)$$

This fact will be important for our discussion of extra dimensions.

B. Evolution in higher dimensions

We now proceed to derive macroscopic evolution equations for string networks in higher dimensions, based on Eqs. (9)–(11) and (16), derived in Sec. II. The result is the EDVOS model.

1. Evolution equation for γ

We start by differentiating the energy equation (16) with respect to time and using the equation of motion (9) for ϵ . This gives

$$\begin{aligned} \dot{E} = & -\frac{1}{N(t)^2} \left[a\dot{a} \left(\langle \dot{\mathbf{x}}^2 \rangle E - \mu N(t) \int \epsilon^{-1} \mathbf{x}^{\prime 2} d\zeta \right) \right. \\ & \left. + b\dot{b} \left(\langle \dot{\mathbf{l}}^2 \rangle E - \mu N(t) \int \epsilon^{-1} \mathbf{l}^{\prime 2} d\zeta \right) \right] \quad (32) \end{aligned}$$

where we have defined the average of a function f along the string as

$$\langle f \rangle \equiv \frac{\int \epsilon f d\zeta}{\int \epsilon d\zeta}. \quad (33)$$

We then use $E \propto \rho a^3 b^{D-3}$ and definition (12) for ϵ to obtain an evolution equation for the energy density of the string network

$$\begin{aligned} -\frac{\dot{\rho}}{\rho} = & \frac{\dot{a}}{a} \left[2 + 2N^{-2} a^2 \langle \dot{\mathbf{x}}^2 \rangle + N^{-2} b^2 \langle \dot{\mathbf{l}}^2 \rangle \right. \\ & \left. + N^{-2} \left\langle \frac{b^2 \mathbf{l}^{\prime 2}}{a^2 \mathbf{x}^{\prime 2} + b^2 \mathbf{l}^{\prime 2}} (N^2 - a^2 \dot{\mathbf{x}}^2 - b^2 \dot{\mathbf{l}}^2) \right\rangle \right] \\ & + \frac{\dot{b}}{b} \left[(D - 4) + 2N^{-2} b^2 \langle \dot{\mathbf{l}}^2 \rangle + N^{-2} a^2 \langle \dot{\mathbf{x}}^2 \rangle \right. \\ & \left. + N^{-2} \left\langle \frac{a^2 \mathbf{x}^{\prime 2}}{a^2 \mathbf{x}^{\prime 2} + b^2 \mathbf{l}^{\prime 2}} (N^2 - a^2 \dot{\mathbf{x}}^2 - b^2 \dot{\mathbf{l}}^2) \right\rangle \right]. \quad (34) \end{aligned}$$

We would normally proceed by assuming that the string network at formation is Brownian with a characteristic length L and writing

$$\rho = \frac{\mu L}{L^D} = \frac{\mu}{L^{D-1}}. \quad (35)$$

Unfortunately, since the metric we are considering is not in general isotropic, we do not expect this Brownian structure to be preserved by the evolution. The amount by which string segments are stretched will depend on their orientation and as time passes, the random walk shape of strings will be distorted by the anisotropic expansion. However, there are two interesting special cases in which the strings can remain Brownian. First, the case of isotropic expansion [$a(t) = b(t)$], corresponding to a generalized $(D + 1)$ -dimensional FRW universe, and second, a situation where the formation of the string network is localized on an isotropic slice. The latter is the case, for example, in brane inflation [27–31] where string formation takes place essentially on a brane and there is an effective three-dimensional description of the evolution. We will consider both cases below.

(i) *Isotropic case.*—Starting with the isotropic case, we set $b(t) = a(t)$ in Eq. (34) to obtain an evolution equation for the energy density of a noninteracting string network in a $(D + 1)$ -dimensional FRW universe

$$\dot{\rho} = -\frac{\dot{a}}{a} [(D - 1) + 2N^{-2} a^2 \langle \dot{\mathbf{x}}^2 \rangle + 2N^{-2} a^2 \langle \dot{\mathbf{l}}^2 \rangle] \rho. \quad (36)$$

As in the usual VOS model we can treat string interactions by introducing a phenomenological loop production term. In the three-dimensional case a string segment of size L , the correlation length, is expected to travel a distance L before encountering another segment and interacting with it in a volume L^{-3} . If the probability of such an interaction producing a loop of length between ℓ and $\ell + d\ell$ is given by the scale invariant function $f(\ell/L)$ then the energy loss due to loop production is [25]

$$\begin{aligned}\dot{\rho}_{\text{loops}} &= -\frac{\mu v}{L^3} \int_0^\infty \frac{d\ell}{L} f(\ell/L) = -\frac{v\rho}{L} \int_0^\infty \frac{d\ell}{L} f(\ell/L) \\ &\equiv -\frac{\tilde{c}v\rho}{L}.\end{aligned}\quad (37)$$

When the number of spatial dimensions $D > 3$, the string segments will generically not interact after moving a distance L . In particular, if the strings have a thickness or capture radius δ (quantifying how close they need to be in order to interact), then the probability of interacting after moving a distance L (in time δt) per volume L^D is

$$\frac{v\delta t}{L} \frac{1}{L^D} \left(\frac{\delta}{L}\right)^{D-3}.\quad (38)$$

Note that this suppression is essentially the intercommuting probability P which we shall further discuss later, though here it is time dependent. The loop production term is then

$$\begin{aligned}\dot{\rho}_{\text{loops}} &= -\frac{\mu v}{L^D} \left(\frac{\delta}{L}\right)^{D-3} \int_0^\infty \frac{d\ell}{L} f(\ell/L) = -\frac{\tilde{c}v\rho}{L} \left(\frac{\delta}{L}\right)^{D-3} \\ &\equiv -\frac{v\rho}{L} \tilde{c}_D\end{aligned}\quad (39)$$

where we have defined

$$\tilde{c}_D = \tilde{c} \left(\frac{\delta}{L}\right)^{D-3}.\quad (40)$$

Including this loop production term in Eq. (36) and setting as in the three-dimensional case $L = \gamma(t)t$, where t is the physical time [$N(t) = 1$] we obtain

$$\gamma^{-1} \frac{d\gamma}{dt} = \frac{1}{(D-1)t} \left[\beta((D-1) + 2v^2) + \frac{v\tilde{c}_D}{\gamma} - (D-1) \right].\quad (41)$$

Here $v^2 = v_x^2 + v_\ell^2$ where v_x and v_ℓ , the rms peculiar velocities of string segments in the x and the ℓ directions, respectively, are defined by

$$v_x^2 = \left\langle \frac{a^2}{N^2} \dot{\mathbf{x}}^2 \right\rangle \quad \text{and} \quad v_\ell^2 = \left\langle \frac{b^2}{N^2} \dot{\mathbf{l}}^2 \right\rangle = \left\langle \frac{a^2}{N^2} \dot{\mathbf{l}}^2 \right\rangle\quad (42)$$

since $b(t) = a(t)$ in the isotropic case.

Note that since \tilde{c}_D is proportional to $L^{-(D-3)} = (\gamma t)^{-(D-3)}$, the loop production term $v\tilde{c}_D/\gamma$ is explicitly time dependent and no scaling solution $\gamma = \text{const}$ exists. In particular, as time increases the loop production term becomes smaller and smaller, reflecting the fact that strings cannot find each other and interact in more than three spatial dimensions. String evolution in this isotropic $D + 1$ -dimensional case will be discussed further in Sec. III B 4.

(ii) *Anisotropic case.*—We now consider the case where $b(t) \neq a(t)$ in (7) but the string network is produced on a FRW slice (the x space) of the spacetime. If the extra

dimensions are larger than the correlation length and the strings are allowed to move in the bulk after formation, then the anisotropic expansion will distort their Brownian structure and they will be more difficult to characterize analytically. We leave this case for subsequent discussion (see the final section). The situation is more straightforward if the extra dimensions are compactified at a size smaller than the correlation length which we shall assume from now on. We can then have an effective three-dimensional description in which the strings will remain Brownian due to isotropy in the three-dimensional FRW space. The presence of the extra dimensions will reduce the intercommuting probability in the effective three-dimensional description, but we shall assume it is nonzero. In principle, we can also have time-varying extra dimensions, as long as their size remains much smaller than the correlation length at all times. However, for simplicity we only consider static extra dimensions and set $b(t) = 1$, assuming stabilization by some unspecified mechanism.

For such cases we can substitute (17) in Eq. (34) to obtain an effective three-dimensional evolution equation for the energy density, which however takes into account velocities in all D spatial dimensions. As before we also introduce a loop production term $-\tilde{c}v_x\rho/L$ and set $L \equiv \gamma(t)t$ where t is the physical time [$N(t) = 1$]. We thus obtain an effective evolution equation for $\gamma(t)$

$$\begin{aligned}\gamma^{-1} \frac{d\gamma}{dt} &= \frac{1}{2t} \left\{ \beta[(2 + w_\ell^2) + (2 - w_\ell^2)v_x^2 + (1 - w_\ell^2)v_\ell^2] \right. \\ &\quad \left. - 2 + \frac{\tilde{c}v_x}{\gamma} \right\}.\end{aligned}\quad (43)$$

In deriving this expression, we have assumed

$$\begin{aligned}\left\langle \frac{\mathbf{l}^2}{a^2 \mathbf{x}^2 + \mathbf{l}^2} (N^2 - a^2 \dot{\mathbf{x}}^2 - \dot{\mathbf{l}}^2) \right\rangle \\ \simeq \left\langle \frac{\mathbf{l}^2}{a^2 \mathbf{x}^2 + \mathbf{l}^2} \right\rangle \langle (N^2 - a^2 \dot{\mathbf{x}}^2 - \dot{\mathbf{l}}^2) \rangle\end{aligned}\quad (44)$$

and we have defined an orientation parameter

$$w_\ell = \sqrt{\left\langle \frac{\mathbf{l}^2}{a^2 \mathbf{x}^2 + \mathbf{l}^2} \right\rangle}\quad (45)$$

which quantifies the degree to which the strings lie in the extra ℓ dimensions.

Note the differences between (43) and the analogous equation for a $3 + 1$ FRW universe (24). Comparing (43) to (24) we see that the coefficients of dilution and x -velocity redshift are modified by w_ℓ^2 , while there is an additional term accounting for the redshifting of velocities in the static ℓ space. We interpret this term as arising from the conservation of momentum: an expanding string segment which is transverse to the ℓ space and moves in the ℓ directions is slowed down because its effective mass increases as the string is stretched in the x direction.

Another important difference with the three-dimensional case is that the loop production parameter \tilde{c} is now suppressed by a factor of order $(\delta/R_\ell)^{(D-3)}$, where R_ℓ is the compactification scale. If R_ℓ was greater than the correlation length L , then the suppression factor would instead be $(\delta/L)^{(D-3)}$, which is time dependent. Thus there would be no scaling solution initially, as in the $(D+1)$ -dimensional FRW case we discussed above (see later discussion for the asymptotic evolution in this case). Here however, we assumed the compactification scale is much less than the correlation length and so the suppression factor can be treated as a constant intercommuting proba-

bility. For a string-theoretic calculation of this probability see Ref. [5].

2. Evolution equations for v_x and v_ℓ

As in the three-dimensional case we also need to know how the average velocities of string segments evolve in time. We begin by considering the case of static extra dimensions $b(t) = 1$. Working in conformal time τ [$N(\tau) = a(\tau)$], we start by differentiating $v_x^2 = \int \dot{\mathbf{x}}^2 \epsilon d\zeta / \int \epsilon d\zeta$ with respect to τ and using the equation of motion (10) for the fields \mathbf{x} . We obtain

$$\begin{aligned} v_x \dot{v}_x = & \frac{1}{2} \left[2 \frac{\dot{a}}{a} \langle (\dot{\mathbf{x}}^2)^2 - \langle \dot{\mathbf{x}}^4 \rangle \rangle + \frac{\dot{a}}{a^3} \langle (\dot{\mathbf{x}}^2)^2 \langle \dot{\mathbf{l}}^2 \rangle - \langle \dot{\mathbf{x}}^2 \dot{\mathbf{l}}^2 \rangle \rangle + \frac{\dot{a}}{a^3} \langle (\langle \dot{\mathbf{x}}^2 \rangle - \dot{\mathbf{x}}^2) \mathbf{l}^2 \epsilon^{-2} \rangle \right] + a \left\langle \dot{\mathbf{x}} \cdot \mathbf{u} \left(1 - \dot{\mathbf{x}}^2 - \frac{\dot{\mathbf{l}}^2}{a^2} \right) \right\rangle \\ & - 2 \frac{\dot{a}}{a} \langle \dot{\mathbf{x}}^2 \rangle + 2 \frac{\dot{a}}{a} \langle \dot{\mathbf{x}}^4 \rangle + \frac{\dot{a}}{a^3} \langle \dot{\mathbf{l}}^2 \dot{\mathbf{x}}^2 \rangle + \frac{\dot{a}}{a} \left\langle \frac{\mathbf{l}^2}{a^2 \mathbf{x}^{\prime 2} + \mathbf{l}^2} \left(1 - \dot{\mathbf{x}}^2 - \frac{\dot{\mathbf{l}}^2}{a^2} \right) \dot{\mathbf{x}}^2 \right\rangle - \left\langle \dot{\mathbf{x}} \cdot \mathbf{x}' \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}' + \dot{\mathbf{l}} \cdot \dot{\mathbf{l}}'/a^2}{\mathbf{x}^{\prime 2} + \mathbf{l}^2/a^2} \right\rangle \end{aligned} \quad (46)$$

where \mathbf{u} is the curvature vector defined by

$$\frac{d^2(a\mathbf{x}, \mathbf{l})}{a^2 ds^2} = \mathbf{u} = \frac{1}{R} \hat{\mathbf{u}} \quad (47)$$

with $ds = \sqrt{(\mathbf{x}^{\prime 2} + \mathbf{l}^{\prime 2}/a^2)} d\zeta$ and R the radius of curvature, equal to the correlation length $L \equiv \gamma t$ for a VOS model Brownian network.

The first three terms in Eq. (46) can be neglected because their only effect is to slightly modify the coefficients of all the remaining terms in the expression (but for the last). Numerical confirmation of the smallness of such terms (in particular the first term in the case of conventional FRW) was presented in [23]. The remaining terms (again except the last one) are of the same form as in the $(3+1)$ -dimensional FRW case but now they also include terms dependent on the velocity in the ℓ space and an extra term which gives corrections of order $\langle \mathbf{l}^2 / (a^2 \mathbf{x}^{\prime 2} + \mathbf{l}^2) \rangle$.

The last term arises from the artificial splitting of the D -dimensional velocity into an x and an ℓ part. It is absent in the usual $(3+1)$ -dimensional case because of the gauge condition $\dot{\mathbf{x}} \cdot \mathbf{x}' = 0$. Here the gauge condition is $\dot{\mathbf{x}} \cdot \mathbf{x}' - \dot{\mathbf{l}} \cdot \mathbf{l}'/a^2 = 0$ and thus $\dot{\mathbf{x}} \cdot \mathbf{x}'$ does not identically vanish. However, we expect its value along the string to change sign randomly with no large-scale correlations. Hence, averaged over the whole string network, $\dot{\mathbf{x}} \cdot \mathbf{x}'$ is expected to vanish so that we can neglect the last term of (46). This was tested numerically for a three-dimensional FRW model, splitting v into a xy and a z part and neglecting the corresponding terms in the xy - and z -velocity equations. The evolution of the system using these equations was found numerically to be very close to the corresponding evolution using the full three-dimensional velocity equation, with the scaling values of v and γ agreeing to three significant figures.

Again using the approximation (44) and switching to physical time t , we can write (46) in a much more elegant and useful form,

$$\begin{aligned} v_x \frac{dv_x}{dt} = & \frac{k_x v_x}{R} (1 - v^2) - (2 - w_\ell^2) H v_x^2 (1 - v^2) \\ & - H v_x^2 v_\ell^2 \end{aligned} \quad (48)$$

where w_ℓ is given in (45) and k_x is defined by

$$\frac{k_x v_x (1 - v^2)}{R} = \left\langle \dot{\mathbf{x}} \cdot \mathbf{u} \left(1 - \dot{\mathbf{x}}^2 - \frac{\dot{\mathbf{l}}^2}{a^2} \right) \right\rangle. \quad (49)$$

Similarly the evolution equation for v_ℓ reads

$$\begin{aligned} v_\ell \frac{dv_\ell}{dt} = & \frac{k_\ell v_\ell}{R} (1 - v^2) - (1 - w_\ell^2) H v_\ell^2 (1 - v^2) \\ & + H v_\ell^2 v_x^2 \end{aligned} \quad (50)$$

with k_ℓ defined in analogy to k_x

$$\frac{k_\ell v_\ell (1 - v^2)}{R} = \left\langle \frac{\dot{\mathbf{l}}}{a} \cdot \mathbf{u} \left(1 - \dot{\mathbf{x}}^2 - \frac{\dot{\mathbf{l}}^2}{a^2} \right) \right\rangle. \quad (51)$$

We see that, as may have been anticipated from (34), the v_x damping term is very different from that of v_ℓ . To demonstrate this we neglect w_ℓ^2 corrections and note that $H v_x^2$ in (48) comes with a factor of $(2 - 2v_x^2 - v_\ell^2)$ but $H v_\ell^2$ in (50) has a factor of only $(1 - 2v_x^2 - v_\ell^2)$ (which can cancel almost to zero). This result will be important for the discussion in the next section.

We can also write down an evolution equation for the total velocity v , by differentiating $v^2 = \langle (\dot{\mathbf{x}}^2, \dot{\mathbf{l}}^2/a^2) \rangle$. The result is simply the sum of (48) and (50) with

$$k_x v_x + k_\ell v_\ell = kv \quad (52)$$

where k has a similar definition to (49) and (51) involving the dot product of $(\dot{\mathbf{x}}, \dot{\mathbf{l}}/a^2)$ with the curvature vector \mathbf{u} .

Equation (52) follows from the linearity of the dot product and the integral.

The momentum parameter k , as in the $(3+1)$ -dimensional case measures the angle between the curvature vector and the velocity of string segments, thus providing a measure of the smoothness of the strings. For smooth strings (the nonrelativistic limit) the velocity of string segments is expected to be more or less parallel to the curvature vector \mathbf{u} corresponding to k of order unity. For relativistic velocities however, small-scale structure accumulates and the strings become wiggly. The parameter k is then expected to approach zero.

Similarly, k_x and k_ℓ provide a measure of the angle between the curvature vector and the x velocity or the ℓ velocity, respectively. Therefore they encode two effects: the wiggleness of the strings and the extent to which the curvature vector \mathbf{u} lies in the x or ℓ subspace.

Following [24] we split $\mathbf{v} \equiv (\dot{\mathbf{x}}, \dot{\mathbf{l}}/a)$ into a ‘‘curvature’’ component \mathbf{v}_c produced during the last correlation time and a ‘‘leftover’’ component \mathbf{v}_p , coming from previous accelerations. We can then derive the following approximate formulae for k , k_x and k_ℓ , valid in the relativistic regime (see the appendix)

$$k \simeq \frac{1 - 8v^6}{(1 + 8v^6)^{1/2}(1 + 8(D-2)v^6)^{1/2}} \quad (53)$$

$$k_x \simeq \frac{v_{xc}}{v_c} \frac{1 - 8v^6}{(1 + 8v^6)^{1/2}(1 + 8(D-2)v^6)^{1/2}} \quad (54)$$

$$k_\ell \simeq \frac{v_{\ell c}}{v_c} \frac{1 - 8v^6}{(1 + 8v^6)^{1/2}(1 + 8(D-2)v^6)^{1/2}}. \quad (55)$$

Note that for flat spacetime, the condition $v^2 = 1/2$ still holds and so $k(1/\sqrt{2}) = 0$. For an expanding universe the velocities of string segments are subject to the constraint

$$v^2 = v_x^2 + v_\ell^2 \leq \frac{1}{2}. \quad (56)$$

Finally, we note that the velocity evolution equation in the case of an isotropic $(D+1)$ -dimensional FRW universe [$b(t) = a(t)$] can be similarly found

$$\frac{dv}{dt} = (1 - v^2) \left(\frac{k}{R} - 2Hv \right). \quad (57)$$

Note that this equation does not depend on w_ℓ . However, in the case $b(t) = 1$ [Eqs. (48) and (50)] there is an explicit dependence on w_ℓ , so we need to know how it evolves. Unfortunately, the search for an evolution equation for w_ℓ is problematic as we discuss below.

3. The w_ℓ equation and higher order terms

Equations (43), (48), and (50) depend on the orientation parameter w_ℓ . We interpret this as the degree to which the strings lie in the extra dimensions, that is, a hidden small-

scale structure parameter. To make a fully closed system of equations we also need an evolution equation for w_ℓ .

In analogy to our treatment of string velocities, we can try to obtain an evolution equation for w_ℓ by differentiating its definition (45) with respect to τ . Unfortunately this produces terms of the form

$$\begin{aligned} \frac{\dot{a}}{a} (\langle \mathbf{x}^{\prime 2} \rangle \langle \dot{\mathbf{x}}^2 \rangle - \langle \mathbf{x}^{\prime 2} \dot{\mathbf{x}}^2 \rangle), \quad \frac{\dot{a}}{a} (\langle \mathbf{x}^{\prime 2} \rangle^2 - \langle \mathbf{x}^{\prime 4} \rangle), \\ \frac{\dot{a}}{a} (\langle \dot{\mathbf{x}}^2 \rangle \langle \mathbf{l}^2 / a^2 \rangle - \langle \dot{\mathbf{x}}^2 \mathbf{l}^2 / a^2 \rangle), \quad \text{etc.} \end{aligned} \quad (58)$$

These purely statistical terms are higher order and cannot be easily determined. We found terms of similar kind in the x -velocity equation (46), but in that case their only effect was to ‘‘renormalize’’ the coefficients of other terms in the equation. Here, however, there are no such terms to be renormalized and the small differences between these undetermined statistical terms contribute at leading order.

However, there are special cases in which an evolution equation for w_ℓ is not needed. For example, the evolution of a string network in a higher-dimensional FRW universe is described by Eqs. (41) and (57), which do not depend on w_ℓ . In particular, isotropy suggests that string segments would have equal probability of moving in any direction and one would expect w_ℓ to be a constant, namely, the square root of the ratio of the number of extra dimensions, $D-3$, over the total number of spatial dimensions D .

For the case of small w_ℓ we note that since it appears in our equations only through factors of $(1 - w_\ell^2)$ and $(2 - w_\ell^2)$ modifying various coefficients, its evolution in time can be ignored as long as it stays small at all times. This is conceivable in cases of FRW universes with static extra dimensions, where the Hubble expansion of the three FRW dimensions will tend to reduce the value of w_ℓ .

Finally, in cases that none of the above are applicable, it may be possible to find an ansatz for w_ℓ as a function of the different velocity components, based on physical arguments. We shall consider all three possibilities in the following discussion.

4. General features of solutions

(i) *Isotropic case.*—We begin by considering the isotropic case $b(t) = a(t)$ corresponding to a $(D+1)$ -dimensional FRW universe. The energy density evolution is described by Eq. (41). Comparing this to the corresponding equation in $3+1$ dimensions (24) we see that the effect of extra FRW dimensions is to modify two of the terms in (24) by a factor of $2/(D-1)$. This reflects the fact that in three spatial dimensions, the density of the network is inversely proportional to the square of the correlation length L [Eq. (17)], whereas in $D > 3$ dimensions it is inversely proportional to the $(D-1)$ th power of L [Eq. (35)]. The energy density in the higher-dimensional case decreases faster, as there are more expanding dimensions [also see Eq. (36)]. Of course there is a second, more

dramatic, effect related to string interactions. As noted in III B 1, the loop production parameter \tilde{c}_D in the higher-dimensional case is much smaller, since it is harder for two strings to find each other and intercommute in $D > 4$. In fact, \tilde{c}_D is proportional to $(\gamma t)^{-(D-3)}$ and as a result Eq. (41) does not have a scaling solution of the form $\gamma = \text{const}$. In particular, the loop production term becomes smaller as time increases and γ will always decrease, so that strings will dominate the energy density of the universe. This can be seen in Fig. 1 where Eqs. (41) and (57) with \tilde{c}_D given by (40) have been solved numerically for $D = 4$.

The asymptotic solution for L in a D -dimensional isotropic model can be easily found to be

$$\frac{L}{L_o} = \left(\frac{t}{t_o}\right)^{\beta D/(D-1)} = \left(\frac{a}{a_o}\right)^{D/(D-1)}, \quad v = \frac{1}{\sqrt{2}}. \quad (59)$$

Contrast this EDVOS model result with simple conformal stretching of the string network $L \propto a$, obtained by assuming $v = 0$. In the EDVOS model if we start with $v \ll 1$ we still find $L \propto a$, but we also discover that v increases according to $v \propto t^{1-\beta}$. Thus, conformal stretching is only a transient solution, which will be followed by one with

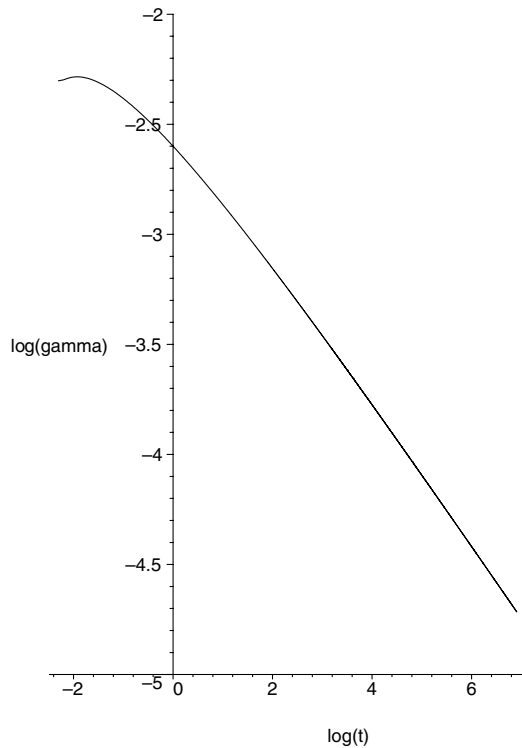


FIG. 1. Evolution of γ in a log-log plot for a string network evolving in a $(4 + 1)$ -dimensional FRW universe. The capture radius δ of the strings has been assumed to be equal to the initial correlation length and the exponent parameter β has been set to $1/2$. There is no scaling solution and at late times $\log(\gamma)$ has a constant slope of $-1/3$, in agreement with our asymptotic solution (59).

non-negligible v . The corresponding scaling result is

$$L \propto t^{\beta[(D+2v^2-1)/(D-1)]} \propto a^{(D+2v^2-1)/(D-1)}, \quad (60)$$

$$v = \text{const}.$$

In flat space v^2 cannot exceed $1/2$ but in an expanding space this upper value is somewhat reduced. Since the horizon is expanding linearly in time, the correlation length quickly falls behind and the cosmological expansion becomes irrelevant so that $v^2 \rightarrow 1/2$. Equation (60) then implies (59) asymptotically.

Note that (60) means that the string energy density scales as $\rho_s \propto L^{-(D-1)} \propto a^{-D+1-2v^2}$, which decays slower than radiation $\rho_{\text{rad}} \propto a^{-(D+1)}$. Therefore, if this regime persists for long enough, the strings will eventually dominate the universe. The Friedmann equation for a string dominated $D + 1$ FRW universe yields

$$a \propto t^{2/(D+2v^2-1)} \equiv t^\beta. \quad (61)$$

Substituting in (60) we finally obtain

$$L \propto t^{2/(D-1)}. \quad (62)$$

The above reproduces and generalizes the recent findings of Ref. [32] for $D = 3$. In particular, we recover nonintercommuting strings in the linear scaling solution $L \propto t$ (i.e. setting $P = 0$ and $D = 3$).

In a situation where the extra dimensions are compactified, but expanding isotropically with scale factor $a(t)$, the solution (59) implies that the correlation length will eventually catch up with the size of the extra dimension R_ℓ and so the string network will become effectively three-dimensional with $\tilde{c}_D = \tilde{c}(\delta/R_\ell)^{D-3}$ where $L > R_\ell \propto a$. Unlike the case of static compact dimensions, which we will discuss, this change is insufficient to prevent string domination over ordinary matter and radiation.

(ii) *Anisotropic case.*—We now turn to the case $b(t) = 1$. The relevant equations are (43) and (48)–(50) with k_x and k_ℓ approximately given by (54) and (55). To demonstrate the general features of solutions, we can keep w_ℓ as a *constant* parameter (in the range $0 \leq w_\ell \leq 1$), whose only effect is to modify the coefficients of damping terms in (43), (48), and (50). We will later consider a simplified version of the EDVOS model, where w_ℓ is not constant, but is given by a physically motivated ansatz, which is a function of v_ℓ and v_x .

If the string network at formation is Brownian in D dimensions, then the curvature vector \mathbf{u} explores the x space as well as the ℓ space and k_x, k_ℓ are comparable. Equation (48) for v_x and (50) for v_ℓ have comparable source terms but v_ℓ has a much weaker damping term. Therefore, we expect v_ℓ to become much greater than v_x and eventually to drive v_x to zero, because of the constraint $v_x^2 + v_\ell^2 \leq 1/2$. We conclude that expansion of the x space together with the fact that the ℓ space is not expanding will halt string motion in the x space. This can be verified

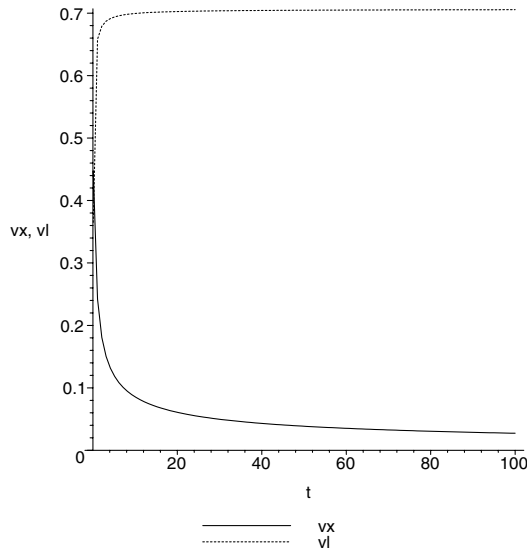


FIG. 2. Velocity evolution for a Brownian network in D spatial dimensions, three of which are expanding. The expansion strongly redshifts the velocities in the expanding dimensions (v_x), resulting in domination of the extradimensional velocities (v_ℓ). The strings will stop moving in the expanding dimensions.

numerically (Fig. 2) by solving (48) and (50) with the approximations $v_{xc}/v_c \approx v_x/v$ and $v_{\ell c}/v_c \approx v_\ell/v$ in (54) and (55). These are valid approximations in both the nonrelativistic and relativistic limits, as long as the curvature vector explores all D spatial dimensions. Thus the strings in this case will soon become noninteracting and, if this regime were to last long enough, dominate the universe irreversibly. However, there are two important caveats in this case, namely, that anisotropic expansion would soon distort the simple Brownian network structure and that the intercommuting probability for a string network in D spatial dimensions would be small and decreasing (as discussed above). For compact extra dimensions, this regime will end when the correlation length becomes larger than the compact dimension $L > R_\ell$, after which point the evolution becomes effectively three-dimensional (we will return to this point later).

The situation is different if the curvature vector effectively lies in the x space, as may be the case in brane inflation (see next section) where the formation of the string network is constrained on an effectively $(3 + 1)$ -dimensional FRW slice. In this case the dot product $\mathbf{u} \cdot \hat{\mathbf{l}}$ is negligible and thus k_ℓ is much smaller than k_x . Equation (50) has essentially no source term. We expect to see $v_x > v_\ell$ with v_ℓ given by its initial condition, weakly damped according to Eq. (50). This was verified numerically (Fig. 3) by setting $v_{xc}/v_c \approx v_x/v$, as before, and $v_{\ell c}/v_c \ll 1$. The effect of increasing $v_{\ell c}/v_c$ was also considered. It was found that there is a critical value of $v_{\ell c}/v_c \approx 0.15$ for which the scaling values of v_x and v_ℓ become equal. Above this critical value v_ℓ eventually dominates and we return to the previous regime, with k_ℓ

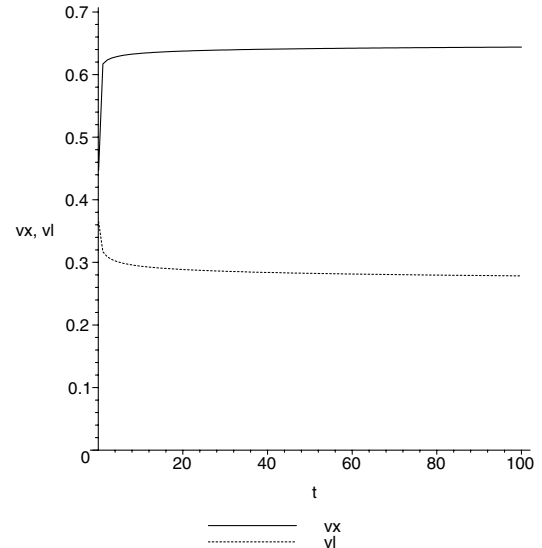


FIG. 3. Velocity evolution for a Brownian network formed on a FRW slice. The curvature vector lies in the space of the three expanding dimensions, so there is no source for the extradimensional velocities v_ℓ . They are given by their initial condition, redshifted weakly by the expansion. The three-dimensional velocity v_x will not vanish but it can be significantly smaller than in the purely three-dimensional case $v \approx 0.7$.

and k_x comparable. In particular, for $v_{\ell c}/v_c \approx 0.5$, v_ℓ soon reaches a “relativistic” value, driving v_x to zero. It may seem that if the strings are free to explore the extra dimensions after formation we may run into the same problems we had before, namely, a practically zero intercommuting probability and the loss of the Brownian structure of the string network. However, the extra dimensions can be small and compact, in which case the probability of intercommuting can still be appreciable and the network is to a good approximation Brownian in three dimensions.

Finally we consider the effect of varying the parameter w_ℓ . We find that γ is insensitive to changes in w_ℓ^2 between 0 and 0.2 but is reduced by approximately 20% for $w_\ell^2 = 0.5$. On the other hand v_ℓ depends on w_ℓ more strongly. In particular, changing w_ℓ^2 from 0 to 0.01 increases the scaling value of v_ℓ by approximately 0.5% but a further increase of w_ℓ^2 to 0.1 increases v_ℓ typically by 30% (this does not strongly affect v_x , which is approximately equal to $\sqrt{1/2 - v_\ell^2}$). As a result, the effective three-dimensional behavior of the evolving network is insensitive to the value of w_ℓ^2 , at least when it is smaller than 0.1 or so. There is a critical value of about 0.2 above which the sign of the damping term in Eq. (50) becomes positive and v_ℓ begins to increase. This apparently unphysical effect signifies the fact that the constant w_ℓ approximation breaks down for relatively large values of w_ℓ . In the following we discuss a simplified version of the extradimensional VOS model with w_ℓ given by a function of v_ℓ and v_x , where this unphysical behavior does not occur.

5. A simplified model

We can obtain a simplified version of Eqs. (43), (48), and (50) by expressing w_ℓ as a function of v_ℓ and v_x . String gradients in one direction are produced by velocities in the same direction so one would expect the network to achieve a sort of equipartition between velocities and gradients in the extra dimensions. This motivates the ansatz

$$w_\ell^2 = \frac{v_\ell^2}{v^2} \quad (63)$$

which mathematically corresponds to

$$\left\langle \frac{\dot{\mathbf{I}}^2}{a^2 \dot{\mathbf{x}}^2 + \dot{\mathbf{I}}^2} \right\rangle = \left\langle \frac{\mathbf{I}^2}{a^2 \mathbf{x}^2 + \mathbf{I}^2} \right\rangle. \quad (64)$$

With this substitution, Eqs. (43) and (48)–(50) become

$$\gamma^{-1} \frac{d\gamma}{dt} = \frac{1}{2t} \left[\beta \left(2 + 2v_x^2 + \frac{v_\ell^2}{v^2} \right) - 2 + \frac{\tilde{c}v_x}{\gamma} \right] \quad (65)$$

$$v_x \frac{dv_x}{dt} = \frac{1}{t} \left[\frac{k_x v_x}{\gamma} (1 - v^2) - \left(2 - 2v_x^2 - \frac{v_\ell^2}{v^2} \right) \beta v_x^2 \right] \quad (66)$$

$$v_\ell \frac{dv_\ell}{dt} = \frac{1}{t} \left[\frac{k_\ell v_\ell}{\gamma} (1 - v^2) - \left(1 - \frac{v_\ell^2}{v^2} \right) (1 - 2v^2) \beta v_\ell^2 \right]. \quad (67)$$

These three equations describe the macroscopic evolution of a Brownian network of strings, produced in a three-dimensional FRW slice, which are then free to explore the extra dimensions. In order for the Brownian structure to be maintained [so that Eq. (65) is valid] the extra dimensions need to be compactified at a size smaller than the initial correlation length at the time of formation.

Note that the unphysical behavior leading to the change of sign of the damping terms in the v_ℓ evolution equation (which is a result of the fact that the constant w_ℓ approximation is not valid for large w_ℓ) is now absent from Eq. (67). The only way the damping terms can change sign is if v^2 exceeds $1/2$, which is not allowed by the constraint (56). Similarly the damping terms in Eq. (66) cannot change sign either.

We can now solve (65)–(67) together with (54) and (55) for k_x and k_ℓ . Since we are considering the case where the structure of the network at formation is essentially three-dimensional, the curvature vector lies mainly in the x space, so that $k_\ell \ll k_x$ [equivalently $v_{\ell c}/v_c \ll 1$ in Eq. (55)]. The solutions are qualitatively the same as the corresponding constant w_ℓ solutions mentioned above. This is because $w_\ell^2 = v_\ell^2/v^2$ is small (and decreasing with time) so its effect in Eqs. (65)–(67) is only to slightly modify coefficients of order unity. We will discuss these solutions in more detail in the next section where we will apply the model to brane inflation.

IV. APPLICATION TO BRANE INFLATION

Recently there has been much activity in trying to derive models of cosmological inflation from string theory [27–31,33–40], in which there are a large number of scalar fields that can potentially serve as inflatons. One of the most interesting scenarios (especially from the point of view of cosmic string production) is brane inflation [27–31], where the role of the inflaton field is played by the distance between two branes or a brane and an antibrane. The branes are initially relatively displaced and move towards each other due to an attractive potential arising from the exchange of bulk Neveu-Schwarz–Neveu-Schwarz and Ramond-Ramond modes. As they come closer together, open string modes stretching between them start contributing more strongly to the inflaton potential. At a critical distance of order the inverse superstring scale M_s^{-1} such modes become tachyonic and inflation ends in a hybrid-inflation-type exit.

Brane stacks carry Chan-Paton gauge groups and hence the tachyonic instability appearing at brane collision corresponds to a symmetry breaking process. This allows the formation of topological defects as lower-dimensional branes if the vacuum manifold is nontrivial [41]. For two brane stacks of N branes each, the vacuum manifold is isomorphic to $U(N)$, whose only nontrivial homotopy groups are the odd ones, π_{2k-1} . Thus the topologically allowed defects have *even* codimension $2k$.

We need to make sure that cosmologically dangerous defects like monopoles or domain walls are not produced. This is fortunately the case (but see [42]) as can be seen by the following argument given in Ref. [1]. In order to describe our $(3+1)$ -dimensional universe the branes must have three infinite spatial dimensions. They can either be $D3$ -branes or Dp -branes with $(p-3)$ dimensions compactified at a size smaller than the horizon [27]. Thus the Kibble mechanism can only take place in the three infinite dimensions so the codimension of the defects must lie in these dimensions (hence it can only be 1, 2 or 3). Since the codimension is even, the defects that are produced have codimension 2, that is they are $D(p-2)$ -branes wrapping the same compact space as the original Dp -branes. These objects will appear as cosmic strings to a three-dimensional observer.

Alternative string production mechanisms (bulk preheating, resonant string formation) have been discussed in [4], resulting in both D -string and F -string networks, or even an interacting $(F-D)$ -string network (also see [43]). String stability was studied in [44]. Various possible cosmologically relevant metastable strings were found in many models, including (p, q) strings that is, bound states of p F strings and q D strings. For string stability also see [45].

Given these possibilities, cosmic string production can be considered as a generic prediction of brane inflation (though models with no cosmic strings have been constructed [46]). It is then natural to ask what the evolution

of the string network would be in these models and whether it is distinctly different from the standard cosmic string evolution, thus providing a potential observational window into superstring physics. We consider this question below.

A. Basics

We have seen that in the context of brane inflation, D strings are in general $D(p-2)$ -branes wrapping the same $(p-3)$ -dimensional compactified space as the initial Dp -branes. Their production is localized on the plane of brane collision, but later they are free to explore the $(9-p)$ compact dimensions transverse to the brane.

For two strings to collide their world sheets must intersect. In the $3+1$ case the number of spacetime dimensions equals the sum of the dimensions of the two world sheets and hence strings will generally collide [47]. For a higher-dimensional spacetime, however, this is no longer the case so strings will generally miss. One then expects to be able to model cosmic string evolution in the context of brane inflation by simply introducing an intercommuting probability P in the usual $(3+1)$ -dimensional evolution equations.

This was done in [3] where an intercommuting probability P (λ/λ_0 in the original paper) was introduced in Eq. (19). The new scaling solution is $\gamma \simeq P$ (cf. $\gamma \sim 1$ in three dimensions), corresponding to an enhancement in the scaling energy density of strings by a factor of P^{-2} . This can be orders of magnitude different than the usual 3D case [5].

There are at least two effects which could significantly alter the above result. The first is related to the small-scale structure of the strings. If they are wiggly, they may have more than one opportunity to intercommute during each crossing time. One expects that a probability of $1/3$ or so should have no impact in the scaling of γ , and thus, contrary to what is sometimes suggested in the literature, strings with such large P could not be observationally distinguished from standard field theory strings. On the other hand, for $P \ll 1$ the scaling value of γ should depend more strongly on P . The dependence of γ on P , taking into account small-scale wiggles can only be modeled numerically. Simulations in flat space suggest a scaling $\gamma \simeq \sqrt{P}$ in the range $0.05 < P < 0.3$ [4,48,49]. To deal with this uncertainty we will introduce an *effective* intercommuting probability $P_{\text{eff}} = f(P)$ as a multiplicative parameter, modifying the phenomenological loop production term of Eq. (65). A numerical study of the functional dependence of P_{eff} on P in an expanding universe will be presented in [50].

The second possibility arises from the observation that the velocities in the three infinite dimensions and in the compact ones, v_x and v_ℓ , respectively, satisfy the constraint $v_x^2 + v_\ell^2 \leq 1/2$. Together with the fact that v_ℓ is very weakly damped compared to v_x (Sec. III B 4), this will slow down the motion of the strings in the three infinite

dimensions. It is even possible that string motion in these dimensions can completely stop, in which case the strings will no longer be able to intercommute. We study this effect in more detail below.

B. Applying the EDVOS model

As we have seen, the spatial structure of the string network at formation is essentially three-dimensional, but the strings evolve in $9-p+4$ spacetime dimensions with $9-p$ of them compactified. The compactification radius is assumed to be stabilized at a size smaller than the horizon [27]. Thus we can model the evolution of the string density using an effective three-dimensional description, where we introduce an intercommuting probability P to account for the fact that strings can “miss” when they evolve in higher than 4 spacetime dimensions. As an evolution equation for γ we therefore use (43) and replace the loop production term $\tilde{c}v_x\gamma^{-1}$ by $P_{\text{eff}}\tilde{c}v_x\gamma^{-1}$, where an effective intercommuting probability $P_{\text{eff}} = f(P)$ has been used to implicitly take into account the effect of string wiggleness on the scaling dependence on P . We also need to include both v_x and v_ℓ , the velocities in the $(3+1)$ -dimensional FRW x space and the $(9-p)$ -dimensional compact ℓ space, respectively, which we evolve using Eqs. (48) and (50). The parameters k_x and k_ℓ are given by (54) and (55) for $D = 9-p+3$ with $v_{x_c}/v_c \simeq v_x/v$ and $v_{\ell_c}/v_c \ll 1$ (see Sec. III B 4).

The formation of the network takes place on the plane of the colliding branes, which has a finite thickness. One may worry that this fuzziness will translate to an uncertainty in the position of the strings in the extra dimensions, which may give rise to a significant structure in the ℓ space just after the strings are formed. This could spoil the assumption that the initial structure of the network can be considered three-dimensional, in which case Eq. (43) would not be valid. Fortunately this thickness is of the order of the inverse superstring scale M_s^{-1} , which is approximately 10^4 times smaller than the correlation length at formation (see below). The network is to a very good approximation three-dimensional even at the time of formation. This justifies the assumption $k_\ell \ll k_x$ or equivalently $v_{\ell_c}/v_c \ll 1$ at early times.

The correlation length at formation, estimated by studying the tachyon potential, is of the same order of magnitude as the expected horizon size at that time [1]

$$L_0 \sim H^{-1} \sim 10^4 M_s^{-1}, \quad (68)$$

where the superstring scale is set by CMB data to a low GUT scale

$$M_s \sim 10^{14} \text{ GeV}. \quad (69)$$

After the strings are formed, the correlation length grows because of the Hubble expansion. On the other hand the dimensions transverse to the brane are compactified at a size a few times the inverse superstring scale,

which is some 10^3 times smaller than the correlation length. Thus, if the initial long string network is Brownian, it will remain so (as in the usual three-dimensional case) and Eq. (43) will be valid at all times.

To simplify the equations we can use the ansatz $w_\ell^2 = v_\ell^2/v^2$ (Sec. III B 5). Equations (43), (48), and (50) are then replaced by

$$\gamma^{-1} \frac{d\gamma}{dt} = \frac{1}{2t} \left[\beta \left(2 + 2v_x^2 + \frac{v_\ell^2}{v^2} \right) - 2 + \frac{P_{\text{eff}} \tilde{c} v_x}{\gamma} \right] \quad (70)$$

$$v_x \frac{dv_x}{dt} = \frac{1}{t} \left[\frac{k_x v_x}{\gamma} (1 - v^2) - \left(2 - 2v_x^2 - \frac{v_\ell^2}{v^2} \right) \beta v_x^2 \right] \quad (71)$$

$$v_\ell \frac{dv_\ell}{dt} = \frac{1}{t} \left[\frac{k_\ell v_\ell}{\gamma} (1 - v^2) - (1 - 2v^2) \left(1 - \frac{v_\ell^2}{v^2} \right) \beta v_\ell^2 \right] \quad (72)$$

where we have explicitly written the loop production parameter in terms of the effective intercommuting probability P_{eff} , that is, we have set

$$\tilde{c} \rightarrow P_{\text{eff}} \tilde{c}. \quad (73)$$

Below we discuss numerical solutions of Eqs. (70)–(72) with different initial conditions for v_x , v_ℓ as well as different values for the effective intercommuting probability P_{eff} in the range between 10^{-3} and 1. For an illustration we first consider the case $P_{\text{eff}} \simeq 0.1$. As in Sec. III B 4 we have that

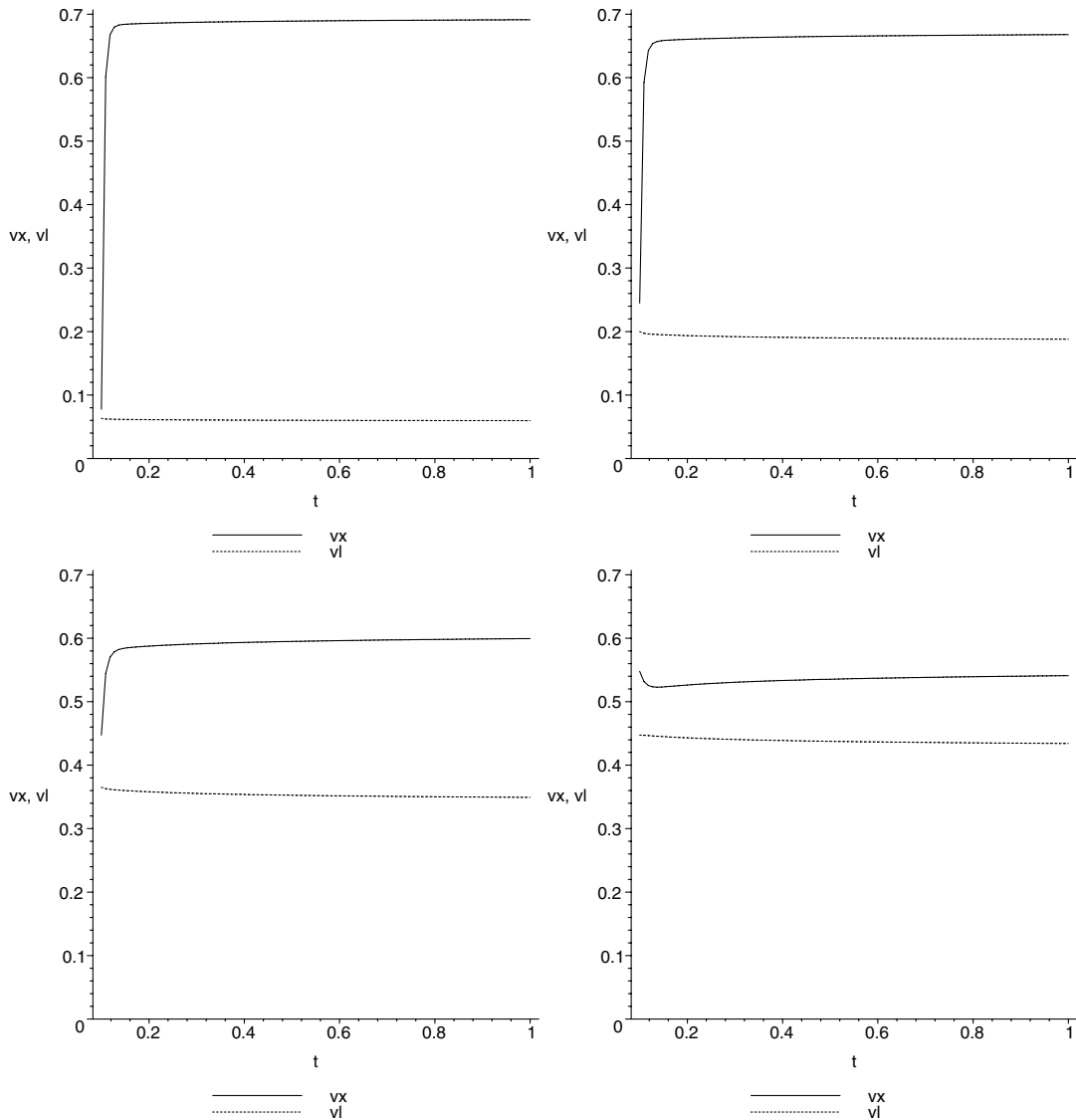


FIG. 4. Velocity evolution in arbitrary time units with initial condition for the total velocity $v_0 = 0.1, 1/\sqrt{10}, 1/\sqrt{3}$ and $1/\sqrt{2}$. We have assumed equipartition of kinetic energies at $t = 0$ and that the number of dimensions transverse to the brane is 2. For small $v_{\ell 0}$ the three-dimensional velocities scale to a value close to the purely three-dimensional result $v \simeq 0.7$. However for $v_{\ell 0} > 0.4$ the value of v_x can be significantly less than 0.6.

since $v_{\ell c}/v_c$ is small, there is essentially no source term for v_ℓ in (72) and thus v_ℓ is just given by its initial condition slowly damped by the expansion. On the other hand v_x is sourced by the string curvature $R \sim L$ and although it is more strongly damped it dominates v_ℓ . There is a critical value of $v_{\ell c}/v_c \simeq 0.15$ above which the curvature (source) term for v_ℓ becomes large enough for v_ℓ to dominate. This value is too large to be relevant in the context of brane inflation: it corresponds to a situation where more than one-tenth of the velocity developed during the last correlation time is in the ℓ dimensions, which would require the curvature vector to have a significant component in the ℓ space. As explained above the correlation length (the typical radius of curvature of strings) at formation is of order 10^3 times the size of the extra dimensions and further grows with the expansion, so $v_{\ell c}/v_c$ is expected to be much less than this critical value. Thus, string propagation in the x space will not stop, but it can be significantly slowed down if the strings are created with enough momentum in the extra dimensions.

Indeed we expect that some of the energy associated with the brane collision will be converted to kinetic energy of string segments in the dimensions transverse to the brane, so the slowing down of string motion in the infinite dimensions could be a significant effect. In Fig. 4 we plot v_x and v_ℓ as functions of time, assuming equipartition of kinetic energies at formation. This assumption is not necessary, and even if $v_x \ll v_\ell$ initially, v_x will increase very fast (Fig. 5) because of the curvature source term in Eq. (71). As long as the constraint $v^2 \leq 1/2$ is satisfied, the important initial condition is v_ℓ at the time of string formation. From Fig. 4 we see that for initial values of v_ℓ

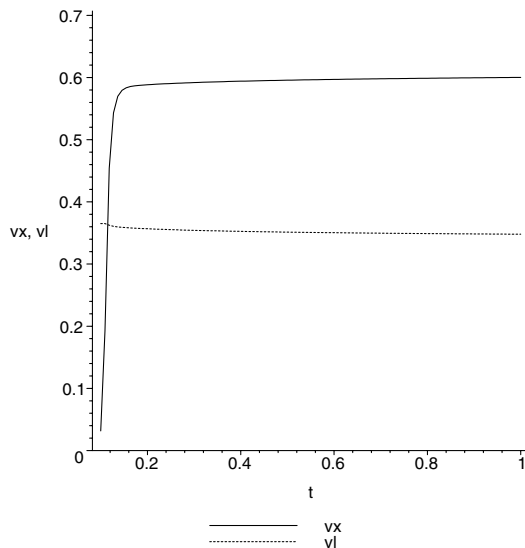


FIG. 5. Velocity evolution in arbitrary time units for $v_x \ll v_\ell$ at $t=0$. Here, v_x rapidly increases and reaches the same equilibrium value as in the case $v_x \sim v_\ell$ at $t=0$. This shows that the assumption of equipartition of kinetic energies at string formation is not necessary. The important initial condition is v_ℓ .

smaller than 0.4 the scaling value of v_x is less than a few percent different from the purely three-dimensional result $v \simeq 0.7$. However for v_ℓ initially greater than this value, the slowing down of strings in the three infinite dimensions is significant. In particular if v_ℓ has a value close to the maximum allowed ($v_\ell \simeq 1/\sqrt{2}$) then v_x will approach a very small value $v_x \simeq \sqrt{1/2 - v_\ell^2}$ (Fig. 6). Such a dramatic reduction of the three-dimensional speed of the strings also has a significant effect on the scaling value of γ (Fig. 7) because the loop production term is proportional to v_x : slowly moving strings will intercommute less often so the final density of the network will be higher, corresponding to a smaller γ .

It is interesting to compare our results with the discussion of Jones, Stoica and Tye [3]. As a first approximation they used a simple one-scale model and assumed a relativistic, constant speed of strings. They found that the scaling value of the energy density of the network is P^{-2} times greater than in the purely three-dimensional case. The leading correction to this result comes from allowing the velocity of strings to be a variable, that is using a VOS model rather than a simple one-scale model. This is equivalent to setting $v_\ell = 0$ in our extradimensional EDVOS model. The effect of this variable velocity is to reduce the density by a factor of 10 or so. We then include velocities in the extra dimensions $v_\ell \neq 0$. If $v_\ell < 0.4$ there is no additional observable effect. If $v_\ell > 0.4$ but considerably smaller than $1/\sqrt{2}$ then these extradimensional velocities have no significant effect on the energy density, though they may lead to a substantial reduction of the three-dimensional string velocities. Finally, if $v_\ell \simeq 1/\sqrt{2}$ the strings will be moving very slowly in the three infinite

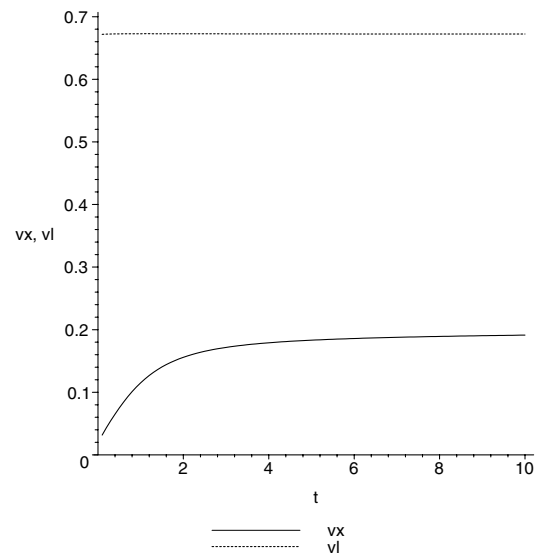


FIG. 6. Velocity evolution in arbitrary time units for $v_\ell = 0.95/\sqrt{2}$ at $t=0$. Because v_ℓ is so high, v_x scales to a low value of about 0.2. Such a low velocity also affects the string density (Fig. 7).

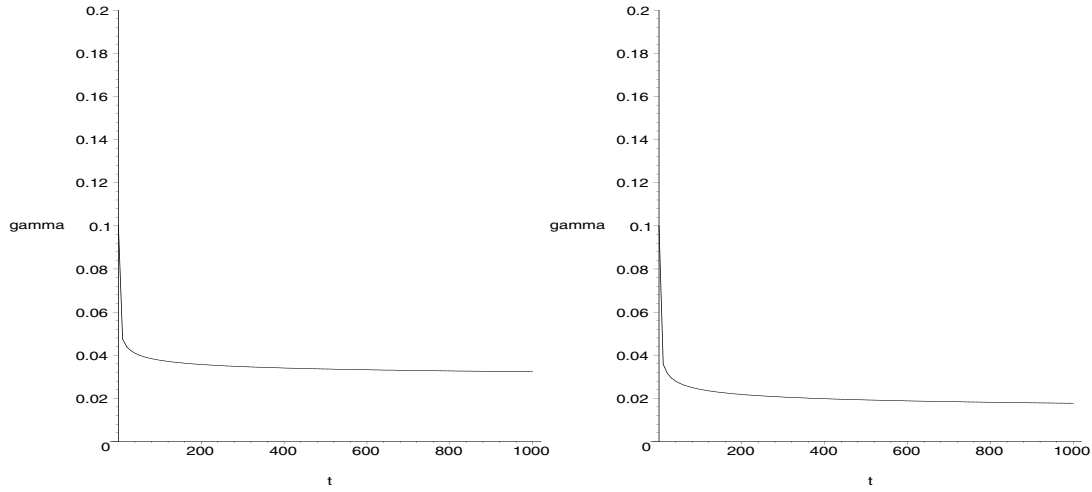


FIG. 7. The effect of slowing down of strings on γ . On the left is the evolution of γ for $v_\ell \approx 0.3$ at formation. On the right is the corresponding evolution for $v_\ell = 0.95/\sqrt{2}$, in which v_x is considerably smaller.

dimensions so that the number of intercommutings will be further reduced. This results in a substantial increase of the energy density of the string network.

Finally we study the effect of varying the intercommuting probability P . For networks of the same type (F - or D -string networks), P is expected to be in the range

$10^{-3} < P < 1$ (in particular $10^{-3} < P < 1$ for F strings and $0.1 < P < 1$ for D strings [5]). However, since the strings can develop significant small-scale structure and become wiggly, they may have more than one opportunity for reconnection in each crossing time. Our one-scale model does not fully take into account such wiggly effects,

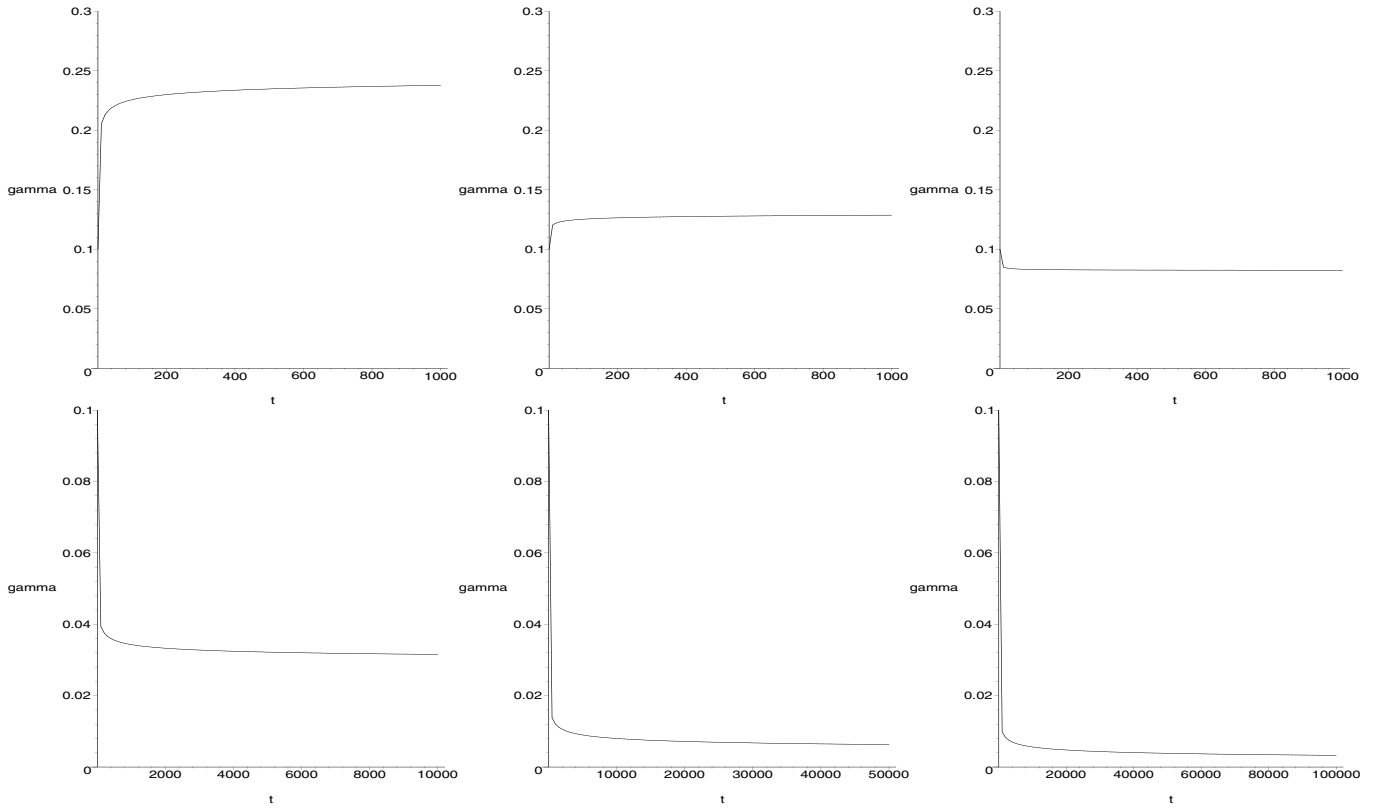


FIG. 8. The dependence of γ on the effective intercommuting probability P_{eff} . From top left to bottom right $P_{\text{eff}} = 1, 0.5, 0.3, 0.1, 0.01, 10^{-3}$. As P_{eff} tends to zero the approximation $\gamma \approx P_{\text{eff}}$ becomes better but the time required to reach scaling increases.

but we have done so implicitly by using the effective intercommuting probability $P_{\text{eff}} = f(P) > P$, as we have discussed above. The effect of small-scale wiggles can only be accounted for numerically and the results of such studies in an expanding space will be presented in [50]. For the present discussion we take P_{eff} in the range $10^{-3} \leq P \leq 1$ and study how the model is affected by changes in P_{eff} . In Fig. 8 we show the behavior of γ for different values of P_{eff} in that range assuming a moderate $v_\ell \simeq 0.36$ at formation. Reducing P_{eff} leads to less intercommutings and therefore greater string energy density, corresponding to a smaller γ .

V. SUMMARY AND DISCUSSION

We have presented an extension of the VOS model that can be used to study the macroscopic evolution of a Brownian network of Nambu-Goto cosmic strings in cosmological spacetimes with extra dimensions (EDVOS model). In order for the Brownian structure of the network to be preserved by time evolution either the spacetime must be isotropic or, in the case of anisotropic expansion, the network must be formed on an isotropic slice of the spacetime. If the strings are not confined on this slice after their formation then the extra dimensions must be compactified at a size smaller than the correlation length, as is the case in models of brane inflation. The evolution has then an effective, three-dimensional description, in which the effect of possible velocities in the extra dimensions can be taken into account. This is a significant factor because extradimensional velocities will act to slow down string motion in the infinite dimensions, reducing the number of string intercommutings while also changing the strings' effective 3D energy per unit length.

First, we applied the model to the case of an isotropic $(D + 1)$ -dimensional FRW universe and found the generic behavior (for $D > 3$) in which $L \propto a^{D/(D-1)}$ (in contrast to the naive conformal stretching obtained without the EDVOS model $L \propto a$). Obviously, these strings do not scale because they find it increasingly difficult to find each other and intercommute, and so they would quickly dominate the energy density of such a universe. However, even if the additional dimensions were compact, this evolution might pertain at early times, as long as the expansion remained isotropic. In this case, the network correlation length L would continue to expand until it inevitably catches up with $R_\ell \propto a$, the scale of the expanding compact dimension. Thereafter, the network would become effectively three-dimensional and, if the compact dimensions were then stabilized, the evolution would turn over towards the alternative FRW 3-brane scenario we discussed earlier. We point out therefore, in the case of static compact extra dimensions that, even if we begin with an initial correlation length smaller than the compact dimension $L_0 < R_\ell$, the correlation length will grow until $L > R_\ell$, at which point the evolution becomes effectively three-

dimensional. Hence, we conclude that the usual assumption $L_0 \gg R_\ell$ is not actually necessary for achieving asymptotically 3D scaling evolution.

The most pertinent application of our model is to a situation where the string network is formed on a FRW 3-brane with the dimensions transverse to it compact and small (we have only considered flat extra dimensions). We have allowed the strings to be able to explore the bulk after formation, as is the case in models of brane inflation. We find that the density of the network in our VOS model (after scaling has been achieved) can be up to a factor of 10 smaller than previous estimates [3]. This correction comes from quantitatively accounting for the role of the string 3D velocity, allowing it to be a variable rather than a constant as in the simplest one-scale models. We also found additional effects arising from the fact that the strings can also move in the extra dimensions. Standard field theory strings can be shown numerically to approach an average velocity of about $1/\sqrt{2}$ (on length scales $\lambda \ll H^{-1}$). This is also the case in our extradimensional VOS model but now a significant amount of this velocity can be trapped in the extra dimensions, with the result that the observable 3D velocity is reduced. Since only the three FRW dimensions are expanding while the size of the extra dimensions is stabilized, there will only be a very weak redshifting of the extradimensional velocities. Thus, if the strings are created with significant velocities in the extra dimensions, these could survive for a long time and result in a slowing down of string motion in the three large dimensions.

One could argue that after reheating, the strings would enter a damping regime where velocities would slow down due to the damping force from a high radiation background density. This could damp velocities in the extra dimensions more strongly than Hubble expansion, in particular, eliminating v_ℓ . However, we note that reheating only takes place on the brane (the bulk remains cold and empty [27]) and it is only the small portion of the string network intersecting the brane which feels this damping. As the strings move, they briefly encounter the brane and pass through it, feeling a damping force, but at any time most of the length of the network is in the bulk, where no friction is felt. There are only a finite number of such brief encounters before cosmological expansion on the brane cools it sufficiently for the friction force to become negligible. The strings also might be oriented such as to intersect and remain in contact with the 3-brane in the longer term, but this only implies that the string will be ‘‘pinned’’ at one point. Taking into account these geometrical effects, it appears that the usual 3D frictional damping terms will be suppressed by factors of order the intercommuting probability P (for both v_ℓ and v_x). The network in higher dimensions will approach its relativistic scaling regime much more quickly than its $3 + 1$ FRW counterpart, though a more quantitative investigation is certainly warranted.

Gravitational backreaction can also be expected to reduce the extradimensional velocities. The effect of gravitational backreaction can be incorporated in the VOS model by including a term $8\Gamma G\mu v^6$ in the evolution equation for the correlation length L [24], where for long strings Γ is a constant of order 10 [24,51]. Even if $G\mu$ has a value close to the upper limit 10^{-6} , this term is small compared to the loop production term $P\tilde{c}v$ so it has little direct influence on the network density. However, its effect on the extradimensional velocity v_ℓ could be more dramatic since gravitational radiation tends to act to eliminate small-scale structure below a certain length scale λ . This length scale was previously thought to be $\lambda \sim \Gamma G\mu t$, but closer investigation has suggested that it was in fact considerably smaller $\lambda \sim (\Gamma G\mu)^\eta t$ with $\eta = 3/2$ in the radiation era and $\eta = 5/2$ in the matter era [52]. The reanalysis demonstrated that only modes of comparable wavelength interact efficiently, a fact which may be relevant in the case with extra dimensions. Here, we could envisage the small-scale modes trapped in the extra dimensions essentially decoupling from the effective 3D evolution. Damping of the velocity v_ℓ then would be primarily through self-interactions between the trapped modes which tend toward slow power law, rather than exponential, suppression. Whatever the outcome of a closer examination of this issue in higher dimensions, it is clear that the extradimensional velocity v_ℓ will have a long lifetime and will influence the network evolution over long time scales. Note however that since the string position in the compact dimensions is described by (world sheet) scalar fields, one expects some stabilization mechanism for these moduli to kick in at low energies. This would render these excitations massive, effectively localizing the string in the extra dimensions [44]. Given the large hierarchy between the preferred inflationary and SUSY breaking scales in brane inflation models (GUT and TeV, respectively), the extradimensional effects we have considered are likely to be relevant for many orders of magnitude in time.

While our EDVOS model can characterize the effect of the velocity v_ℓ on the overall large-scale 3D network properties, uncertainties remain as to its magnitude at network formation, the subsequent (weak) rate of damping, and whether it can be sourced in any way during the subsequent evolution. Given the brane collision out of which the string network forms, it seems plausible that the string network will be released from the brane with a significant velocity v_ℓ with which to traverse the extra bulk dimensions. If v_ℓ is relativistic, then the average 3D velocity v_x will be smaller and the network density will be higher than previously expected. However, this presumes the absence of any significant source terms for v_ℓ , for example, due to string reconnections or other dynamical effects. By including such source terms, we have demonstrated that there is, in principle, the pathological possibility that the Hubble-damped 3D velocity v_x becomes

negligible because of continuous contributions to the undamped v_ℓ . In this case the string network would dominate over radiation in the early universe. But we have also shown that there is a threshold ($k_\ell \sim 0.1k_x$) below which this does not happen. For brane inflation scenarios in which the string curvature is predominantly three-dimensional at formation, we expect this criteria to be fulfilled, with the EDVOS model predicting a scaling solution in which $L = \gamma t$ and $\gamma \sim P_{\text{eff}}$. We stress that the statement $L \propto P$ should not be taken as a prediction of our model. Instead, we have used an effective intercommuting probability P_{eff} , which is a phenomenological parameter of the model, to be determined by string network simulations or a deeper statistical analysis. The functional dependence of $P_{\text{eff}} = f(P)$ on the *actual* intercommuting probability P will be the subject of a forthcoming publication [50].

Another interesting issue which we can use our EDVOS model to investigate is the evolution of closed strings (loops), and, in particular, the possibility that such loops could wrap around the compact dimensions [4]. If the compact manifold admits nontrivial one-cycles, this would give rise to stable monopolelike objects (from the 3D point of view) which could dominate the universe or even provide a dark matter candidate. We denote these objects *cycloops*, that is, loops wrapping around nontrivial cycles. We leave the study of the formation, properties and consequences of cycloops for a different publication [53].

The purpose of the present paper was not to exhaustively explore all possible avenues or to investigate specific brane inflation models, but rather to provide a broad picture of the key dynamical effects which will influence a cosmic string network emerging from a higher-dimensional theory. We believe the generalized VOS model we have presented here is an important step in developing a more quantitative description of cosmic string evolution in brane inflation and other contexts. There are a host of further issues to explore. We have, for example, only considered the ‘‘Abelian’’ case where there is only one particular type of string, which simply exchanges partners and reconnects when it meets another (albeit with a reduced probability for such an encounter). Similar methods can be applied to the richer structures created by F and D strings in more complex (true) string networks (see Ref. [32]) and this deserves further investigation.

ACKNOWLEDGMENTS

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APPENDIX: APPROXIMATE FORMULAS FOR k , k_x AND k_ℓ

Defining $\dot{\mathbf{y}} = (\dot{\mathbf{x}}, \dot{\mathbf{t}}/a)$ we split the velocity $\dot{\mathbf{y}}$ into a ‘‘curvature’’ component $\dot{\mathbf{y}}_c$ produced during the last correlation time and a leftover component $\dot{\mathbf{y}}_p$ (coming from previous accelerations) by writing $\dot{\mathbf{y}} = \dot{\mathbf{y}}_c + \dot{\mathbf{y}}_p$. We interpret the first as the velocity induced on large scales by the present correlation length of the string, whereas the second is the velocity remaining from previous correlation times, generally on small scales. These two components are uncorrelated so we have $\langle \dot{\mathbf{y}}_c \cdot \dot{\mathbf{y}}_p \rangle = 0$. We also have the gauge condition $\dot{\mathbf{y}} \cdot \mathbf{y}' = 0$ and we assume that both components $\dot{\mathbf{y}}_c$ and $\dot{\mathbf{y}}_p$ separately satisfy this, i.e. we have $\dot{\mathbf{y}}_c \cdot \mathbf{y}' = 0$ and $\dot{\mathbf{y}}_p \cdot \mathbf{y}' = 0$.

Hence, since the curvature vector \mathbf{u} is normal to the tangent \mathbf{y}' we have

$$\hat{\mathbf{u}} = A\hat{\mathbf{y}}_c - B\hat{\mathbf{y}}_p + \sum_{i=1}^{D-2} C_i \hat{\mathbf{y}}_i \quad (\text{A1})$$

where $\hat{\mathbf{y}}_c, \hat{\mathbf{y}}_p$ are the unit vectors in the direction of $\dot{\mathbf{y}}_c, \dot{\mathbf{y}}_p$, respectively, and $\hat{\mathbf{y}}_i$ are unit vectors spanning the $(D-3)$ -dimensional subspace normal to $\dot{\mathbf{y}}_c, \dot{\mathbf{y}}_p$ and \mathbf{y}' . Dotting with $\dot{\mathbf{y}}$ and taking the average along the string we have

$$\begin{aligned} \langle \hat{\mathbf{u}} \cdot \dot{\mathbf{y}} \rangle &= \langle A|\dot{\mathbf{y}}_c| - B|\dot{\mathbf{y}}_p| \rangle \\ &= \left\langle A|\dot{\mathbf{y}}_c| \left(1 - \frac{B}{A} \frac{|\dot{\mathbf{y}}_p|}{|\dot{\mathbf{y}}_c|} \right) \sqrt{\dot{\mathbf{y}}^2 (\dot{\mathbf{y}}_c^2 + \dot{\mathbf{y}}_p^2)^{-1/2}} \right\rangle \\ &= \left\langle \sqrt{\dot{\mathbf{y}}^2} \frac{A \left(1 - \frac{B}{A} \frac{|\dot{\mathbf{y}}_p|}{|\dot{\mathbf{y}}_c|} \right)}{\left(1 + \frac{\dot{\mathbf{y}}_p^2}{\dot{\mathbf{y}}_c^2} \right)^{1/2}} \right\rangle. \end{aligned} \quad (\text{A2})$$

Taking the modulus of (A1) gives $1 = A(1 + (B^2/A^2) + \sum_{i=1}^{D-3} (C_i^2/A^2))^{1/2}$, which we use to substitute for A in (A2). Remembering that $k v \simeq \langle \hat{\mathbf{u}} \cdot \dot{\mathbf{y}} \rangle$ we find

$$k \simeq \frac{1 - \frac{B}{A} \frac{v_p}{v_c}}{\left(1 + \frac{v_p^2}{v_c^2} \right)^{1/2} \left(1 + \frac{B^2}{A^2} + \sum_{i=1}^{D-3} \frac{C_i^2}{A^2} \right)^{1/2}}. \quad (\text{A3})$$

For small velocities ($v \ll 1$) we have that $v \simeq v_c$ and the ratio v_p/v_c is much less than unity. However, as the

velocity tends to relativistic values, the v_p contribution becomes more and more significant. Assuming that this relative contribution is proportional to some power of the total velocity, we set $v_p/v_c = f v^\alpha$ with f, α constants (clearly $\alpha > 1$). Similarly in the low velocity limit, the curvature vector \mathbf{u} is parallel to $\dot{\mathbf{y}}_c$ so $B/A, C_i/A \ll 1$ but as v increases, B and C_i become comparable to A . Assuming a power law dependence we set $B/A = C_i/A = g v^\beta$. Equation (A3) becomes

$$k \simeq \frac{1 - f g v^{\alpha+\beta}}{\left(1 + f^2 v^{2\alpha} \right)^{1/2} \left(1 + (D-2) g^2 v^{2\beta} \right)^{1/2}}. \quad (\text{A4})$$

Comparison with the helicoidal string solution in $D = 3$ flat space gives $\alpha + \beta = 3$ [24]. The limit $k(1/\sqrt{2}) = 0$ (which can be shown to hold analytically [21,23]) gives $f g = 8$. Then, comparison with the three-dimensional case (30) suggests $f = g, \alpha = \beta$. The approximate formula for k is therefore

$$k \simeq \frac{1 - 8v^6}{\left(1 + 8v^6 \right)^{1/2} \left(1 + 8(D-2)v^6 \right)^{1/2}}. \quad (\text{A5})$$

For k_x we form $\langle \hat{\mathbf{u}} \cdot \dot{\mathbf{x}} \rangle$ and work as above to find

$$k_x \simeq \frac{v_{xc}}{v_c} \frac{1 - \frac{B}{A} \frac{v_{xp}^2}{v_{xc}^2} \frac{v_c}{v_p}}{\left(1 + \frac{v_{xp}^2}{v_{xc}^2} \right)^{1/2} \left(1 + \frac{B^2}{A^2} + \sum_{i=1}^{D-3} \frac{C_i^2}{A^2} \right)^{1/2}} \quad (\text{A6})$$

where we have used that $\langle \hat{\mathbf{y}}_c \cdot \dot{\mathbf{x}} \rangle = \langle (1/|\dot{\mathbf{y}}_c|) \dot{\mathbf{y}}_c \cdot \dot{\mathbf{x}} \rangle = \langle (1/|\dot{\mathbf{y}}_c|) \dot{\mathbf{x}}_c \cdot \dot{\mathbf{x}} \rangle = \langle |\dot{\mathbf{x}}_c|^2 / |\dot{\mathbf{y}}_c| \rangle$.

Assuming that v_{xp}/v_{xc} has the same velocity dependence as v_p/v_c we obtain the following approximate formula for k_x

$$k_x \simeq \frac{v_{xc}}{v_c} \frac{1 - 8v^6}{\left(1 + 8v^6 \right)^{1/2} \left(1 + 8(D-2)v^6 \right)^{1/2}}. \quad (\text{A7})$$

The corresponding equation for k_ℓ is

$$k_\ell \simeq \frac{v_{\ell c}}{v_c} \frac{1 - 8v^6}{\left(1 + 8v^6 \right)^{1/2} \left(1 + 8(D-2)v^6 \right)^{1/2}}. \quad (\text{A8})$$

Equations (A5), (A7), and (A8) are indeed the approximate formulas (53)–(55) given in Sec. III.

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