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Gauss-Bonnet dark energy

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We propose the Gauss-Bonnet dark energy model inspired by string/M-theory where standard gravity with scalar contains additional scalar-dependent coupling with a Gauss-Bonnet invariant. It is demonstrated that the effective phantom (or quintessence) phase of the late universe may occur in the presence of such a term when the scalar is phantom or for nonzero potential (for canonical scalar). However, with the increase of the curvature, the Gauss-Bonnet term may become dominant so that the phantom phase is transient and the $w = -1$ barrier may be passed. Hence, the current acceleration of the universe may be caused by a mixture of scalar phantom and/or potential or stringy effects. It is remarkable that scalar-Gauss-Bonnet coupling acts against the big rip occurrence also in phantom cosmology.

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I. INTRODUCTION

It became clear recently that late-time dynamics of the current accelerated universe is governed by the mysterious dark energy. The interpretation of the astrophysical observations indicates that such dark energy fluid (if it is fluid) is characterized by the negative pressure, and its equation of state parameter w lies very close to -1 (most probably below it). It is quite possible that it may be oscillating around -1. It is extremely difficult to present the completely satisfactory theory of the dark energy (also due to lack of all required astrophysical data), especially in the case of (oscillating) w less than -1. (For instance, thermodynamics is quite strange there with possible negative entropy [1].)

The successful dark energy theory may be searched in string/M-theory. Indeed, it is quite possible that some unusual gravity-matter couplings predicted by the fundamental theory may become important at the current, low-curvature universe (being not essential in intermediate epoch from strong to low curvature). For instance, in the study of string-induced gravity near initial singularity, the role of Gauss-Bonnet (GB) coupling with scalar was quite important for the occurrence of nonsingular cosmology [2,3] (for an account of dilaton and higher order corrections near initial singularity, see also [4]). The present paper is devoted to the study of the role of GB coupling with the scalar field to the late-time universe. It is explicitly demonstrated that such a term itself cannot induce the effective phantom late-time universe if the scalar is canonical in the absence of the potential term. It may produce the effective quintessence (or phantom) era, explaining the current acceleration only when the scalar is phantom or when the scalar is canonical with nonzero potential. It is interesting that it may also have the important impact to the big rip singularity [5], similarly to quantum effects [6,7], preventing it in the standard phantom cosmology. Note that we concentrate mainly on the exponential scalar-GB coupling and exponential scalar potential, while the consideration of other types of such functions and their role in late-time cosmology will be considered elsewhere.

II. THE ACCELERATED UNIVERSE FROM SCALAR-GB GRAVITY

We consider a model of the scalar field ϕ coupled with gravity. As a stringy correction, the term proportional to the GB invariant G is added:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{1}$$

The starting action is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)G\right\}. \tag{2}$$

Here $\gamma = \pm 1$. For the canonical scalar, $\gamma = 1$ but at least when the GB term is not included, the scalar behaves as phantom only when $\gamma = -1$ [8] showing in this case the properties similar to a quantum field [9]. In analogy with model [10] where also nontrivial coupling of a scalar

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Lagrangian with some power of curvature was considered, one may expect that such a GB coupling term may be relevant for the explanation of dark energy dominance.

By the variation over ϕ, we obtain

$$0 = \gamma\nabla^2\phi - V'(\phi) + f'(\phi)G. \tag{3}$$

On the other hand, the variation over the metric $g_{\mu\nu}$ gives

$$\begin{aligned} 0 = {} & \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \gamma\left(\frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right) + \frac{1}{2}g^{\mu\nu}(-V(\phi) + f(\phi)G) - 2f(\phi)RR^{\mu\nu} \\ & + 2\nabla^\mu\nabla^\nu(f(\phi)R) - 2g^{\mu\nu}\nabla^2(f(\phi)R) + 8f(\phi)R^\mu{}_\rho R^{\nu\rho} - 4\nabla_\rho\nabla^\mu(f(\phi)R^{\nu\rho}) - 4\nabla_\rho\nabla^\nu(f(\phi)R^{\mu\rho}) \\ & + 4\nabla^2(f(\phi)R^{\mu\nu}) + 4g^{\mu\nu}\nabla_\rho\nabla_\sigma(f(\phi)R^{\rho\sigma}) - 2f(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4\nabla_\rho\nabla_\sigma(f(\phi)R^{\mu\rho\sigma\nu}). \end{aligned} \tag{4}$$

By using the identities obtained from the Bianchi identity

$$\begin{gathered} \nabla^\rho R_{\rho\tau\mu\nu} = \nabla_\mu R_{\nu\tau} - \nabla_\nu R_{\mu\tau}, \qquad \nabla^\rho R_{\rho\mu} = \frac{1}{2}\nabla_\mu R, \qquad \nabla_\rho\nabla_\sigma R^{\mu\rho\nu\sigma} = \nabla^2 R^{\mu\nu} - \frac{1}{2}\nabla^\mu\nabla^\nu R + R^{\mu\rho\nu\sigma}R_{\rho\sigma} - R^\mu{}_\rho R^{\nu\rho}, \\ \nabla_\rho\nabla^\mu R^{\rho\nu} + \nabla_\rho\nabla^\nu R^{\rho\mu} = \frac{1}{2}(\nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R) - 2R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 2R^\mu{}_\rho R^{\nu\rho}, \qquad \nabla_\rho\nabla_\sigma R^{\rho\sigma} = \frac{1}{2}\Box R, \end{gathered} \tag{5}$$

one can rewrite (4) as

$$\begin{aligned} 0 = {} & \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \gamma\left(\frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right) + \frac{1}{2}g^{\mu\nu}(-V(\phi) + f(\phi)G) - 2f(\phi)RR^{\mu\nu} \\ & + 4f(\phi)R^\mu{}_\rho R^{\nu\rho} - 2f(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4f(\phi)R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2(\nabla^\mu\nabla^\nu f(\phi))R - 2g^{\mu\nu}(\nabla^2 f(\phi))R \\ & - 4(\nabla_\rho\nabla^\mu f(\phi))R^{\nu\rho} - 4(\nabla_\rho\nabla^\nu f(\phi))R^{\mu\rho} + 4(\nabla^2 f(\phi))R^{\mu\nu} + 4g^{\mu\nu}(\nabla_\rho\nabla_\sigma f(\phi))R^{\rho\sigma} - 4(\nabla_\rho\nabla_\sigma f(\phi))R^{\mu\rho\nu\sigma}. \end{aligned} \tag{6}$$

The above expression is valid in arbitrary spacetime dimensions. In four dimensions, the terms proportional to $f(\phi)$ without derivatives are canceled with each other and vanish since the GB invariant is a total derivative in four dimensions.

The starting Friedmann-Robertson-Walker universe metric is

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}(dx^i)^2, \tag{7}$$

where

$$\begin{gathered} \Gamma^t_{ij} = a^2H\delta_{ij}, \qquad \Gamma^i_{jt} = \Gamma^i_{tj} = H\delta^i{}_j, \qquad R_{itjt} = -(\dot{H} + H^2)\delta_{ij}, \qquad R_{ijkl} = a^4H^2(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}), \\ R_{tt} = -3(\dot{H} + H^2), \qquad R_{ij} = a^2(\dot{H} + 3H^2)\delta_{ij}, \qquad R = 6\dot{H} + 12H^2, \qquad \text{other components} = 0 \end{gathered} \tag{8}$$

(here the Hubble rate H is defined by $H = \dot{a}/a$). Assuming ϕ only depends on time, the $(\mu, \nu) = (t, t)$ component in (4) has the following simple form:

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{\gamma}{2}\dot{\phi}^2 + V(\phi) - 24\dot{\phi}f'(\phi)H^3. \tag{9}$$

On the other hand, Eq. (2) becomes

$$0 = -\gamma(\ddot{\phi} + 3H\dot{\phi}) - V'(\phi) + 24f'(\phi)(\dot{H}H^2 + H^4). \tag{10}$$

We now consider the case that $V(\phi)$ and $f(\phi)$ are given as exponents with the constant parameters V_0, f_0, and ϕ_0:

$$V = V_0\mathrm{e}^{-[(2\phi)/\phi_0]}, \qquad f(\phi) = f_0\mathrm{e}^{(2\phi)/\phi_0}. \tag{11}$$

Assume that the scale factor behaves as $a = a_0t^{h_0}$ (power law). In the case that h_0 is negative, this scale factor does not correspond to an expanding universe but it corresponds to a shrinking one. If one changes the direction of time as $t \to -t$, the expanding universe whose scale factor is given by $a = a_0(-t)^{h_0}$ emerges. In this expression, however, since h_0 is not always an integer, t should be negative so that the scale factor should be real. To avoid the apparent difficulty, we may further shift the origin of the time as $t \to -t \to t_s - t$. Then the time t can be positive as long as $t < t_s$. Hence, we can propose

$$H = \frac{h_0}{t}, \qquad \phi = \phi_0\ln\frac{t}{t_1}, \tag{12}$$

when $h_0 > 0$ or

$$H = -\frac{h_0}{t_s - t}, \qquad \phi = \phi_0\ln\frac{t_s - t}{t_1}, \tag{13}$$

when $h_0 < 0$, with an undetermined constant t_1. By the assumption (12) or (13), one obtains

$$0 = -\frac{3h_0^2}{\kappa^2} + \frac{\gamma\phi_0^2}{2} + V_0 t_1^2 - \frac{48 f_0 h_0^3}{t_1^2}, \tag{14}$$

from (9) and

$$0 = \gamma(1 - 3h_0)\phi_0^2 + 2V_0 t_1^2 + \frac{48 f_0 h_0^3}{t_1^2}(h_0 - 1), \tag{15}$$

from (10). Using (14) and (15), it follows

$$V_0 t_1^2 = -\frac{1}{\kappa^2(1+h_0)}\left\{3h_0^2(1-h_0) + \frac{\gamma\phi_0^2\kappa^2(1-5h_0)}{2}\right\},$$
$$\frac{48 f_0 h_0^2}{t_1^2} = -\frac{6}{\kappa^2(1+h_0)}\left(h_0 - \frac{\gamma\phi_0^2\kappa^2}{2}\right). \tag{16}$$

The second equation in (16) shows that if $-1 < h_0 < 0$ and $\gamma = 1$, f_0 should be negative. Without the GB term, that is, $f_0 = 0$, a well known result follows:

$$h_0 = \frac{\gamma\phi_0^2\kappa^2}{2}. \tag{17}$$

Since the equation of state parameter w is given by

$$w = -1 + \frac{2}{3h_0}, \tag{18}$$

if $h_0 < 0$ ($h_0 > 0$), $w < -1$ ($w > -1$). Equation (16) indicates that even if $\gamma = 1$, with the proper choice of parameters h_0 can be negative or $w < -1$. Even if $\gamma > 0$ when $h_0 < -1$, V_0 is positive, which means that the potential $V(\phi)$ is bounded below. As a special case, we consider

$$\phi_0^2 = -\frac{6h_0^2(1-h_0)}{\gamma(1-5h_0)\kappa^2}, \tag{19}$$

which gives $V(\phi) = 0$. In order that ϕ_0 could be real, one has

$$\frac{1}{5} < h_0 < 1 \quad \text{when } \gamma = 1,$$
$$\text{or } h_0 > \frac{1}{5} \text{ or } h_0 \geq 1 \quad \text{when } \gamma = -1. \tag{20}$$

In the case of Eq. (19), the scalar field ϕ is canonical ($\gamma = 1$), and there is no potential $V(\phi) = 0$. Even if we include the term proportional to the GB invariant, we cannot obtain the effective phantom cosmological solution with $h_0 < 0$ or $w < -1$. Equation (16) tells, however, when $\gamma = 1$ and $V_0 > 0$, even if V_0 is arbitrarily small, if we choose f_0 properly, we may obtain the effective phantom. The qualitative behavior of $\gamma\phi_0^2$ versus h_0 when $V_0 = 0$ is given in Fig. 1. There is one positive solution, which may mimic the effective matter with $1/5 < h_0 < 1$ when $\gamma = 1$. We also find that, when $\gamma = -1$, there are always three solutions for h_0 from (19): one is given by $h_0 < 0$ and describes the phantom cosmology; another is $h_0 > 1$ and describes the quintessence cosmology; the final one corresponds to the matter with $0 < h_0 < 1/5$. Then, even if $\gamma = -1$, there appear the solutions describing nonphantom cosmology corresponding to quintessence or matter.

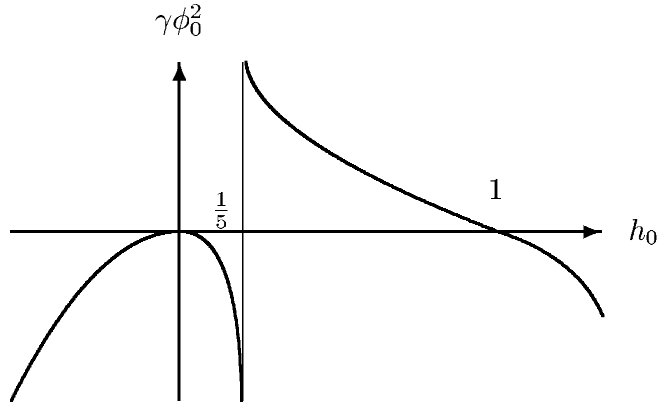

FIG. 1. The qualitative behavior of ϕ_0^2 versus h_0 from (20).

As an example, we consider the case that

$$h_0 = -\frac{80}{3} < -1, \tag{21}$$

which gives, from (18),

$$w = -1.025. \tag{22}$$

This is consistent with the observational bounds for effective w (for a recent discussion and a complete list of references, see [11]). Then, from (16), one obtains

$$V_0 t_1^2 = \frac{1}{\kappa^2}\left(\frac{531\,200}{231} + \frac{403}{154}\gamma\phi_0\kappa^2\right),$$
$$\frac{f_0}{t_1^2} = -\frac{1}{\kappa^2}\left(\frac{9}{49\,280} + \frac{27}{7\,884\,800}\gamma\phi_0\kappa^2\right). \tag{23}$$

Therefore even starting from the canonical scalar theory with positive potential before introducing the term proportional to the GB invariant, we may obtain a solution which reproduces the observed value of w as in (22).

In the case of the model induced from the string theory [2], we have $V_0 = 0$ ($V(\phi) = 0$) and

$$\phi_0^2 = \frac{2}{\kappa^2}, \tag{24}$$

in (11). Then Eq. (19) reduces as

$$3h_0^3 - 3h_0^2 + 5h_0 - 1 = 0, \tag{25}$$

which has only one real solution,

$$h_0 = 0.223\,223. \tag{26}$$

The solution gives

$$w = 1.986\,54. \tag{27}$$

There is another solution of (9) and (10) with (11). In the solution, ϕ and H are constants,

$$\phi = \varphi_0, \qquad H = H_0, \tag{28}$$

what corresponds to de Sitter space. Using (9) and (10) with (11), one finds

$$H_0^2 = -\frac{e^{-[(2\varphi_0)/\phi_0]}}{8f_0\kappa^2}. \tag{29}$$

Therefore in order for the solution to exist, we may require $f_0 < 0$. In (29), φ_0 can be arbitrary. Hence, the Hubble rate $H = H_0$ might be determined by an initial condition.

In the case of the model (11), the term including the GB invariant always gives the contribution in the same order with those from other terms even if the curvature is small. This is due to the factor $f(\phi)$, which enhances the contribution when the curvature is small.

III. LATE-TIME ASYMPTOTIC COSMOLOGY IN SCALAR-GB GRAVITY AND BIG RIP AVOIDANCE

In the following, another model, which is slightly different from (11), may be considered:

$$V(\phi) = V_0 e^{-[(2\phi)/\phi_0]}, \qquad f(\phi) = f_0 e^{(2\phi)/(\alpha\phi_0)}, \qquad (\alpha > 1). \tag{30}$$

Different from model (11), model (30) will not be solved exactly. We can only find the asymptotic qualitative behavior of the solutions. Nevertheless, the asymptotic behavior suggests the existence of the cosmological solution, where the value of w could vary with time (oscillation) and/or could depend on the curvature.

Assuming the solution behaves as (12) or (13), when the curvature is small, that is t in (12) or $t_s - t$ in (13) is large, the GB term becomes small and could be neglected since it behaves like $1/t^{-(2/\alpha)+4}$ or $1/(t_s - t)^{-(2/\alpha)+4}$. When the curvature is small, the solution could be given by (17), then the effective phantom phase with $w < -1$ could appear only in case $\gamma = -1 < 0$. On the other hand, when the curvature is large, that is t in (12) or $t_s - t$ in (13) is small, the classical potential could be neglected. Without the classical potential, by assuming, instead of (12)

$$H = \frac{h_0}{t}, \qquad \phi = \alpha\phi_0 \ln\frac{t}{t_1}, \tag{31}$$

when $h_0 > 0$ or

$$H = -\frac{h_0}{t_s - t}, \qquad \phi = \alpha\phi_0 \ln\frac{t_s - t}{t_1}, \tag{32}$$

when $h_0 < 0$, the following equations replace (14) and (15)

$$0 = -\frac{3h_0^2}{\kappa^2} + \frac{\gamma\alpha^2\phi_0^2}{2} - \frac{48f_0h_0^3}{t_1^2}, \qquad 0 = \gamma(1 - 3h_0)\alpha^2\phi_0^2 + \frac{48f_0h_0^3}{t_1^2}(h_0 - 1). \tag{33}$$

By deleting f_0 in the above two equations, one gets

$$\phi_0^2 = -\frac{6h_0^2(1 - h_0)}{\gamma\alpha^2(1 - 5h_0)\kappa^2}, \tag{34}$$

which corresponds to (19). Then when $\gamma = 1$, the solutions of (34) are not qualitatively changed from those of (19), and there is only one solution, $1/5 < h_0 < 1$. On the other hand, when $\gamma = -1$, since the sign of the right-hand side of (19) is changed from the $\gamma = 1$ case, as is clear from Fig. 1, there are three solutions, corresponding to the phantom $h_0 < 0$ or $w < -1$, the quintessence $h_0 > 1$ or $-1 < w < -1/3$, and the matter with $0 < h_0 < 1/5$ or $w > 7/3$. Then if the term proportional to the GB invariant in the case $\gamma < 0$ (which corresponds to a scalar phantom solution without the GB term) is included, the effective w becomes larger than -1 and the big rip singularity might be avoided (see [6] for quantum effects account to the escape of the big rip). That is, in the case $\gamma < 0$, when the curvature is small as in the current universe, the GB term is negligible and the potential term dominates, which gives the cosmic acceleration with $w < -1$. Then the curvature increases gradually and the universe seems to tend to the big rip singularity [5]. However, when the curvature is large, the GB term becomes dominant and might prevent the singularity. Hence, in the case $\gamma < 0$, the GB term may work against the big rip singularity occurrence, like quantum effects [6]. After the GB term dominates when $\gamma < 0$, the curvature turns to become smaller. Then the potential term dominates again. This might tell that the behavior of the universe might approach the de Sitter space with $w = -1$ by the damped oscillation. In fact, even in the model (30), if (28) is assumed, there is a de Sitter solution corresponding to (28):

$$H_0^2 = -\frac{e^{-[(2\phi_0)/(\alpha\phi_0)]}}{8f_0\kappa^2}, \qquad \varphi_0 = \frac{\alpha\phi_0}{2(1-\alpha)} \ln\left(-\frac{8V_0f_0\kappa^2}{3}\right). \tag{35}$$

In (30), we have assumed $\alpha > 1$. If we consider the case that $V(\phi) = V_0 e^{-(2\phi)/\phi_0}$ and $f(\phi) = f_0 e^{[(2\phi)/(\alpha\phi_0)]}$ as in (30) but $0 < \alpha < 1$, there appears a solution where the term including the GB invariant becomes dominant even if the curvature is small. By assuming (31) or (32), Eq. (33) is obtained again. Hence, when $0 < \alpha < 1$, the solution where h_0 is positive or $w > -1$ even if $\gamma < 0$ (scalar phantom) appears.

On the other hand, if $\gamma > 0$, there is no accelerated universe solution with $w < -1$. The parameter w may change with time but w is larger than $-1/3$.

IV. DISCUSSION

We considered essentially two models with exponential couplings given by (11) and (30). The model (11) may be considered as the special case corresponding to $\alpha = 1$. The main results can be summarized as follows:

(1) $\alpha = 1$ case: exactly solvable.
   (a) $V = 0$ case: When $\gamma = 1$, there is only one solution, $-1/3 < w < 7/3$. On the other hand, when $\gamma = -1$, there are three solutions, corresponding to $w < -1$, $-1 < w < -1/3$, and $w > 7/3$.

(2) $\alpha > 1$ case: the potential term dominates for a small curvature and the GB term for a large one.
   (a) $\gamma > 0$: The value of w may be time dependent, but there is no solution describing the acceleration of the universe.
   (b) $\gamma < 0$: There might appear the big rip singularity, but there might be a solution asymptotically approaching the de Sitter space.

(3) $0 < \alpha < 1$ case: the potential term dominates for a large curvature and the GB term for a small one.
   (a) $\gamma > 0$: There is no solution describing the acceleration of the universe.
   (b) $\gamma < 0$: There appears the big rip singularity.

For the models (11) and (30), in case $V_0 = 0$ (that is, when the potential vanishes), by replacing $\alpha\phi_0$ with ϕ_0, it follows that the two models are equivalent. Especially the $\gamma = -1$ case has been well studied and it has been shown that there are always three effective cosmological phases corresponding to the phantom with $h_0 < 0$ or $w < -1$, the quintessence with $h_0 > 1$ or $-1 < w < -1/3$, and the matter with $0 < h_0 < 1/5$ or $w > 7/3$. Even if $V_0 \neq 0$, the model (11) can be solved exactly and the solutions where h_0 and therefore w are constants may be found. On the other hand, if $V_0 \neq 0$, in the model (30) there exist the solutions where the values of h_0 and therefore of w are time dependent. There could emerge a cosmology, which behaves as a phantom one with $w < -1$ when the curvature is small and as a usual matter dominated universe with $w > -1$ when the curvature is large. Moreover, big rip singularity does not occur.

Our study indicates that current acceleration may be caused by stringy/M-theory effects (terms) which somehow became relevant quite recently (in a cosmological sense). It remains a challenge to construct the consistent dark energy universe model from string/M-theory.

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Note added in proofs.—After the first version of this paper (with some errors) appeared in hep-th, a related study, for instance, of the influence of the scalar-GB term on big rip has appeared in Ref. [12].

APPENDIX: STABILITY OF PHANTOM COSMOLOGY

In this appendix, we check the stability of the above solutions. The following quantities are convenient to introduce:

$$X \equiv \frac{\dot{\phi}}{H}, \qquad Z \equiv H^2 f'(\phi), \qquad \frac{d}{dN} \equiv a\frac{d}{da} = \frac{1}{H}\frac{d}{dt}. \tag{A1}$$

For simplicity, we also put κ^2 to be unity. Then by using (9) and (10) with (11), one finds

$$\frac{dX}{dN} = \frac{\gamma^2\phi_0 X^3 + 2\gamma X(8X^2Z - \phi_0(3 + 52XZ)) + 4(\frac{2V_0f_0}{\phi_0 Z} + \frac{24V_0f_0X}{\phi_0} + 12Z(\phi_0 + 16\phi_0XZ - 8X^2Z))}{2\phi_0(\gamma + 6\gamma XZ + 96Z^2)}, \tag{A2}$$

$$\frac{dZ}{dN} = \frac{Z(-\gamma^2\phi_0X^2 - 16Z(\frac{2V_0F_0}{\phi_0 Z} + 12(\phi_0 - X)Z) + 2\gamma(X + 16\phi_0XZ))}{\phi_0(\gamma + 6\gamma XZ + 96Z^2)}. \tag{A3}$$

For the solution (12) or (13), it follows

$$X = X_0 \equiv \frac{\phi_0}{h_0}, \qquad Z = Z_0 \equiv \frac{2f_0h_0^2}{\phi_0 t_1^2}. \tag{A4}$$

In terms of X_0 and Z_0, Eqs. (14) and (15) can be rewritten as

$$0 = -\frac{3}{\kappa^2} + \frac{\gamma X_0^2}{2} + \frac{2V_0f_0}{\phi_0 Z_0} - 24Z_0X_0, \tag{A5}$$

$$0 = \gamma X_0^2 - 3\phi_0\gamma X_0 + \frac{4V_0f_0}{\phi_0 Z_0} + 24\phi_0 Z_0 - 24Z_0X_0. \tag{A6}$$

For the solution (A4), by using (A5) and (A6), the right-hand sides of Eqs. (A2) and (A3) vanish consistently. We now consider the perturbation around the solution (A4):

$$X = X_0 + \delta X, \qquad Y = Y_0 + \delta Y. \tag{A7}$$

We now check only the stability for $V = 0$ ($V_0 = 0$) case. Using (A2) and (A3), one obtains

$$\frac{d}{dN}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = M\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix}, \qquad M = \begin{pmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{pmatrix}. \tag{A8}$$

Here

$$\begin{aligned}
\tilde{A} &\equiv \frac{3\gamma^2\phi_0X_0^2 + 48\gamma X_0^2Z_0 - 6\gamma\phi_0 - 208\gamma\phi_0X_0Z_0 + 768(\phi_0 - X_0)Z_0^2}{2\phi_0(\gamma + 8\gamma X_0Z_0 + 96Z_0^2)},\\
\tilde{B} &\equiv \frac{16\gamma X_0^3 - 104\gamma\phi_0X_0^2 + 48\phi_0 + 1536\phi_0X_0Z_0 - 768X_0^2Z_0}{2\phi_0(\gamma + 8\gamma X_0Z_0 + 96Z_0^2)},\\
\tilde{C} &\equiv \frac{2Z_0(-\gamma^2\phi_0X_0 + 96Z_0^2 + \gamma + 16\gamma\phi_0Z_0)}{\phi_0(\gamma + 8\gamma X_0Z_0 + 96Z_0^2)},\\
\tilde{D} &\equiv \frac{32Z_0(-12(\phi_0 - X_0)Z_0 + \gamma\phi_0X_0)}{\phi_0(\gamma + 8\gamma X_0Z_0 + 96Z_0^2)}.
\end{aligned} \tag{A9}$$

If the real parts of all the eigenvalues of the matrix M are negative, the perturbation becomes small and the system is stable. Then the condition of the stability is given by

$$\tilde{A} + \tilde{D} < 0, \qquad \tilde{A}\,\tilde{D} - \tilde{B}\,\tilde{C} > 0. \tag{A10}$$

By using (A4)–(A6), we find

$$X_0^2 = \frac{\phi_0^2}{h_0^2} = -\frac{6(h_0 - 1)}{\gamma(5h_0 - 1)}, \qquad Z_0^2 = -\frac{\gamma(3h_0 - 1)^2}{96(h_0 - 1)(5h_0 - 1)}, \qquad X_0Z_0 = -\frac{3h_0 - 1}{4(5h_0 - 1)}. \tag{A11}$$

In order that X_0^2 and Z_0^2 are positive, it follows

$$\frac{1}{5} < h_0 < 1, \quad \text{when } \gamma > 0, \quad \text{or} \quad h_0 < \frac{1}{5} \quad \text{or} \quad h_0 > 1, \quad \text{when } \gamma < 0 \tag{A12}$$

By using (A11), $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ in (A9) can be expressed in terms of h_0:

$$\begin{aligned}
\tilde{A} &= \frac{(h_0 - 1)(9h_0^2 - 4h_0 + 1)}{h_0(5h_0^2 - 4h_0 + 1)}, \qquad \tilde{B} = \frac{24(h_0 - 1)(3h_0^2 - 2h_0 + 1)}{\gamma h_0(5h_0^2 - 4h_0 + 1)},\\
\tilde{C} &= \frac{\gamma(3h_0 - 1)(3h_0^2 - 1)}{12(h_0 - 1)(5h_0^2 - 4h_0^2 + 1)}, \qquad \tilde{D} = -\frac{2(h_0 - 1)^2(3h_0 - 1)}{h_0(5h_0^2 - 4h_0 + 1)}.
\end{aligned} \tag{A13}$$

Then we obtain very simple results:

$$\tilde{A} + \tilde{D} = -\frac{3(h_0 - 1)}{h_0}, \tag{A14}$$

$$\tilde{A}\,\tilde{D} - \tilde{B}\,\tilde{C} = \frac{2(1 - 3h_0)}{h_0^2}. \tag{A15}$$

Therefore Eq. (A10) is satisfied and the system is stable if and only if

$$h_0 < 0. \tag{A16}$$

Then the case corresponding to phantom cosmology with $h_0 < 0$ is stable.

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