$\langle A^2 \rangle$ condensate, Bianchi identities, and chromomagnetic field degeneracy in SU(2) Yang-Mills theory

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We consider the non-Abelian Bianchi identities in SU(2) pure Yang-Mills theory in $D = 3, 4$ focusing on the possibility of their violation and the significance of the chromomagnetic fields degeneracy points. We show that the recently proposed non-Abelian Stokes theorem allows one to formulate the Bianchi identities in terms of the physical fluxes and their relative color orientations. Then the violation of Bianchi identities becomes a well defined concept ultimately related to the degeneracy points. The locality and gauge invariance of our approach allows one to study the problem numerically. We present evidence that in $D = 4$ the suppression of the Bianchi identities violation is likely to destroy confinement, while the removal of the degeneracy points drives the theory to the topologically nontrivial sector. However, confronting the results obtained in three and four dimensions, we argue that it is the mass dimension two condensate $\langle A^2_{\min} \rangle$ which probably explains our findings.

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I. INTRODUCTION

Gauge theories are usually formulated in terms of the gauge potentials A^a_μ taking values in the Lie algebra of the corresponding gauge group. Provided that the gauge coupling is small, this description is indeed adequate and provides local functionally independent coordinates on the configuration space. However, in the strongly coupled gauge theories, the potentials themselves obtain a separate physical meaning. Here we mean the nonperturbative dimension 2 condensate $\langle A^2_{\min} \rangle$ introduced in Refs. [1,2], which received particular attention in recent years (see, e.g., Ref. [3] for review and further references).

The original motivation of this work was the analysis of various possible contributions to the $\langle A^2_{\min} \rangle$ condensate. Note that the central point of Ref. [1] was, in fact, the consideration of the Abelian Bianchi identities and their ultimate relation to $\langle A^2_{\min} \rangle$. As far as the Abelian theory is concerned, the nontriviality of the $\langle A^2_{\min} \rangle$ condensate is essentially equivalent to the Bianchi identities violation. Therefore, in the non-Abelian case it seems natural to start from the corresponding Bianchi identities and investigate their role in the $\langle A^2_{\min} \rangle$ condensate formation. However, the literature on the subject turns out to be scarce. In particular, as is well known from the Abelian models, the rigorous treatment of the Bianchi identities requires nonperturbative (say, lattice) regularization. But we were unable to find papers devoted to this problem in the non-Abelian case.

On the other hand, the investigation of the non-Abelian Bianchi identities is important in its own right. Without mentioning all the aspects of the problem, let us note that the $\langle A^2_{\min} \rangle$ condensate is certainly connected with the non-Abelian Bianchi identities. Moreover, it was emphasized in Refs. [4–6] that the Bianchi identities and the possibility of their violation are ultimately related to the confinement problem. Then the logic suggests to consider whether the $\langle A^2_{\min} \rangle$ condensate is relevant for confinement as well, the question which was discussed in Refs. [2,5] (see also [7]). Therefore, we see that all these problems are in fact indispensable from each other and cannot be considered separately. We decided to focus on the Bianchi identities in this paper; the connection with the quantities such as $\langle A^2_{\min} \rangle$ is discussed in due course. Throughout the paper, we work with Euclidean three- and four-dimensional SU(2) gluodynamics, keeping in mind the lattice regularization of the theory, although we nowhere rely exclusively on the lattice. The paper is reasonably self-contained; the results which we are using are briefly reviewed. Note that the similar in spirit but in no way identical treatment could be found in Refs. [8,9].

The primary tool of our analysis is the non-Abelian Stokes theorem [10] derived recently by one of us. The advantage is that it allows one to work directly in terms of the gauge invariant quantities like magnitudes of the elementary fluxes and their relative orientations. As might be expected, the non-Abelian Bianchi identities could be reduced to the application of the above theorem to the infinitesimal closed surfaces. However, in this case the non-Abelian Stokes theorem does not necessarily give zero; the answer, in fact, is proportional to the integer number. Since every step in the derivation is gauge invariant, this integer is gauge invariant as well and in the continuum language corresponds to the non-Abelian Bianchi identities violation.

The non-Abelian nature of the theory manifests itself in the complicated geometry underlying the Bianchi identities. We consider all these questions in detail and show that the careful but purely geometrical treatment leads to the consideration of the special degenerate points in the con-

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figuration space at which particular determinants constructed from chromoelectric and chromomagnetic fields vanish. Finally, we show that the investigation of the non-Abelian Bianchi identities is indispensable from the study of these degenerate points. Therefore, the framework outlined above naturally extends to include the degeneracy points, the relevance of which for both confinement and chiral symmetry breaking was discussed in Refs. [6,11].

The locality and gauge invariance of our construction allow us to study the problem numerically. We investigate the effects due to the Bianchi identities violation and the degenerate points in the numerical simulations. As might be *a priori* expected, the suppression of the degenerate points always leads to the violation of the reflection positivity. Moreover, in $D = 4$ one could easily pinpoint the origin of the reflection positivity violation: It is caused by rapidly rising global topological charge. Thus, in $D = 4$ the suppression of the degenerate points shifts the vacuum to the nontrivial topological sector.

As far as the Bianchi identities are concerned, the results depend crucially on the space-time dimensionality. In $D = 3$ the suppression of the Bianchi identities violation does not change the theory in any notable way. However, in $D = 4$ the effect is different: It seems that the suppression of the Bianchi identities violation is likely to destroy confinement while other measured characteristics of the theory remain qualitatively unchanged. At least, this is so for the lattices and coupling constants we have considered. Note that the problem still requires a careful numerical investigation; in particular, we had not studied yet the volume dependence of our results. The corresponding analysis will be published elsewhere.

Finally, we argue that it would be misleading to interpret our results as the statement that confinement is caused by the Bianchi identities violation. Confronting the results obtained in three and four dimensions, we show that it is the $\langle A^2_{\mathrm{min}} \rangle$ condensate which is probably relevant for confinement. Although the argumentation is not rigorous, it seems to be the only one which matches our findings.

II. FORMULATION OF THE PROBLEM

The primary object of our investigation is the Bianchi identities for SU(2) gauge fields in four space-time dimensions. Thus, we will analyze the equations

$$\partial_\mu \tilde{F}^a_{\mu\nu} + \varepsilon^{abc} A^b_\mu \tilde{F}^c_{\mu\nu} = 0, \qquad \tilde{F}^a_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F^a_{\lambda\rho} \quad [D = 4], \tag{1}$$

having in mind eventually Euclidean lattice regularization of SU(2) pure Yang-Mills theory. Here $F^a_{\mu\nu}$ is the conventional continuum field-strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu; \tag{2}$$

Greek and Latin indexes run through $0, \ldots, 3$ and $1, \ldots, 3$, respectively. Our treatment also applies in three dimensions where Bianchi identities are as follows:

$$\partial_i B^a_i + \varepsilon^{abc} A^b_i B^c_i = 0, \qquad B^a_i = \tfrac{1}{2}\varepsilon_{ijk} F^a_{jk} \qquad [D = 3]. \tag{3}$$

However, it turns out that the three-dimensional case is physically quite different from $D = 4$ and we will comment on that in due course.

In this section, we give qualitative continuum arguments which show that, at least at some points in the configuration space, the Bianchi identities (1) and (3) should be considered with care.

A. Chromomagnetic fields degeneracy

It has been known for a long time that in non-Abelian gauge theories two or more gauge inequivalent potentials could produce the same field strength [12]. This phenomenon, known as Wu-Yang ambiguity, had received great attention in the past (see, e.g., [13–15]), and it was noted long ago [16–18] that in $D = 4$ the Bianchi identities constitute an algebraic obstruction for the ambiguity to exist. Namely, for given chromoelectric $E^a_i = F^a_{0i}$ and chromomagnetic $B^a_i = 1/2\varepsilon_{ijk} F^a_{jk}$ fields, Eq. (1) is a linear algebraic system of 12 equations for 12 unknown A^a_μ. Therefore, away from the set of points where the matrix $T^{ab}_{\mu\nu} = \varepsilon^{abc}\tilde{F}^c_{\mu\nu}$ degenerates

$$\det T = 0, \tag{4}$$

Bianchi identities allow one to express the gauge potentials as local single-valued functions of E^a_i and B^a_i. On the other hand, there is no physical principle or symmetry which could keep the sign of $\det T$ fixed. Indeed, in the weak coupling perturbation theory the sign of $\det T$ changes wildly, and, therefore, the degeneracy of chromomagnetic fields, Eq. (4), is, in a sense, generic. Note that the situation is quite different in $D = 3$ since Eq. (3) formally constitutes 3 equations for 9 unknown variables. Therefore, in three dimensions the Bianchi identities do not constrain the gauge potentials at all and the Wu-Yang ambiguity problem is much more severe (see, e.g., Refs. [19,20] for discussion). Unfortunately, we are not aware of any conclusive considerations of the degenerate points (4) in the literature. It is true that Eq. (4) by itself has been known for a long time [18,21,22], but most of the analysis performed so far considered it in the context of dual formulation of gluodynamics [23–27], from which the information about original Yang-Mills fields is hard to extract. Reference [28] seems to be the only exception where it was argued that physical wave functionals should vanish at the points of degeneracy. We will see below that equations similar to (4) arise naturally in the construction of the Bianchi identities. Moreover, the points of degeneracy seem to be relevant for gauge fields dynamics.

What we have said so far is in accordance with the general expectation that in the non-Abelian gauge theories there is no unique way to express A_μ^a in terms of the corresponding field strength (apart from the usual gauge ambiguity, of course). At this point, one could give an example of special gauges (complete axial, coordinate, contour gauges; see [29,30] for review) in which the gauge potentials are always explicit single-valued functions of the field strength. Is there any contradiction? Although this question is not directly related to our work, we note that all the gauges mentioned above are consistent only if Bianchi identities (1) and (3) are satisfied identically [31]. In particular, in the Abelian case one notices [29] that the presence of elementary magnetic charges forces the potentials in contour gauge to depend upon the arbitrary contour prescription. Of course, this is a manifestation of the famous Wu-Yang ambiguity, which in this case certainly arises because pointlike monopoles violate the Bianchi identities. We conclude, therefore, that the possibility of Bianchi identities violation should not be excluded *a priori*. Moreover, the very existence of Wu-Yang ambiguous potentials hints at the violation of (1) and (3).

B. Bianchi identities violation

The possibility that the right-hand side of Eqs. (1) and (3) might be nonzero was considered long ago (see, e.g., [31]), but as far as we know this approach had never been actively developed. This is mostly because the study of Bianchi identities violation requires a particular regularization, which should correctly respect the global structure of the gauge group. It turns out that for our purposes the lattice formulation is distinguished (see Refs. [32,33] for discussion). Therefore, consider the basic SU(2) gauge theory observable, which is also the fundamental object on the lattice, the Wilson loop in spin 1/2 representation

$$W(C, x_0) = \mathrm{P}\exp i\sigma^a \oint_{C(x_0)} A_\mu^a dx^\mu, \qquad W(C) = \tfrac{1}{2}\mathrm{Tr}W(C, x_0). \tag{5}$$

Here σ^a are the Pauli matrices, C is some closed contour with marked point $x_0 \in C$ from which the path ordered integral starts, and P ordering is defined from left to right. Note the unusual normalization of SU(2) generators which we take for future convenience. By definition, the operator $W(C, x_0)$ measures the non-Abelian flux $\Phi(C, x_0)$ penetrating the contour

$$W(C, x_0) = e^{i\sigma^a \Phi^a(C,x_0)}, \qquad W(C) = \cos\Phi(C), \qquad \Phi(C) = \sqrt{\Phi^a(C, x_0)\Phi^a(C, x_0)}, \tag{6}$$

where the flux [34] $\Phi(C)$ is gauge invariant and does not depend on x_0. Equation (5) will be thoroughly analyzed later, but now we note that the physically observable flux is always bounded $0 < \Phi(C) < \pi$ due to periodicity (compactness) of the gauge action. Moreover, there exists no physically meaningful experiment which could distinguish the fluxes $\Phi(C)$ and $\Phi(C) + 2\pi$, and this observation applies equally well to the infinitesimal contours which constitute the lattice definition of the field strength. On the other hand, there is no trace whatsoever of the gauge action compactness in the continuum expression (2). In this respect, the SU(2) gluodynamics is similar to the compact U(1) gauge model [35] (see Ref. [36] for review). In fact, some consequences of the compactness of the non-Abelian gauge theories were already discussed in the past [37]. Note, however, that we are not saying that singular fluxes are important in the continuum limit of lattice formulation. After all, this is a dynamical question which cannot be studied with simple arguments above. Rather, we point out that the very definition of $F_{\mu\nu}^a$ on the lattice is *a priori* different from the continuum one (2), and, therefore, the validity of (1) and (3) in the lattice context should be considered anew. We stress that our arguments are purely kinematical and follow directly from the gauge invariance alone. Whether or not the violation of Bianchi identities is physically relevant is a dynamical issue which we investigate (at least partially) later on.

To conclude, we note that nowadays there exist both theoretical arguments [4–6] and the experimental lattice data [38] which favor the nonvanishing right-hand side of Eqs. (1) and (3) in the continuum limit of lattice gauge models. Although the approaches of these papers are quite different, the conclusion is essentially the same: The non-Abelian Bianchi identities are indeed violated in the scaling (continuum) limit, and this fact is related to the problem of confinement.

III. LATTICE BIANCHI IDENTITIES

A. Preliminaries

In this section, we briefly summarize what has been known so far about the non-Abelian Bianchi identities on the lattice and comment on the strategy we employ in this paper. Surprisingly enough, the literature on the subject seems to be very scarce (contrary to the Abelian case, which we do not consider, however) and the most relevant references for our discussion are Refs. [39–41] (see also [37]). Historically, the Bianchi identities explicitly appeared first in the context of plaquette (field-strength) formulation of lattice QCD [40,41]. In particular, it was noted that the strong coupling expansion can be obtained as an expansion towards restoring the lattice Bianchi identities.

It turns out that the formulation of Ref. [39] is the most appropriate for our purposes. Essentially, it consists in the observation that any lattice gauge field configuration could be interpreted as a homomorphism from the lattice edge path group into the gauge group (see Ref. [42] for definitions). It follows from the definition of homomorphic mapping that

$$U(C_{xy} \circ C_{xy}^{-1}) = U(C_{xy})U^{-1}(C_{xy}) = 1, \tag{7}$$

where C_{xy} is an arbitrary path connecting the points x and y, and the composite path $C_{xy} \circ C_{xy}^{-1}$ is usually referred to as null-homotopic. In fact, Eq. (7) looks rather obvious for everyone familiar with lattice formulation. However, the assertion of Ref. [39] is that Eq. (7) constitutes the most general form of lattice Bianchi identities and indeed just that: an identity. Note that Eq. (7) looks quite different from what is expected in the continuum. To establish the relation between (7) and (1) and (3), consider the path C_{xx} shown in Fig. 1. It follows trivially that the equality $U(C_{xx} \circ C_{xx}^{-1}) = 1$ is equivalent to

$$U(R_1)U(R_2)U(R_3)U(R_4)U(R_5)U(R_6) = 1, \tag{8}$$

which in the naive continuum limit reduces to the conventional Bianchi identities (1) and (3). Moreover, Eq. (8) is the particular case of the so-called operator non-Abelian Stokes theorem [43–46] (see, e.g., [29] for review) which allows one to represent (rather formally, though) the path ordered exponent as the surface ordered integral

$$\mathrm{P}\exp i\int_{C=\delta S_C} A_\mu dx^\mu = \mathrm{P}_S \exp\frac{i}{2}\int_{S_C} \mathcal{F}_{\mu\nu} d^2\sigma^{\mu\nu}, \tag{9}$$

where $\mathcal{F}$ is a nonlocal covariantly transformed field strength, the concrete form of which is not important for what follows. The surface S_C is arbitrary, and consistency requires the representation (9) to be independent on S_C as long as $\delta S_C = C$. In particular, the right-hand side of Eq. (9) being applied to closed surface S_0, $\delta S_0 = 0$, should always give the identity

$$\mathrm{P}_S \exp\frac{i}{2}\int_{S_0, \delta S_0 = 0} \mathcal{F}_{\mu\nu} d^2\sigma^{\mu\nu} = 1. \tag{10}$$

In fact, Eq. (8) is the special case of (10) in which S_0 is the boundary of an elementary lattice cube. Therefore, it seems to be legitimate to formulate the non-Abelian Bianchi identities as the requirement of surface independence of the non-Abelian Stokes theorem.

Equations (7)–(10) are the starting point of our considerations below. However, before going into detail, let us comment a bit on our strategy. We note first that the identity on the right-hand side of Eqs. (8) and (10) could, in general, be written as

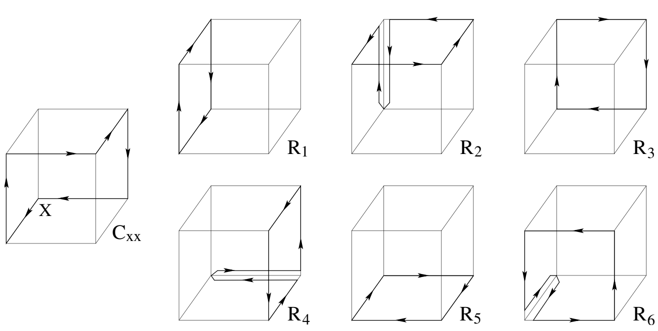

FIG. 1. Graphical representation of lattice Bianchi identities.

$$1 = e^{i\vec{\sigma}\,\vec{n}\cdot 2\pi q}, \qquad \vec{n}^2 = 1, \qquad q \in Z. \tag{11}$$

The color direction $\vec{n}$ is gauge variant and will not concern us here. Suppose that we are able to give an unambiguous gauge invariant meaning to the integer q and that it is nonzero for some S_0 in a given gauge background. Then this would certainly mean that there is a point [47] somewhere inside S_0 at which the continuum Bianchi identities are violated. Here the argumentation is essentially the same as in the well known Abelian case. So the problem is to make sense of q, which should be well defined and gauge invariant. From now on, we refer to the integer q as the "magnetic charge," whatever it is. In particular, neither charge conservation nor any other usual properties of the magnetic charge are assumed. Second, Eqs. (8) and (10) are not quite suitable to analyze the Bianchi identities. This is precisely because neither (8) nor (10) make, in fact, direct reference to the non-Abelian field strength. This is in sharp contrast with the Abelian theory, in which the Bianchi identities even on the lattice explicitly refer to physical fluxes. It turns out that the solution of the second problem simultaneously solves the first; namely, the non-Abelian Stokes theorem being expressed in terms of the physical field strength provides the definition of q for which we are looking.

B. Chromomagnetic fields on the lattice

The distinguished feature of the lattice regularization is that the gauge theory is formulated in terms of the Wilson loops alone, and, strictly speaking, the lattice does not need to introduce the notion of the field strength. Chromomagnetic fields appear only in the limit of vanishing lattice spacing; otherwise, one should rather think in terms of the non-Abelian fluxes which are defined by Eqs. (5) and (6). Therefore, consider the Wilson loop

$$\begin{gathered} W(C,t) = \mathrm{P}\exp i\int_t^{T+t} A(\tau) d\tau = e^{i\vec{\sigma}\,\vec{n}(C,t)\cdot\Phi(C)}, \\ A(\tau) = \sigma^a A^a_\mu(x)\dot{x}^\mu(\tau), \qquad \vec{n}^2(C,t) = 1, \\ W(C) = \tfrac{1}{2}\mathrm{Tr}W(C,t) = \cos\Phi(C), \end{gathered} \tag{12}$$

defined for some closed contour $C = \{x(t), 0 \le t \le T, x(0) = x(T)\}$ (our presentation is similar but not identical to that of Ref. [10]; see also Ref. [48]). We assume that $W(C) \neq \pm 1$, and then it is convenient to parametrize the Wilson loop in terms of the flux magnitude $\Phi(C) \in (0; \pi)$ and the instantaneous flux direction in color space $\vec{n}(C,t)$ which explicitly depends on t. It is clear that $\Phi(C)$ is gauge invariant while $\vec{n}(C,t)$ rotates as a three-dimensional vector under the gauge transformations at point $x(t)$. Consider now another contour C' which touches (or intersects) C at point $x(t_0) = x'(t'_0)$. Evidently, while both $\vec{n}(C, t_0)$ and $\vec{n}(C', t'_0)$ are gauge variant, their relative orientation (angle in between) is gauge independent. Moreover, the construction could be iterated: For any number of contours intersecting at one point, the relative orientation of instantaneous fluxes at that point is gauge invariant. It is amusing to note that the relative orientation of elementary fluxes received almost no attention in the past. While the magnitude of various fluxes had been discussed and measured in various circumstances (see, e.g., Ref. [49] and references therein), it seems that only Refs. [50,51] studied their relative orientations.

Consider next the behavior of the flux parametrized by Eq. (12) under the change of contour orientation. Physically, one expects that the total flux should change sign when contour is followed in the opposite direction

$$\Phi^a(C^{-1}, t) = \Phi(C^{-1}) n^a(C^{-1}, t) = -\Phi^a(C, t). \tag{13}$$

The parametrization (12) respects the intuition, and, indeed, the flux direction changes sign while the flux magnitude is orientation independent

$$\vec{n}(C^{-1}, t) = -\vec{n}(C, t), \qquad \Phi(C^{-1}) = \Phi(C). \tag{14}$$

Here we come to the important point concerning the determination of physical field strength from the infinitesimal fluxes. Suppose that we measure the elementary flux twice, first with an oriented area element $\delta\sigma^{\mu\nu}$ and then with reversed orientation $\delta\sigma^{\nu\mu} = -\delta\sigma^{\mu\nu}$. Evidently, the corresponding Wilson loops are conjugated to each other

$$W(\delta\sigma^{\mu\nu}) = W^\dagger(\delta\sigma^{\nu\mu}). \tag{15}$$

On the other hand, the expansion in powers of lattice spacing a reads

$$W(\delta\sigma^{\mu\nu}) = 1 + a^2 i\vec{\sigma}\vec{F}_{\mu\nu}\delta\sigma^{\mu\nu} + O(a^4),$$
$$W(\delta\sigma^{\nu\mu}) = 1 + a^2 i\vec{\sigma}\vec{F}_{\nu\mu}\delta\sigma^{\nu\mu} + O(a^4) = W(\delta\sigma^{\mu\nu})$$

and disagrees with (15). This simple exercise which applies equally in the Abelian case shows that the lattice area element $dx^\mu dx^\nu$ is, in fact, unoriented $dx^\mu dx^\nu = dx^\nu dx^\mu$ contrary to the usual continuum relation $\delta\sigma^{\mu\nu} = dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Therefore, in order to define the field strength on the lattice, a canonical orientation of all elementary squares (plaquettes) should be fixed first. Otherwise, the field strength will suffer from sign ambiguity on different plaquettes. In fact, the canonical ordering is well known in the lattice community, and the conventional agreement is to consider $\delta\sigma^{\mu\nu}$ with $\mu < \nu$ only. However, the orientation conventions are crucial for the interpretation of lattice equations below in the continuum terms. From now on, we always assume that the infinitesimal fluxes are constructed with canonically oriented plaquettes.

It is convenient to generalize the representation (12) in order to gain a simple physical interpretation. Namely, it is natural to describe the instantaneous flux direction by means of a fictitious (iso)spin 1/2 particle living on the contour. The spinor wave function is given by a two-component normalized complex quantity

$$\langle z| = [z_1; z_2], \qquad \langle z \mid z\rangle = |z_1|^2 + |z_2|^2 = 1, \tag{16}$$

which is bra-vector in accordance with our left to right P-ordering convention. The defining equation for the Wilson loop becomes the Schrödinger equation for spinor

$$\langle z(t)|(i\overleftarrow{\partial}_t + A) = 0, \qquad \langle z(t)| = \langle z(0)| \cdot \mathrm{P}\exp i\int_0^t A(\tau)d\tau. \tag{17}$$

Therefore, the Wilson loop (12) is the quantum mechanical evolution operator for spin degrees of freedom. As is usual in quantum mechanics, the state vectors could be arbitrary rephased

$$\langle z(t)| \to e^{i\theta(t)}\langle z(t)|. \tag{18}$$

The particular choices $\mathrm{Im} z_1 = 0$, $\mathrm{Im} z_2 = 0$ lead to well known families of (anti)holomorphic spin coherent states [52] (see, e.g., [53] for review). Following the quantum mechanical analogy [54,55], one could argue that the eigenstate of the evolution operator $W(C, 0)$,

$$\langle z(0)|W(C, 0) = e^{i\Phi(C)}\langle z(0)|, \tag{19}$$

is of special importance and is usually referred to as a cyclic state. In particular, the state $\langle z(0)|$ being the eigenstate of $W(C, 0)$ at $t = 0$ remains the eigenstate of $W(C, t)$ during the evolution (17). It follows immediately that the cyclic state (19) is best suited to describe the instantaneous flux direction. Indeed, it is a matter of one-line calculation to show that $n^a(C, t) = \langle z(t)|\sigma^a|z(t)\rangle$. In other words, the flux direction $\vec{n}(C, t)$ and the ratio $z_2(t)/z_1(t)$ of cyclic state components are related to each other by standard stereographic projection. In particular, the flux magnitude is given by

$$\Phi(C) = \arg[\langle z(t)|W(C, t)|z(t)\rangle] \tag{20}$$

and is t independent. Moreover, if contour C is subdivided into N segments, then

$$\Phi(C) = \arg\prod_{k=0}^{N-1}\langle z(t_k)|\mathrm{P}\exp i\int_{t_k}^{t_{k+1}} A(\tau)d\tau|z(t_{k+1})\rangle, \tag{21}$$

where the identification $t_0 = t_N$ is assumed. As far as the relative orientation of fluxes is concerned, it is tempting to consider the quantities such as $\arg\langle z \mid \zeta\rangle$. However, it is not invariant under (18) because $\langle z|$ and $\langle\zeta|$ could be rephased independently. Nevertheless, the equations we will get do indeed include the products such as $\langle z \mid \zeta\rangle$ yet respect the U(1) invariance (18).

It remains only to consider the multivaluedness of the cyclic state defining Eq. (19). Indeed, there exist two solutions of Eq. (19) while we discussed only one of them. The second eigenstate is obtainable from the first one by substitution

$$z_2 \to z_1^*, \qquad z_1 \to -z_2^*. \tag{22}$$

It is clear that Eq. (19) corresponds to the "spin-up" wave function for which the spin is aligned with the magnetic field, while the second eigenstate (22) is the "spin-down" state which has spin antialigned. Our original goal was to describe the direction of instantaneous flux, and, therefore, the antialigned state should be discarded since it corresponds to the inverted flux direction. Note also that the flux magnitude $\Phi(C)$ is positive by definition, but with the antialigned state we get $\Phi(C) < 0$. We conclude, therefore, that for given contour orientation there is no ambiguity in Eq. (19) and the appropriate family of cyclic states $\langle z(t)|$ is uniquely defined. The second spin-down eigenstate describes the flux direction for inverted contour orientation, and, therefore, Eq. (22) corresponds to the time reversal operation for spinors in quantum mechanics.

The above considerations apply immediately on the lattice. The only difference with the continuum is that the gauge potentials are unknown; we have only the parallel transporters along the elementary links. But this is actually enough: The Wilson loop is constructed by direct matrix multiplication and then Eq. (19) applies literally. The instantaneous flux direction is determined via (19) or (17) at lattice sites passed by Wilson loop. The flux magnitude is given by Eqs. (20) and (21).

To summarize, every Wilson loop [56] $W(C, t)$ is characterized by the magnitude of the flux $\Phi(C)$ and the instantaneous flux direction $\vec{n}(C, t)$, $\vec{n}^2 = 1$, which varies along the contour and is reversed on changing contour orientation. The quantum mechanical language is adequate to describe both $\Phi(C)$ and $\vec{n}(C, t)$: There is a fictitious spin $1/2$ particle living on C, the polarization of which gives exactly $\vec{n}(C, t)$; the wave function of the particle is defined for a given gauge background uniquely up to the phase, and change of contour orientation is equivalent to the time reversal operator applied to the spinor; the particle evolution along C is cyclic; initial and final states differ only by phase, and this phase is the magnitude of the flux penetrating C. On the lattice the difference is that the flux direction (wave function of a spinning particle) is known only at lattice sites $x \in C$. Moreover, the orientation of all elementary plaquettes is fixed to be the canonical one.

C. Non-Abelian Stokes theorem

The last ingredient which we need to complete the program outlined in Sec. III A is the non-Abelian Stokes theorem derived recently by one of us [10]. Although the results of Ref. [10] are applicable almost literally, let us review them in order to introduce the notations and comment on the differences with the present work.

Therefore, consider the Wilson loop $W(C)$, a segment of which is shown by the straight horizontal line in Fig. 2, and the surface S_C bounded by C, which is to the top of the contour in the same figure. According to what had been said above, we assign to every plaquette $p \in S_C$ and Wilson loop itself the corresponding flux magnitudes $\Phi(p)$, $\Phi(C)$ and the instantaneous flux directions $\langle z_k(p)|$, $\langle \zeta_k(C)|$ correspondingly [plaquette vertices are followed according to the orientation induced by C, while the states $\langle z_k(p)|$ are constructed in accordance with the canonical orientation]. It is convenient to use the graphical ribbonlike representation in which all plaquettes and Wilson loop contour are slid apart (Fig. 2). Let us denote

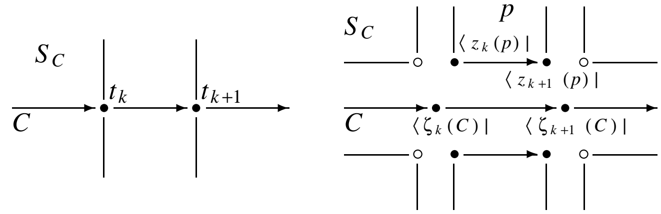

FIG. 2. Segment of the Wilson loop $W(C)$ in the original and ribbonlike representation. The operator in between solid blobs is $U_{k,k+1}$, Eq. (23).

$$U_{k,k+1} = \mathrm{P}\exp i \int_{t_k}^{t_{k+1}} A(\tau) d\tau \tag{23}$$

and consider the matrix element

$$\langle \zeta_k(C)| U_{k,k+1} | \zeta_{k+1}(C)\rangle = \mathrm{const} \cdot e^{i\phi_{k,k+1}(C)}, \tag{24}$$

where const is some real positive number which is irrelevant. According to (21),

$$\Phi(C) = \left[\sum_k \phi_{k,k+1}(C)\right] \mathrm{mod} 2\pi. \tag{25}$$

The important observation of Ref. [10] is that the matrix element (24) might be calculated in the $\langle z_k(p)|$ basis, provided that the relative orientation of plaquette and Wilson loop fluxes is taken into account

$$\begin{aligned}\langle \zeta_k(C)| U_{k,k+1} | \zeta_{k+1}(C)\rangle = \mathrm{const} \cdot \langle \zeta_k(C) | z_k(p)\rangle \\ \times \langle z_k(p)| U_{k,k+1} | z_{k+1}(p)\rangle \\ \times \langle z_{k+1}(p) | \zeta_{k+1}(C)\rangle.\end{aligned} \tag{26}$$

The equality (26) was shown in Ref. [10] in the matrix form. Here we note that Eq. (26) follows from its invariance under (18) and the unitarity of the evolution operator (23). In fact, the relations similar to (26) are well known in quantum mechanics [57] (see, e.g., [54,55,58] for details). In particular, Refs. [55,58] showed the importance and physical significance of the geodesic interpolation used in Ref. [10].

Applying Eqs. (24)–(26) repeatedly for every link of S_C, one gets the non-Abelian Stokes theorem

$$\Phi(C) = \sum_{p\in S_C} I(p)\Phi(p) + \sum_{x\in S_C} \Omega_x + \sum_{x\in C} \gamma_x + 2\pi k(S_C),$$
$$k(S_C) \in Z, \tag{27}$$

where $\Phi(p)$ is the plaquette flux, $1/2\,\mathrm{Tr}W(p) = \cos\Phi(p)$, and the factors $I(p) = \pm 1$ are analogous to the usual incidence numbers in the differential geometry [42]: $I(p) = 1$ if vertices of the plaquette p are followed in the canonical order, and $I(p) = -1$ otherwise. The remaining terms are illustrated in Fig. 3. In particular,

$$\Omega_x = \arg[\langle z_1 | z_2 \rangle \langle z_2 | z_3 \rangle \langle z_3 | z_4 \rangle \langle z_4 | z_1 \rangle] \mathrm{mod} 2\pi \qquad (28)$$

is the oriented area of spherical quadrilateral polygon [59] (solid angle) between the flux directions on the plaquettes $p_1, \ldots, p_4$. It is known in quantum mechanics as the Bargmann invariant [60] for the particle's wave functions (see, e.g., [61,62] for review). Physically, Ω_x accounts for the difference of flux orientations on the plaquettes sharing the same point x. The third term

$$\gamma_x = \arg[\langle \zeta | z_1 \rangle \langle z_1 | z_2 \rangle \langle z_2 | \zeta \rangle] \mathrm{mod} 2\pi \qquad (29)$$

equals the oriented area of a spherical triangle constructed from the Wilson loop flux direction at x and the flux orientations of two plaquettes $p_1, p_2 \in S_C$ touching C and sharing the point x. Equation (29) is again the Bargmann invariant for the wave functions of three particles living on C, p_1, and p_2.

Note that we have omitted the mod 2π operation on the right-hand side of Eq. (27) and wrote instead the additional $2\pi k(S_C)$ term, such that $\Phi(C) \in (0; \pi)$. It is clear that $k(S_C)$ is not vanishing, in general, and is analogous to the Dirac string contribution in the Abelian Stokes theorem applied for compact U(1) gauge fields [35,63] (see [36,64] for review and further references). This is in accordance with the discussion in Sec. II B, where we noted that the SU(2) gauge model is intrinsically compact and is similar to compact photodynamics in this respect. However, in the non-Abelian case the nonzero $k(S_C)$ could come from any of three terms in Eq. (27). In particular, the Dirac string contribution $k(S_C) \neq 0$ does not necessarily correspond to the singular elementary non-Abelian flux (singular field strength). It could equally come from Ω_x, γ_x terms which are genuine non-Abelian contributions.

Note that Eq. (27) is not only invariant under SU(2) gauge transformations, it also remains intact with respect to local (gauge) rephasing (18) [this U(1) gauge symmetry is crucial for the dual representation considered in Ref. [33]]. We are in haste to add, however, that this does not concern the $2\pi k(S_C)$ term. As might be expected, the Dirac string contribution is not invariant with respect to either of the symmetries. Equation (27) could be illustrated nicely in the particular case of a pure Abelian gauge background. In the Abelian limit, all fluxes become aligned, but their directions could be opposite. For anti-aligned flux directions, the Bargmann invariants (28) and (29) become, strictly speaking, undefined. For instance, the area of the spherical triangle (29) is undefined when two of its vertices are at the north pole of the two-dimensional sphere while the third one is at the south pole. However, we could avoid this degenerate case by changing simultaneously the sign of both $\vec{n}(p,t)$ and $\Phi(p)$, which does not affect the parametrization (12). The flux magnitude becomes not positively definite and the incidence coefficients could be absorbed into the definition of $\Phi(p)$. Then the second and third terms, which account for the flux rotation in color space, vanish and Eq. (27) becomes identical to the usual Abelian Stokes theorem.

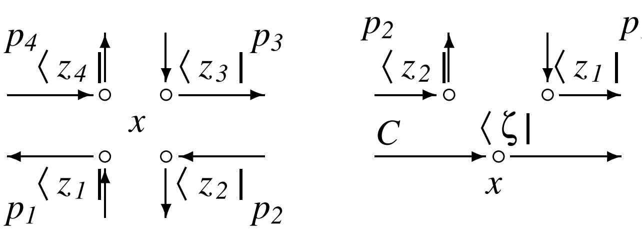

FIG. 3. Ω_x (left) and γ_x (right) terms in Eq. (27). Arrows correspond to the orientation induced by C.

To summarize, the flux $\Phi(C)$ could be represented almost entirely in terms of local physically observable contributions coming from the arbitrary surface S_C bounded by C. The point of crucial importance is that all these terms are "almost total differentials": Without mod2π operation, both the plaquette flux (25) and the Bargmann invariants (28) and (29) would become exact 2-forms. The adequate graphical language to account for all terms is the ribbonlike representation in which all plaquettes and Wilson contour are slid apart. The only troublesome contribution is the last one in Eq. (27), which explicitly depends upon the color orientation of the flux $\Phi(C)$ itself. In the next section, we analyze the arbitrariness of S_C and γ-angle dependence of Eq. (27).

D. Non-Abelian Bianchi identities

To complete the program outlined in Sec. III A, consider the surface independence of the non-Abelian Stokes theorem (27). As one could expect, the requirement of surface independence reduces to Eq. (10). On the other hand, the non-Abelian Stokes theorem (27) applied formally to closed surface S_0 gives

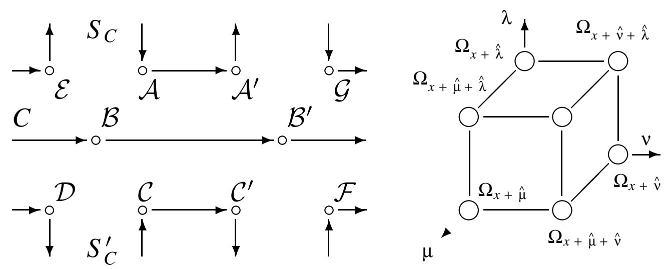

FIG. 4. Left: The non-Abelian Stokes theorem in application to the closed contour C which bounds two distinct surfaces S_C, S'_C, $\delta S_C = \delta S'_C = C$. Arrows indicate the order of plaquette vertices induced by the orientation of C. Right: The non-Abelian Bianchi identities for a single lattice cube (see the text).

$$\sum_{p\in S_0} I(p)\Phi(p) + \sum_{x\in S_0}\Omega_x = 2\pi q(S_0), \tag{30}$$

where the integer $q(S_0)$ is not vanishing, in general, and is discussed below. Since Eq. (30) is one of the central points of our work, let us explicitly rederive it starting from Eqs. (7) and (27).

Consider Eq. (7) for some closed contour C

$$U(C\circ C^{-1}) = U(C)U(C^{-1}) = U(C)U^{-1}(C) = 1, \tag{31}$$

part of which is shown in Fig. 4. There are two distinct surfaces S_C, S'_C shown to the top and bottom of the contour with orientations induced by C. The non-Abelian Stokes theorem (27) applied for S_C and S'_C leads to

$$\Phi_{S_C} = \sum_{p\in S_C} I(p)\Phi(p) + \sum_{x\in S_C}\Omega_x + \sum_{x\in C}\gamma_x(S_C) + 2\pi k(S_C) \tag{32}$$

and analogous equation for $\Phi_{S'_C}$. The surface independence requires that $\Phi_{S_C} = \Phi_{S'_C}$ and, therefore,

$$\sum_{p\in S_0} I(p)\Phi(p) + \sum_{\substack{x\in S_0\\ x\notin C}}\Omega_x + \sum_{x\in C}[\gamma_x(S_C) - \gamma_x(S'_C)] = 2\pi[k(S'_C) - k(S_C)] = 2\pi q(S_0). \tag{33}$$

Here $S_0 = S_C \cup \tilde{S}'_C$ and $\tilde{S}'_C$ is just the S'_C taken with reversed orientation due to which the terms $\sum_{p\in S'_C} I(p)\Phi(p)$, $\sum_{x\in S'_C}\Omega_x$ changed sign in Eq. (33). Consider the γ-angle contribution in (33) coming from points $\mathcal{B}, \mathcal{B}' \in C$ and let $\Delta(\mathcal{A}\mathcal{B}\mathcal{C})$ denote the Bargmann invariant (29) for spinor wave functions at the points $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. In particular, $\gamma_{\mathcal{B}}(S_C) = \Delta(\mathcal{A}\mathcal{E}\mathcal{B})$ and similarly for other γ angles. We note that one and the same unitary operator transforms $\mathcal{A}\to\mathcal{A}'$, $\mathcal{B}\to\mathcal{B}'$, $\mathcal{C}\to\mathcal{C}'$. In other words, the color directions of the fluxes at these points are rotated by one and the same rotation matrix. However, the Bargmann invariant being the area of the spherical triangle is unchanged when the sphere is rotated. Therefore, the following identity holds:

$$\Delta(\mathcal{A}\mathcal{B}\mathcal{C}) - \Delta(\mathcal{A}'\mathcal{B}'\mathcal{C}') = 0. \tag{34}$$

It is clear that when Eq. (34) taken for each link of C is added to the left-hand side of (33) the total γ-angle contribution becomes

$$\sum_{x\in C}[\gamma_x(S_C) - \gamma_x(S'_C)] = \sum_{x\in C}\Omega_x, \tag{35}$$

where the orientation change of S'_C in the inclusion $S_0 = S_C \cup \tilde{S}'_C$ is crucial. For instance, $\Omega(\mathcal{B})$ is given by $\Delta(\mathcal{A}\mathcal{E}\mathcal{D}\mathcal{C})$ and does not depend at all on contour C. We conclude, therefore, that Eq. (30) is the consistency requirement for the non-Abelian Stokes theorem (27) to be independent on the surface. But the point is that Eq. (30) is more than the consistency condition. As we have argued in Sec. III A, Eq. (30) being applied to the infinitesimal cube is, in fact, the lattice implementation of the non-Abelian Bianchi identities and is illustrated in Fig. 4 (right). It is clear that the integer $q(S_0)$ is the magnetic charge discussed in Sec. III A. Therefore, the non-Abelian Stokes theorem (27) which refers explicitly to the physically observable field strength allows one to formulate the non-Abelian Bianchi identities on the lattice and to study their violation in gauge invariant terms.

E. Discussions

This section is devoted to general notes concerning the Bianchi identities and the magnetic charge definition. We do not pretend on the exhaustive treatment, of course. However, the following items seem to be worth mentioning:

(i) The SU(2) gauge invariance of the magnetic charge is evident from the fact that each term on the left-hand side of (30) is SU(2) gauge invariant by construction. The U(1) gauge invariance (18) of Eq. (30) is also obvious. One could argue that this Abelian symmetry is artificial and is due only to our intent to represent the non-Abelian flux direction in terms of the fictitious spinning particle. However, we do think that the U(1) invariance of (30) might be relevant. Indeed, the interpretation of the Wilson loop defining Eq. (17) in quantum mechanical language is natural and forces us to concentrate on the phase differences of wave functions [see, e.g., Eqs. (19), (28), and (29)], not on their concrete phases. Moreover, it allows one to use the machinery related to the line bundle structure of quantum mechanics, mathematical foundations of geometrical phases, and Bargmann invariants. In this respect, the U(1) symmetry appears naturally and is inherent to our approach (it had been also discussed, although in a different context, in Refs. [32,33]).

(ii) What was also crucial for our construction is the canonical orientation of elementary lattice plaquettes. We discussed this in detail in Sec. III B and concluded that, in order to deduce the field strength from the infinitesimal Wilson loops, some canonical ordering must be introduced. It is true that in most cases the concrete ordering prescription does not matter, since the usually considered quantities do not depend on it. For instance, the gauge action is insensitive to plaquette orientations, but this is certainly because the action is even in the field strength. As far as the magnetic fields are concerned, their unambiguous definition is possible only with some canonical ordering prescription; otherwise, the components of $F_{\mu\nu}$ could be determined only up to the sign even in the Abelian theory. However, it is clear that the ordering is not unique, and, although there are only few possibilities to choose from, the dependence of Eq. (30) on the particular choice should be investigated separately. In this work, we stuck with the conventional canonical ordering described above; the ordering dependence will be investigated elsewhere.

(iii) As we have noted already, it is natural to describe the non-Abelian Stokes theorem (27) in the ribbonlike graphical representation in which the theorem becomes essentially Abelian-like. In other words, the non-Abelian nature of the theory is traded for the complicated geometry. Therefore, the ribbonlike representation is actually not only a convenience. Once we could unambiguously assign each term in Eqs. (27) and (30) to a particular geometrical object, it is natural to ask whether these objects form a self-contained cell complex. For the non-Abelian Stokes theorem, the answer is "no" because each Wilson contour requires the introduction of its own set of triangles (e.g., $\mathcal{ABE}$ in Fig. 4) to which the γ angles are to be ascribed. But the non-Abelian Bianchi identities do indeed allow the introduction of a specific cell complex in which every term on the left-hand side of Eq. (30) is unambiguously assigned to the particular two-dimensional cell. Moreover, Eq. (30) could then be interpreted as a usual coboundary operator acting on 2-cochains. Note that the above reasoning resembles slightly the dual gravitylike representation of SU(2) gluodynamics [23–27]. We stress that this approach is not only a mathematical convenience. In fact, it is the only way to analyze the structure of Eq. (30) at finite lattice spacing. In particular, it allows one to show that the magnetic charge is closely related to the degenerate points (4) mentioned in Sec. II A (this is the topic of the next section). Here we note that the cell complex underlying Eq. (30) is described in the appendix, the results of which are used in the next section.

(iv) It seems to be instructive to start from Eq. (30), expand it in powers of the lattice spacing, and get the Bianchi identities (1) and (3) in the continuum limit. However, we failed to implement this program. As far as we can see, the reason is twofold. First, the original problem (11) was posed quite differently from what could be expected in the continuum. Indeed, our primary goal was to determine the magnetic charge, and we intentionally refused to consider its gauge dependent color orientation. The manifestation of this could be seen by comparing Eqs. (1) and (3) with (30): While the former is in the adjoint representation and is vector in the color space, the latter is gauge invariant and is just one equation. Therefore, it is *a priori* unclear how one could get (1) and (3) from (30) even in the limit of vanishing lattice spacing. On the other hand, Eq. (30) follows rigorously from (7), and we have no doubt that Eq. (30) indeed expresses the Bianchi identities on the lattice. Second, as we argue in the item below (see also the next section), the discussion of Eq. (30) in the continuum limit is indispensable from the consideration of the degenerate points (4).

(v) Let us qualitatively consider what happens with the magnetic charge (30) in the extreme weak coupling limit. The plaquette fluxes do not play any role since they are highly suppressed by the action. Therefore, Eq. (30) simplifies

$$\sum_{x\in\delta c} \Omega_x = 2\pi q(c), \tag{36}$$

where c is elementary lattice cube. Note that the magnetic charge is not directly suppressed by the action, and, therefore, there seems to be no reason for it to die out in the continuum limit. Moreover, it is clear from (36) that the nonzero $q(c)$ is due to the particular distribution of the chromomagnetic field directions and is almost insensitive to the magnitude of the elementary fluxes. Indeed, each Ω_x depends only on the flux directions and not on their magnitudes. In the next section, we show that the nonzero right-hand side of Eq. (36) in the continuum limit indicates that at this point the chromomagnetic fields are degenerated, and the particular determinant constructed from E^a_i, B^a_i vanishes.

IV. CHROMOMAGNETIC FIELDS DEGENERACY

In this section, we analyze the points of chromomagnetic fields degeneracy introduced in Sec. II A. First, we review the essential facts known in the continuum and then turn to the lattice definitions.

A. Preliminaries

In four dimensions, the points of degeneracy of the chromomagnetic fields are defined by

$$\det T = 0, \tag{37}$$

$$T^{ab}_{\mu\nu} = \varepsilon^{abc}\tilde{F}^c_{\mu\nu} = \tfrac{1}{2}\varepsilon^{abc}\varepsilon_{\mu\nu\lambda\rho}F^c_{\lambda\rho}. \tag{38}$$

As we noted in Sec. II A the physical significance of the points (37) crucially depends on the dimensionality. Indeed, in $D = 3$ the operator coupled to the gauge potentials A^a_k in the Bianchi identities (3) is

$$T^{ab}_{k(3D)} = \varepsilon^{abc}B^c_k = \tfrac{1}{2}\varepsilon^{abc}\varepsilon_{kij}F^c_{ij} \tag{39}$$

and, in fact, is a 3×9 matrix for which the determinant is undefined. We could at best consider the rank of the matrix (39) and clearly

$$\mathrm{rank}\, T_{(3D)} < 3, \tag{40}$$

since B^a_k is always the eigenvector with zero eigenvalue. We conclude, therefore, that in $D = 3$ the very notion of chromomagnetic fields degeneracy is uncertain.

In four dimensions, the $\det T$ was calculated long ago [16,18,22]:

$$\det T \propto \det K, \tag{41}$$

$$K_{\mu\nu} = K_{\nu\mu} = \tfrac{1}{3}\varepsilon^{abc}\tilde{F}^a_{\mu\rho}F^b_{\rho\lambda}\tilde{F}^c_{\lambda\nu}, \tag{42}$$

$$K_{00} = 2\det B, \qquad K_{ik} = \tfrac{1}{2}B^a_{\{i}Q^a_{k\}},$$

$$K_{0i} = B^a_i\varepsilon^{abc}E^b_kB^c_k, \qquad E^a_i = F^a_{0i}, \qquad B^a_i = \tfrac{1}{2}\varepsilon_{ijk}F^a_{jk},$$

$$Q^a_k = \varepsilon^{abc}\varepsilon_{kij}E^b_iE^c_j,$$

where curly braces denote symmetrization $B_{\{i}Q_{k\}} = B_iQ_k + B_kQ_i$. It is important that each element of $K_{\mu\nu}$ is gauge invariant determinant constructed in terms of E^a_i and B^a_i. In particular, the off-diagonal elements are of the form $\det[E_iE_kB_k]$, $\det[B_iE_kB_k]$, where no summation in k is implied, and $\det[e_1e_2e_3]$ is understood as the determinant of the column matrix constructed from color vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Note that the off-diagonal elements vanish identically in (anti)self-dual sectors [21]. However, our aim is not to analyze Eqs. (37)–(42) in their generality. Rather, we would like to show that the lattice Bianchi identities naturally lead to the same determinants (42). In particular, in the next section we show that the magnetic charge (30) is ultimately related to the zeros of these determinants and hence to the degenerate points (37).

B. $\det B = 0$ on the lattice

In this section, we consider first the three-dimensional case, which is much simpler geometrically. The results remain valid in four dimensions, but in $D = 4$ there are important differences as well.

It was noted in Sec. III E that the only reliable and rigorous way to analyze Eq. (30) at finite lattice spacing is to consider the specially crafted cell complex for which Eq. (30) is the coboundary operation. The existence and structure of this cell complex could be inferred by noting that the non-Abelian Stokes theorem and the Bianchi identities on the lattice are described most naturally in the ribbonlike graphical representation. Starting from this, the cell complex could be completely constructed (see the appendix). The advantage of this approach is that it is rather formal. Once we were able to assign gauge invariant numbers (magnitudes of elementary fluxes and their relative orientations expressed in terms of Bargmann invariants) to each two-dimensional cell, all we have to do is to consider the coboundary operator $d:\mathbb{C}^2 \to \mathbb{C}^3$, where $\mathbb{C}^k$ is the k skeleton. For every lattice cube, the action of d is equivalent to the Bianchi identities (30) by construction, and, hence, d assigns the corresponding magnetic charge to each lattice cube. However, the additional Ω-angle contribution implies that the geometry of the cell complex is not (hyper)cubical. In particular, the 3-skeleton $\mathbb{C}^3$ is larger than the union of lattice 3-cubes.

On the other hand, there is formally no difference between different 3-cells of the complex. In particular, one can show that $d:\mathbb{C}^2 \to \mathbb{C}^3$ always assigns an integer number to every 3-cell. It is true that some of these "new" 3-cells are trivial and the corresponding magnetic charge is always zero. However, there exist the nontrivial cases as well (see the appendix), one of which (and the only one in $D = 3$) is illustrated in Fig. 5 (right).

Consider some point x on the original $D = 3$ lattice together with 12 plaquettes and 8 cubes which share this point. Equation (30) applied to each cube forces us to take into account 8 triangles at the cube's corners (cf. Fig. 4)

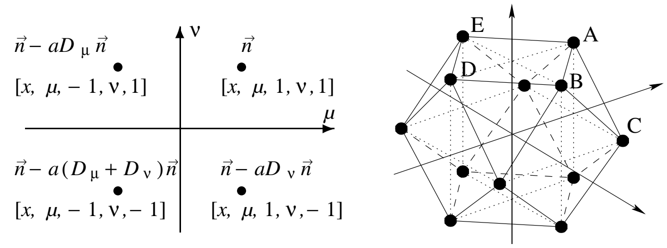

FIG. 5. Left: The flux directions in the plane (μ, ν) around point x in the weak coupling limit, Eq. (44). Right: The only nontrivial 3-cell in $D = 3$ (see the text).

and to assign the corresponding Bargmann invariants Ω_i, $i = 1, \ldots, 8$ to each triangle. Figure 5 (right) shows the triangles around point x coming from different cubes. Note that all 8 triangles are properly oriented. By the same token, one concludes that 6 squares, e.g., $ABDE$, are also valid 2-cells of the cell complex and are equipped with the corresponding Bargmann invariants $\Delta_i(x)$, $i = 1, \ldots, 6$. Then it is clear that the application of $d:\mathbb{C}^2 \to \mathbb{C}^3$ to the set of 2-cells in Fig. 5 assigns a well defined and gauge invariant integer number to the 3-cell shown in that figure:

$$2\pi\tilde{q}(x) = \sum_{i=1}^{8} \Omega_i(x) + \sum_{i=1}^{6} \Delta_i(x). \tag{43}$$

Formally, it is just the same magnetic charge we have considered so far, but now it is ascribed to the site of the original lattice. We are confident that the magnetic charges in the lattice cubes correspond to the Bianchi identities violation. But what is violated in the lattice sites?

To answer this question, we expand (43) in powers of the lattice spacing. However, it is worth mentioning that this expansion is not the usual one. In particular, it would be plainly wrong to look for $O(a^3)$ terms, since the integer number on the left-hand side of Eq. (43) does not depend at all on the lattice spacing. Therefore, in the weak coupling expansion we should look for a-independent contributions or, better to say, to look for the conditions for a-independent terms to appear.

In fact, all necessary relations were derived in Ref. [10]. In particular, consider four plaquettes in the same plane which share the point x, Fig. 5 (left). To the leading order, the color directions of the fluxes at point x are given by

$$\begin{aligned} &x + \hat{\mu} + \hat{\nu}:\vec{n} + O(a^2), \\ &x + \hat{\mu} - \hat{\nu}:\vec{n} - aD_\nu\vec{n} + O(a^2), \\ &x - \hat{\mu} + \hat{\nu}:\vec{n} - aD_\mu\vec{n} + O(a^2), \\ &x - \hat{\mu} - \hat{\nu}:\vec{n} - a(D_\mu + D_\nu)\vec{n} + O(a^2), \end{aligned} \tag{44}$$

where we have denoted $\vec{n} = \vec{n}_{(\mu\nu)}(x)$ for brevity. We conclude, therefore, that in the weak coupling limit the three points A, B, C (Fig. 5, right) are distinguished: The flux directions assigned to them are, in general, independent and coincide with the color direction of the particular component of $\vec{F}_{\mu\nu}$. The flux directions in all other vertices are obtainable by infinitesimal variation of the flux direction in one of the points A, B, C.

Recall now that the Bargmann invariant assigned to each triangle and square is the oriented solid angle between the corresponding flux directions. It follows then that the contribution of all squares is always of order $O(a^3)$ and is negligible. As far as the triangles are concerned, they also give terms of order $O(a^3)$ unless the fluxes at points A, B, C become linearly dependent. In this case, the corresponding Bargmann invariant could be $\pm\pi + O(a)$, and the order $O(a)$ variation of the fluxes at various vertices is enough to change it by 2π. It is clear that only in this degenerate case is the nonzero left-hand side of (43) at all possible. On the other hand, the flux directions at the points A, B, C in the weak coupling limit are given by the corresponding chromomagnetic field components $\vec{B}_k$. We conclude, therefore, that the nonvanishing magnetic charge (43) implies that the chromomagnetic fields are degenerate at this point

$$\tilde{q}(x) \neq 0 \Rightarrow \det B(x) = 0. \tag{45}$$

Note that the statement could not be reversed. For instance, in the case $\vec{B}_1 = \vec{B}_2 = \vec{B}_3$ both $\det B$ and $\tilde{q}$ vanish.

Equation (45) remains valid in four dimensions as well. The only distinction is that now we have 4 different magnetic charges $\tilde{q}_\mu(x)$ labeled by the direction $\hat{\mu}$ dual to a given three-dimensional slice. In particular, the nonzero $\tilde{q}_\mu(x)$ implies that one of the determinants $\det[B_1 B_2 B_3]$, $\det[B_1 E_2 E_3]$, $\det[E_1 B_2 E_3]$, $\det[E_1 E_2 B_3]$ vanishes. Note that these determinants are the diagonal entries of $K_{\mu\nu}$, Eq. (42), and, therefore,

$$\tilde{q}_\mu(x) \neq 0 \Rightarrow K_{\mu\mu} = 0 \qquad (\text{no sum over } \mu). \tag{46}$$

By symmetry considerations, one expects that there should exist 3-cells for which the magnetic charge indicates the zeros of $\det[E_1 E_2 E_3]$, $\det[E_1 B_2 B_3]$, $\det[B_1 E_2 B_3]$, $\det[B_1 B_2 B_3]$. It turns out that these cells are $\mathcal{D}^{(1)}(x, \mu, d_\mu)$ (see the appendix). Indeed, the structure of $\mathcal{D}^{(1)}$ cells is such that the argumentation leading to (45) applies literally. Then the inspection of the flux directions assigned to vertices of $\mathcal{D}^{(1)}(x, \mu, d_\mu)$ shows that the nonzero magnetic charge of one of these 3-cells is the sufficient condition for the particular determinant above to vanish.

As far as the off-diagonal elements of $K_{\mu\nu}$ are concerned, they are highly sensitive to the topological properties of the gauge fields. For instance, $K_{12} = \det[B_2 E_2 E_3] - \det[B_1 E_1 E_3]$ vanishes in the (anti)self-dual sectors. It is possible to identify the 3-cells which are related to the off-diagonal entries of the $K_{\mu\nu}$ matrix. Indeed, consider the diamondlike 3-cells $\mathcal{D}^{(2)}$ (see the appendix). In the weak coupling limit, the flux directions assigned to 4 plaquette corners become essentially the same and coincide with the corresponding component of $\vec{F}_{\mu\nu}$. Then the flux orientations ascribed to 3 pairs of opposite vertices of $\mathcal{D}^{(2)}$ are given by $\vec{E}_k$, $\vec{B}_k$, $k = 1, 2, 3$. Geometrically, it is clear that for $\vec{E}_k = \pm\vec{B}_k$ the 3-cells $\mathcal{D}^{(2)}$ are highly degenerated, and there is a good chance for the coboundary operator $d: \mathbb{C}^2 \to \mathcal{D}^{(2)}$ to give a nonzero magnetic charge. However, we are still lacking the rigorous argumentation here. One could only say (see also Sec. V C) that the $\mathcal{D}^{(2)}$ cells are indeed closely connected to the topological properties of the gauge background. The relation of the present approach to the gauge fields topology goes beyond the scope of the present publication and will be investigated elsewhere.

To summarize, the non-Abelian Bianchi identities (30) could be interpreted as the coboundary operator $d: \mathbb{C}^2 \to \mathbb{C}^3$ for the specific cell complex, the complicated geometry of which is the direct consequence of the non-Abelian nature of the theory. Moreover, the operator d considered in its generality necessitates the consideration of gauge invariant magnetic charges associated with various 3-cells. While the nonvanishing magnetic charge in a three-dimensional cube implies the violation of the Bianchi identities, in other 3-cells it is the sufficient condition for the particular determinant constructed from E_i^a, B_i^a to vanish. At finite lattice spacing, these two types of magnetic charges are almost independent and should be considered as such, especially since they are geometrically distinct: The former are ascribed to the lattice cubes; the latter are assigned to the sites of the original lattice. However, at vanishing lattice spacing the two types of magnetic charges become closely interrelated [cf. Eq. (36)]: Once the flux magnitude on the elementary plaquettes becomes negligible everywhere, the non-Abelian Bianchi identities could be violated only at the degenerate points (46).

V. NUMERICAL EXPERIMENTS

It is true that the relevance of the above construction for the dynamics of the Yang-Mills fields is not evident from the preceding presentation. However, we specifically kept in mind from the very beginning the possibility to apply our approach in real lattice experiments. In this section, we describe the results of our numerical simulations. The problem to be considered is whether the violation of the Bianchi identities and the degeneracy of the chromomagnetic fields are physically significant.

The general setup is as follows. We simulate the SU(2) lattice gluodynamics in three and four dimensions on the symmetric lattices with periodic boundary conditions. The action we adopt initially (see below) is the standard Wilson action. Until Sec. V B, the lattices we used are 16^3 and 10^4 with corresponding β ranges $[5.0; 9.0]$ and $[2.2; 2.8]$. Note that these parameters are partially unphysical. The purpose is to consider the behavior of the magnetic charges (30), (45), and (46) in various circumstances, in particular, across the finite-volume phase transition.

The simplest and instructive quantities to study are the densities $\rho(\beta)$, $\tilde{\rho}(\beta)$ of the magnetic charges (30) and (46). The density $\rho(\beta)$ is defined irrespectively of the space-time dimensionality

$$\rho(\beta) = \frac{1}{N_c} \sum_c |q(c)|, \tag{47}$$

where summation is over all lattice 3-cubes and N_c is their total number. Evidently, $\rho(\beta)$ measures the fraction of points at which the non-Abelian Bianchi identities are violated. The definition of $\tilde{\rho}(\beta)$ differs in $D = 3$ and $D = 4$. In three dimensions, we have

$$\tilde{\rho}(\beta) = \frac{1}{N_s} \sum_s |\tilde{q}(s)|, \tag{48}$$

where s is the lattice site, N_s is the total lattice volume, and $\tilde{q}(s)$ was defined in Sec. IV B. In $D = 4$ there are several types of the magnetic charges $\tilde{q}$, and, therefore, the definition (48) is ambiguous. We take the symmetric definition which looks similar to (48): s denotes the 3-cell which is not the lattice cube and N_s is the total number of these cells. Physically, $\tilde{\rho}(\beta)$ is the fraction of the lattice volume occupied by zeros of various determinants, e.g., (45) and (46).

The dependence of ρ and $\tilde{\rho}$ on β is shown in Fig. 6. One can see that both densities are numerically similar in three and four dimensions and are almost β independent in accordance with general arguments of Sec. III E. Indeed, the β independence of $\tilde{\rho}$ is certainly expected since there is no symmetry which could keep the sign of the determinants (46) fixed. In particular, the perturbation theory gives the dominant contribution to the density $\tilde{\rho}(\beta)$. On the other hand, the β independence of ρ follows from the fact that the violation of Bianchi identities is closely related to the zeros of the above determinants. Therefore, we come to the paradoxical conclusion that the perturbation theory also saturates the density $\rho(\beta)$.

To resolve the problem, we note that in the continuum limit the Bianchi identities are formulated for elementary 3-volumes, while the determinants are defined at any particular point. The corresponding construction on the lattice is essentially the same: The Bianchi identities and the magnetic charge q are ascribed to the elementary 3-cubes, while the degeneracy points and the charge $\tilde{q}$ are assigned to the lattice sites. It is important that these charges are geometrically distinct on the lattice: At arbitrary small but nonzero spacing, there is $O(a)$ distance between them and they are defined on different 3-cells. It turns out that on the lattice the magnetic charge at 3-cube and anticharge at the neighboring site may coexist with almost no additional action penalty [cf. Eqs. (36) and (43)]. Moreover, one can show that there could be no mechanism to prevent the creation of these ultraviolet (UV) q-$\tilde{q}$ pairs, since it would violate the gauge invariance. Indeed, although the

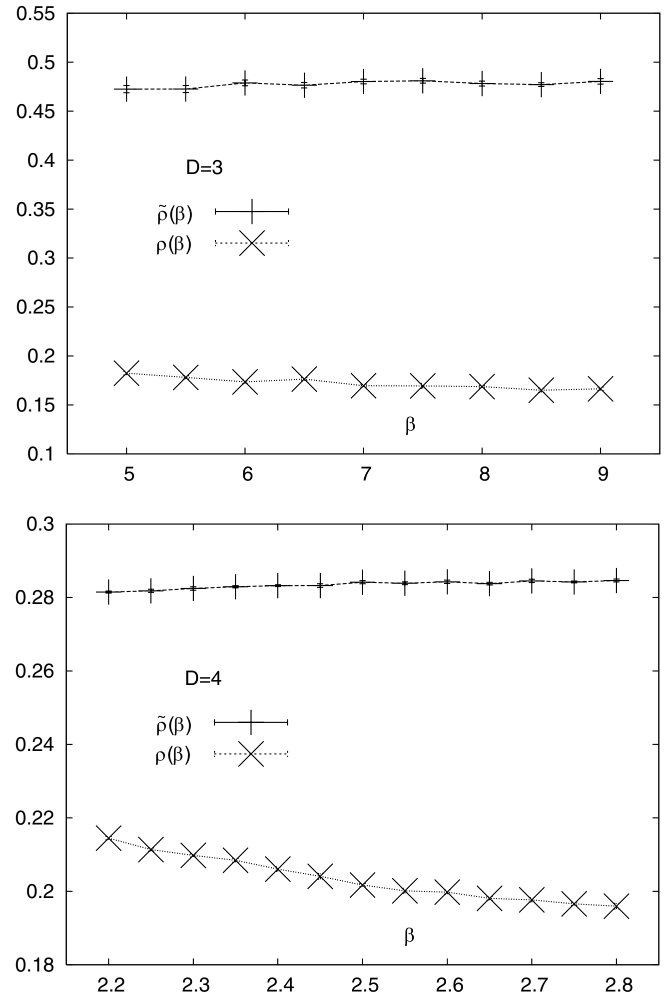

FIG. 6. The densities (47) and (48) versus β coupling. The lines are drawn to guide the eye.

relative orientation of the fluxes is formally gauge invariant, any restriction of it will effectively squeeze the non-Abelian fluxes into one particular color direction. Then it would be hardly possible to call the resulting theory non-Abelian [65]. Note that the UV pairs above are irrelevant from the continuum viewpoint. Indeed, there is no trace whatsoever of the ultraviolet q-$\tilde{q}$ pairs on the blocked lattice with lattice spacing $N \cdot a$. At the same time, the densities $\rho(\beta)$, $\tilde{\rho}(\beta)$ account for all the charges q, $\tilde{q}$ on equal footing and, therefore, are dominated by the UV fluctuations.

We conclude, therefore, that the densities $\rho(\beta)$ and $\tilde{\rho}(\beta)$ are not the appropriate observables on the unblocked lattices. They are dominated by the ultraviolet noise which is only due to the mismatch in the domain of definition of the Bianchi identities and the degenerate points. It seems that the only way to make sense of the densities ρ, $\tilde{\rho}$ is to consider them on the blocked configurations for which the ultraviolet noise is gradually removed. However, our approach to the problem is different and is described below.

A. Modification of the action

As follows from the above presentation, the dynamics of q and $\tilde{q}$ magnetic charges is highly UV sensitive and the dominant configurations are small (at the scale of UV cutoff) q-$\tilde{q}$ pairs. It seems that this observation forbids the discussion of the significance of the Bianchi identities violation and the points of degeneracy, since it is impossible to separate the UV noise from physically relevant excitations. Essentially, the same problem exists in usual field theories, where the vacuum condensates are commonly used to parametrize the nonperturbative effects. The well known example is the gluon condensate $\langle\alpha_s(F^a_{\mu\nu})^2\rangle$, which is perturbatively divergent but its nonperturbative part is nonvanishing and is known to be of major phenomenological importance. The subtraction of the perturbative tail of various condensates is challenging, and the usual approach is to subtract it order by order in the coupling constant. However, we do not see any tractable way to do this in our case.

On the other hand, it is possible to reformulate slightly the original problem. Instead of trying to isolate the effects due to the UV q-$\tilde{q}$ pairs, we could equally ask what happens when the magnetic charges are partially removed from the vacuum. Indeed, the definition of q and $\tilde{q}$ charges is local and gauge invariant. Therefore, nothing prevents us from modifying the Wilson action to include the additional terms which could influence the dynamics of q, $\tilde{q}$ charges. Since it is hardly possible to invent the additional well defined terms which are sensitive to the UV dynamics only, we will study the following simplest modification:

$$S = -\beta\sum_p \frac{1}{2}\,\mathrm{Tr}U_p + \gamma\sum_c |q(c)| + \tilde{\gamma}\sum_s |\tilde{q}(s)|, \qquad (49)$$

where the first term is the standard Wilson action and c denotes the elementary lattice cubes. The last term in Eq. (49) has different interpretation in three and four dimensions. In $D = 3$, s denotes the lattice sites and $\tilde{q}(s)$ is given by Eq. (43). In four dimensions, the last term *a priori* depends on the concrete definition of the magnetic charges $\tilde{q}$. As in the previous section, we take the symmetric definition: s denotes the 3-cells which are not the lattice cubes, and $\tilde{q}(s)$ is the corresponding magnetic charge. It turns out that our results are almost insensitive to the particular choice of the last term in Eq. (49); see Sec. V C.

The modified action is local and SU(2) gauge invariant. Indeed, from the defining equations (30) and (43) one can see that (49) intertwines the links which are at most two lattice spacings apart, while the gauge invariance follows by construction. Then the universality suggests that the continuum limit of the model defined by (49) should be the same as one for the model with the conventional Wilson action (see also Sec. VI for discussions). On the other hand, the additional coupling constants γ, $\tilde{\gamma}$ allow one to study the effects which are due to the Bianchi identities violations and the degeneracy points. The particular limit $\gamma \to \infty$ is of special interest, since it corresponds to the theory with nowhere violated Bianchi identities. As far as the $\tilde{\gamma}$ coupling is concerned, we are not so confident that the limit $\tilde{\gamma} \to \infty$ corresponds to a sensible theory. For instance, in $D = 3$ the nowhere vanishing $\det B = \det[B_1 B_2 B_3]$ implies that it is of the same sign everywhere, which contradicts the perturbative expectations [66] and probably violates CP symmetry. At the same time, the point $\gamma = \tilde{\gamma} = 0$ is certainly equivalent to the conventional lattice gluodynamics.

In the next two sections, we study the model (49) along the lines $\tilde{\gamma} = 0$ and $\gamma = 0$ in the $(\gamma, \tilde{\gamma})$ parameter space at a fixed value of the gauge coupling β. The simulations were performed on 20^3 and 12^4 lattices at $\beta = 6.0$ and $\beta = 2.4$, correspondingly. Note that this choice of parameters is based on the experience with pure Yang-Mills (YM) theory, in which these β values and volumes correspond to the physical scaling regime [67,68]. While the point $\gamma = \tilde{\gamma} = 0$ was simulated with standard overrelaxed heatbath updating, away from it we implemented the Metropolis algorithm, which is the only one available at nonzero γ, $\tilde{\gamma}$. The procedure turns out to be very time consuming, especially in $D = 4$. Indeed, the one link update step requires one to take into account the magnetic charges q, $\tilde{q}$ in all neighboring cells, the number of which is much larger in $D = 4$ (see the appendix). Because of this, we were unable to thoroughly scan the ample range of γ couplings; only the following points were considered in detail:

$$\begin{aligned}(\gamma, \tilde{\gamma})_{3D} &= \{(0, 0); (4, 0), (7, 0), (9, 0); (0, 4)\},\\ (\gamma, \tilde{\gamma})_{4D} &= \{(0, 0); (4, 0), (6, 0), (8, 0); (0, 4)\}.\end{aligned} \qquad (50)$$

In particular, the complexity of the algorithm precludes us from studying the phase diagram of the model (49) (see below) and investigate the finite-volume effects. Below it is silently assumed that the chosen volumes are large enough even at nonzero γ, $\tilde{\gamma}$ couplings. At each γ point, we generated about 100 statistically independent gauge samples separated by $\sim 10^3$ Monte Carlo sweeps. The observables of primary importance are the planar Wilson loops from which we extracted the heavy quark potential (see, e.g., Ref. [69] for details) and the correlator of the Polyakov lines $\langle P(0)P(R)\rangle$, $P(\vec{x}) = 1/2\,\mathrm{Tr}\prod_t U_0(\vec{x} + t)$. To improve the statistics, the standard spatial smearing [70] and hypercubic blocking [71] for temporal links were used. In $D = 4$ we also monitored the topological charge Q, the topological susceptibility $\chi = \langle Q^2\rangle/V$ defined by means of the overlap Dirac operator [72] (see, e.g., Ref [73] for details and further references).

B. $\tilde{\gamma} = 0$ line

Here we study the effect of the gradual removal of the points in which the Bianchi identities are violated. Let us consider first the behavior of the densities (47) and (48) with rising γ coupling. It turns out that $\tilde{\rho}$ stays almost constant (Fig. 7, upper panel) in both three and four dimensions, monotonically varying from 0.477(1) to 0.435(1) in $D = 3$ and from 0.2831(1) to 0.2441(2) in $D = 4$ in the entire γ range considered. On the other hand, the density ρ falls down exponentially with γ and becomes of order $O(10^{-4})$ in $D = 3$ [$O(10^{-3})$ in $D = 4$]. We note in passing that the mean plaquette $\langle 1/2\, \mathrm{Tr} U_p \rangle$ is also almost insensitive to the γ coupling (Fig. 7, bottom) rising in $D = 3$ from 0.8248(1) to 0.8263(3) when γ is changed in the entire range [the corresponding change in $D = 4$ is from 0.6301(2) to 0.6548(2)].

A few comments are now in order. First, the constancy of $\tilde{\rho}$ and the simultaneous falloff of ρ by orders of magnitude implies that the above picture of dominating ultraviolet q-$\tilde{q}$ pairs is greatly oversimplified. It seems that the UV fluctuations are indeed dominating, but their structure is much more involved. In particular, it has little to do with the model of tightly bounded q-$\tilde{q}$ dipoles; rather, it is some complicated mixture of various charge-anticharge configurations which probably do not form the dipolelike pairs at all.

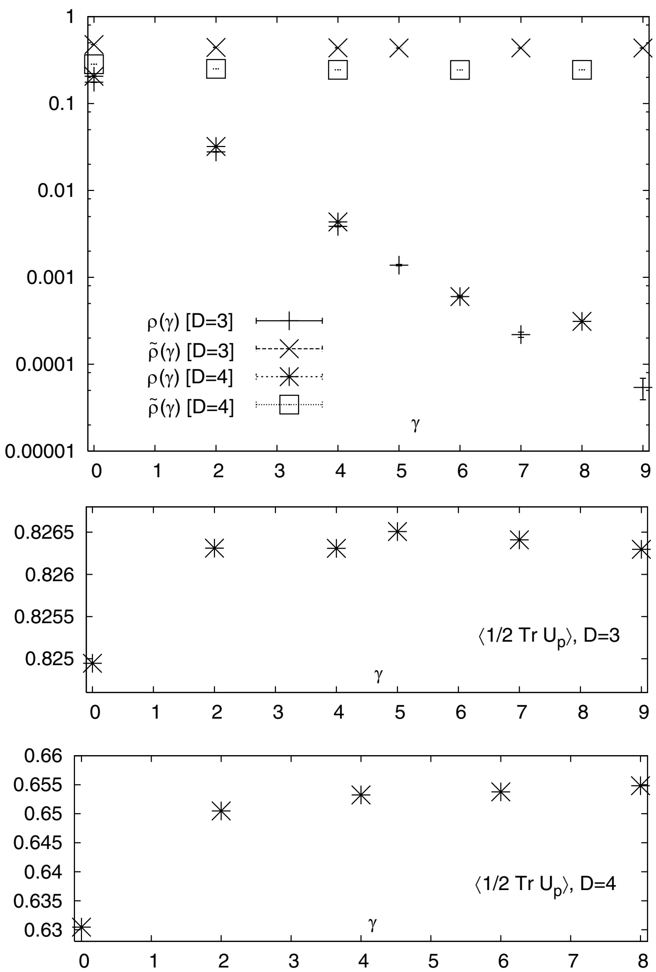

FIG. 7. The densities (47) and (48) and the mean plaquette $\langle 1/2\, \mathrm{Tr} U_p \rangle$ on the line $\tilde{\gamma} = 0$ as functions of γ.

Turn now to the behavior of the heavy quark potential and the Polyakov lines correlation function with rising γ coupling. In three dimensions (Fig. 8), both the Wilson loops and the correlator $\langle P(0)P(R) \rangle$ show almost no sign of γ coupling dependence; in particular, the asymptotic string tension at large γ is equal to its value in the pure Yang-Mills theory. However, the situation changes drastically in $D = 4$. One can see from Fig. 9 that the correlation function $\langle P(0)P(R) \rangle$ tends to nonzero positive value at large separations when γ coupling becomes of order few units

$$\lim_{R\to\infty} \langle P(0)P(R) \rangle_{\gamma \gtrsim 1} = \mathrm{const} > 0. \qquad (51)$$

The heavy quark potential extracted from Wilson loops is shown in Fig. 10 (upper panel), and for $\gamma \gtrsim 1$ is indeed flattening at large distances

$$\lim_{R\to\infty} V_{\gamma \gtrsim 1}(R) = \mathrm{const}. \qquad (52)$$

Note that it is hardly possible to conclude firmly from Fig. 10 alone that the asymptotic string tension is indeed vanishing; however, Eq. (51) and Fig. 9 are incompatible with its nonzero value.

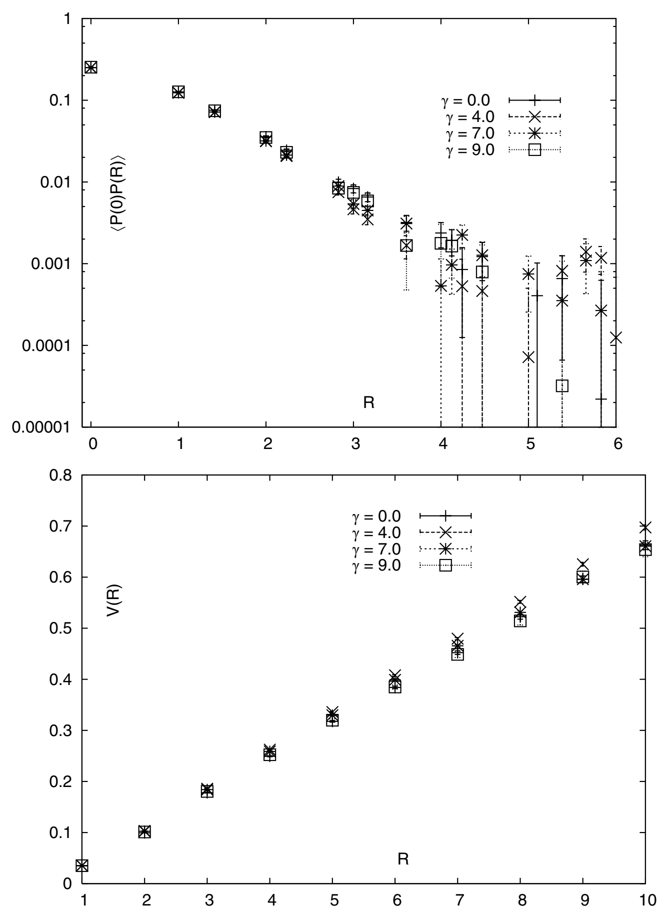

FIG. 8. The correlation function of Polyakov lines and the heavy quark potential in $D = 3$ at $\tilde{\gamma} = 0$ and various γ.

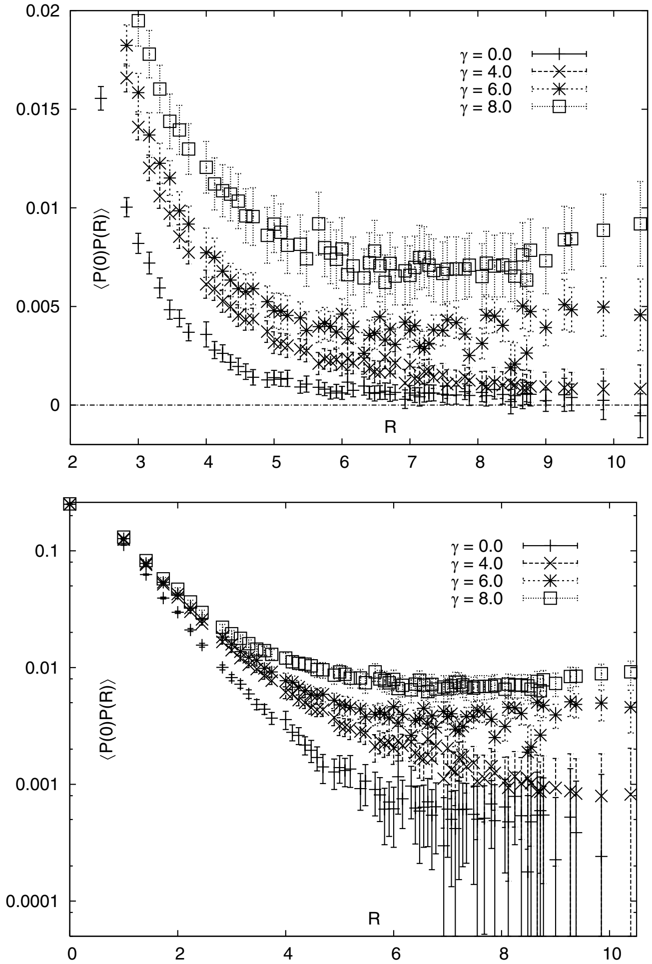

FIG. 9. The Polyakov lines correlator in $D = 4$ at $\tilde{\gamma} = 0$ and various γ.

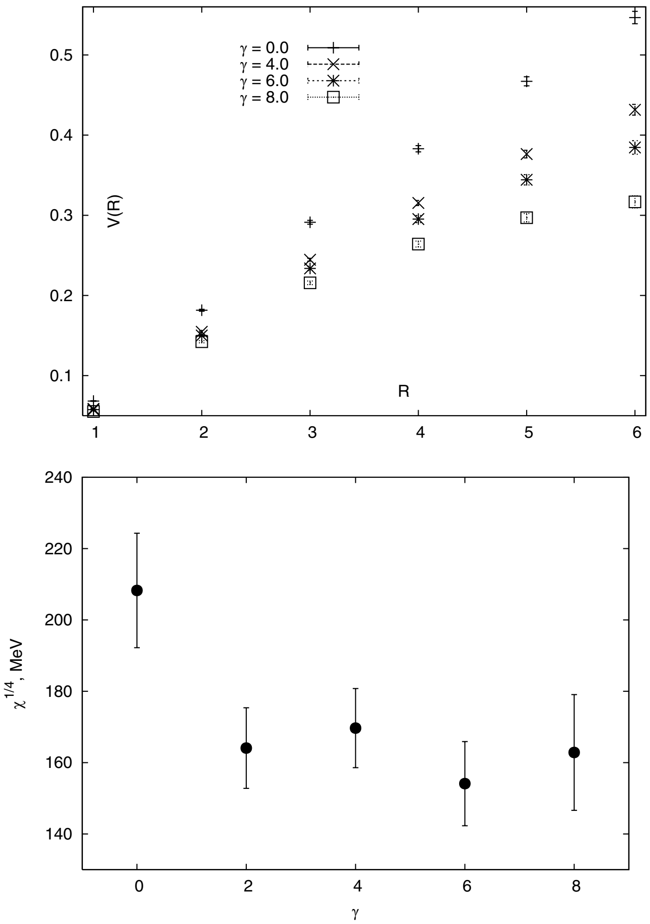

FIG. 10. The heavy quark potential $V(R)$ and the topological susceptibility χ in $D = 4$ at $\tilde{\gamma} = 0$ and various γ.

The other measured observables do not show strong dependence on γ coupling. In particular, the topological charge Q stays at zero in average value, albeit with slightly narrower distribution. As is clear from the bottom panel of Fig. 10, the topological susceptibility $\chi = \langle Q^2 \rangle / V$ diminishes at $\gamma \approx 1$ by approximately 25%, and the estimation of its limiting value is

$$\lim_{\gamma \to \infty} \chi^{1/4}(\gamma) = 163(8)\ \text{MeV}, \tag{53}$$

which should be compared [68] with $\chi^{1/4}(0) = 212(3)$ MeV in pure YM theory, where the physical units are fixed by the string tension $\sqrt{\sigma} = 440$ MeV.

The discussion of the results presented above is postponed until Sec. VI. Here we note only that the dynamics of YM fields in $D = 3$ seems to be almost insensitive to whether or not the Bianchi identities are violated. In particular, the complete suppression of the magnetic charges which indicates the violation of the Bianchi identities has almost no consequences for the correlators we considered. However, the four-dimensional case appears to be quite different. Our results indicate that the suppression of the Bianchi identities violation is likely to destroy confinement, while other measured characteristics of the theory remain essentially unchanged.

C. Suppressing the degenerate points

Consider the response of the theory on the suppression of the degenerate points. The qualitative difference in the behavior of the system along the lines $\gamma = 0$ and $\tilde{\gamma} = 0$ could be seen already on the simplest observables such as ρ, $\tilde{\rho}$. We have checked that the falloff of the degenerate points fraction $\tilde{\rho}(\tilde{\gamma})$ is indeed exponential with $\tilde{\gamma}$ in both three and four dimensions; the relevant numbers are $\tilde{\rho}_{3D}(0) = 0.477(1)$, $\tilde{\rho}_{3D}(4) = 0.0098(2)$ and $\tilde{\rho}_{4D}(0) = 0.2831(1)$, $\tilde{\rho}_{4D}(4) = 0.045(3)$. However, the fraction of points at which the Bianchi identities are violated also notably diminishes with $\tilde{\gamma}$. The falloff of $\rho(\tilde{\gamma})$ in $D = 3$ is not so pronounced [$\rho(0) = 0.1758(4)$, $\rho(4) = 0.1010(4)$], and, starting from $\tilde{\gamma} \approx 1$, it is numerically larger than $\tilde{\rho}$. It is surprising, however, that in $D = 4$ the inequality $\rho < \tilde{\rho}$ holds for all $\tilde{\gamma}$ values considered, and, in fact, the fraction of points at which the Bianchi identities are violated is diminished by the order of magnitude

[0.2059(2) at $\tilde{\gamma} = 0$ versus 0.024(3) at $\tilde{\gamma} = 4$]. As far as the mean plaquette energy is concerned, its behavior is similar to that on the $\tilde{\gamma} = 0$ line. In particular, in $D = 3$ it essentially stays constant, while in four dimensions it changes from 0.6301(2) to 0.655(1).

As we noted already, the suppression of the degenerate points (45) and (46) might not be physically meaningful. For instance, in three dimensions the orientation of the triple $(\vec{B}_1, \vec{B}_2, \vec{B}_3)$, although being gauge invariant, is not fixed by any symmetry or physical principle. The attempt to fix the sign of $\det B$ everywhere probably will lead to physically unacceptable results. Indeed, closer inspection of the Polyakov lines correlator reveals that it is an oscillating function of the distance. Hence, the lattice reflection positivity is lost and the theory seems to be pathological at nonzero $\tilde{\gamma}$.

In four dimensions, the suppression of the degenerate points leads to qualitatively the same results which, however, are much more pronounced. For instance, the Wilson loops $\langle W(R, T)\rangle$ measured at $\tilde{\gamma} \neq 0$ are notably oscillating at fixed R and varying T (Fig. 11, top panel). However, unlike the three-dimensional case, we can easily pinpoint the origin of the reflection positivity violation. Indeed, it is well known that in the fixed topological sector the theory certainly violates CP, and it is natural then to ask what the typical topological charge of the configurations at nonzero $\tilde{\gamma}$ is.

The bottom panel of Fig. 11 shows the Monte Carlo history of the topological charge on 8^4 lattice at $\beta = 2.30$, $\gamma = 0$, $\tilde{\gamma} = 0.1, 0.5$ when the starting configuration was thermalized at $\gamma = \tilde{\gamma} = 0$ (note that we changed the lattice geometry and the β coupling for reasons to be explained shortly). In view of the observed reflection positivity violation at $\tilde{\gamma} \neq 0$, it is not surprising that Q indeed stays away from zero in average. What is surprising, however, is that the average topological charge $\langle Q\rangle$ turns out to be always positive and extremely large for $\tilde{\gamma} > 0$. In particular, for $0 < \tilde{\gamma} \ll 1$ the mean topological charge is shifted only slightly from zero being of order few units. However, once the $\tilde{\gamma}$ coupling becomes comparable with unity, Q flows away from zero during Monte Carlo updating towards extremely large positive values with almost constant and very high rate. In fact, it quickly becomes too large to be technically accessible for us, and this was essentially the reason to consider such small lattices here. The volume dependence of $\langle Q\rangle$ could be inferred by noting that the last term in (49) responsible for the rapid increase of the topological charge is the bulk quantity. Therefore, $\langle Q\rangle$ seems to be proportional to the volume at fixed $\tilde{\gamma}$, although we had not thoroughly investigated this dependence numerically. We have checked that the behavior of Q at nonzero $\tilde{\gamma}$ is always similar to that in Fig. 11, irrespectively to the concrete meaning of the last term in the action (49). It does not matter which particular type of magnetic charges $\tilde{q}$ is suppressed by the $\tilde{\gamma}$ coupling; we always see the violation of the reflection positivity which is due to the rapid increase of the global topological charge. This problem is discussed in next section.

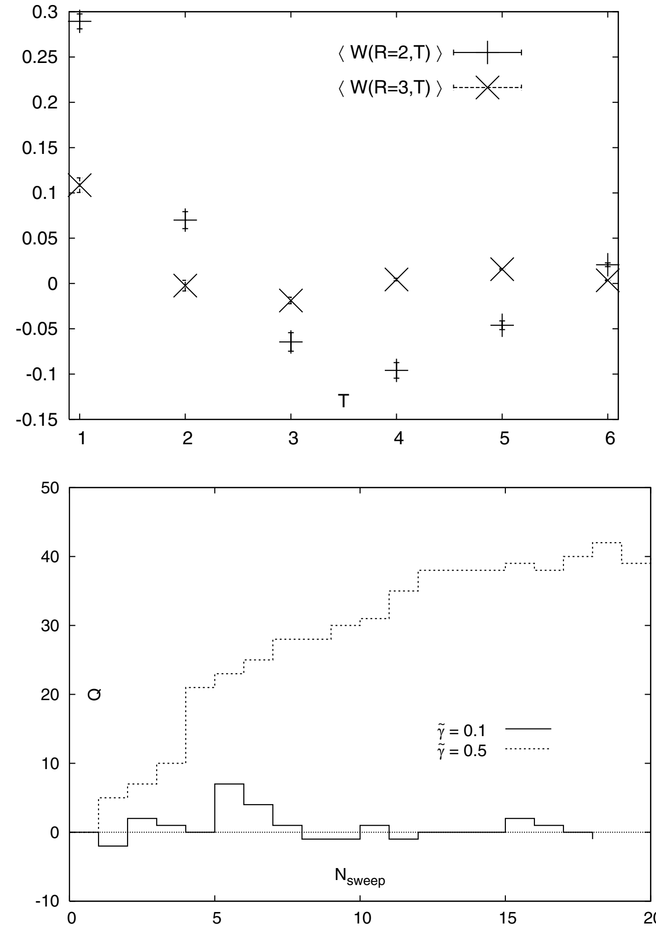

FIG. 11. The oscillations of Wilson loops $\langle W(R, T)\rangle$ at $\gamma = 0$, $\tilde{\gamma} = 4$ and the Monte Carlo history of the topological charge at $\gamma = 0$ in $D = 4$.

VI. DISCUSSIONS

The interpretation of the results we achieved so far may not be simple and straightforward. Here we discuss a few particular points which are essential for our work.

First of all, we do see that the physical significance of the Bianchi identities is quite different in $D = 3$ and $D = 4$. The three-dimensional theory turns out to be insensitive to the suppression of the Bianchi identities violation. Even the complete removal of q charges from the vacuum does not change the theory in any notable way. The four-dimensional theory seems to be different in this respect.

The suppression of the Bianchi identities violation is likely to destroy confinement liberating color charges in the fundamental representation. It is tempting to conclude then that the confinement phenomenon is due to the field configurations for which the right-hand side of Eq. (1) is nonvanishing. This conclusion looks natural for the following reasons. First, it matches the known confinement mechanism in the simple Abelian models. Second, it could explain why in the continuum considerations confinement is missing, since usually the Bianchi identities with vanishing right-hand side are taken for granted. Third, it qualitatively matches the phenomenological lattice observations that the geometrically thin linelike or stringlike objects (Abelian monopoles, P vortices) might be relevant for confinement (see, e.g., Refs. [64,74] for review and further references). And, finally, it does not look hopeless from the field-theoretical point of view since, as we argued above, at vanishing lattice spacing the mechanism of the Bianchi identities violation has little to do with singular fields; rather, it is related in some complicated way to the points of chromomagnetic fields degeneracy.

However, the striking difference between three- and four-dimensional theories with respect to the suppression of the Bianchi identities violation shows that this conclusion is probably misleading. If the confinement phenomenon is indeed due to the Bianchi identities violation, then it should disappear also in $D = 3$ at large γ coupling. But this does not happen, and, hence, we come to the unnatural conclusion that the confinement mechanism has little in common in $D = 3$ and $D = 4$.

However, we could take a different point of view. Namely, there is indeed a great physical difference between the Bianchi identities in three and four dimensions. As we discussed in Sec. II, Eq. (1) constitutes the algebraic restriction on the gauge potentials for a given distribution of the chromomagnetic fields. Away from the degeneracy points [75], the gauge potentials could be completely reconstructed just from the Bianchi identities alone. In this respect, the violation of the Bianchi identities could be seen as the source of the gauge potentials ambiguities, and the suppression of the nonzero right-hand side of Eq. (1) effectively restricts the gauge inequivalent A^a_μ, which are to be taken into account in the functional integral. It is crucial that in three dimensions the analogous argumentation fails, and, in fact, Eq. (3) does not restrict the gauge potentials in any notable way irrespective of whether or not it is violated.

The natural and probably the only available quantity which is sensitive to the gauge potentials ambiguities is the $\langle A^2_{\min} \rangle$ condensate. Therefore, the following qualitative scenario emerges. It is the nonperturbative $\langle A^2_{\min} \rangle$ condensate which seems to be relevant for confinement. In four dimensions the Bianchi identities are the tool which allows one to restrict the $\langle A^2_{\min} \rangle$ condensate. Moreover, the suppression of the nonzero right-hand side of Eq. (1) makes $\langle A^2_{\min} \rangle$ vanish. Clearly, the same approach does not work in three dimensions, because the Bianchi identities do not constrain A^a_μ in $D = 3$. Note that this is only the qualitative picture. In particular, the dependence of $\langle A^2_{\min} \rangle$ on the q charges density could be very complicated, especially because of the dominating perturbative contributions.

To reiterate the point, we note that the justification to consider the modified action (49) is that the second term in (49) is local and preserves all the symmetries of the original action. Then the universality suggests that the continuum limit of the model (49) should be γ coupling independent (we take $\tilde{\gamma} = 0$ for definiteness). At the same time, our results indicate that this is probably not the case. If we would accept the Bianchi identities violation as the primary reason for confinement, then we would be faced with serious universality problems. However, the above scenario based on the $\langle A^2_{\min} \rangle$ condensate seems to avoid (at least formally) this issue.

In fact, the dependence of the $\langle A^2_{\min} \rangle$ condensate on the γ coupling could be measured directly. Namely, we could measure the quantity $\langle A^2 \rangle$ in the Landau gauge, and its drop with rising γ coupling gives an estimate for the behavior of the $\langle A^2_{\min} \rangle$ condensate when the violation of the Bianchi identities is gradually removed. Moreover, this could be compared with the results of Ref. [2], where the same Landau gauge $\langle A^2 \rangle$, albeit with different normalization, was measured across the finite-temperature deconfinement phase transition. Note that the quantity $\langle A^2 \rangle$ in the Landau and Coulomb gauges was already introduced in Refs. [76,77]. The details of our measurements are as follows. The gauge potentials are defined in terms of the link matrices $U_\mu(x)$

$$A^a_\mu = \mathrm{Tr} \frac{\sigma^a}{2ia}[U_\mu(x) - U^\dagger_\mu(x)], \tag{54}$$

where a is the lattice spacing. The Landau gauge was fixed by minimizing $\sum_{x,\mu}(A^a_\mu(x))^2$ with an overrelaxation algorithm until the magnitude of $\partial_\mu A^a_\mu$ becomes everywhere less than 10^{-6}. The results are presented in Fig. 12. One can see that in three dimensions $\langle A^2 \rangle$ is indeed almost insensitive to the γ coupling, confirming the qualitative scenario outlined above. On the other hand, in four dimensions $\langle A^2 \rangle$ drops down with increasing γ by essentially the same amount which was reported in Ref. [2]. Note that the relative drop of $\langle A^2 \rangle$ is expected to be small [1,2]. Indeed, on general grounds we have

$$\langle A^2 \rangle = \frac{1}{a^2}\left(\sum_n b_n \alpha_s^n + a^2 \langle A^2_{\min} \rangle\right),$$

and clearly the Landau gauge $\langle A^2 \rangle$ is dominated by the perturbative tail at weak coupling. However, the drop in $\langle A^2 \rangle$ across the phase transition is believed to be entirely due to the nonperturbative condensate $\langle A^2_{\min} \rangle$.

The next comment concerns the behavior of the topological charge with respect to the degenerate points suppression. It is true that the violation of the reflection positivity with rising $\tilde{\gamma}$ coupling is to be expected on general grounds. Moreover, it is also expected that in $D = 4$ the nonzero in average global topological charge is the origin of the reflection positivity violation. However, the following questions remain: Why is the topological charge

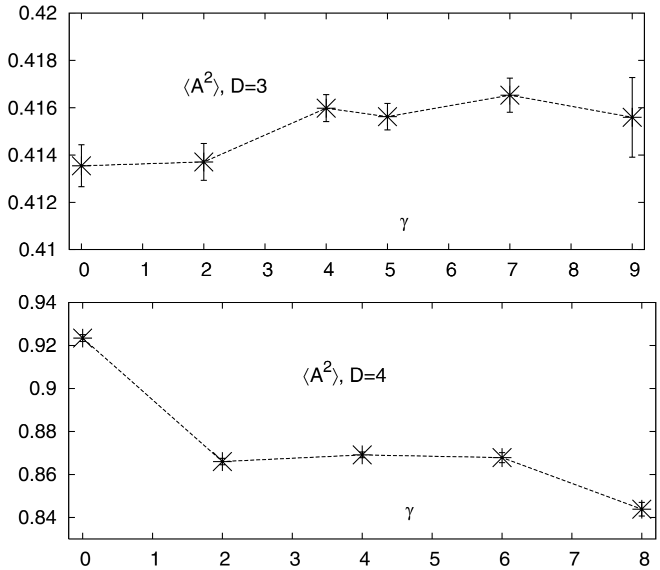

FIG. 12. The quantity $\langle A^2 \rangle$ in the Landau gauge at $\tilde{\gamma} = 0$ as function of γ in three and four dimensions.

always positive and rises so rapidly with respect to the Monte Carlo updating? What is the relation between Q and the magnetic charges q, $\tilde{q}$?

Evidently, the nonzero in average value of the topological charge requires it to be mostly either positive or negative. At the same time, our experience with various possible definitions of the last term in the action (49) shows that the positivity of Q at $\tilde{\gamma} \neq 0$ is seemingly built into our approach from the very beginning. As far as we can see, the only place which distinguishes between $Q \gtrless 0$ is the canonical orientation of the elementary plaquettes which we accepted (see Sec. III B). Indeed, the construction of the magnetic charges q, $\tilde{q}$ depends on the particular canonical orientation which is not uniquely defined. Although there are only a few possibilities to choose from, different choices could discriminate the sign of the topological charge. Indeed, in the fermionic language the sign of Q distinguishes the left and the right chiralities (orientations) analogously to the canonical orientation, which discriminates the left and the right coordinate systems. Thus, we expect that the sign of Q at nonzero $\tilde{\gamma}$ will change with inequivalent choice of the canonical orientation. As far as the rapid growth of Q is concerned, it seems that the only possible explanation is that the nonzero $\tilde{\gamma}$ coupling lifts the degeneracy of different topological sectors. The situation is reminiscent to the quantum mechanical problem of the periodic potential on which the constant $\tilde{\gamma}$-dependent electric field is superimposed.

It would be instructive to have the explicit expression for Q in terms of the q, $\tilde{q}$ magnetic charges distribution. However, the usual approaches available in the literature (see, e.g., Refs. [78,79]) seemingly lead to erroneous results. For instance, the treatment of Ref. [79] applies almost literally in our case. The outcome is that the topological charge is given by the linear combination of the q, $\tilde{q}$ magnetic charges and, hence, vanishes when q, $\tilde{q}$ are highly suppressed. This seems to contradict the observed rapid growth of Q with rising γ, $\tilde{\gamma}$ couplings when all the q, $\tilde{q}$ charges are suppressed on equal footing. Therefore, either the results of Ref. [79] should be modified in our case or we should look for a different definition of the topological charge. The definition of the topological charge, which closely follows the approach of the present work, could be given along the lines of Refs. [80,81] (see also Ref. [82] for an excellent introduction). Instead of studying the evolution of a single spinor along the closed contours, we could consider the corresponding evolution of the degenerate two-level system for which the resulting non-Abelian geometrical phase describes the YM instanton. This approach is under investigation and will be published elsewhere.

VII. CONCLUSIONS

In this paper, we considered the non-Abelian Bianchi identities in SU(2) pure Yang-Mills theory, focusing on the physical significance of the chromomagnetic fields degeneracy points and the possibility of Bianchi identities violation. These questions necessitate regularization, and we specifically kept in mind the lattice formulation. It had been known for a long time that the Bianchi identities, in general, are the requirement that the gauge holonomy for any null-homotopic path equals unity. The main achievement of this paper is the reformulation of the above requirement in terms of the physical elementary fluxes (field strength). Our approach is based on the non-Abelian Stokes theorem that appeared recently and allows one to give an explicit gauge invariant expression for the Bianchi identities on the lattice. Simultaneously, it allows one to formulate the notion of the non-Abelian Bianchi identities violation in gauge invariant and local form.

As a further development of our approach, we showed that the study of the lattice Bianchi identities naturally leads to the consideration of the chromomagnetic fields degeneracy points at which particular determinants constructed from E^a_i, B^a_i vanish. It turns out that the violation of the Bianchi identities and the degenerate points are closely related to each other. In particular, in the weak coupling regime the Bianchi identities violation is not related generically to the singular fields; rather, it is due to the existence of the degenerate points.

As is clear from the above presentation, the main advantage of our approach is that the non-Abelian nature of the theory had been traded for the complicated geometry which, however, allows the pure geometrical Abelian-like treatment. Then both the Bianchi identities violation and the degeneracy points formally appear as usual magnetic charges. However, we stress that the term "magnetic charge" and, in fact, the entire Abelian analogy, is only formal. In particular, the physical interpretation of q, $\tilde{q}$

charges is completely different; there is no magnetic charge conservation whatsoever on the original (hyper)-cubical lattice. Nevertheless, the Abelian-like representation is invaluable for the analysis presented above.

The locality and gauge invariance of the definition of the Bianchi identities and the chromomagnetic fields degeneracy points permits us to modify the original gauge action and to study the effects of gradual removal of these objects from the vacuum. It turns out that, in the four-dimensional case, the suppression of the Bianchi identities violation seems to be relevant for confinement: The heavy quark potential extracted from Wilson loops flattens at large distances and the correlator of the Polyakov lines tends to nonzero constant at large separations. At least, this is the case on the lattices we have studied. At the same time, other correlation functions which we measured had not been changed considerably. The situation in $D = 3$ turns out to be just the opposite. Namely, the theory is almost insensitive to the suppression of the Bianchi identities violation. However, in $D = 4$ the complexity of the numerical simulations precluded us from studying the relevant issues such as the phase diagram of the modified model, the volume dependence of our results, etc. We hope to address these questions elsewhere.

As far as the degenerate points are concerned, any attempt to remove them from the vacuum results in the reflection positivity violation. Moreover, in $D = 4$ this violation is due to the extremely large positive global topological charge which grows rapidly during Monte Carlo updating. This observation could be relevant for studying the gluodynamics in the topologically nontrivial sectors.

Confronting the results obtained in $D = 3, 4$, we argued that it is probably misleading to consider the violation of the Bianchi identities as the primary cause of confinement. Instead, the correct picture would be to interpret the Bianchi identities as an algebraic constraint on the gauge potentials and to relate the confinement phenomenon to the existence of the nonperturbative $\langle A^2_{\min} \rangle$ condensate. This scenario seems to be in agreement with universality expectations, works the same in both three and four dimensions, and does not contradict our findings.

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APPENDIX

Here we describe the cell complex underlying the Bianchi identities (30). We start from a single plaquette and note that the application of the non-Abelian Stokes theorem (27) assigns spinor wave function $\langle z|$ to each plaquette corner. This could be represented by 4 points belonging to this plaquette and shifted from the corners towards the plaquette center. The totality of these points constitutes the 0-skeleton $\mathbb{C}^0$ of the cell complex and it is convenient to parametrize $s \in \mathbb{C}^0$ by the point x of the original lattice and by two shifts with corresponding shift directions (see Fig. 5, left):

$$s = s(x, \mu, d_\mu, \nu, d_\nu) = s(x, \nu, d_\nu, \mu, d_\mu), \qquad \mu \neq \nu, \quad d_\mu, d_\nu = \pm 1. \tag{A1}$$

In total there are $2D(D-1) \cdot V$ sites, where V is the lattice volume.

Turn now to the 1-skeleton $\mathbb{C}^1$ which consists of two types of links. The first group contains the original links

$$s_i = s(x, \mu, d_\mu, \nu, d_\nu) \rightarrow s_f = s(x + \hat{\mu}, \mu, -d_\mu, \nu, d_\nu), \tag{A2}$$

which carry the matrix element $\langle z(s_i)| U_\mu(x) | z(s_f)\rangle$. Links from the second group

$$s_i = s(x, \mu, d_\mu, \nu, d_\nu) \rightarrow s_f = s(x, \lambda, d_\lambda, \nu, d_\nu) \quad \mu \neq \lambda \tag{A3}$$

are ascribed with the matrix element $\langle z(s_i) | z(s_f)\rangle$.

As far as the 2-skeleton $\mathbb{C}^2$ is concerned, its structure is different in three and four dimensions. As a consequence, $\mathbb{C}^k$, $k > 2$, also differ considerably and are described separately below.

$D = 3$

Here $\mathbb{C}^2$ contains three types of 2-cells. First, there are original plaquettes p, the boundary of which consists of the links (A2). Moreover, the standard coboundary operator $d{:}\mathbb{C}^1 \rightarrow \mathbb{C}^2$ acts in accordance with Eq. (25) and assigns the flux magnitude $\Phi(p)$ to the plaquette. Second, a group of 2-cells contains various squares $S^{(1)}(x, \mu, d_\mu)$ constructed from links (A3). Namely, $S^{(1)}(x, \mu, d_\mu)$ has the following four points in its boundary:

$$S^{(1)}(x, \mu, d_\mu){:} s(x, \mu, d_\mu, \nu, \pm d_\nu), s(x, \mu, d_\mu, \lambda, \pm d_\lambda), \quad \mu \neq \nu \neq \lambda. \tag{A4}$$

The coboundary operator $d{:}\mathbb{C}^1 \rightarrow \mathbb{C}^2$ assigns the Bargmann invariant (28) to the particular square. The argumentation of Sec. III D allows one to show that

$$S^{(1)}(x, \mu, 1) = S^{(1)}(x + \hat{\mu}, \mu, -1), \tag{A5}$$

where we have denoted the values assigned to each square by the same symbol "$S^{(1)}$," hoping that this will not lead to confusion. A third type of 2-cells contains various triangles $\mathcal{T}^{(1)}$ constructed from links (A3); there are three points in the boundary of $\mathcal{T}^{(1)}$

$$\mathcal{T}^{(1)}: s(x, \mu, d_\mu, \nu, d_\nu), s(x, \nu, d_\nu, \lambda, d_\lambda), s(x, \lambda, d_\lambda, \mu, d_\mu), \quad \mu \neq \nu \neq \lambda. \tag{A6}$$

The operator $d: \mathbb{C}^1 \to \mathbb{C}^2$ assigns the corresponding Bargmann invariant to the triangle. Note that the last group of 2-cells is formed by a mixture of links (A2) and (A3) and need not be considered; in fact, by Eq. (26) the phase associated with them is always zero. It is important that the value assigned by d to every 2-cell is always taken modulo 2π and is rather similar to $\theta_{\rm plaq} = [d\theta]_{2\pi}$ in the language of a compact U(1) gauge model. In other words, it is silently assumed that only gauge invariant quantities are ascribed to every 2-cell.

As far as the 3-skeleton $\mathbb{C}^3$ is concerned, it contains essentially two types of 3-cells. First, the original lattice cubes which look as in Fig. 4 (right); each cube contains 6 plaquettes and 8 triangles at its corners. The coboundary operator $d: \mathbb{C}^2 \to \mathbb{C}^3$ considered for any particular cube is identical to Eq. (30) by construction. The 3-cells of the second group are constructed entirely from triangles $\mathcal{T}^{(1)}$ and squares $S^{(1)}$ above and are illustrated in Fig. 5 (right). The physical meaning of the corresponding magnetic charge is analyzed in Sec. IV B.

Thus, the consideration of the three-dimensional case is completed. Note that geometrically there is one more type of 3-cells, which, however, need not be taken into account. These 3-cells are formed by two squares (A4) and four links (A2) connecting them. It follows from (26) and (A5) that $d: \mathbb{C}^2 \to \mathbb{C}^3$ always gives zero on these cells.

$D = 4$

In four dimensions the consideration of the cell complex underlying the lattice Bianchi identities (30) becomes cumbersome. In particular, we do not give the full list of cells forming $\mathbb{C}^k$, $k = 2, 3, 4$; only cells relevant to the considerations in Secs. IV B and V are presented.

First, we note that the $D = 3$ construction applies directly in $D = 4$. In particular, the 2-skeleton includes the plaquettes, squares (A4), and triangles (A6), trivially generalized to four dimensions. In the 3-skeleton $\mathbb{C}^3$ we identify then the usual 3-cubes and 3-cells shown in Fig. 5 (right).

However, it is clear that in $D = 4$ the $\mathbb{C}^2$, $\mathbb{C}^3$ are not exhausted by the above 2- and 3-cells. In particular, the 2-skeleton contains now an additional set of triangles $\mathcal{T}^{(2)}$ with vertices

$$\mathcal{T}^{(2)}: s(x, \mu, d_\mu, \nu, d_\nu), s(x, \mu, d_\mu, \lambda, d_\lambda), s(x, \mu, d_\mu, \rho, d_\rho), \quad \mu \neq \nu \neq \lambda \neq \rho, \tag{A7}$$

and squares $S^{(2)}$, the vertices of which are

$$S^{(2)}: s(x, \mu, d_\mu, \nu, d_\nu), s(x, \nu, d_\nu, \lambda, d_\lambda), s(x, \lambda, d_\lambda, \rho, d_\rho), s(x, \rho, d_\rho, \mu, d_\mu), \qquad \mu \neq \nu \neq \lambda \neq \rho. \tag{A8}$$

All these 2-cells are constructed from links (A3) and, therefore, are ascribed with the appropriate Bargmann invariants.

In the 3-skeleton $\mathbb{C}^3$ the new diamondlike cells $\mathcal{D}$ consisting of 6 vertices and 8 triangles appear. In turn, these 3-cells could be subdivided into two groups.

$\mathcal{D}^{(1)}(x, \mu, d_\mu)$.—The 3-cells in this group are constructed from 8 triangles (A7) and are similar to those considered in $D = 3$, Fig. 5 (right). In particular, one can show that [cf. Eq. (A5)]

$$\mathcal{D}^{(1)}(x, \mu, 1) = \mathcal{D}^{(1)}(x + \hat{\mu}, \mu, -1); \tag{A9}$$

see the note following Eq. (A5). The physical interpretation of the corresponding magnetic charge is discussed in Sec. IV B.

$\mathcal{D}^{(2)}(x, d_\mu, d_\nu, d_\lambda, d_\rho)$.—These 3-cells are built from both types of triangles (A6) and (A7). The corresponding vertices are constructed by fixing a particular combination of shift directions d_μ: There are 6 distinct planes passing through a given lattice site in $D = 4$, and $s(x, \mu, d_\mu, \nu, d_\nu)$,

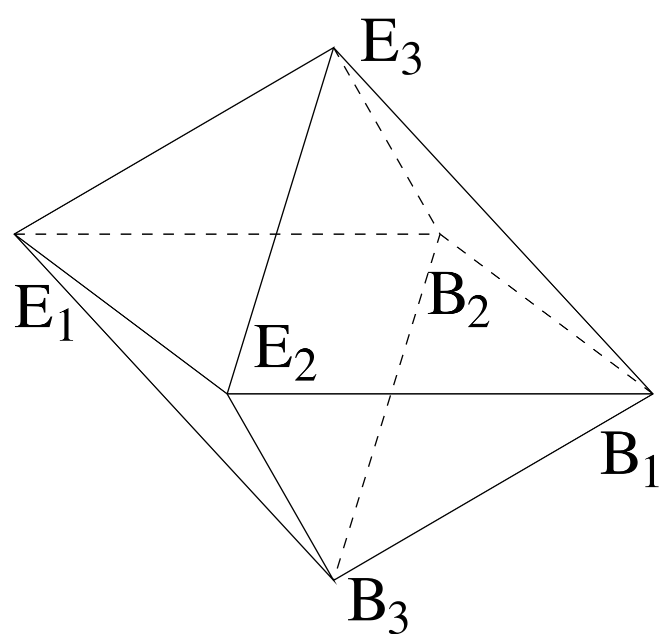

FIG. 13. $\mathcal{D}^{(2)}$ 3-cell in $D = 4$ (see the text).

$\mu \neq \nu$ is one of the six vertices of the $\mathcal{D}^{(2)}(x, d_\mu, d_\nu, d_\lambda, d_\rho)$ cell. The total number of these 3-cells per lattice site is $2^4 = 16$. Note the specific pattern of the flux directions assigned to the vertices of $\mathcal{D}^{(2)}(x, d_\mu, d_\nu, d_\lambda, d_\rho)$, which is radically different from what we have encountered so far. In the weak coupling limit the opposite vertices are ascribed with the same components of chromoelectric and chromomagnetic fields (see Fig. 13).

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