Divergent chiral condensate in the quenched Schwinger model

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We calculate numerically the eigenvalue distribution of the overlap Dirac operator in the quenched Schwinger model on a lattice. The distribution does not fit any of the three universality classes of spontaneous chiral-symmetry breaking, and its strong volume dependence indicates that the chiral condensate in the quenched theory is an ill-defined and divergent quantity. When we reweight configurations with the Dirac determinant to study the theory with $N_f = 1$, we obtain a distribution of eigenvalues that is well-behaved and consistent with the theory of explicit symmetry breaking due to the anomaly.

I. INTRODUCTION

Quantum electrodynamics in $(1 + 1)$-dimensions, the Schwinger model [1], continues to play an important role as testing ground for field theory ideas. In this paper we use the Schwinger model to study the quenched approximation for the chiral condensate. The quenched chiral condensate has long been believed, from various indirect arguments, to be an ill-defined quantity in gauge theories in any number of dimensions. Calculations in the Schwinger model are much easier than in higher-dimensional gauge theories, and this is our motivation for the present study. One must keep in mind, however, the fact that spontaneous chiral-symmetry breaking is prohibited in two dimensions by the Coleman-Mermin-Wagner theorem [2]. For this reason we also consider the unquenched theory, where chiral symmetry is broken explicitly by the anomaly so that the chiral condensate should be well defined.

The quenched approximation, in which the fermion determinant is discarded when generating the gauge field configurations, was first discussed analytically in this theory by van den Doel [3]. Some of the subtleties were subsequently discussed in Refs. [4,5]. As we shall review in the next section, the indications of disease in the quenched Schwinger model very much resemble those in higher-dimensional quenched theories, analyzed by means of quenched chiral perturbation theory [6,7]. This may not be surprising, since the bosonized form of the Schwinger model [8] is a two-dimensional analogue of a chiral Lagrangian. The trouble with the quenched chiral condensate then stems, in both contexts, from the famous double pole in the singlet correlation function [6,9]. When analyzed in the finite-volume ϵ-regime [10,11], the quenched chiral condensate is seen to be plagued with a "quenched finite-volume logarithm" at one-loop order in four dimensions [12]. Taken at face value, that is, if one were to push the expansion beyond its region of validity, this could indicate a divergent condensate. The analysis of the quenched chiral condensate in the ϵ-regime has guided us in our present finite-volume calculations.

There have been many Monte Carlo calculations of the chiral condensate in the Schwinger model, both quenched and unquenched (see [13–20] for recent work). The first to study the quenched condensate via the distribution of the lowest Dirac operator eigenvalues were Farchioni *et al.* [15]. They found quite an odd result—agreement with one universality class of spontaneous chiral-symmetry breaking at small volumes, and with another at larger volumes. We shall return to this issue in detail below. Kiskis and Narayanan [18] reconsidered the problem recently and found evidence for a *divergent* quenched chiral condensate, with Dirac eigenvalues that did not appear to fit any of the three possible chiral-symmetry breaking classes. We shall see how these last two papers can be reconciled. In the process we shall consider volumes far exceeding what has been studied earlier.

Our paper is organized as follows. In the next section we briefly review the issues surrounding the quenched chiral condensate, particularly in this two-dimensional setting, and show how analytical arguments favor an ill-defined, divergent quantity. In Sec. III we turn to our Monte Carlo simulations of the quenched theory. We compare numerical results for the distribution of Dirac operator eigenvalues with distributions based on the three possible classes of spontaneous chiral-symmetry breaking. We show how the distributions drift as the volume is changed, and how this explains the results of Ref. [15]. Moreover, we find that the distributions appear not to converge to any fixed limit, indicating that the spectral density $\rho(\lambda)$ does not attain a finite value at $\lambda = 0$. By way of contrast, we present in Sec. IV numerical results for the unquenched theory ($N_f = 1$), obtained by reweighting the path integral with the Dirac determinant. Our results there are consistent with the detailed predictions of the ϵ-regime based on the explicit breaking of chiral symmetry due to the anomaly [11]. Section V contains a brief summary of our results.

II. THE QUENCHED SCHWINGER MODEL

Analytically, one can treat quenching by means of the replica method that considers N_f identical fermion copies, and then sends $N_f \to 0$ at the end. This can be done trivially in perturbation theory at the fundamental level; in more than two dimensions it can be done in chiral perturbation theory [7]. Because the fermion determinant is exactly calculable in two dimensions, it can also be done beyond perturbation theory in the Schwinger model. This was first realized by van den Doel [3], and we shall here briefly review his calculation (see also [4,5]). At the same time we shall make contact with the very similar calculation in four-dimensions, in quenched chiral perturbation theory based on the replica formulation.

In the continuum theory, we consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \sum_{i=1}^{N_f}\bar{\psi}_i(i\slashed{\partial} + m - g\slashed{A})\psi_i$$
$$- \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

with gauge group U(1), coupling g, and N_f species of fermions. A two-dimensional θ-term has been included as well. Since the fermion determinant is exactly calculable in two dimensions, the theory is in large measure soluble. A convenient representation of the model is its bosonized form [8] where the Lagrangian density, upon an exact integration over the gauge potential, takes the form

$$\mathcal{L} = \sum_{j=1}^{N_f}\frac{1}{2}\partial_\mu\phi_j\partial^\mu\phi_j - \frac{g^2}{2\pi}\left(\sum_{j=1}^{N_f}\phi_j + \frac{\theta}{2\sqrt{\pi}}\right)^2$$
$$+ cm^2\sum_{j=1}^{N_f}\mathcal{N}\cos(2\sqrt{\pi}\phi_j). \qquad (2)$$

Here $\mathcal{N}$ denotes normal ordering and $c = e^\gamma/2\pi$ where γ is Euler's constant.

Although there is no spontaneous chiral-symmetry breaking here, the bosonized action (2) bears a strong resemblance to chiral Lagrangians in four dimensions, with $f = 1/\sqrt{\pi}$ playing the role of a (dimensionless) pion decay constant [21]. This becomes particularly clear when we consider the theory in an expansion around static (zero-momentum) modes. If we define the partition function in a sector of fixed topological charge ν by means of [11]

$$\mathcal{Z}_\nu \equiv \frac{1}{2\pi}\int_0^{2\pi}d\theta e^{i\nu\theta}Z(\theta), \qquad (3)$$

the integral over θ can be performed exactly. Upon defining the $N_f \times N_f$ matrix $U \equiv \text{diag}\{\exp(i2\sqrt{\pi}\phi_j)\}$, the terms that survive in the static limit yield a Boltzmann factor

$$(\det U)^\nu \exp[-\frac{c}{2}Vm^2\mathcal{N}\text{Tr}(U + U^\dagger)],$$

in analogy with the result in 4 dimensions [11] (here V denotes the finite two-dimensional volume). Note, however, that the mass term is proportional to m^2 rather than to m as in four dimensions. This is crucial for understanding the difference with respect to spontaneous breaking of chiral symmetry.

The analogy to a four-dimensional chiral Lagrangian holds to any order in chiral perturbation theory. In particular it is useful for understanding the quenched limit of the theory. The mass term and its normal-ordering prescription complicate matters slightly, and it suffices to consider the massless limit. In that limit we read off the diagonal scalar propagator from Eq. (2),

$$\mathcal{G}(p^2) = \frac{1}{p^2} - \frac{g^2}{\pi}\frac{1}{p^2(p^2 + N_f g^2/\pi)}, \qquad (4)$$

which reduces to the classic Schwinger result of an ordinary massive propagator when $N_f = 1$ [4]; in the quenched limit ($N_f = 0$) a double pole develops,

$$\mathcal{G}(p^2) = \frac{1}{p^2} - \frac{g^2}{\pi}\frac{1}{(p^2)^2} \qquad (N_f = 0), \qquad (5)$$

as first noted by van den Doel [3]. This phenomenon is completely analogous to what happens in quenched chiral perturbation theory in four dimensions when formulated in terms of replicas; see the appropriate replica Feynman rule in Eq. (7) of Ref. [7]. The only difference is the replacement of the Schwinger mass parameter $\mu^2 = g^2/\pi$ by what is there commonly normalized as μ^2/N_c, where N_c is the number of colors. (The double pole had of course been observed much earlier in quenched chiral perturbation theory by means of a supersymmetric extension [6,9].)

The above description of the diagonal scalar propagator glosses over the fact that in two dimensions the propagation of massless degrees of freedom requires an infrared regularization. This is true even when the Schwinger mass $\mu = g/\sqrt{\pi}$ is taken into account because of the remaining $1/p^2$-poles in the propagator (4) when $N_f \to 0$. After regularizing this infrared divergence by an additional mass parameter m_{IR}, the calculation of the chiral condensate requires integration over a closed loop of the propagator

$$\tilde{\mathcal{G}}(p^2) = \frac{p^2 + (N_f - 1)\mu^2}{(p^2 + m_{IR}^2)(p^2 + N_f\mu^2 + m_{IR}^2)} - \frac{1}{p^2 + M^2}, \qquad (6)$$

where M is the arbitrary mass defining the normal-ordering prescription [8]. From here the quenched chiral condensate has been computed [3–5] to give

$$\langle \bar{\psi}\psi \rangle = -\lim_{\substack{m_{IR} \to 0 \\ N_f \to 0}} cM \langle \mathcal{N} \cos[2\sqrt{\pi}\phi_k] \rangle$$

$$= -\lim_{\substack{m_{IR} \to 0 \\ N_f \to 0}} cm_{IR} \left(1 + \frac{N_f \mu^2}{m_{IR}^2} \right)^{1/2N_f}. \qquad (7)$$

For fixed infrared cutoff m_{IR}, the limit $N_f \to 0$ yields [4]

$$\langle \bar{\psi}\psi \rangle = -\lim_{m_{IR} \to 0} cm_{IR} e^{\mu^2/2m_{IR}}, \qquad (8)$$

which is infrared divergent. This ordering of limits seems closest to the actual "physical" (i.e., computational) definition of the quenched theory, and we shall view it as the simplest manifestation of the difficulty with defining the quenched Schwinger model. Since the result is divergent, and since the computation has been done based on an expansion around the massless theory, one can question to what extent the resulting divergence is a truly reliable prediction. More detailed computations at fixed V can be found in Refs. [10,22].

III. MONTE CARLO ANALYSIS

We consider a lattice with volume $(La)^2$ and employ a noncompact formulation for the gauge field. In a sector with topological charge Q, the gauge field A_μ may be decomposed [23] as

$$A_1(x) = \partial_2^* \phi(x) + \frac{2\pi}{L} h_1 - \partial_1 \alpha(x), \qquad (9)$$

$$A_2(x) = -\partial_1^* \phi(x) + \frac{2\pi}{L} h_2 - \partial_2 \alpha(x) - \frac{2\pi Q}{L^2}. \qquad (10)$$

Here $\phi(x)$ is a real periodic function with no zero mode, and h_μ are two real constants in the interval $(-1/2, 1/2)$ that parametrize the two Polyakov loops on the 2-d lattice. ∂_μ and ∂_μ^* are forward and backward finite differences; $\alpha(x)$ represents the gauge degree of freedom. The gauge action is then

$$S_G = \frac{1}{4g^2} \sum_x F_{\mu\nu}^2(x) \qquad (11)$$

$$= \frac{1}{2g^2} \sum_x [\Delta \phi(x)]^2 + \frac{2\pi^2 Q^2}{(gL)^2}, \qquad (12)$$

where

$$\Delta \phi(x) = \sum_\mu [\phi(x+\mu) + \phi(x-\mu) - 2\phi(x)]. \qquad (13)$$

We denote the gauge coupling by $\beta = 1/(2g^2 a^2)$. The continuum limit may be taken at fixed finite g and at fixed volume by taking β and L to infinity. Alternatively, we can keep g and L fixed and vary the physical volume by changing β.

Our choice of the noncompact action is motivated mainly by the ease with which we can generate independent gauge configurations with the desired topological charge. Successive configurations are generated by a heat bath so that there is no autocorrelation. We restrict ourselves to the sector with zero topological charge.

We employ lattices of linear size L ranging from 8 to 60, and fix $\beta = 2$. We also have one data set with $\beta = 1$ and $L = 48$; if we renormalize at fixed g as discussed above then this is equivalent to $L = 48\sqrt{2} \simeq 68$ at $\beta = 2$, allowing us a larger physical volume at modest additional cost.

We use the massless overlap-Dirac operator [24] for the fermions,

$$D = \frac{1}{2}[1 + \gamma_5 \epsilon(H_W)]. \qquad (14)$$

Here $H_W = \gamma_5 D_W(-1)$ is the hermitian Wilson–Dirac operator with mass parameter set to -1. With this normalization the eigenvalues of the hermitian overlap-Dirac operator $H = \gamma_5 D$ lie in the interval $[-1, 1]$, but there is a wave-function renormalization $Z_\psi = 2$ with respect to the conventional normalization of the Dirac operator [25] that one needs to keep in mind.

For $L \leq 32$ we diagonalize H_W exactly by a Householder transformation followed by QL iteration [26]. From this we construct H, whose eigenvalues we obtain similarly. For the larger lattices we compute the lowest eigenvalues of H^2 with the Ritz variational algorithm of Kalkreuter and Simma [27]. Here the action of $\epsilon(H_W)$ on a vector is obtained using a modified version of the two-pass algorithm [28]. In the first pass, the Lanczos algorithm is used to obtain a tridiagonal matrix, T_W, that is a good approximation to H_W. Then $\epsilon(T_W)$ is obtained by an exact diagonalization of T_W, and the second pass is used to compute the action of $\epsilon(T_W)$ on a vector.

Previous papers have compared the eigenvalue distributions to the predictions of random matrix theory (RMT), and we shall proceed to do the same. In higher dimensions, where spontaneous chiral-symmetry breaking is allowed, it is by now well known that the lowest eigenvalues of the Dirac operator in the ϵ-regime are distributed according to universal finite-size scaling distributions that can be derived either from RMT [29,30] or directly from the chiral Lagrangian framework [31,32]. There is ample evidence that the universality class of these distributions is dictated by the way the fermions transform under the gauge group. This has been demonstrated for three different gauge group representations with overlap fermions [33], and for a variety of exotic representations with staggered fermions [34].

According to this classification, if chiral-symmetry could be spontaneously broken in the Schwinger model,

where the fermions transform as a complex representation, the relevant universality class in RMT terms would be that of the chiral unitary ensemble (chUE). It is by no means clear, however, whether these predictions are of any relevance to the quenched Schwinger model, where spontaneous chiral-symmetry breaking is prohibited [2]. Reference [15] included such a comparison, but the results were not easy to understand. For small lattices the eigenvalue distributions appear to fall in with the chiral symplectic ensemble (chSE), which should be of relevance to *real* fermion representations. For larger lattices there appears to be a switch to the perhaps more natural chUE. With our greater statistics and our larger lattices, we are now able to see that such surprising results do not really hold.

We begin with attempts to fit the distribution of the lowest eigenvalue to the form predicted by RMT for the quenched ($N_f = 0$) chUE [35],

$$p_1(\xi_1) = \frac{1}{2}\xi_1 e^{-\xi_1^2/4}, \qquad (15)$$

where $\xi_1 = \lambda_1 \Sigma L^2$ and the condensate Σ is the fit parameter. We compare to histograms with 20 bins in ξ_1. The comparisons are shown in Fig. 1 for nine volumes, and the results of the fits to the RMT prediction are listed in Table I.

$L = 24$ is the case that comes closest to agreement with the chUE prediction, but even here our fit gives a χ^2/dof of 58.1/19 corresponding to a confidence level of 7.6×10^{-6}. Our high-statistics data enable us to rule out the chUE scenario here. As can be seen from Table I, for all other lattice sizes the RMT fits to the chUE predictions are ruled out even more thoroughly.

Farchioni *et al.* [15] found agreement with the chUE prediction for large volumes, in particular, for $L = 16$ and $\beta = 1$ (see their Fig. 6; their definition of β is twice ours).
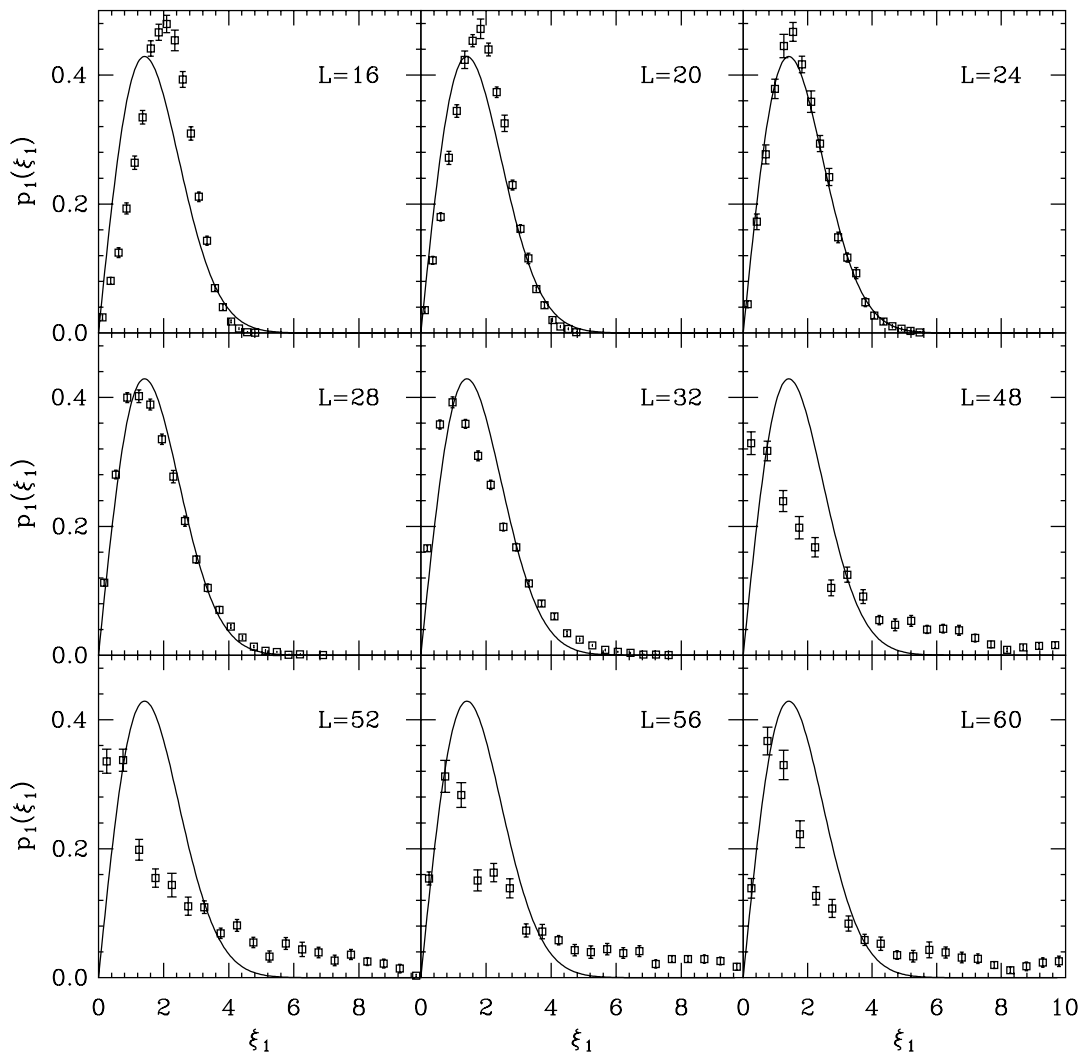

FIG. 1. Comparison of the distribution of the rescaled lowest eigenvalue of the quenched Schwinger model with the prediction from RMT for the quenched chUE. All ensembles here have $\beta = 2$.

TABLE I.   Number of configurations studied for each volume $V = L^2$, the condensate Σ, the χ^2, number of degrees of freedom, and confidence level from fits of the lowest eigenvalue distribution to the RMT form of the chUE with 20 histogram bins. The last line is for $\beta = 1$, all others for $\beta = 2$.

L	N_{meas}	Σ	χ^2	dof	CL
8	1000	0.2221(25)	415.5	19	2.6×10^{-76}
12	10000	0.1705(6)	3278.	19	$<10^{-100}$
16	10000	0.1752(4)	2801.	19	$<10^{-100}$
20	12440	0.1632(5)	1024.	19	$<10^{-100}$
24	4880	0.1679(11)	58.14	19	7.6×10^{-6}
28	9360	0.1879(11)	204.9	18	1.1×10^{-33}
32	12320	0.2242(14)	1223.	19	$<10^{-100}$
48	1660	0.828(19)	945.4	17	$<10^{-100}$
52	1280	1.164(16)	714.9	17	$<10^{-100}$
56	1320	2.115(29)	894.7	18	$<10^{-100}$
60	1020	2.36(11)	929.8	19	$<10^{-100}$
48	860	3.921(42)	457.4	17	1.7×10^{-86}

This is the same physical volume as $L \simeq 23$ and $\beta = 2$. As we stated above, we rule out the chUE at this volume and at all other volumes studied.[1] For smaller volumes, Farchioni *et al.* favor the chSE, claiming, in particular, a good fit at $L = 16$ for $\beta = 2$. A fit of our high-statistics $L = 16$ data to the quenched chSE distribution gives $\chi^2/\text{dof} = 436/19$, ruling it out as well.

As seen in Fig. 1, the peak of the distribution of the lowest eigenvalue moves downward faster than $1/L^2$, the RMT prediction, while a sizable tail of the distribution persists. The RMT scaling law is based on the assumption of a finite eigenvalue density $\rho(0)$ at the origin; a scaling faster than $1/L^2$ thus entails a divergence in $\rho(0)$ as $L \rightarrow \infty$. By the Banks-Casher relation $\Sigma = \pi\rho(0)$, this in turn implies a divergent chiral condensate. Indeed, from Table I we see that the fitted Σ grows quite rapidly with increasing L.

We now elaborate on this last point. Let $\lambda_i(L, \beta)$ be the i-th lowest nonzero eigenvalue of H on an L^2 lattice at a coupling β. For a finite chiral condensate to form we expect that

$$f_i(L, \beta) \equiv \frac{1}{L^2 \langle \lambda_i(L, \beta) \rangle} \qquad (16)$$

approaches an i-dependent constant as $L \rightarrow \infty$,

$$\lim_{L \rightarrow \infty} f_i(L, \beta) = f_i(\beta). \qquad (17)$$

In lattice units we expect $f_i(\beta) \propto \Sigma$ [11]. Furthermore, the products $f_i(\beta)\sqrt{\beta}$ should approach finite continuum limits for $\beta \rightarrow \infty$.

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[1]Reference [15] describes a simulation of a compact gauge action, however, so the comparison between our calculations cannot be made precise.

The scaled variables $f_1(L, \beta)\sqrt{\beta}$ and $f_2(L, \beta)\sqrt{\beta}$ are plotted as functions of the physical size $L/\sqrt{\beta}$ in Fig. 2. (We combine data for $\beta = 1$. with data for $\beta = 2$.) We see that these quantities do not approach L-independent constants as one would expect on the basis of the existence of a finite chiral condensate Σ.

The distribution $p_2(\xi_2)$ of the *second* scaled eigenvalue (with Σ taken from Table I) for some large volumes are shown in Fig. 3 and compared with the predictions for the $(N_f = 0)$ chUE [35],

$$p_2(\xi_2) = \frac{1}{4} \xi_2 e^{-\xi_2^2/4} \int_0^{\xi_2} du\, u [I_2^2(u) - I_1(u)I_3(u)]. \quad (18)$$

Much like the distributions of ξ_1 above, these clearly do not fall in the universality class of the chUE. We can eliminate the scale $\Sigma(L, \beta)$ in these comparisons, by plotting the distribution of $r = \lambda_1(\beta, L)/\lambda_2(\beta, L)$. We compare this to the prediction from the chUE [35,36]

$$p(r) = \frac{1}{4} \frac{r}{(1 - r^2)^2} \int_0^{\infty} du\, u^3 \exp\left(-\frac{u^2}{4(1 - r^2)}\right)$$
$$\times [I_2^2(u) - I_1(u)I_3(u)], \qquad (19)$$

in Fig. 4. Again, the data do not fall in the universality class of the chUE.

Do the $p_i(\xi_i)$ reach some limiting distributions as $L \rightarrow \infty$? It is difficult to answer this question based on the data shown in Figs. 1 and 3. It is conceivable that $p_1(\xi_1)$ approaches a function peaked at zero while $p_2(\xi_2)$ reaches a limiting form peaked away from zero, but we have no real evidence for this. Whatever the answer, however, it is possible that the distribution $p(r)$ of the ratio *does* have a limiting distribution. Even though λ_1 and λ_2 go to zero much faster than $1/L^2$, we can discern some level repulsion that favors a ratio r between 0 and 1. Figure 5 shows
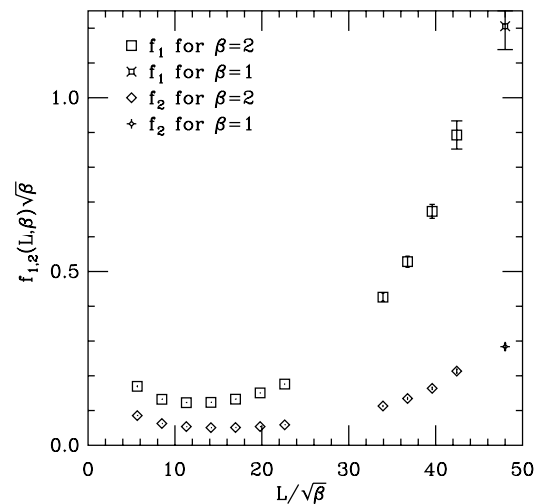

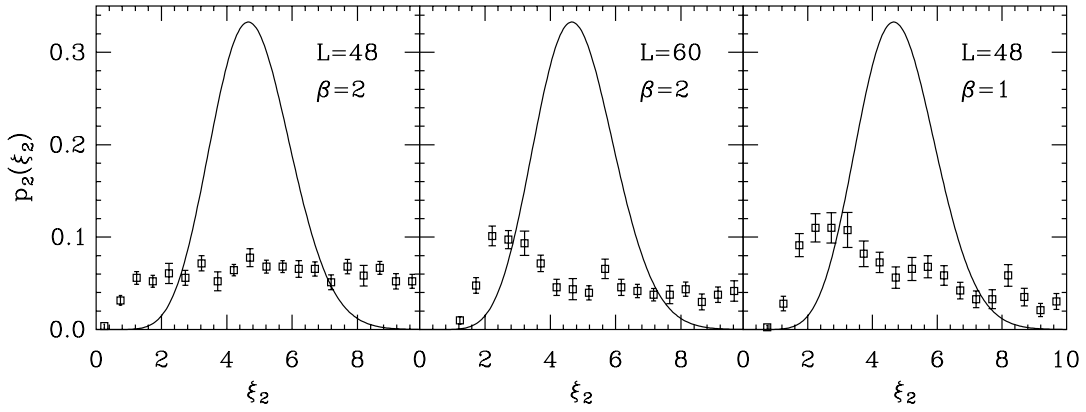FIG. 2.   Evidence for a diverging condensate in the quenched Schwinger model.

FIG. 3.   Distribution of the scaled second eigenvalue of the quenched Schwinger model.
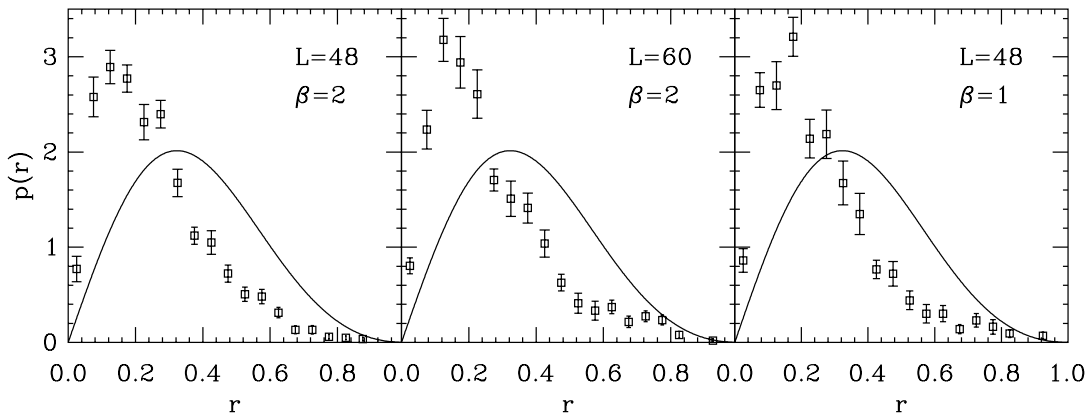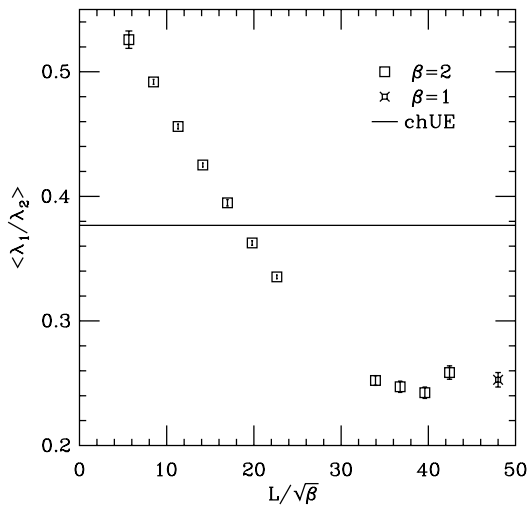

FIG. 4.   The distribution of the ratio of the two lowest eigenvalues in the quenched Schwinger model shows that they do not obey the universal distribution given by the chUE.

FIG. 5.   Expectation value of $r = \lambda_1(L, \beta)/\lambda_2(L, \beta)$ in the quenched Schwinger model as a function of the physical size. The data are not consistent with chUE but do seem to approach a finite limit as $L \to \infty$.

$\langle r \rangle$ as a function of the physical size and this average seems to approach a finite limit as $L \to \infty$.

IV. THE UNQUENCHED THEORY

When we compute all the eigenvalues of the Dirac operator on each pure gauge configuration, we can calculate observables in the theory with $N_f \neq 0$ by reweighting with the fermion determinant. We restrict ourselves to $N_f = 1$ since statistical fluctuations are worse when the target ensemble is farther from the original quenched ensemble. In the continuum $N_f = 1$ is of course Schwinger's original model [1], which is exactly soluble. In particular, the infinite-volume chiral condensate can be computed analytically, [8]

$$\Sigma = \frac{e^{\gamma} \mu}{2\pi} = \frac{g e^{\gamma}}{2\pi^{3/2}} = (0.1599\ldots)g, \tag{20}$$

in the conventional normalization, where $\mu = g/\sqrt{\pi}$ is the Schwinger mass and γ is Euler's constant. Because chiral symmetry is broken explicitly, the analysis of Leutwyler and Smilga [11] and the whole RMT analysis

TABLE II. Number of configurations studied for each volume $V = L^2$ with $\beta = 2$, reweighted to $N_f = 1$, the condensate Σ, the χ^2, number of degrees of freedom and confidence level from fits of the lowest eigenvalue distribution to the RMT form with 20 histogram bins.

L	N_{meas}	Σ	χ^2	dof	CL
12	10000	0.2168(6)	2158.	19	$<10^{-100}$
16	10000	0.2019(5)	2067.	19	$<10^{-100}$
20	12440	0.1781(5)	791.8	19	$<10^{-100}$
24	4880	0.1694(12)	60.48	19	3.2×10^{-6}
28	9360	0.1671(11)	45.98	18	3.0×10^{-4}
32	12320	0.1648(11)	16.73	19	0.61

for the $N_f = 1$ theory apply directly here. We therefore know the complete microscopic spectrum of the Dirac operator in the ϵ-regime and it belongs to the universality class of the chUE.

Once again, the simplest quantity with which to compare is the distribution of the smallest (nonzero) Dirac eigenvalue [35],

$$p_1(\xi_1) = \frac{1}{2}\xi_1 I_2(\xi_1)e^{-\xi_1^2/4}, \qquad (21)$$

here restricted to the massless case. We were able to reweight ensembles on lattices up to size $L = 32$. Again, we made fits to the RMT prediction, with Σ the fit parameter, using histograms with 20 bins. The fits are detailed in Table II and shown in Fig. 6. The agreement with the chUE is quite good for the two largest lattices. The second eigenvalue (Fig. 7) tells a similar story. We show the averaged ratio $\langle r \rangle$ in Fig. 8, which may be compared to Fig. 5 for the quenched theory. The result for $L = 32$ agrees with the prediction of the chUE, $\langle r \rangle = 0.5044$ [36].

Converting the result from Table II for the largest lattice size to the conventional normalization, we find
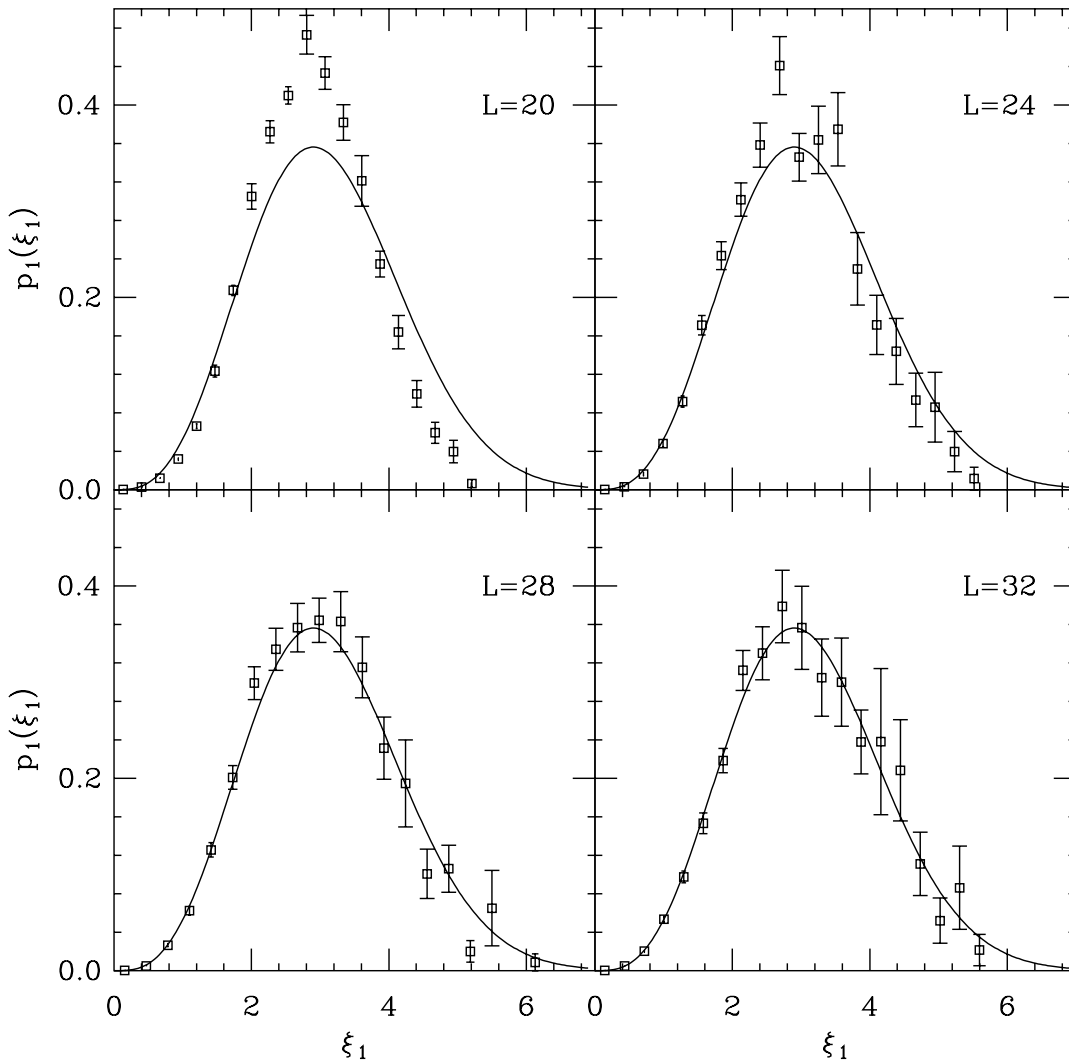

FIG. 6. Comparison of the distribution of the rescaled lowest eigenvalue for $N_f = 1$ with the prediction of RMT for the $N_f = 1$ chUE.
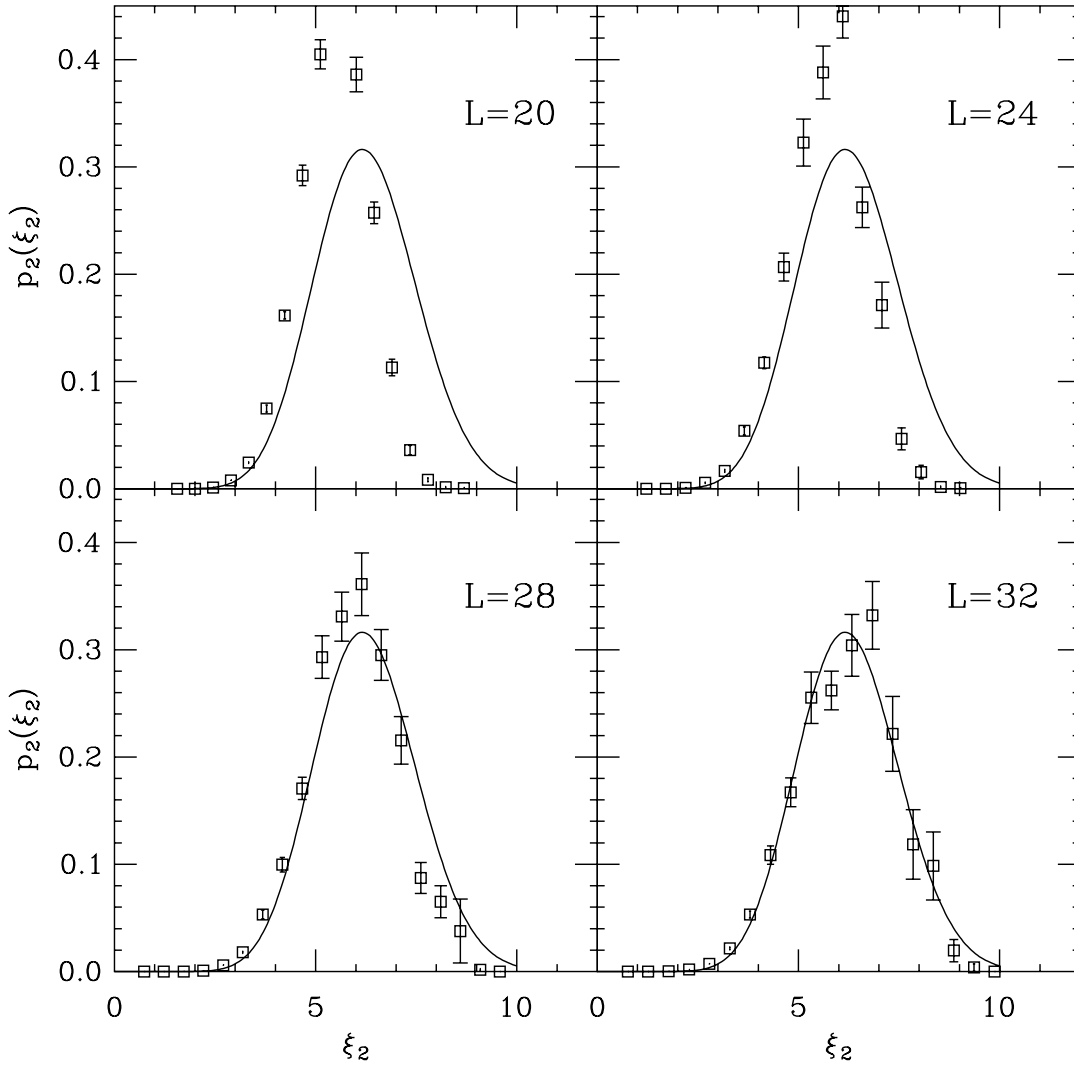
FIG. 7. Rescaled second eigenvalue for $N_f = 1$, compared with the prediction of RMT for the $N_f = 1$ chUE [see Eq. (15) in Ref. [36]].
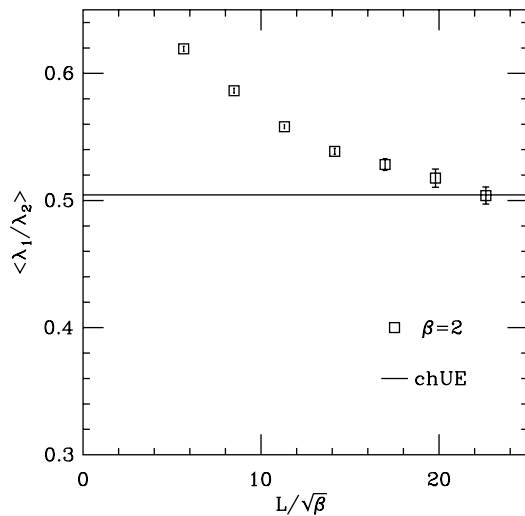

FIG. 8. Expectation value of $r = \lambda_1(L, \beta)/\lambda_2(L, \beta)$ for $N_f = 1$ as a function of the physical size.

$\Sigma/g = 0.1648(11)$, within 3% of the continuum value given in Eq. (20); finite lattice spacing corrections, expected to be of $\mathcal{O}(1/\beta)$, could easily explain the difference (see also [13]).[2] We note, however, that fits to the RMT distribution of the lowest eigenvalue do not work as well for the smaller lattices, $L \leq 24$, giving unacceptably small confidence levels. It would be interesting to see whether the agreement with RMT persists on larger lattices and whether the continuum value for Σ is correctly reproduced as $\beta \to \infty$. Unfortunately, reweighting becomes prohibitive for larger volumes. A direct numerical simulation of the $N_f = 1$ theory is probably needed.

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[2]The fit to $p_2(\xi_2)$ for $L = 32$ gives a value of Σ that is 7% larger than the value shown in Table II. This is consistent with wide experience that higher eigenvalues suffer larger finite-volume effects.

V. SUMMARY

As we have shown, the quenched Schwinger model does not fall into any of the three universality classes of chiral-symmetry breaking. In view of the Coleman-Mermin-Wagner theorem, which forbids spontaneous breaking of a continuous symmetry in two dimensions, this in itself is not very surprising. The only possible loophole out of this argument would be to note that the quenched theory is nonunitary and thus it might not satisfy some assumptions of the theorem.

The application of the Coleman-Mermin-Wagner theorem to the quenched theory may be considered in two ways. In the so-called supersymmetric formulation of quenching, spontaneous chiral-symmetry breaking is associated with quenched Goldstone bosons and Goldstone fermions. In the replica formulation, the spontaneous breaking is associated with Goldstone bosons alone. In both ways of considering quenching, the theorem seems to exclude rigorously the possibility of a nonzero condensate. In that light it is perhaps surprising that the numerical evidence now points towards an ill-defined, divergent chiral condensate, rather than a vanishing condensate.

The Schwinger model with $N_f = 1$ is on an entirely different footing due to the explicit breaking of chiral symmetry by the anomaly. Here we have unambiguous analytical predictions for the behavior of Dirac operator spectra near the origin, and our $N_f = 0$ results reweighted with the Dirac determinant to simulate the $N_f = 1$ theory are consistent with these analytical predictions. For the *massive* Schwinger model with $N_f = 1$, a disagreement with RMT must appear as the mass is taken to infinity. Our statistics have not been good enough to attempt the even more ambitious reweighting to simulate the $N_f = 2$ theory.

Also here unusual results should appear, presumably with the Dirac operator eigenvalues being strongly repelled by the origin so as to produce a vanishing $\rho(0)$. For $N_f = 2$ analytical calculations [10] suggest a behavior

$$\rho(\lambda) \sim \lambda^{1/3}, \tag{22}$$

and an interesting question is whether such behavior at the rescaled level has a universal distribution from ''critical'' Random Matrix Theories (the precise behavior (22) is actually realized in a very simple chiral matrix model [37]) or analogous eigenvalue models [38]. This could be an interesting topic for future investigations.

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