Note on the nonexistence of σ-model solitons in the 2 + 1 dimensional AdS gravity

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We show that the gravitating static soliton in the 2 + 1 dimensional $O(3)$ σ model does not exist in the presence of a negative cosmological constant, contrary to the claim made by Kim and Moon in [Phys. Rev. D **58**, 105013 (1998).].

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I. INTRODUCTION

The 2 + 1 dimensional $O(3)$ σ model coupled to gravity is a wave map $X:M \to N$ from a 2 + 1 dimensional spacetime (M, g_{ab}) into a two-sphere S^2 with the round metric G_{AB} defined by the action

$$S = \int_M \left(\frac{R + 2\Lambda}{16\pi G} + L_{WM} \right) dv_g \tag{1}$$

with the Lagrangian density

$$L_{WM} = -\frac{f_\pi^2}{2} g^{ab} \partial_a X^A \partial_b X^B G_{AB}. \tag{2}$$

Here Λ is a cosmological constant, G is the Newton constant, and f_π^2 is the wave map coupling constant. The product $\alpha = 8\pi G f_\pi^2$ is dimensionless. The field equations derived from (1) are the wave map equation

$$\Box_g X^A + \Gamma^A_{BC}(X) \partial_a X^B \partial_b X^C g^{ab} = 0, \tag{3}$$

where $\Gamma^A_{BC}(X)$ are the Christoffel symbols of the target metric G_{AB} and $\Box_g$ is the wave operator associated with the metric g_{ab}, and the Einstein equations $R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab}$ with the stress-energy tensor

$$T_{ab} = f_\pi^2 \left(\partial_a X^A \partial_b X^B - \frac{1}{2} g_{ab} (g^{cd} \partial_c X^A \partial_d X^B) \right) G_{AB}. \tag{4}$$

In polar coordinates $X^A = (F, \Phi)$ on the target S^2 the metric takes the form

$$G_{AB} dX^A dX^B = dF^2 + \sin^2 F d\Phi^2. \tag{5}$$

For the domain manifold we take a spherically symmetric 2 + 1 dimensional spacetime and parametrize the metric using areal coordinates

$$g_{ab} dx^a dx^b = -e^{-2\delta} A dt^2 + A^{-1} dr^2 + r^2 d\phi^2, \tag{6}$$

where δ and A are functions of (t, r). Next, we assume that the wave maps are corotational, that is

$$F = F(t, r), \qquad \Phi = \phi. \tag{7}$$

Equation (3) reduces then to the single semilinear wave equation (hereafter, primes and dots denote derivatives with respect to r and t, respectively)

$$-e^{\delta} (e^{\delta} A^{-1} \dot{F})^{\cdot} + \frac{e^{\delta}}{r} (r e^{-\delta} A F')' = \frac{\sin(2F)}{2r^2}, \tag{8}$$

and the Einstein equations become

$$\dot{A} = -\alpha r A \dot{F} F', \tag{9}$$

$$\delta' = -2\Lambda r - \alpha r (F'^2 + A^{-2} e^{2\delta} \dot{F}^2), \tag{10}$$

$$A' = -\alpha r \left(A F'^2 + A^{-1} e^{2\delta} \dot{F}^2 + 2 \frac{\sin^2 F}{r^2} \right). \tag{11}$$

The studies of the initial value problem for this system in the case of zero cosmological constant, performed first in the flat spacetime ($\alpha = 0$) [1] and recently also for $\alpha > 0$ [2], showed that the scale-free static solution plays a crucial role in the process of singularity formation, namely, singularities form via the static solution shrinking adiabatically to zero size. In fact, Struwe showed that for equivariant wave maps in the flat spacetime singularities must form in this way [3], in other words nonexistence of a nontrivial static solution implies global regularity. Thus, it seems interesting to see how the inclusion of a negative cosmological constant affects the structure of static solutions.

II. STATIC SOLUTIONS FOR $\Lambda = 0$

Before looking at the static solutions of Eqs. (8)–(11) with $\Lambda < 0$, in this section we review some well-known facts about static solutions for $\Lambda = 0$. We first consider the case $\alpha = 0$ which corresponds to the flat spacetime $A = 1$, $\delta = 0$ so Eq. (8) reduces to

$$\frac{1}{r} (r F')' = \frac{\sin(2F)}{2r^2}. \tag{12}$$

The trivial constant solutions of (12) are $F = 0$ and $F = \pi$; geometrically these are maps into the north and the south pole of S^2, respectively. The energy of these maps

$$E(F) = \pi \int_0^\infty \left(F'^2 + \frac{\sin^2 F}{r^2} \right) r dr \tag{13}$$

attains the global minimum $E = 0$. Note that the requirement that energy be finite imposes a boundary condition at spatial infinity $F(\infty) = k\pi$ $(k = 0, 1, \ldots)$ which compactifies $\mathbb{R}^2$ into S^2 and thereby breaks the phase space into infinitely many disconnected topological sectors labeled by the degree k of the map $S^2 \to S^2$.

The fact that Eq. (12) is scale invariant does not exclude nontrivial regular solutions with finite energy (Derrick's argument is not applicable) and, in fact, such solutions are well known both in the mathematical literature as harmonic maps from $\mathbb{R}^2$ into S^2 and in the physics literature as instantons in the two-dimensional Euclidean sigma model. One way to derive them is to use the Bogomol'nyi identity

$$E(F) = \pi \int_0^\infty \left(F'^2 + \frac{\sin^2 F}{r^2} \right) r dr = \pi \int_0^\infty \left(\sqrt{r} F' - \frac{\sin F}{\sqrt{r}} \right)^2 dr - 2\pi \cos F|_0^\infty. \tag{14}$$

It follows from (14) that in the topological sector $k = 1$ the energy attains the minimum, $E = 4\pi$, on the solution of the first order equation $rF' = \sin F$, which is

$$F_S(r) = 2 \arctan(r/\lambda), \tag{15}$$

where λ is a nonzero constant. This solution is a well-known harmonic map from R^2 to S^2. We remark in passing that this solution can be alternatively obtained in an elegant geometric way by composing the identity map between two spheres with the inverse stereographic projection.

It has been known for long that the solution (15) persists if one couples gravity with zero cosmological constant [4]. To see this let us consider Eqs. (8)–(11) and assume that the fields are time independent. We get

$$\frac{1}{r} e^\delta (A e^{-\delta} r F')' = \frac{\sin(2F)}{2r^2}, \tag{16}$$

and (assuming that $\Lambda = 0$)

$$\delta' = -\alpha r F'^2, \tag{17}$$

$$A' = -\alpha r \left(A F'^2 + \frac{\sin^2 F}{r^2} \right). \tag{18}$$

For regular solutions the boundary conditions at $r = 0$ are

$$A(0) = 1, \qquad \delta(0) = 0, \qquad F(0) = 0, \qquad F'(0) = b, \tag{19}$$

where b is a free parameter. We want a finite energy degree-one solution so we require that $A(r)$ and $\delta(r)$ tend to constants at infinity and $F(\infty) = \pi$. Such a solution can be found explicitly as follows. Let $B = \exp(-2\delta)A$. Then, from (17) and (18) we obtain

$$B' = \alpha r \left(A F'^2 - \frac{\sin^2 F}{r^2} \right) e^{-2\delta}. \tag{20}$$

Using the boundary conditions (19), this implies that $B(r) \equiv 1$, hence $A = \exp(2\delta)$. Substituting this into (16) we get

$$\frac{1}{r} e^\delta (r e^\delta F')' = \frac{\sin(2F)}{2r^2}. \tag{21}$$

Using the new coordinate ρ defined by

$$r e^\delta \frac{d}{dr} = \rho \frac{d}{d\rho}, \tag{22}$$

one can rewrite Eq. (21) in the form of the flat space Eq. (12)

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dF}{d\rho} \right) = \frac{\sin(2F)}{2\rho^2}, \tag{23}$$

which, as we showed above, is solved by $F_S(\rho) = 2\arctan(\rho/\lambda)$. Inserting this solution into Eq. (17) and integrating we get

$$e^\delta = 1 - \frac{2\alpha\rho^2}{\lambda^2 + \rho^2}. \tag{24}$$

This yields the metric

$$ds^2 = -dt^2 + (\lambda^2 + \rho^2)^{-2\alpha} (d\rho^2 + \rho^2 d\phi^2) \tag{25}$$

which has the deficit angle equal to $4\alpha\pi$, hence this solution exists only for $\alpha < 1/2$.

III. NONEXISTENCE OF STATIC SOLUTIONS FOR $\Lambda < 0$

For a nonvanishing cosmological constant, the static equations (16) and (17) do not change while Eq. (18) picks up an additional term

$$A' = -2\Lambda r - \alpha r \left(A F'^2 + \frac{\sin^2 F}{r^2} \right). \tag{26}$$

Assuming that $\Lambda < 0$, by rescaling, without loss of generality, we set hereafter $\Lambda = -1$. Using (17) we eliminate δ from (16) and get the following system

$$A' = 2r - \alpha r \left(A F'^2 + \frac{\sin^2 F}{r^2} \right), \tag{27}$$

$$F'' + \frac{1}{r} F' + \frac{2r^2 - \alpha \sin^2 F}{A r} F' = \frac{\sin(2F)}{2 A r^2}. \tag{28}$$

The boundary conditions at $r = 0$ are

$$F(r) \sim br, \qquad A(r) \sim 1 - (\alpha b^2 - 1) r^2. \tag{29}$$

We want a solution for which $A \sim r^2$ at infinity and $F(\infty) = \pi$. Such a solution was claimed to exist and constructed numerically in [5]. We shall show now that this claim was erroneous.

Let us define a function

$$H = \cos^2 F + r^2 A F'^2. \tag{30}$$

We claim that $H(r)$ is monotone decreasing. To prove this, using the field equations, we compute

$$H' = -2r^3 F'^2 - \alpha r^3 A F'^4 + \alpha r \sin^2 F F'^2. \tag{31}$$

It is convenient to rewrite Eq. (31) in the form

$$H' = -rF'^2 G(r), \qquad G(r) = 2r^2 + \alpha r^2 A F'^2 - \alpha \sin^2 F. \tag{32}$$

From the boundary conditions (29) $G(r) \sim 2r^2 > 0$ near $r = 0$, independent of b and α. Now we shall show that $G(r) = 0$ implies $G'(r) \geq 0$. To this end we compute

$$G'(r) = -2\alpha r^3 F'^2 - \alpha^2 r^3 A F'^4 + 4r + \alpha^2 r \sin^2 F F'^2. \tag{33}$$

To evaluate G' when $G = 0$ we solve $G = 0$ for $\sin^2 F$ and substitute that value into Eq. (33). We get

$$G'_{G=0} = 4r. \tag{34}$$

Thus $G'(r) > 0$ for $r > 0$, and therefore $H'(r) \leq 0$ for all r. Since $H(0) = 1$, this implies that $H(r) < 1$ for all $r > 0$. This excludes existence of a solution having $\lim_{r\to\infty} F(r) = \pi$ because that would mean $\lim_{r\to\infty} H(r) \geq 1$.

In view of Struwe's result mentioned above, the nonexistence of a static nonconstant soliton suggests (but by no means proves) that the negative cosmological constant might act as a cosmic censor in this model.

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