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Noncommutative field theory: Nonrelativistic fermionic field coupled to the Chern-Simons field in 2 + 1 dimensions

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We study a noncommutative nonrelativistic fermionic field theory in 2 + 1 dimensions coupled to the Chern-Simons field. We perform a perturbative analysis of the model and show that up to one loop the ultraviolet divergences are canceled and the infrared divergences are eliminated by the noncommutative Pauli term.

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I. INTRODUCTION

The noncommutative Aharonov-Bohm (AB) effect for scalar particles has been studied in the quantum mechanical [1,2] and in the field theory contexts [3]. As in the commutative case, in the latter case the effect was simulated by a nonrelativistic field theory of spin zero particles interacting through a Chern-Simons (CS) field. Differently from its commutative counterpart, however, the model turns out to be renormalizable even without a quartic self-interaction of the scalar field (the quartic self-interaction is however necessary if a smooth commutative limit is required). It is known that, in the commutative situation, the Pauli term plays for the spin $1/2$ AB scattering [4] the same role as the quartic interaction plays for the case of scalar particles [5]. One may then conjecture that a noncommutative Pauli interaction is also not necessary at least as a prerequisite for renormalizability. In this brief note we will prove that this conjecture indeed holds, the Pauli term being necessary to obtain a smooth result in the commutative limit but not to fix the ultraviolet renormalizability of the model. Our analysis is based on the (2 + 1) dimensional model described by the action

$$\begin{aligned} S[A,\psi] = \int d^3x\bigg[&\frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}\Big(A_\mu * \partial_\nu A_\lambda + \frac{2ig}{3}A_\mu * A_\nu * A_\lambda\Big) \\ &- \frac{1}{2\xi}\partial_i A^i * \partial_j A^j + i\psi^\dagger * D_t\psi \\ &- \frac{1}{2m}(D_i\psi)^\dagger * (D_i\psi) + \lambda\psi^\dagger * B * \psi \\ &+ \partial^i\bar{c} * \partial_i c + ig\partial^i\bar{c} * [A_i, c]_*\bigg], \end{aligned} \tag{1.1}$$

where the Pauli term is the one explicitly involving the magnetic field $B = -F_{12} = \partial_1 A^2 - \partial_2 A^1 - ig[A^1, A^2]_*$ with $[A^1, A^2]_* = A^1 * A^2 - A^2 * A^1$ and the one-component fermion field ψ. The fermion field, depending on the sign of λ, represents either a spin-up or a spin-down particle.

In these expressions $\phi_1(x) * \phi_2(x)$ denotes the Moyal product of $\phi_1(x)$ and $\phi_2(x)$:

$$\phi_1(x) * \phi_2(x) = \lim_{y\to x} e^{(i/2)\Theta^{\mu\nu}[\partial/(\partial y^\mu)][\partial/(\partial x^\nu)]}\phi_1(y)\phi_2(x), \tag{1.2}$$

where the constant and antisymmetric matrix $\Theta_{\mu\nu}$ gives a measure of the noncommutativity strength. To evade possible unitarity and/or causality problems [6], we will keep time local by imposing $\Theta_{0i} = 0$ (other noncommutative aspects of nonrelativistic fermions interacting with the CS field were considered in [7,8]). We set also $\theta^{ij} = \theta\varepsilon^{ij}$ with ε^{ij} being the two-dimensional Levi-Cività symbol, normalized as $\varepsilon^{12} = 1$.

Observe that a Coulomb-type gauge fixing and the corresponding Faddeev-Popov terms have already been included in the action (1.1). The covariant derivatives in Eq. (1.1) are given by

$$D_t\psi = \partial_t\psi + igA_0 * \psi, \qquad D_i\psi = \partial_i\psi + igA_i * \psi, \tag{1.3}$$

so that the above action is invariant under the small Becchi-Rouet-Stora-Tyutin transformation,

$$\delta\psi = ig\epsilon c * \psi, \tag{1.4}$$

$$\delta\psi^\dagger = ig\psi^\dagger * c\epsilon, \tag{1.5}$$

$$\delta A_\mu = -\epsilon D_\mu c = -\epsilon(\partial_\mu c + ig[A_\mu, c]_*, \tag{1.6}$$

$$\delta c = 0, \tag{1.7}$$

$$\delta\bar{c} = \frac{\epsilon}{\xi}\partial_i A^i, \tag{1.8}$$

where ϵ is a position-independent anticommuting parameter. For convenience, we will work in a strict Coulomb gauge obtained by letting $\xi \to 0$. As is well known, one Moyal product (one asterisk) may be eliminated in each term under the action integral; therefore, the quadratic

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terms in the action and the corresponding propagators are the same as in the commutative model.

We will use a graphical notation where the CS field, the matter field, and the ghost field propagators are represented by wavy, continuous, and dashed lines, respectively. The graphical representation for the Feynman rules is given in Fig. 1 and the corresponding analytical expressions are:

(i) (i) The matter field propagator:

$$S(p) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{2m} + i\epsilon}. \tag{1.9}$$

(ii) (ii) The ghost field propagator:

$$G(p) = -\frac{i}{\mathbf{p}^2}. \tag{1.10}$$

(iii) (iii) The gauge field propagator in the limit $\xi \to 0$ is

$$D_{\mu\nu}(k) = \frac{\varepsilon_{\mu\nu\lambda}\bar{k}^\lambda}{\kappa\mathbf{k}^2}, \tag{1.11}$$

where $\bar{k}^\lambda = (0, \mathbf{k})$.
Because of the explicit appearance of the B field in the Pauli term, it is convenient to have at hand the mixed propagator

$$\Delta_{A_0B}(p) = -\frac{i}{\kappa}, \tag{1.12}$$

obtained directly from Eq. (1.11).

(iv) (iv) The analytical expressions associated with the vertices are

$$\Gamma^0(p, p') = -ige^{ip\theta p'}, \tag{1.13}$$

$$\Gamma^i(p, p') = \frac{ig}{2m}(p + p')^i e^{ip\theta p'}, \tag{1.14}$$

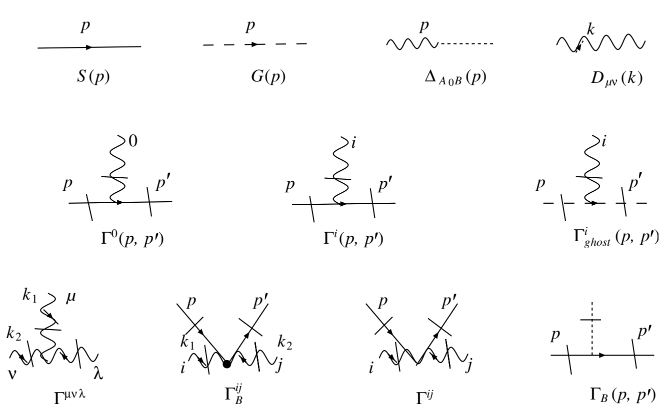

FIG. 1. Feynman rules for the action (1.1). The black blob on the Γ^{ij}_B vertex (which is associated with the noncommutative Pauli term) is to distinguish it from the Γ^{ij} vertex.

$$\Gamma^i_{\mathrm{ghost}}(p, p') = -2gp'^i \sin(p\theta p'), \tag{1.15}$$

$$\Gamma^{\mu\nu\lambda}(k_1, k_2) = 2ig\kappa\varepsilon^{\mu\nu\lambda}\sin(k_1\theta k_2), \tag{1.16}$$

$$\Gamma^{ij}_B(k_1, k_2, p, p') = 2ig\lambda\sin(k_1\theta k_2)e^{ip\theta p'}\varepsilon^{ij}, \tag{1.17}$$

$$\Gamma^{ij}(k_1, k_2, p, p') = -\frac{ig^2}{m}\cos(k_1\theta k_2)e^{ip\theta p'}\delta^{ij}, \tag{1.18}$$

$$\Gamma^B(p, p') = i\lambda e^{ip\theta p'}. \tag{1.19}$$

In these expressions we have defined $k_1\theta k_2 \equiv \frac{1}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}$.

II. PARTICLE-PARTICLE SCATTERING

A. Tree level scattering

In the tree approximation and in the center-of-mass frame, the two body scattering amplitude, depicted in Fig. 2(a), is given by

$$\mathcal{A}^0_a(\varphi) = -\frac{2ig^2(\mathbf{p}_1 \wedge \mathbf{p}_3)}{m\kappa}\left[\frac{e^{i(p_1\theta p_3 + p_2\theta p_4)}}{(\mathbf{p}_1 - \mathbf{p}_3)^2} - \frac{e^{-i(p_1\theta p_3 + p_2\theta p_4)}}{(\mathbf{p}_1 + \mathbf{p}_3)^2}\right], \tag{2.1}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, $\mathbf{p}_4$ are, respectively, the incoming and outgoing momenta. Since $\theta_{ij} = \theta\varepsilon_{ij}$, the phase is $p_1\theta p_3 + p_2\theta p_4 = \theta(\mathbf{p}_1 \wedge \mathbf{p}_3) = \theta\mathbf{p}^2\sin\varphi = \bar{\theta}\sin\varphi$, where we have defined $\bar{\theta} \equiv \theta\mathbf{p}^2$ and φ is the scattering angle. Therefore, Eq. (2.1) can be rewritten as

$$\mathcal{A}^0_a(\varphi) = -\frac{ig^2}{m\kappa}\left[\frac{e^{i\bar{\theta}\sin\varphi}}{1 - \cos\varphi} - \frac{e^{-i\bar{\theta}\sin\varphi}}{1 + \cos\varphi}\right]\sin\varphi. \tag{2.2}$$

For the graph in Fig. 2(b), we have

$$\mathcal{A}^0_b(\varphi) = -\frac{4g\lambda}{\kappa}\cos(\bar{\theta}\sin\varphi). \tag{2.3}$$

Thus, the full tree level amplitude is

FIG. 2. Tree level scattering. Another contributing diagram similar to (b) but with the ends of the Δ_{A_0B} exchanged was not drawn.

$$\mathcal{A}^0(\varphi) = -\frac{ig^2}{m\kappa}[\cot(\varphi/2)e^{i\bar{\theta}\sin\varphi} - \tan(\varphi/2)e^{-i\bar{\theta}\sin\varphi}] - \frac{4g\lambda}{\kappa}\cos(\bar{\theta}\sin\varphi), \tag{2.4}$$

furnishing up to first order in the parameter $\bar{\theta}$,

$$\mathcal{A}^{\,0}(\varphi) = -\frac{2ig^2}{m\kappa}(\cot\varphi + i\bar{\theta}) - \frac{4g\lambda}{\kappa} + \mathcal{O}(\bar{\theta}^2). \tag{2.5}$$

Notice that the noncommutative contribution is isotropic although energy dependent.

B. One-loop scattering

The one-loop contribution to the scattering amplitude is depicted in Fig. 3. Two other diagrams, corresponding to graphs 3(b) and 3(c) with the upper and bottom fermionic lines exchanged, are not explicitly shown. All other possible one-loop graphs vanish. The expressions for the contributions of the box, triangle, and trigluon graphs, shown in Figs. 3(a)–3(c), are the same as in the scalar case [3] so that we just quote the results:

$$\mathcal{A}_a(\varphi) = -\frac{g^4}{2\pi m\kappa^2}[\ln(2\sin\varphi) + i\pi] - \frac{i\bar{\theta}g^4\sin\varphi}{\pi m\kappa^2}\ln\left[\tan\left(\frac{\varphi}{2}\right)\right] + \mathcal{O}(\bar{\theta}^2), \tag{2.6}$$

for the total contribution of the box graph,

$$\mathcal{A}_b^{np}(\varphi) = \frac{g^4}{2\pi m\kappa^2}[\ln(\bar{\theta}/2) + \gamma] + \frac{g^4}{2\pi m\kappa^2}\ln(2\sin\varphi) + \frac{i\bar{\theta}\sin\varphi g^4}{2\pi m\kappa^2}\ln[\tan(\varphi/2)] + \mathcal{O}(\bar{\theta}^2) \tag{2.7}$$

and

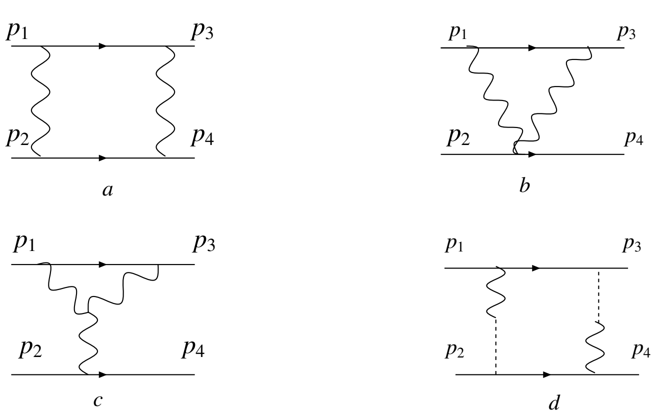

FIG. 3. Typical one-loop scattering diagrams.

$$\mathcal{A}_b^{\,p}(\varphi) = -\frac{g^4}{4\pi m\kappa^2}\Bigg[\cos(\bar{\theta}\sin\varphi)\ln\left(\frac{\Lambda^2}{\mathbf{p}^2}\right) - \ln|2\sin(\varphi/2)|e^{i\bar{\theta}\sin\varphi} - \ln|2\cos(\varphi/2)|e^{-i\bar{\theta}\sin\varphi}\Bigg], \tag{2.8}$$

for the nonplanar and planar parts of the triangle graph,

$$\mathcal{A}_c^{\,np}(\varphi) = \frac{3g^4}{2\pi m\kappa^2}[\ln(\bar{\theta}/2) + \gamma] + \frac{3g^4}{2\pi m\kappa^2}\ln(2\sin\varphi) + \frac{3i\bar{\theta}\sin\varphi g^4}{2\pi m\kappa^2}\ln[\tan(\varphi/2)] + \frac{2g^4}{\pi m\kappa^2} + \mathcal{O}(\bar{\theta}^2) \tag{2.9}$$

and

$$\mathcal{A}_c^{\,p}(\varphi) = \frac{g^4}{4\pi m\kappa^2}\Bigg[\cos(\bar{\theta}\sin\varphi)\left[\ln\left(\frac{\Lambda^2}{\mathbf{p}^2}\right) + 1\right] - \ln|2\sin(\varphi/2)|e^{i\bar{\theta}\sin\varphi} - \ln|2\cos(\varphi/2)|e^{-i\bar{\theta}\sin\varphi}\Bigg], \tag{2.10}$$

for the nonplanar and planar parts of the trigluon graph, where Λ is an ultraviolet cutoff and γ is the Euler-Mascheroni constant.

The graphs containing the noncommutative Pauli vertex are depicted in Figs. 3(d), 4(a), and 4(b).

The contribution of the graph in Fig. 3(d), which is purely nonplanar, is given by

$$\mathcal{A}_d^{\,np}(\varphi) = \frac{4mg^2\lambda^2}{\kappa^2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left[\frac{e^{2iq\theta k} + e^{2iq'\theta k}}{(\mathbf{k}^2 - \mathbf{p}^2 - i\epsilon)}\right]. \tag{2.11}$$

This integral can be evaluated by using the result [9]

$$\int\frac{d^n k}{(2\pi)^n}\frac{e^{ik_\alpha p^\alpha}}{[k^2 - M^2]^\lambda} = i(-1)^\lambda\frac{M^{n/2-\lambda}}{2^{\lambda-1}(2\pi)^{n/2}\Gamma[\lambda]} \times \frac{K_{n/2-\lambda}(\sqrt{-M^2p^2})}{(-p^2)^{n/2-\lambda}}, \tag{2.12}$$

and yields

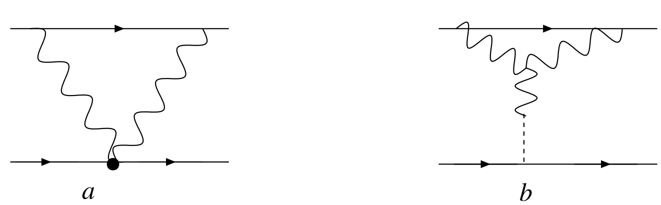

FIG. 4. Typical contributions from the noncommutative Pauli term.

$$\mathcal{A}_d^{np}(\varphi) = -\frac{4mg^2\lambda^2}{\pi\kappa^2}\left[\ln\left(\frac{\bar{\theta}}{2}\right) + \gamma\right] - \frac{2mg^2\lambda^2}{\pi\kappa^2}\ln[2\sin\varphi] + \frac{2img^2\lambda^2}{\kappa^2} + \mathcal{O}(\bar{\theta}^2), \tag{2.13}$$

for small θ. As can be easily verified, the contributions from the other two graphs, shown in Figs. 4(a) and 4(b) cancel among themselves.

Summing all the contributions, we get the total one-loop amplitude:

$$\begin{aligned}\mathcal{A}_{\text{one-loop}}(\varphi) &= \mathcal{A}^p_{\text{one-loop}}(\varphi) + \mathcal{A}^{np}_{\text{one-loop}}(\varphi) + \mathcal{A}_a(\varphi)\\ &= -\frac{2ig^2}{m\kappa}\cot\varphi - \frac{4g\lambda}{\kappa} + \frac{2\bar{\theta}g^2}{m\kappa} + \frac{9g^4}{4\pi m\kappa^2} - \frac{ig^4}{2m\kappa^2} + \frac{2img^2\lambda^2}{\kappa^2} + \left(\frac{3g^4}{2\pi m\kappa^2} - \frac{2mg^2\lambda^2}{\pi\kappa^2}\right)\ln[2\sin\varphi]\\ &\quad + \frac{i\bar{\theta}g^4\sin\varphi}{\pi m\kappa^2}\ln[\tan(\varphi/2)] + \frac{4mg^2}{\pi\kappa^2}\left(\frac{g^2}{2m^2} - \lambda^2\right)[\ln(\bar{\theta}/2) + \gamma] + \mathcal{O}(\bar{\theta}^2).\end{aligned} \tag{2.14}$$

For $\lambda = \pm\frac{g}{\sqrt{2}m}$, the limit $\bar{\theta} \rightarrow 0$ is analytical and conformally invariant. This result also shows that, up to one-loop order, the scattering amplitude does not present ultraviolet divergences and that, by conveniently adjusting the noncommutative Pauli term, the would-be infrared logarithmically divergences may be eliminated.

In this work we have studied the two body scattering amplitude when the colliding particles both have either spin-up or spin-down. If the particles in colliding beams have opposite spins, the contributions of the noncommutative Pauli terms cancel. In this case, designating by ψ and ϕ the fermionic fields associated with the particles in the beams, to get a smooth commutative limit it will be necessary to include a quartic term $\phi^\dagger * \phi * \psi^\dagger * \psi$. In fact, in the commutative situation one such term is induced if one starts from the relativistic theory [10] and integrates over the high energy modes to get an effective nonrelativistic field theory.

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