Unruh effect in the general light-front frame

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We study the phenomenon of Unruh effect in a massless scalar field theory quantized on the light front in the general light-front frame. We determine the uniformly accelerating coordinates in such a frame and through a direct transformation show that the propagator of the theory has a thermal character in the uniformly accelerating coordinate system with a temperature given by Tolman's law. We also carry out a systematic analysis of this phenomenon from the Hilbert space point of view and show that the vacuum of this theory appears as a thermal vacuum to a Rindler observer with the same temperature as given by Tolman's law.

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I. INTRODUCTION

It has been observed in recent years that a statistical description of theories quantized on the light front [1,2] prefers a general coordinate frame [3–5]. Denoting the Minkowski coordinates by $x^\mu = (t, x, y, z)$ and the coordinates of the general light-front frame by $\bar{x}^\mu = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$, the relation between the two can be written as [6]

$$\bar{t} = t + z, \qquad \bar{z} = At + Bz, \qquad \bar{x} = x, \qquad \bar{y} = y, \tag{1}$$

where A, B are arbitrary constants with the restriction that $|B| \geq |A|$ which arises if we require $\bar{t}$ to correspond to the time variable. The metric in the general light-front frame (GLF) has the form (We refer the reader to [6] for details on notations as well as various other properties of GLF.)

$$\bar{g}^{(\mathrm{GLF})}_{\mu\nu} = \begin{pmatrix} \frac{A+B}{B-A} & 0 & 0 & -\frac{1}{B-A} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{B-A} & 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

so that the line element is given by

$$\mathrm{d}\tau^2 = \frac{A+B}{B-A}\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{x}^2 - \mathrm{d}\bar{y}^2 - \frac{2}{B-A}\mathrm{d}\bar{t}\mathrm{d}\bar{z}. \tag{3}$$

For $A = -B = 1$, Eq. (1) represents the conventional light-front frame (CLF) while for $A = 0$, $B = 1$ we have the oblique light-front frame (OLF) where most of the discussions of statistical mechanics have been carried out thus far [3–5,7,8]. In general, if a quantum field theory quantized at equal time t is at temperature T_{M} in the Minkowski space, then by Tolman's law [9], the corresponding temperature for the theory quantized at equal $\bar{t}$ in the generalized light-front frame is given by

$$\frac{T_{\mathrm{GLF}}}{\sqrt{\bar{g}^{(\mathrm{GLF})}_{00}}} = \frac{T_{\mathrm{M}}}{\sqrt{\eta_{00}}} = T_{\mathrm{M}} \quad \text{or} \qquad T_{\mathrm{GLF}} = \sqrt{\bar{g}^{(\mathrm{GLF})}_{00}}\, T_{\mathrm{M}} = \sqrt{\frac{A+B}{B-A}}\, T_{\mathrm{M}}. \tag{4}$$

This shows that in the conventional light-front frame where $A = -B = 1$

$$T_{\mathrm{CLF}} = 0, \tag{5}$$

for any finite T_{M} so that any finite temperature in the Minkowski frame is mapped to zero temperature in the conventional light-front frame and a statistical description is not possible. On the other hand, in the oblique light-front frame where $A = 0$, $B = 1$,

$$T_{\mathrm{OLF}} = T_{\mathrm{M}}, \tag{6}$$

and the temperature coincides with that corresponding to conventional quantization. As is clear from (4), statistical mechanical description for a theory quantized on the light front is possible as long as $|B| \neq |A|$. In the general light-front frame, the temperature will be related to that in the Minkowski frame simply through a scale factor.

It has also been understood for sometime now that equal time quantum field theories exhibit Unruh effect [10] when viewed from a uniformly accelerating coordinate frame. More specifically, a uniformly accelerating observer sees the vacuum of the equal time quantum theory to correspond to a thermal vacuum with temperature given by (We note that because of the isotropy of Minkowski space, the direction of acceleration is not important.)

$$T_{\mathrm{M}} = \frac{\alpha}{2\pi}, \tag{7}$$

where α represents the constant proper acceleration of the observer. It is interesting to study the phenomenon of Unruh effect within the context of quantum field theories quantized on the light front (equal $\bar{t}$) in GLF [6] for the following reasons. Since the statistical density matrix in

light-front field theories does not correspond to a naive generalization of the known density matrix [3–5], further support for the structure of this density matrix in GLF can be obtained from studying the thermal behavior of the vacuum in an accelerated coordinate system. This, of course, immediately raises this interesting issue, namely, since the form of the line element in (3) shows that there are now two distinct possibilities for acceleration (unlike the Minkowski frame), it is not *a priori* clear whether this would lead to two distinct temperatures for the Unruh effect (corresponding to the two directions for acceleration) and how this will be compatible with the unique temperature of the GLF description following from Tolman's law in (4).

In this paper, we study these issues systematically. In Sec. II , we work out the uniformly accelerating coordinates for the two cases of acceleration along the $\bar{x}$ axis and the $\bar{z}$ axis. In Sec. III, we show that even though the uniformly accelerating coordinates are different for the two cases, the zero temperature propagator of a massless scalar field theory quantized on the light front in GLF corresponds to a thermal propagator with the unique temperature given by (4) when transformed to the accelerating coordinates [11]. In Sec. IV we carry out the Hilbert space analysis for this theory systematically and show that a Rindler observer (uniformly accelerating along $\bar{z}$) perceives the GLF vacuum of the theory as a thermal vacuum with the same temperature as in (4). We present a brief summary in Sec. V.

II. UNIFORMLY ACCELERATING COORDINATES

Let us recall that the line element in the general light-front frame has the form

$$\mathrm{d}\tau^2 = \frac{A+B}{B-A}\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{x}^2 - \mathrm{d}\bar{y}^2 - \frac{2}{B-A}\mathrm{d}\bar{t}\mathrm{d}\bar{z}. \quad (8)$$

In this case, the Lorentz contraction factor, in general, has the form

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}\tau} = \bar{\gamma} = \frac{1}{\sqrt{\frac{A+B}{B-A} - \bar{v}_{\bar{x}}^2 - \bar{v}_{\bar{y}}^2 - \frac{2}{B-A}\bar{v}_{\bar{z}}}}, \quad (9)$$

with appropriate restrictions on the velocity components. Here the coordinate velocities are defined as

$$\bar{v}_{\bar{x}} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}}, \qquad \bar{v}_{\bar{y}} = \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{t}}, \qquad \bar{v}_{\bar{z}} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}}. \quad (10)$$

From (8) we note that there is an obvious symmetry between the coordinates $\bar{x}$, $\bar{y}$. Therefore, in studying acceleration in such a frame, there are two distinct cases to consider.

A. Motion along $\bar{x}$

Let us first consider the case where a particle is moving (and being uniformly accelerated) along the $\bar{x}$ axis [12]. Therefore, neglecting the $\bar{y}$ and $\bar{z}$ coordinates, we can define the four velocity of the particle as

$$\bar{u}^{\mu} = \bar{\gamma}(1, \bar{v}_{\bar{x}}, 0, 0), \quad (11)$$

where we have identified [see (9)]

$$\bar{v}_{\bar{y}} = 0 = \bar{v}_{\bar{z}}, \qquad \bar{\gamma} = \frac{1}{\sqrt{\frac{A+B}{B-A} - \bar{v}_{\bar{x}}^2}}. \quad (12)$$

Defining the proper acceleration as

$$\bar{a}^{\mu} = \frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}\bar{t}}{\mathrm{d}\tau}\frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\bar{t}} = \bar{\gamma}\frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\bar{t}}, \quad (13)$$

we obtain after some algebra

$$\bar{a}^{\mu} = \left(\bar{v}_{\bar{x}}, \frac{A+B}{B-A}, 0, 0\right)\bar{\gamma}^4\frac{\mathrm{d}\bar{v}_{\bar{x}}}{\mathrm{d}\bar{t}}. \quad (14)$$

It follows from this that

$$\bar{a}^2 = \bar{g}^{(\mathrm{GLF})}_{\mu\nu}\bar{a}^{\mu}\bar{a}^{\nu} = -\left(\sqrt{\frac{A+B}{B-A}}\bar{\gamma}^3\frac{\mathrm{d}\bar{v}_{\bar{x}}}{\mathrm{d}\bar{t}}\right)^2 = -\alpha^2, \quad (15)$$

where α is known as the proper acceleration. In terms of the proper acceleration α, we can write

$$\bar{a}^{\mu} = \frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\tau} = \left(\sqrt{\frac{B-A}{A+B}}\bar{\gamma}\bar{v}_{\bar{x}}, \sqrt{\frac{A+B}{B-A}}\bar{\gamma}, 0, 0\right)\alpha. \quad (16)$$

For constant α, we can now solve the dynamical equations

$$\frac{\mathrm{d}^2\bar{x}^{\mu}}{\mathrm{d}\tau^2} = \frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\tau} = \bar{a}^{\mu}, \quad (17)$$

to determine the trajectory

$$\bar{\gamma}(\tau) = \sqrt{\frac{B-A}{A+B}}\cosh\alpha\tau, \qquad \bar{v}_{\bar{x}}(\tau) = \sqrt{\frac{A+B}{B-A}}\tanh\alpha\tau,$$
$$\bar{t}(\tau) = \sqrt{\frac{B-A}{A+B}}\frac{1}{\alpha}\sinh\alpha\tau, \qquad \bar{x}(\tau) = \frac{1}{\alpha}\cosh\alpha\tau, \quad (18)$$

corresponding to the initial conditions $\bar{v}_{\bar{x}}(\tau=0) = 0 = \bar{t}(\tau=0)$, $\bar{x}(\tau=0) = \frac{1}{\alpha}$. It follows now that the trajectory with a constant proper acceleration defines the hyperbola

$$\bar{x}^2 = \bar{g}^{(\mathrm{GLF})}_{\mu\nu}\bar{x}^{\mu}\bar{x}^{\nu} = -\frac{1}{\alpha^2}, \quad (19)$$

with (18) providing the uniformly accelerating coordinates for the present case. These are quite similar to the case of uniformly accelerating coordinates in the Minkowski frame except for normalization factors.

B. Motion along $\bar{z}$

Let us consider next the case where the particle is moving (and being uniformly accelerated) along the $\bar{z}$

axis. In this case, setting $\bar{x} = \bar{y} = 0$, we can write

$$\bar{u}^{\mu} = \frac{d\bar{x}^{\mu}}{d\tau} = (\bar{\gamma}, 0, 0, \bar{\gamma}\bar{v}_{\bar{z}}), \tag{20}$$

where, in the present case, [see (9)]

$$\bar{v}_{\bar{x}} = 0 = \bar{v}_{\bar{y}}, \qquad \bar{\gamma} = \sqrt{\frac{B-A}{A+B-2\bar{v}_{\bar{z}}}}. \tag{21}$$

In this case, the acceleration

$$\bar{a}^{\mu} = \frac{d\bar{u}^{\mu}}{d\tau}, \tag{22}$$

can be shown, with a little bit of algebra to have the form

$$\bar{a}^{\mu} = (1, 0, 0, (A + B - \bar{v}_{\bar{z}}))\bar{\gamma}\frac{d\bar{\gamma}}{d\bar{t}}, \tag{23}$$

which leads to

$$\bar{a}^{2} = \bar{g}^{(\mathrm{GLF})}_{\mu\nu}\bar{a}^{\mu}\bar{a}^{\nu} = -\left(\frac{d\bar{\gamma}}{d\bar{t}}\right)^{2} = -\alpha^{2}, \tag{24}$$

with α representing the proper acceleration. In terms of this, we can write

$$\bar{a}^{\mu} = (1, 0, 0, (A + B - \bar{v}_{\bar{z}}))\bar{\gamma}\alpha. \tag{25}$$

For constant α, we can solve the dynamical equations

$$\frac{d^{2}\bar{x}^{\mu}}{d\tau^{2}} = \frac{d\bar{u}^{\mu}}{d\tau} = \bar{a}^{\mu}, \tag{26}$$

to obtain the trajectory

$$\begin{aligned}
\bar{\gamma}(\tau) &= \sqrt{\frac{B-A}{A+B}}e^{\alpha\tau}, \\
\bar{v}_{\bar{z}}(\tau) &= \frac{A+B}{2}(1 - e^{-2\alpha\tau}), \\
\bar{t}(\tau) &= \sqrt{\frac{B-A}{A+B}}\frac{1}{\alpha}e^{\alpha\tau}, \\
\bar{z}(\tau) &= \mathrm{sgn}(B-A)\frac{\sqrt{B^{2}-A^{2}}}{\alpha}\cosh\alpha\tau,
\end{aligned} \tag{27}$$

where, for simplicity, we have assumed the initial conditions $\bar{v}_{\bar{z}}(\tau = 0) = 0$, $\bar{t}(\tau = 0) = \sqrt{\frac{B-A}{A+B}}\frac{1}{\alpha}$, and $\bar{z}(\tau = 0) = \mathrm{sgn}(B-A)\frac{\sqrt{B^{2}-A^{2}}}{\alpha}$. (We note here, for later use, that when $|B| > |A|$, it follows that $\mathrm{sgn}(B-A) = \mathrm{sgn}(A+B)$.) It is straightforward to check that the trajectory (27) with a constant proper acceleration defines the hyperbola

$$\bar{x}^{2} = \bar{g}^{(\mathrm{GLF})}_{\mu\nu}\bar{x}^{\mu}\bar{x}^{\nu} = -\frac{1}{\alpha^{2}}, \tag{28}$$

and (27) defines the uniformly accelerating coordinates in the present case.

III. TRANSFORMATION OF THE GREEN'S FUNCTION

Let us next consider a massless scalar field theory quantized on the light front (at equal $\bar{t}$) in the general light-front frame. In this case, the field expansion takes the form (we refer the reader to [6] for details)

$$\phi(\bar{x}) = \int \frac{d^{2}\bar{k}_{\perp}}{(2\pi)^{3/2}}\int_{0}^{\infty}\frac{d\bar{k}_{3}}{2\bar{k}_{3}}(e^{-i\tilde{\bar{k}}\cdot\bar{x}}a(\bar{k}) + e^{i\tilde{\bar{k}}\cdot\bar{x}}a^{\dagger}(\bar{k})), \tag{29}$$

where $\bar{k}_{\perp}$ denotes the transverse components of the momenta (namely, $\bar{k}_{1}, \bar{k}_{2}$) and we have identified ($i = 1, 2$)

$$\tilde{\bar{k}}_{\mu} = (\bar{k}_{0}, \mathrm{sgn}(A-B)\bar{k}_{i}, \mathrm{sgn}(A-B)\bar{k}_{3}), \tag{30}$$

with

$$\bar{k}_{0} = \bar{\omega} = \frac{\bar{k}_{i}^{2} + (B^{2}-A^{2})\bar{k}_{3}^{2}}{2|A-B|\bar{k}_{3}} > 0, \tag{31}$$

for $\bar{k}_{3} > 0$. When quantized on the light front, the creation and the annihilation operators satisfy the commutation relation

$$[a(\bar{k}), a^{\dagger}(\bar{k}')] = 2\bar{k}_{3}\delta^{3}(\bar{k} - \bar{k}'). \tag{32}$$

The Feynman propagator in the momentum space has the form

$$iD(\bar{k}) = \lim_{\epsilon\to 0}\frac{i}{\bar{k}^{2} + i\epsilon}, \tag{33}$$

while in the coordinate space it takes the form [13]

$$\begin{aligned}
\langle\phi(\bar{x}_{1})\phi(\bar{x}_{2})\rangle &= \frac{1}{(2\pi)^{3}}\int d^{2}\bar{k}_{\perp}\int_{0}^{\infty}\frac{d\bar{k}_{3}}{2\bar{k}_{3}}e^{-i\tilde{\bar{k}}\cdot(\bar{x}_{1}-\bar{x}_{2})} \\
&= \lim_{\epsilon\to 0} -\frac{1}{(2\pi)^{2}}\frac{1}{(\bar{x}_{1}-\bar{x}_{2})^{2} - i\epsilon},
\end{aligned} \tag{34}$$

where we are assuming that $\bar{x}_{1}^{0} - \bar{x}_{2}^{0} > 0$.

In the rest frame of the heat bath, the propagator for this theory at a temperature T can be obtained easily in the momentum space both in the imaginary time formalism as well as in the real time formalism. In the imaginary time formalism [14,15], it has the form

$$D^{(T_{\mathrm{GLF}})}(\bar{k}) = \frac{1}{2(A-B)\bar{k}_{0}\bar{k}_{3} - \bar{k}_{\perp}^{2} - (B^{2}-A^{2})\bar{k}_{3}^{2}}, \tag{35}$$

with $\bar{k}_{0} = 2i\pi n T_{\mathrm{GLF}}$ where n represents an integer. In the real time formalism [15], on the other hand, the $++$ component of the propagator can be written as

$$iD^{(T_{\mathrm{GLF}})}_{++}(\bar{k}) = \lim_{\epsilon\to 0}\frac{i}{\bar{k}^{2}} + 2\pi n(|\bar{k}_{0}|)\delta(\bar{k}^{2}), \tag{36}$$

where $n(|k_{0}|)$ represents the Bose-Einstein distribution function (the Boltzmann constant is assumed to be unity)

$$n(|\bar{k}_0|) = \frac{1}{e^{\frac{|\bar{k}_0|}{T_{\rm GLF}}} - 1}. \tag{37}$$

Here $T_{\rm GLF}$ denotes the temperature in the general light-front frame. We can take the Fourier transform of the finite temperature propagator in either the imaginary time formalism or the real time formalism with respect to the time variable to obtain the coordinate space representation of the thermal propagator (which we have calculated). Alternatively, we can simply evaluate this directly from the field expansion given in (29). For a theory quantized on the light front [see (32)], we note that

$$\langle a^\dagger(\bar{k}) a(\bar{k})\rangle_{T_{\rm GLF}} = 2\bar{k}_3 n(\bar{k}_0), \tag{38}$$

$$\langle a(\bar{k}) a^\dagger(\bar{k})\rangle_{T_{\rm GLF}} = 2\bar{k}_3 (1 + n(\bar{k}_0)), \tag{39}$$

where $\bar{k}_0$ is defined in (31). Recalling that we are in the rest frame, we have

$$\langle \phi(\bar{t}_1)\phi(\bar{t}_2)\rangle_{T_{\rm GLF}} = \int \frac{d^2\bar{k}_\perp}{(2\pi)^3} \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3} (n(\bar{k}_0) e^{i\bar{k}_0(\bar{t}_1 - \bar{t}_2)} + (1 + n(\bar{k}_0)) e^{-i\bar{k}_0(\bar{t}_1 - \bar{t}_2)}), \tag{40}$$

where we are assuming that $\bar{t}_1 - \bar{t}_2 > 0$. The integral can be easily done using standard tables [16] and recalling that in the rest frame, the proper time is related to the coordinate time as [see (8)]

$$\bar{t}_1 - \bar{t}_2 = \sqrt{\frac{B-A}{A+B}}(\tau_1 - \tau_2) = \sqrt{\frac{B-A}{A+B}}\tau, \tag{41}$$

we obtain the coordinate representation of the thermal propagator to be

$$\langle \phi(\bar{\tau}_1)\phi(\bar{\tau}_2)\rangle_{T_{\rm GLF}} = -\frac{1}{(2\pi)^2}\left(\pi T_{\rm GLF}\sqrt{\frac{B-A}{A+B}}\right)^2 \times \mathrm{cosech}^2\left(\pi T_{\rm GLF}\sqrt{\frac{B-A}{A+B}}\tau\right). \tag{42}$$

Let us next look at the zero temperature propagator (34) in the accelerating coordinate system. As we have seen, there are two cases to consider. When the motion is along the $\bar{x}$ axis, we can set $\bar{y}_1 = \bar{y}_2$, $\bar{z}_1 = \bar{z}_2$. Furthermore, using the accelerating coordinate system in (18), we can write

$$\begin{aligned} \bar{t}_1 - \bar{t}_2 &= \sqrt{\frac{B-A}{A+B}}\frac{1}{\alpha}(\sinh\alpha\tau_1 - \sinh\alpha\tau_2), \\ \bar{x}_1 - \bar{x}_2 &= \frac{1}{\alpha}(\cosh\alpha\tau_1 - \cosh\alpha\tau_2), \end{aligned} \tag{43}$$

where we are assuming that $\tau_1 - \tau_2 = \tau > 0$. It follows now that the invariant length is given by

$$\begin{aligned} (\bar{x}_1 - \bar{x}_2)^\mu (\bar{x}_1 - \bar{x}_2)_\mu &= \frac{A+B}{B-A}(\bar{t}_1 - \bar{t}_2)^2 - (\bar{x}_1 - \bar{x}_2)^2 \\ &= \frac{4}{\alpha^2}\sinh^2\frac{\alpha\tau}{2}. \end{aligned} \tag{44}$$

As a result, in the coordinate system (18) uniformly accelerating along $\bar{x}$, the zero temperature propagator (34) takes the form

$$\langle \phi(\bar{x}_1)\phi(\bar{x}_2)\rangle = -\frac{1}{(2\pi)^2}\left(\frac{\alpha}{2}\right)^2 \mathrm{cosech}^2\frac{\alpha\tau}{2}. \tag{45}$$

Comparing with (42), we conclude that an observer in the frame accelerating along the $\bar{x}$ axis sees the propagator of the light-front field theory as a thermal propagator with temperature

$$\frac{\alpha}{2} = \pi T_{\rm GLF}\sqrt{\frac{B-A}{A+B}} \quad \text{or}$$
$$T_{\rm GLF} = \sqrt{\frac{A+B}{B-A}}\frac{\alpha}{2\pi} = \sqrt{\bar{g}_{00}^{(\rm GLF)}}\frac{\alpha}{2\pi}. \tag{46}$$

The other case to consider is when the motion is along the $\bar{z}$ axis. In this case, we can set $\bar{x}_1 = \bar{x}_2$, $\bar{y}_1 = \bar{y}_2$. In the uniformly accelerating coordinates (27), we note that we can write

$$\begin{aligned} \bar{t}_1 - \bar{t}_2 &= \sqrt{\frac{B-A}{A+B}}\frac{1}{\alpha}(e^{\alpha\tau_1} - e^{\alpha\tau_2}), \\ \bar{z}_1 - \bar{z}_2 &= \mathrm{sgn}(B-A)\frac{\sqrt{B^2 - A^2}}{\alpha}(\cosh\alpha\tau_1 - \cosh\alpha\tau_2), \end{aligned} \tag{47}$$

where we are again assuming that $\tau_1 - \tau_2 = \tau > 0$. It follows now that

$$\begin{aligned} (\bar{x}_1 - \bar{x}_2)^\mu (\bar{x}_1 - \bar{x}_2)_\mu &= \frac{A+B}{B-A}(\bar{t}_1 - \bar{t}_2)^2 \\ &\quad - \frac{2}{B-A}(\bar{t}_1 - \bar{t}_2)(\bar{z}_1 - \bar{z}_2) \\ &= \frac{4}{\alpha^2}\sinh^2\frac{\alpha\tau}{2}, \end{aligned} \tag{48}$$

so that in the uniformly accelerating coordinates, the zero temperature propagator (34) takes the form

$$\langle \phi(\bar{x}_1)\phi(\bar{x}_2)\rangle = -\frac{1}{(2\pi)^2}\left(\frac{\alpha}{2}\right)^2 \mathrm{cosech}^2\frac{\alpha\tau}{2}. \tag{49}$$

Comparing with (42), we conclude that an observer in the frame accelerating along the $\bar{z}$ axis sees the propagator of the light-front field theory as a thermal propagator with temperature

$$\frac{\alpha}{2} = \pi T_{\rm GLF}\sqrt{\frac{B-A}{A+B}} \quad \text{or}$$

$$T_{\rm GLF} = \sqrt{\frac{A+B}{B-A}}\frac{\alpha}{2\pi} = \sqrt{\bar{g}_{00}^{(\rm GLF)}}\,\frac{\alpha}{2\pi}. \tag{50}$$

In other words, even though there are two distinct possibilities for acceleration in the GLF, both cases lead to a thermal character for the Green's function with the same temperature. Furthermore, recalling that the Unruh effect predicts $T_{\rm M} = \frac{\alpha}{2\pi}$ for a conventionally quantized scalar field theory in Minkowski space (7), we recover (4) in both the cases.

IV. HILBERT SPACE ANALYSIS

In the last section, we carried out a very simple analysis where we transformed the propagator of a massless scalar field quantized on the light front to a uniformly accelerating coordinate system and thereby showed that it behaves like a thermal propagator with temperature $T_{\rm GLF} = \sqrt{\bar{g}_{00}^{(\rm GLF)}}\,\frac{\alpha}{2\pi}$ independent of whether the acceleration is along the $\bar{x}$ ($\bar{y}$) axis or along the $\bar{z}$ axis. This simple analysis, however, does not bring out many important aspects of the Hilbert space structure of the theory in the present case which we will like to investigate systematically in this section. We will do this only for the case where the acceleration is along the $\bar{z}$ axis for simplicity. A parallel analysis for the case where the acceleration is along the $\bar{x}$ axis can be carried out exactly along the lines to be discussed in this section and does not lead to any new information. The Hilbert space analysis shows that a Rindler observer would perceive the vacuum of the theory to correspond to a thermal vacuum with temperature given in (4). Such a result is much more powerful in showing that any matrix element of the theory would appear as a thermal amplitude to an observer in the accelerating frame.

Let us first define various relevant coordinate systems associated with this problem. First, we note that if we define new coordinates as ($\bar{x} = \bar{y} = 0$)

$$\bar{t} = \sqrt{\frac{B-A}{A+B}}Xe^{T},$$
$$\bar{z} = \mathrm{sgn}(B-A)\sqrt{B^2-A^2}X\cosh T, \tag{51}$$

then, the line element (8) can be written as

$$\mathrm{d}\tau^2 = X^2\mathrm{d}T^2 - \mathrm{d}X^2. \tag{52}$$

Here X, T define the Rindler coordinates [12] for the present case and we have

$$\bar{x}^2 = -X^2, \tag{53}$$

so that for constant X, they define the hyperbola of constant acceleration $\alpha = \frac{1}{X}$.

The null geodesics for the theory defined on the general light-front frame are given (in the present case) by

$$\bar{t} = \text{constant}, \qquad \bar{t} - \frac{2}{A+B}\bar{z} = \text{constant}, \tag{54}$$

which, in turn, allow us to define the null coordinates

$$\begin{aligned} u &= \bar{t} - \frac{2}{A+B}\bar{z} = -\sqrt{\frac{B-A}{A+B}}Xe^{-T} \\ &= -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-a(\eta-\xi)} = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-aU}, \\ v &= \bar{t} = \sqrt{\frac{B-A}{A+B}}Xe^{T} \\ &= \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{a(\eta+\xi)} = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{aV}, \end{aligned} \tag{55}$$

where we have identified [17]

$$\begin{aligned} T &= a\eta, & X &= \frac{e^{a\xi}}{a}, \\ U &= \eta - \xi, & V &= \eta + \xi, \end{aligned} \tag{56}$$

with a representing an arbitrary positive constant. The line element (8) or (52) can now be written as

$$\mathrm{d}\tau^2 = e^{2a\xi}(\mathrm{d}\eta^2 - \mathrm{d}\xi^2), \tag{57}$$

which makes it clear that η, ξ define the conformal coordinates for the system and that

$$\bar{g}_{\eta\eta}^{(\rm conformal)} = -\bar{g}_{\xi\xi}^{(\rm conformal)} = e^{2a\xi}. \tag{58}$$

Furthermore, in these coordinates, we have

$$\bar{x}^2 = -\left(\frac{e^{a\xi}}{a}\right)^2, \tag{59}$$

so that for a constant ξ, we have the hyperbola corresponding to acceleration

$$\alpha = ae^{-a\xi}. \tag{60}$$

The Rindler wedges, in the present case, are defined by

$$\mathrm{R}: \quad 0 \le \bar{t} \le \frac{2}{A+B}\bar{z}, \qquad \mathrm{L}: \quad \frac{2}{A+B}\bar{z} \le \bar{t} \le 0. \tag{61}$$

In the wedge labeled "R," we have (as we have pointed out earlier $\mathrm{sgn}(A+B) = \mathrm{sgn}(B-A)$ when $|B| > |A|$)

$$\begin{aligned} \bar{t} &= \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{aV} = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{a(\eta+\xi)}, \\ \bar{z} &= \mathrm{sgn}(A+B)\sqrt{B^2-A^2}\,\frac{1}{2a}(e^{aV} + e^{-aU}) \\ &= \mathrm{sgn}(A+B)\frac{\sqrt{B^2-A^2}}{2a}(e^{a(\eta+\xi)} + e^{-a(\eta-\xi)}). \end{aligned} \tag{62}$$

It follows from this that in this wedge, we can write

$$u = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-aU} = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-a(\eta-\xi)},$$
$$v = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{aV} = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{a(\eta+\xi)}, \tag{63}$$

so that we have

$$U = -\frac{1}{a}\ln\left(-\sqrt{\frac{A+B}{B-A}}au\right), \qquad V = \frac{1}{a}\ln\left(\sqrt{\frac{A+B}{B-A}}av\right). \tag{64}$$

On the other hand, in the other wedge labeled "L," we have

$$\bar{t} = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{aV} = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{a(\eta+\xi)},$$
$$\bar{z} = -\mathrm{sgn}(A+B)\sqrt{B^2-A^2}\,\frac{1}{2a}(e^{aV}+e^{-aU}) \tag{65}$$
$$= -\mathrm{sgn}(A+B)\frac{\sqrt{B^2-A^2}}{2a}(e^{a(\eta+\xi)}+e^{-a(\eta-\xi)}),$$

so that we can write

$$u = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-aU} = \sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{-a(\eta-\xi)},$$
$$v = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{aV} = -\sqrt{\frac{B-A}{A+B}}\frac{1}{a}e^{a(\eta+\xi)}. \tag{66}$$

In the "L" wedge, therefore, we have

$$U = -\frac{1}{a}\ln\left(\sqrt{\frac{A+B}{B-A}}au\right), \qquad V = \frac{1}{a}\ln\left(-\sqrt{\frac{A+B}{B-A}}av\right). \tag{67}$$

Let us next consider the quantization of a massless scalar field on the light front (equal $\bar{t}$) in the general light-front frame. The field decomposition can be written as

$$\phi(\bar{t},\bar{z}) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{2k}(e^{-ik\bar{t}}a_1(k) + e^{-ik(\bar{t}-[2/(A+B)]\bar{z})}a_2(k) + \text{Hermitian conjugate})$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{2k}(e^{-ik\bar{v}}a_1(k) + e^{-ik\bar{u}}a_2(k) + \text{Hermitian conjugate}), \tag{68}$$

where $a_1(k), a_2(k)$ can be thought of as the annihilation operators for the null modes of the field components. The field $\phi(\bar{t},\bar{z})$ can easily be checked to satisfy light-front quantization conditions provided

$$[a_1(k), a_1^\dagger(k')] = 2k\delta(k-k') = [a_2(k), a_2^\dagger(k')], \tag{69}$$

and the vacuum of the theory satisfies

$$a_1(k)|0\rangle_{\mathrm{GLF}} = 0 = a_2(k)|0\rangle_{\mathrm{GLF}}. \tag{70}$$

On the other hand, we can also quantize the theory in the two Rindler wedges. Here, using the conformal coordinates, we can write the field expansion as

$$\phi(\eta,\xi) = \int_0^\infty dK(g_K^{(\mathrm{R})}(U)b_1(K) + g_K^{(\mathrm{R})}(V)b_1(-K) + g_K^{(\mathrm{L})}(V)b_2(K) + g_K^{(\mathrm{L})}(U)b_2(-K) + \text{Hermitian conjugate}), \tag{71}$$

where we have defined the basis functions $g^{(\mathrm{R})}, g^{(\mathrm{L})}$ in the two wedges as

$$g_K^{(\mathrm{R})}(U) = \begin{cases} \frac{e^{-iK(\eta-\xi)}}{\sqrt{2\pi 2K}} = \frac{e^{-iKU}}{\sqrt{2\pi 2K}} & \text{in R} \\ 0 & \text{in L} \end{cases},$$
$$g_K^{(\mathrm{R})}(V) = \begin{cases} \frac{e^{-iK(\eta+\xi)}}{\sqrt{2\pi 2K}} = \frac{e^{-iKV}}{\sqrt{2\pi 2K}} & \text{in R} \\ 0 & \text{in L} \end{cases},$$
$$g_K^{(\mathrm{L})}(U) = \begin{cases} 0 & \text{in R} \\ \frac{e^{iK(\eta-\xi)}}{\sqrt{2\pi 2K}} = \frac{e^{iKU}}{\sqrt{2\pi 2K}} & \text{in L} \end{cases},$$
$$g_K^{(\mathrm{L})}(V) = \begin{cases} 0 & \text{in R} \\ \frac{e^{iK(\eta+\xi)}}{\sqrt{2\pi 2K}} = \frac{e^{iKV}}{\sqrt{2\pi 2K}} & \text{in L} \end{cases}. \tag{72}$$

We can think of $b_1(K), b_1(-K)$ as the annihilation operators for the two modes in the wedge "R" while $b_2(K), b_2(-K)$ correspond to the annihilation operators in the wedge "L." It is easy to check that with the commutation relations

$$[b_1(K), b_1^\dagger(K')] = 2K\delta(K-K') = [b_1(-K), b_1^\dagger(-K')]$$
$$[b_2(K), b_2^\dagger(K')] = 2K\delta(K-K') = [b_2(-K), b_2^\dagger(-K')], \tag{73}$$

the fields satisfy the conventional commutation relation for a theory quantized on the light front. The Rindler vacuum, which will be the product of the vacua for the theories on the L and the R wedges satisfies

$$b_1(K)|0\rangle_{\mathrm{Rindler}} = b_1(-K)|0\rangle_{\mathrm{Rindler}} = 0,$$
$$b_2(K)|0\rangle_{\mathrm{Rindler}} = b_2(-K)|0\rangle_{\mathrm{Rindler}} = 0. \tag{74}$$

From the definition of the basis functions $g^{(\mathrm{R})}, g^{(\mathrm{L})}$ in (72), we see that they are not analytic and, therefore, we cannot compare the Rindler vacuum to the GLF vacuum directly. In fact, we note that while the annihilation operators in (68) are the coefficients of positive frequency eigenfunctions of the $\bar{P}_0$ operator (Hamiltonian), those in (71) correspond to coefficients of positive frequency eigenfunctions of the boost operator $\bar{K}_3$ along $\bar{z}$ [We note that $\partial_\eta = a(\bar{t}\partial_{\bar{t}} + ((A+B)\bar{t}-\bar{z})\partial_{\bar{z}})$ is proportional to the boost operator along the $\bar{z}$ axis]. In order to compare the two vacua, let us define a new set of basis functions (for the expansion in the Rindler wedges) that are analytic,

$$\begin{aligned} F_K(U) &= \cosh\theta_K g_K^{(\mathrm{R})}(U) + \sinh\theta_K g_K^{(\mathrm{L})*}(U), \\ F_K(V) &= \cosh\theta_K g_K^{(\mathrm{R})}(V) + \sinh\theta_K g_K^{(\mathrm{L})*}(V), \\ G_K(U) &= \cosh\theta_K g_K^{(\mathrm{L})}(U) + \sinh\theta_K g_K^{(\mathrm{R})*}(U), \\ G_K(V) &= \cosh\theta_K g_K^{(\mathrm{L})}(V) + \sinh\theta_K g_K^{(\mathrm{R})*}(V), \end{aligned} \tag{75}$$

with

$$\tanh\theta_K = e^{-\pi K/a}. \tag{76}$$

It is easy to check that this new basis functions are analytic. For example, we note that

$$F_K(U) = \cosh\theta_K(g_K^{(\mathrm{R})}(U) + \tanh\theta_K g_K^{(\mathrm{L})*}(U)) = \frac{\cosh\theta_K}{\sqrt{2\pi}2K}\begin{cases} e^{-iKU} = (-\sqrt{\frac{A+B}{B-A}}au)^{iK/a} & \text{in R} \\ e^{-iK(U-i\pi/a)} = (-\sqrt{\frac{A+B}{B-A}}au)^{iK/a} & \text{in L} \end{cases}, \tag{77}$$

and so on, where u in the wedge L is assumed to lie slightly above the principal branch of the logarithm.

The field expansion in the two wedges in (71) can now be expressed in terms of this basis function as

$$\begin{aligned} \phi(\eta, \xi) = \int_0^\infty \mathrm{d}K &(F_K(U)c_1(K) + F_K(V)c_1(-K) \\ &+ G_K(V)c_2(K) + G_K(U)c_2(-K) \\ &+ \text{Hermitian conjugate}), \end{aligned} \tag{78}$$

where we have defined new field operators resulting from the unitary change in the basis as

$$\begin{aligned} c_1(K) &= \cosh\theta_K b_1(K) - \sinh\theta_K b_2^\dagger(-K), \\ c_1(-K) &= \cosh\theta_K b_1(-K) - \sinh\theta_K b_2^\dagger(K), \\ c_2(K) &= \cosh\theta_K b_2(K) - \sinh\theta_K b_1^\dagger(-K), \\ c_2(-K) &= \cosh\theta_K b_2(-K) - \sinh\theta_K b_1^\dagger(K). \end{aligned} \tag{79}$$

We note that the new creation and annihilation operators can be easily seen to be related to the old ones through a Bogoliubov transformation. Defining the formally unitary operator [15,18,19]

$$U(\theta) = e^{-iG(\theta)}, \tag{80}$$

where

$$\begin{aligned} G(\theta) = -i\int_0^\infty \mathrm{d}K \frac{\theta_K}{2K}&((b_1(K)b_2(-K) - b_2^\dagger(-K)b_1^\dagger(K)) \\ &+ (b_1(-K)b_2(K) - b_2^\dagger(K)b_1^\dagger(-K))), \end{aligned} \tag{81}$$

it is easy to check using the commutation relations (73) that we can write

$$\begin{aligned} c_1(K) &= U(\theta)b_1(K)U^{-1}(\theta), \\ c_1(-K) &= U(\theta)b_1(-K)U^{-1}(\theta), \\ c_2(K) &= U(\theta)b_2(K)U^{-1}(\theta), \\ c_2(-K) &= U(\theta)b_2(-K)U^{-1}(\theta). \end{aligned} \tag{82}$$

We note that the basis functions F_K, G_K are positive frequency with respect to GLF coordinates and hence the expansion in (78) can be directly compared with the field expansion in (68). In particular, we note that the GLF vacuum satisfying (70) can also be written as

$$\begin{aligned} c_1(K)|0\rangle_{\mathrm{GLF}} &= c_1(-K)|0\rangle_{\mathrm{GLF}} = 0, \\ c_2(K)|0\rangle_{\mathrm{GLF}} &= c_2(-K)|0\rangle_{\mathrm{GLF}} = 0. \end{aligned} \tag{83}$$

On the other hand, using (82) as well as (74), it follows now that we can relate the GLF vacuum with the Rindler vacuum as

$$|0\rangle_{\mathrm{GLF}} = U(\theta)|0\rangle_{\mathrm{Rindler}}. \tag{84}$$

Furthermore, from (79) we note, for example, that

$$c_1(K)|0\rangle_{\mathrm{GLF}} = \cosh\theta_K(b_1(K) - e^{-\pi K/a}b_2^\dagger(-K))|0\rangle_{\mathrm{GLF}} = 0, \tag{85}$$

which implies that

$$b_1(K)|0\rangle_{\mathrm{GLF}} = e^{-\pi K/a}b_2^\dagger(-K)|0\rangle_{\mathrm{GLF}}. \tag{86}$$

This shows [15,18,19] that the Rindler observer perceives the GLF vacuum as a thermal vacuum at a temperature

$$T_{\eta\xi} = \frac{a}{2\pi}. \tag{87}$$

The corresponding temperature in the GLF frame can then be obtained from Tolman's law to be

$$\begin{aligned} \frac{T_{\mathrm{GLF}}}{\sqrt{\bar{g}_{00}^{(\mathrm{GLF})}}} &= \frac{T_{\eta\xi}}{\sqrt{\bar{g}_{\eta\eta}^{(\text{conformal})}}} \quad \text{or} \\ T_{\mathrm{GLF}} &= \sqrt{\frac{A+B}{B-A}}\frac{ae^{-a\xi}}{2\pi} = \sqrt{\frac{A+B}{B-A}}\frac{\alpha}{2\pi}, \end{aligned} \tag{88}$$

where we have used (60). This shows through a systematic analysis from the Hilbert space point of view that a uniformly accelerating observer would perceive the GLF vacuum to correspond to a thermal vacuum with a temperature (in the GLF frame) given by (4). This is, of course, consistent with the results of the earlier section, but as mentioned earlier is useful in showing that any matrix element of the theory would appear as a thermal amplitude to the accelerating observer.

V. SUMMARY

In this paper, we have investigated the phenomenon of Unruh effect for a massless scalar field theory quantized on the light front in the general light-front frame. In this case, there are two possible directions for acceleration (as opposed to the Minkowski frame which is isotropic) and we have determined the uniformly accelerating coordinates for both the possible accelerations. By transforming the Green's function for the massless scalar field quantized on the light-front to the uniformly accelerating coordinate systems, we have shown that it has a thermal character corresponding to a unique temperature given by Tolman's law (4) (independent of the direction of acceleration). We have also carried out a systematic analysis of this phenomenon from the point of view of the Hilbert space and have shown that a Rindler observer finds the vacuum of the theory to correspond to a thermal vacuum with the temperature given by Tolman's law, which in turn shows that any amplitude of the theory would appear to be a thermal amplitude to such an observer.

Finally, we note from the results obtained from our analysis that it is an interesting question to determine whether the vacuum of a quantum field theory quantized on equal time surface is equivalent (through some Bogoliubov transformation or otherwise) to that of the theory quantized on the light front. *A priori* there is no reason for such an equivalence, but the fact that *physical amplitudes* in perturbation theory in the two theories agree on a case by case basis, both at zero as well as finite temperature, (there is no proof that this should happen in general and the agreement at zero temperature depends crucially on the regularizations used since the power counting arguments in the two theories are quite distinct) makes it a worthwhile topic of study. Any direct relation between the two vacua will lead to a better understanding of many aspects of both (equal time and light front) the theories.

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[13] We take this opportunity to point out a couple of typos in Ref. [6]. Namely, the exponent in the propagator in Eq. (38) should have a $\bar{\bar{k}}$ and not $\bar{k}$ as it appears in the paper. Similarly, the last term on the right hand side of eq. (23) should have $\bar{\partial}_3\lambda$ instead of $\bar{\partial}_3\phi$.

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