# Center-symmetric 1/N expansion

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The free energy of $U(N)$ gauge theory is expanded about a center-symmetric topological background configuration with vanishing action and vanishing Polyakov loops. We construct this background for $SU(N)$ lattice gauge theory and show that it uniquely describes center-symmetric minimal action orbits in the limit of infinite lattice volume. The leading contribution to the free energy in the $1/N$ expansion about this background is of $\mathcal{O}(N^0)$ rather than $\mathcal{O}(N^2)$ as one finds when the center symmetry is spontaneously broken. The contribution of planar 't Hooft diagrams to the free energy is $\mathcal{O}(1/N^2)$ and subleading in this case. The change in behavior of the diagrammatic expansion is traced to Linde's observation that the usual perturbation series of non-Abelian gauge theories suffers from severe infrared divergences [A. Linde, Phys. Lett. B 96, 289 (1980).]. This infrared problem does not arise in a center-symmetric expansion. The 't Hooft coupling $\lambda = g^2N$ is found to decrease $\propto 1/\ln(N)$ for large $N$. There is evidence of a vector-ghost in the planar truncation of the model.

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## I. INTRODUCTION

Confinement can be defined as the absence of asymptotic states in nontrivial multiplets of the global gauge group. Since the number of singlet states does not increase proportional to $N$, the free energy of a $U(N)$ gauge theory in a confining phase should be of order $N^0$ [1]. Perturbatively, the adjoint multiplet of gauge bosons and the fundamental fermion multiplets contribute to the free energy density of $U(N)$ gauge theory in $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$. A direct application of 't Hooft's $1/N$ expansion [2,3] apparently also gives a free energy density of order $N^2$ even at low temperatures. Using the Dyson-Schwinger equations of the lattice and Migdal's factorization condition for planar diagrams, Gocksch and Neri [4] on the other hand found that the free energy density in the confining phase does not depend on the temperature at $N = \infty$.

A leading contribution to the free energy of $U(N)$ gauge theory of order $N^2$ is not compatible with the result of Gocksch and Neri [4]. It is more reasonable to assume that the coefficient of the $N^2$-term in the $1/N$-expansion of the free energy vanishes in the confining phase and that the model defined by planar diagrams is a topological theory without dynamical degrees of freedom. It turns out that this is the case in the $N \rightarrow \infty$ limit only. Factorization for large $N$ in the confining phase implies that $U(N = \infty)$ is described by a matrix model that depends on space-time parametrically [5–8] only.

The confining phase of pure Yang-Mills models is characterized by a global center symmetry. This symmetry also is essential in the formulation of reduced models [6–8] at $N = \infty$. The objective here is to construct an $1/N$ expansion that preserves this center symmetry in every order. Since the pure Yang-Mills action is invariant, one can achieve this by expanding about a center-symmetric topological field configuration.

We show the absence of contributions to the free energy proportional to $N^2$ and $N$ when $SU(N)$ gauge theory is expanded about such a center-symmetric orbit. In this expansion, planar 't Hooft diagrams contribute to the free energy in order $1/N^2$. The present analysis systematizes and extends the result of Gocksch and Neri [4] in several ways. The center-symmetric $1/N$ expansion is possible for all $N$. It not only gives the order of planar contributions to the free energy but also of higher genus 't Hooft diagrams. Although fields in the fundamental representation explicitly break the center symmetry, there are no contributions to the free energy of order $N$ in this expansion—planar diagrams with a single fundamental color loop contribute in $\mathcal{O}(1/N^3)$. The leading temperature dependent contributions to the free energy are of order $N^0$. These nonplanar contributions survive the large $N$ limit—as might be expected if the masses of asymptotic singlet states have a finite limit [3].

In the topological sector with vanishing instanton number that interests us here, the classical action of a gauge theory vanishes at field configurations with minimal action. However, this classical field is not necessarily a pure gauge configuration. Since the local curvature of the configuration vanishes, the possibly nontrivial gauge-invariant quantities are noncontractible Wilson loops. [These noncontractible loops in general are sensitive to global symmetries of the action and thus can distinguish different phases of the model.] At a finite temperature $T$ and infinite volume $\mathcal{V}$, configurations with vanishing curvature are characterized by their Polyakov loops, noncontractible Wilson loops in the Euclidean temporal direction.

Specifically, consider the Polyakov loop of an $SU(N)$ gauge theory at finite temperature $T$ with periodic boundary conditions for the connection,

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$$\mathcal{L}(\mathbf{x}) = \mathrm{Tr}U(x)$$
$$\text{with } U(x) = \mathcal{P}\exp\left[ i\int_{x_4}^{x_4+1/T} V_4(\mathbf{x},\tau)d\tau \right]. \tag{1}$$

In Eq. (1) $\mathcal{P}$ denotes ordering of the exponential along the path and $V_\mu$ is the gauge connection in the fundamental representation. On the lattice, $U(x)$ is the ordered product of the links in the periodic temporal direction, beginning with the link at $x$. One can choose a gauge in which $V_4(\mathbf{x},\tau)$ does not depend on the Euclidean time $\tau$ and is diagonal. On the lattice this may be achieved in three steps: one first uses the gauge freedom to set all temporal links apart from those on the $x_4 = 0$ time slice to unity. [The nontrivial temporal links of this representative configuration then are the $U(\mathbf{x}, x_4 = 0)$ of the Polyakov loop.] One next uses time-independent gauge transformations to diagonalize the remaining nontrivial temporal links. Since the permutation group is a subgroup of $SU(N)$, the phases in addition can be ordered so that the temporal links of the $x_4 = 0$ time slice are of the form,

$$\begin{aligned} &U(\mathbf{x}, x_4 = 0) = \mathrm{diag}(e^{i\theta_1(\mathbf{x})}, \ldots, e^{i\theta_N(\mathbf{x})}),\\ &\text{with } \sum_{j=1}^{N}\theta_j(\mathbf{x}) = 0,\\ &\text{and } -\pi \le \theta_1(\mathbf{x})\ldots \le \theta_j(\mathbf{x}) \le \theta_{j+1}(\mathbf{x})\ldots\\ &\qquad \le \theta_N(\mathbf{x}) < \pi, \end{aligned} \tag{2}$$

and all other temporal links are unity. The Abelian invariant subgroup of the configuration is enhanced to a non-Abelian one when some of the phases in Eq. (2) are degenerate. The corresponding continuum configuration in this case may have a nontrivial monopole number [9]. [Since all lattice configurations are contractible, the usual topological classification of smooth continuum configurations cannot be used, but degenerate configurations that are invariant under a non-Abelian subgroup of $SU(N)$ can also be found on the lattice.]

One finally may use time-dependent Abelian gauge transformations to evenly distribute the $U(\mathbf{x}, 0)$ of Eq. (2) in temporal direction. In the continuum limit, the resulting configuration corresponds to a temporal component of the connection $V_4(\mathbf{x})$ that does not depend on the Euclidean time $x_4$ and is Abelian.

The perturbation series can be constructed about any configuration of minimal classical action. Although such a configuration generally will not correspond to a minimum of the effective action, the perturbation series nevertheless yields some information about the configuration space in its vicinity. We will see that the confining phase extends to arbitrary small values of the 't Hooft coupling for sufficiently large $N$. That the effective coupling may become weak in the confining phase at sufficiently large $N$ was previously observed [10] by exploiting the analogy with string theory. We show that the perturbative analysis leads to the same conclusion.

In the topologically trivial sector, the local curvature of a minimal action orbit vanishes. The previous construction implies that the temporal links of periodic lattice configurations with *minimal* Wilson action can be chosen Abelian *and* constant across the whole lattice. Since every plaquette-action of a minimal action configuration vanishes and all temporal links apart from those on a particular time slice can be set to unity by a gauge transformation, the spatial links of a minimal action configuration do not depend on time in such a gauge. Periodicity of the configuration in time then requires that the eigenphases of two spatially adjacent temporal Abelian links are the same: since *all* plaquette-actions vanish we must also have that $ga = a'g$, or $gag^\dagger = a'$ for two equal spatial links $g \in SU(N)$ and two adjacent temporal links $a$ and $a'$ on the $x_4 = 0$ time-slice. The previous procedure shows that $a$ and $a'$ can be chosen to lie in the Abelian subgroup of $SU(N)$. $a$ and $a'$ thus are the same up to a permutation of their eigenphases. Taking into account that the eigenphases have been ordered, one concludes that $a = a'$ in this particular gauge.

All temporal links on the $x_4 = 0$ time slice of this representative of an orbit with minimal Wilson action thus are Abelian and the *same*—all other temporal links are unity. We, in particular, have that minimal action configurations of a time-periodic $SU(N)$-lattice are characterized by a Polyakov loop that does not depend on the chosen spatial point. A spatially constant Abelian gauge transformation can be used to evenly distribute the temporal links of the $x_4 = 0$ time-slice in temporal direction. One thus obtains a representative of any orbit with *minimal* Wilson action that is described by a temporally and spatially constant Abelian connection $V_4$. When none of the eigenphases of the temporal links are degenerate, spatial links in this gauge also have to be in the Abelian subgroup and do not depend on Euclidean time.

## II. TOPOLOGICAL CONFIGURATIONS AND CENTER SYMMETRY

The minimal action configurations of $SU(N)$ are further characterized by their transformation under a global $Z_N$ symmetry of the Wilson action. This so-called center symmetry is generated by multiplying every temporal link on a particular time slice by an element of the center of $SU(N)$—possibly followed by a (periodic) gauge transformation of the configuration. This transformation multiplies the Polyakov loops of any configuration by a root of unity, but does not change the Wilson action. The center symmetry therefore maps minimal (Wilson) action configurations onto themselves. It allows to distinguish between minimal action orbits that are invariant under this discrete global symmetry and those that are not.

### A. The center-symmetric topological configuration

Since any Polyakov loop is multiplied by a root of unity, an orbit is center-symmetric only if its Polyakov loops vanish. The $N$ eigenphases of $U(x) \in SU(N)$ in Eq. (1) therefore sum to zero and their product is $\det U(x) = 1$. The discussion in the introduction shows that one may choose $U(\mathbf{x}, 0)$ constant and in the Abelian subgroup. The constant $\theta_j^{(0)}$ on the $x_4 = 0$ time slice of such a center-symmetric minimal action configuration thus are,

$$\theta_j^{(0)}(\mathbf{x}) = \pi(2j - N - 1)/N, \quad \text{for } j = 1, 2, \ldots, N. \quad (3)$$

A center transformation simply permutes the phases in Eq. (3) and the previous ordering can be restored by a time-independent $SU(N)$ gauge transformation (of which the permutations are a subgroup). The fact that the eigenphases in Eq. (3) are equidistant was recently exploited to define an order parameter for the center-symmetric phase [11].

None of the eigenphases of a center-symmetric configuration with minimal action are degenerate. The spatial links therefore do not depend on time and are Abelian as well. On a lattice that is periodic in every direction they can in fact be chosen Abelian and constant. To see this, one may proceed as follows. Using time-independent Abelian gauge transformations only, all (already time-independent) spatial links in $x_3$-direction apart from those on the $x_3 = 0$ slice may be set to unity. This time-independent Abelian gauge transformation does not change the temporal links. Since this is an Abelian minimal action configuration on a lattice that is periodic in $x_3$, the links in the $x_3$-direction on the $x_3 = 0$ slice in fact must all be equal. The remaining Abelian links in the $x_2$- and $x_1$-directions at this stage do not depend on $x_3$ (nor on $x_4$). Using an Abelian gauge transformation that depends on $x_3$ only, the links in $x_3$-direction on the $x_3 = 0$ slice can be distributed evenly in the $x_3$-direction. The result is a gauge equivalent configuration with constant Abelian links in $x_4$- and $x_3$- directions and Abelian links in $x_2$- and $x_1$- directions that do not depend on $x_3$ nor on $x_4$. The procedure is repeated with $x_4$ and $x_3$-independent Abelian gauge transformation to also make the links in $x_2$-direction constant (links in $x_1$-direction at this point do not depend on $x_2, x_3$ nor $x_4$). Abelian gauge transformations that depend only on $x_1$ can finally be used to obtain a configuration with Abelian links in each direction that do not depend on space or time.

In general there are inequivalent center-symmetric minimal action orbits that differ in the eigenphases of the spatial links. However, this distinction is critical at finite volume only. The above construction implies that the phases of the constant spatial links of the final configuration can be chosen to all fall in the interval $(-\pi/L, \pi/L]$, where $L$ is the spatial lattice dimension in lattice units. In the limit $L \to \infty$, the spatial links of the configuration all tend to unity. The arbitrarily small deviations from unity can only be observed by noncontractible Wilson loops that wrap around the whole *spatial* extent of the lattice. These are not observables in the infinite volume limit and a center-symmetric orbit of minimal action in this sense is *unique*.

In the infinite volume limit at a finite temperature $T$ any center-symmetric orbit with vanishing curvature can be represented by a constant Abelian connection. Using Eq. (3) and the previous observation that spatial links of this representative tend to unity for large spatial volume, this center-symmetric Abelian background connection is,

$$\begin{aligned} g\bar{V}_4^{(0)} &= ga_4 \\ &= T\mathrm{diag}(\theta_1^{(0)}, \ldots, \theta_N^{(0)}) \\ &= 2\pi T \mathrm{diag}\left(\frac{-N+1}{2N}, \frac{-N+3}{2N}, \ldots, \frac{N-3}{2N}, \frac{N-1}{2N}\right) \\ &\quad \times g\bar{V}_i^{(0)} = ga_i \qquad (4) \\ &= 0, \quad \text{for } i = 1, 2, 3. \end{aligned}$$

Quadratic fluctuations about the center-symmetric configuration of temporal links have been considered previously (see for instance [12,13] and (for $N \to \infty$) [10]). We here will examine the perturbation series about the configuration Eq. (4) to all orders in the $1/N$ expansion of the free energy.

### B. Topological configurations that break the center symmetry

If any Polyakov loop of a topological configuration does *not* vanish, it necessarily belongs to a multiplet of minimal action configurations. If $N$ is not prime, the configuration may break a subgroup of $Z_N$ only. However, the flat connection $V_4 = 0$ breaks the $Z_N$-group completely. It is one of the $N$ Abelian configurations of the form,

$$g\bar{V}_4^{(q)} = \frac{2\pi T q}{N} \mathrm{diag}((1-N), 1, \ldots, 1, 1); q = 1, 2, \ldots, N \quad (5)$$

These configurations have degenerate eigenphases and one cannot argue that the spatial links of such a minimal action configuration are Abelian. Contrary to the center-symmetric case, it is not clear that the index $q$ uniquely identifies a minimal action orbit in the infinite volume limit.

The configurations of Eq. (5) have been studied extensively [14]. They correspond to minima of the free energy at high temperatures $T$ when corrections proportional to the coupling $g^2(T)$ are negligible. [However, the homogeneous vacua of Eq. (5) do *not* solve the infrared problem of the high-temperature expansion observed by Linde [15]—the high-temperature phase probably [12] can be described by domains of such vacua with different index $q$.]

Minimal action configurations that break the center-symmetry to a subgroup of $Z_N$ also can be constructed for nonprime $N$. They could play a rôle in the (perhaps rather complex) phase structure of an $SU(N)$-model with nonprime $N$. Minimal action solutions that break the $Z(N)$-symmetry correspond to perturbative minima of the free energy. They are not the minima of the free energy in a center-symmetric (confining) phase.

## III. LARGE $N$ EXPANSION IN A CENTER-SYMMETRIC BACKGROUND

We are interested in the expansion of the free energy of a $U(N)$-model at finite temperature for large values of $N$. We shall argue that the model is in a confining phase as long as the center-symmetric background is stable. [More specifically, the free energy density $F$ of a $U(N)$ gauge theory expanded about the center-symmetric background is $\mathcal{O}(N^0)$ rather than $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ as one expects when asymptotic states form multiplets of the adjoint, respectively, fundamental, representation of $SU(N)$.]

The center-symmetric background of Eq. (4) is a *maximum* of the 1-loop free energy [14], whose minima are at the configurations of Eq. (5) that spontaneously break the center symmetry. It was recently found [16] that the non-perturbative contribution from calorons [17], can make the minima of the 1-loop free energy unstable at low temperatures. Near the deconfinement transition, calorons with nontrivial holonomy have been observed by cooling $SU(2)$ and $SU(3)$ lattice configurations [18]. However, classical solutions of finite action generally are suppressed in the limit of large $N$ ($g^2N$ finite). A semiclassical mechanism for restoring the center symmetry thus appears unlikely at large $N$.

Lattice studies at relatively small $N$ see a distribution of values for the Polyakov loop in the confining phase, rather than a strong concentration near $\mathcal{L}(\mathbf{x}) = \mathbf{0}$. This can also be seen by studying the strong-coupling expansion in a gauge where all temporal links except those on the $x_4 = 0$ time slice are set to unity. To leading order, the measure for the eigenphases $\theta_j(\mathbf{x})$ of Eq. (2) in this case is given by the Vandermonde determinant of the eigenvalues of the nontrivial temporal link,

$$[dU] \to \prod_{i=1}^{N} d\theta_i \prod_{i>j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right). \tag{6}$$

This measure is gauge invariant and vanishes when any two eigenphases coincide[1]. It is maximal when the $N$ phases are evenly distributed over the circle $[0, 2\pi]$. For small values of $N$, the dependence on the eigenphases of Eq. (6) is rather weak. However, for $N \sim \infty$ the support of the measure Eq. (6) becomes restricted to the immediate vicinity of the configuration Eq. (3). The strong-coupling limit of $U(N \sim \infty)$ lattice gauge theory thus is an example for the more general conjecture [19] that fluctuations are suppressed at large $N$.

Since its expectation vanishes, the usual factorization argument fails for the Polyakov loop in a center-symmetric phase. However, at strong coupling one can explicitly show [20] that the distribution of $\mathcal{L} = \mathrm{Tr}U = \sum_j \exp(i\theta_j)$ converges to a standard normal in the limit $N \to \infty$. The fact that the variance of the Polyakov loop does not grow with $N$ implies that the standard deviation of the eigenphases is of order $1/\sqrt{N}$ only. This also is apparent from Eq. (6). $<|\mathcal{L}|^2> = 1$ furthermore is consistent with usual $1/\sqrt{N}$ counting, which suggests that the standdard deviation of the eigenphases is $\mathcal{O}(1/\sqrt{(N)})$ in the confining phase even when the strong-coupling limit does not apply.

The orbit described by Eq. (4) is invariant under spatial translations and rotations and minimizes the free energy density of lattice gauge theory in the strong-coupling limit. It also is the only center-symmetric candidate for a perturbative vacuum orbit. Since fluctuations of the temporal links are expected to be small at large $N$, we now consider the perturbative expansion about the background of Eq. (4) at large $N$.

The Euclidean time derivative of a minimally coupled field in a nontrivial representation of the group occurs through the covariant derivative only. In the background of Eq. (4) the time derivative of a field $\Phi$ of the adjoint representation thus is replaced by,

$$\begin{aligned} \partial_4 \Phi^a_b \to \bar{\mathcal{D}}_4 \Phi^a_b &= \partial_4 \Phi^a_b + ig[a_4, \Phi]^a_b \\ &= \partial_4 \Phi^a_b + \frac{2\pi i T}{N}(a-b)\Phi^a_b; \\ a, b &= 1, \ldots, N. \end{aligned} \tag{7}$$

The time derivative of fields $\Psi$ in the fundamental representation is similarly replaced by,

$$\begin{aligned} \partial_4 \Psi^a \to \bar{\mathcal{D}}_4 \Psi^a &= \partial_4 \Psi^a + ig a_4^{ab} \Psi^b \\ &= \partial_4 \Psi^a + \frac{2\pi i T}{N}\left(a - \frac{N+1}{2}\right)\Psi^a; \\ a &= 1, \ldots, N. \end{aligned} \tag{8}$$

Physical correlation functions are colorless. All color indices are summed over. At any finite temperature and for any $N$ Eq. (7) and (8) imply that we can associate a discrete "color momentum",

$$\xi(a) = \frac{2\pi T}{N}(a - (N+1)/2); \quad a = 1, \ldots, N, \tag{9}$$

with a color index of the fundamental representation. For sufficiently large $N$ one is tempted to replace sums over color indices by integrals and neglect the error due to the fact that $\xi(a)$ only takes discrete values,

$$\sum_{a=1}^{N} \to \frac{N}{2\pi T} \int_{-\pi T}^{\pi T} d\xi(a) + \mathcal{O}(1/N). \tag{10}$$

[1]Degenerate configurations that are invariant under a non-Abelian subgroup of $U(N)$ thus have vanishing weight at strong coupling.

Note that color momentum is in the compact interval $[-\pi T, \pi T]$ that does not depend on $N$. Loop integrals over color momentum do *not* induce new UV-divergences. [Something rather similar occurs in solid-state physics where momenta are restricted to a single Brillioun-cell—the associated space is an infinite (periodic) lattice of points. There are no UV-divergences in this case, since the smallest distance is the lattice spacing.] The factors of $N$ (see Eq. (10)) from the loop integrals over color momentum can almost all be absorbed by redefining the coupling,

$$g^2 N \to \lambda. \tag{11}$$

Contrary to the $1/N$-expansion in the broken phase, the reduced coupling $\lambda$ does depend on $N$. We argue in Sec. IV B that the remaining dependence is logarithmic only.

In his seminal work on large $N$ [2] 't Hooft has shown that the contribution of a connected vacuum diagram in ordinary perturbation theory is proportional to a power of $N$ that depends on the difference in the number of color- and momentum- loops of the diagram only. Using 't Hooft's doubleline notation one obtains a topological expansion of $U(N)$ gauge theory in terms of the genus of perturbative diagrams. For a background configuration of Eq. (5) that breaks the center symmetry, the topological expansion coincides with an expansion in powers of $1/N$ (in powers of $1/N^2$ when there are no fundamental representations). In the broken phase, contributions to the free energy of leading order in $N$ are given by planar 't Hooft diagrams that have the topology of a two-sphere, $S_2$.

Some of the characteristics of the usual $1/N$ expansion are retained by an expansion of the model about the center-symmetric but $N$-dependent configuration of Eq. (4). Since the background is diagonal in color, one can still follow the color flow using 't Hooft's doubleline notation (see Fig. 1).

One therefore still has a topological expansion in the genus of the 2-dimensional surfaces described by 't Hooft diagrams. However, *this topological expansion in general no longer coincides with an expansion in* $1/N$. If one could neglect the error due to the discreteness of color momentum in Eq. (10), each color loop indeed would contribute a factor of $N$ only. One then reaches the same conclusions

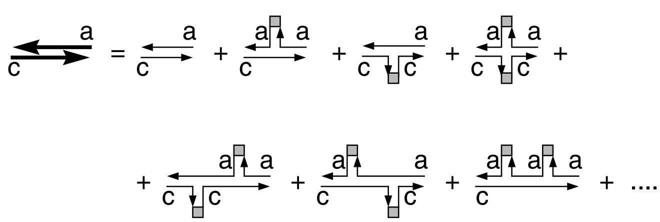


FIG. 1. 'T Hooft's doubleline notation for the "dressed" gluon propagator: insertions of the diagonal Abelian background of Eq. (4) (depicted by shaded squares) do not change the color of a line!

about the order of a diagram as in the broken case. However, due to the discretization error, diagrams of a given genus in the topological expansion may also contribute to higher orders of the $1/N$ expansion. The genus of a diagram thus only gives the *lowest* (superficial) order in the $1/N$ expansion to which it may contribute. This has some interesting consequences for the $1/N$ expansion of the free energy.

We show below that the contribution of planar diagrams to the free energy of a $U(N)$ gauge theory without fundamental fields is of order $1/N^2$ in the center-symmetric background of Eq. (4). [Although they did not specify the order in $1/N$, Gocksch and Neri [4] also found that planar diagrams do not contribute at $N = \infty$.] The leading gluonic contribution to the free energy is of order $N^0$ and given by 't Hooft diagrams with the topology of a torus.

### A. Planar $U(N)$ at finite temperature

The flow of color momentum in planar diagrams is closely associated with that of ordinary momentum. Consider gluonic (vacuum) diagrams without external legs in the doubleline notation of 't Hooft [2]. The number of momentum loops, $L_p$, of a vacuum diagram with $E$ gluon propagators and $V$ interaction vertices is,

$$L_p = E - V + 1. \tag{12}$$

If the diagram is "planar", it has the topology of a 2-sphere [2], $S_2$. Gluon propagators are the edges of cells and the Euler number, $\chi$, of a diagram is,

$$\chi = V - E + L_c. \tag{13}$$

Here $L_c$ is the number of faces, $V$ is the number of vertices and $E$ is the number of edges of the complex. 'T Hooft's doubleline notation shows that the number of faces, $L_c$, is just the number of independent traces over fundamental color indices, that is the number of loops over color momentum. Equation (12) and (13) with $\chi(S_2) = 2$ imply that,

$$L_c = L_p + 1 \text{ for planar vacuum diagrams.} \tag{14}$$

As indicated in Fig. 2(a), the loops over ordinary momentum can be chosen to coincide with the color traces in planar diagrams. One can enforce ordinary momentum conservation at each vertex by writing the momentum of a gluon propagator as the *difference* of two loop momenta associated with each face of the oriented cells the propagator is an edge of. In vacuum diagrams one ends up with just as many loop momenta as color traces. However, one of these loop momenta amounts to an overall translation of all other momenta and is redundant. We again arrive at Eq. (14).

This association between color- and momentum- loops in planar diagrams can be exploited. In equilibrium at finite temperature, gluons are periodic fields in Euclidean time with period $1/T$. Their Matsubara frequency $\omega_n$ therefore

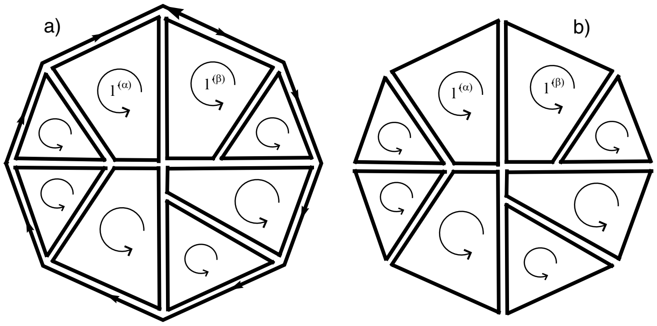

FIG. 2. (a) A typical planar gluonic vacuum diagram that superficially is of order $N^2$. The flow of color *and* of ordinary momentum on each of the faces $\alpha, \beta, \ldots$ is given by composite loop momenta $l^{(\alpha)}, l^{(\beta)} \ldots$. The trace over color for the perimeter loop results in a factor of $N$. (b) A typical planar vacuum diagram with one fundamental loop that superficially is of order $N$. The flow of color *and* ordinary momentum again is captured by composite loop momenta but there is no trace over color only.

is an integer multiple of the fundamental frequency $2\pi T$,

$$\omega_n = 2\pi T n, n \in \mathbf{Z}. \tag{15}$$

We may enforce momentum conservation at a vertex (also at fermion-gluon vertices) by writing the Matsubara frequency of a gluon as the difference of the temporal components of two *half-integer* loop momenta associated with the faces (say $\alpha$ and $\beta$) it is an edge of,

$$\begin{aligned}\omega_{n-m} &= k_4^{(\alpha)}(n) - k_4^{(\beta)}(m) \\ &= 2\pi T((n + 1/2) - (m + 1/2)) \\ &= 2\pi T(n - m).\end{aligned} \tag{16}$$

For a planar vacuum diagram the loop momenta $k_\mu^{(\alpha)}$ can be chosen to run along the color loops. We thus can combine the time-component of loop momentum $k_4^{(\alpha)}$ with the color momentum $\xi^{(\alpha)}$ to the temporal component of a *single* composite loop momentum $l_4^{(\alpha)}$,

$$\begin{aligned}l_4^{(\alpha)}(n, a) &= k_4^{(\alpha)}(n) + \xi^{(\alpha)}(a) \\ &= \frac{2\pi T}{N}(Nn + a - 1/2); \\ & n \in \mathbf{Z}, \\ & a = 1, \ldots, N.\end{aligned} \tag{17}$$

The conservation of the time component of ordinary loop momentum *and* of color at a vertex, thus is equivalent to the conservation of the integer $j = Nn + a$, i.e. the time component of composite loop momentum $l_4$. Note that the sum over the temporal component of composite momentum extends over all half-integers and that the temperature effectively is $T/N$ in planar $U(N)$. In purely gluonic planar vacuum diagrams, every summation over a composite loop index apart from one (the "peripheral" color loop) is accompanied by a factor of $T$. For the peripheral loop of a gluonic planar vacuum diagram the summation is over color only. It amounts to a translation of all other composite loop momenta by a half-integer between $1/2$ and $N + 1/2$. This changes all other summations over half-integer composite loop momenta to summations over *integer* composite loop momenta and in addition yields an overall factor of $N$ [since the expression for the diagram in fact does not depend on finite shifts of all composite loop momenta by integer multiples of $2\pi T/N$].

A planar gluonic vacuum diagram with $L_p$ momentum loops is of perturbative order $(g^2)^{L_p - 1}$ and is proportional to a factor $NT^{L_p}$ due to $L_p$ summations over integer composite loop momenta and the trace over color of the peripheral loop. For the background of Eq. (4) , the regularized[2] planar contributions to the free energy density, $F_{S_2}(T)$, scale as,

$$F_{S_2}(T, g^2, N) = NTf_{S_2}(T/N, \lambda) = T^4/N^2 \tilde{f}_{S_2}(\lambda). \tag{18}$$

After the UV-regularization is removed $\tilde{f}(\lambda)$ is a dimensionless function of the reduced physical coupling $\lambda(T/\Lambda)$, where $\Lambda$ is the appropriate asymptotic scale parameter of the renormalization scheme (see Sec. IV B).

Equation (18) shows that there is no contribution of order $N^2$ to the free energy from planar diagrams in the center-symmetric background of Eq. (4). This implies the absence of asymptotic states in the adjoint representation of the group, that is of (constituent) gluons, in center-symmetric $U(N)$. The free energy of the model otherwise would have to be proportional to $N^2$, the degeneracy of such a multiplet. The result also eliminates the possibility of asymptotic states in higher dimensional representations. Equation (18) suggests that the leading contribution to the free energy of gluonic and center-symmetric $U(N)$ is of order $N^0$ and given by diagrams with the topology of a doughnut $T_2$.

Although this is more or less what one would expect for the confining phase of the model, some omissions and apparent contradictions have to be addressed. Any explicit calculation requires the specification of a gauge and an appropriate regularization procedure. We have to show the existence of a gauge that is compatible with the background of Eq. (4) and does not invalidate the previous argument. We also still have to verify Eq. (18) for the (planar) contribution to the free energy of $U(N)$ of order $\lambda^0$. This "1-loop" contribution to the free energy is a Casimir energy that does not correspond to an evaluation of vacuum diagrams like those discussed above. The following sections support the above argument in important ways.

[2]The superficially quartic ultraviolet divergence of the free energy can be reduced to the superficially quadratic divergence of the specific heat. It then is sufficient to regulate the spatial integrals (see Sec. IV B). The severe infrared divergences of perturbation theory observed by Linde [15] are absent in the present case (see Sec. VI).

## IV. GAUGE FIXING AND RENORMALIZATION

### A. Background Gauge

The background configuration of Eq. (4) is in the maximal Abelian subgroup of $U(N)$ and a crucial point of the previous argument was that all fields couple minimally to it. Covariant Maximal Abelian gauges (MAG) satisfy this requirement and furthermore can be defined [21] on the lattice[3]. The Abelian Ward Identity of MAG implies that the background $ga_\mu$ does not renormalize in these gauges [22]. It therefore is sensible to set this background connection proportional to the physical temperature $T$ in MAG.

However, the fact that MAG distinguishes between diagonal and off-diagonal components of the connection gives rise to additional vertices at which the color flow is *constrained*. In diagrams containing such vertices, not all color loops are independent. This leads to apparent modifications of the $1/N$-expansion and complicates the $1/N$-counting considerably: due to cancellations, gauge-invariant combinations of diagrams can be of different order in $1/N$ than the connected diagrams are individually.

However, the free energy density of $U(N)$ is a gauge-invariant quantity and its expansion in $1/N$ should not depend on the particular gauge. For the purpose of $1/N$-counting, background gauges [23] in fact are much easier to use than covariant MAG. Contrary to MAG one cannot define the BRST-symmetry of background gauges on the lattice [24] since the lattice gauge group is compact [21]. But these are renormalizable gauges that are well defined to all orders in perturbation theory [23,25]. This suffices for our purpose. Background gauges and MAG share the crucial properties that the background $ga_\mu$ does not renormalize [23,25] and that it couples minimally to the fields. Since background gauges are linear, they do not constrain the color flow and do not change the $1/N$ counting of a diagram.

The background gauge in our case is defined by a gauge-fixing part of the Lagrangian of the form,

$$\mathcal{L}^{b.g.}_{GF} = \frac{1}{2\alpha}[\bar{\mathcal{D}}_\mu V^a_{\mu b}][\bar{\mathcal{D}}_\mu V^b_{\mu a}] - \bar{C}^b_a \bar{\mathcal{D}}_\mu (\mathcal{D}_\mu C)^a_b. \quad (19)$$

$\bar{\mathcal{D}}_\mu$ and $\mathcal{D}_\mu$ in Eq. (19) are, respectively, the background covariant derivative (with the connection $ga_\mu$ defined in Eq. (4)) and the ordinary covariant derivative (with connection $gV_\mu$). $C$ and $\bar{C}$ denote the ghost and antighost fields and $\alpha$ is the gauge parameter. Upon shifting the gauge field $V_\mu$ by the constant and Abelian background $a_\mu$, the premise that all time derivatives occur as background covariant derivatives holds in these gauges.

Apart from rigorously defining perturbative propagators and introducing a set of adjoint ghost fields, there are no constraints on the color summations in the background gauge fixing of Eq. (19). These gauges therefore do not modify any of the previous arguments with regard to the order in $N$ of a perturbative diagram.

[3]The lattice in this case is just a theoretical framework for defining the regularized model, and not a very convenient numerical tool.

### B. Regularization and Renormalization

Background gauges are renormalizable to all orders in perturbation theory [25]. We nevertheless have to show that the previous scaling argument is not spoiled by the renormalization procedure. Although the free energy density superficially diverges quartically, the specific heat capacity at constant volume ($C_\mathcal{V} = -T\partial_T^2 F$) is only quadratically divergent. $C_\mathcal{V}$ may, for instance, be regularized by analytic continuation in spatial dimensions only.

The free energy density is recovered by integration of the specific heat with the boundary conditions that the specific entropy, $-\partial_T F$, and the free energy density, $F$, vanish at $T = 0$. This is equivalent to subtracting from the free energy density any contribution that is linear in the temperature. For $D = 3 - \varepsilon$ spatial dimensions, a dimensionally regularized perturbative contribution to the specific heat is of the form,

$$C_\mathcal{V}(T, N, \hat{g}^2; \varepsilon, \mu) = T\partial_T^2 G(T, N, \hat{g}^2; \varepsilon, \mu), \quad (20)$$

where $G(T, N, \hat{g}^2; \varepsilon)$ is the formal expression of the vacuum graph in $D$ spatial dimensions, $\hat{g}^2 = g^2\mu^{-\varepsilon} = \hat{\lambda}/N$ is the renormalized dimensionless coupling and $\mu$ is the renormalization scale. The diagrammatic argument of Sec. III implies that the contributions of planar gluonic vacuum graphs in the center-symmetric background depend on $T$ and $N$ in the particular combination,

$$G_{S_2}(T, N, \hat{g}^2; \mu, \varepsilon) = NTf_{S_2}(T/N, \hat{\lambda}; \varepsilon, \mu) = \frac{T^4}{N^2}\tilde{f}_{S_2}(\hat{\lambda}; N\mu/T, \varepsilon). \quad (21)$$

The subtraction of a constant term and of a term proportional to $T$ from $G$ amounts to the subtraction from $f$ of a term proportional to $N/T$ and of a $T$-independent constant. Possibly divergent terms from planar vacuum diagrams that are proportional to $N^2$ and $N$ thus do not contribute to the specific heat nor to the free energy density.

Further, since the free energy is a physical quantity, $f(\hat{\lambda}; N\mu/T, \varepsilon)$ does not depend on the renormalization point $\mu$. In the renormalization scheme (RS), $f(\hat{\lambda}; N\mu/T, \varepsilon \to 0^+)$ therefore is a function of the renormalization group invariant effective coupling $\lambda(T/\Lambda_{RS})$ only.

The free energies of center-symmetric planar $U(N)$ for different $N$ are proportional only if the temperature is measured in terms of an asymptotic scale parameter, $\Lambda_{RS}$, that does not depend on $N$. To determine this finite renormalization, it is sufficient to, for instance, demand that the deconfinement temperature $T_d(N)$ of planar $U(N)$ be the same for all $N$. The scaled planar free energy

$N^2F_{S_2}(T, N)$ then does not depend on $N$ at any temperature below $T_d$.

Equation (21) shows that the coupling $\lambda(N\mu)$ is a function of $N\mu$ rather than of the renormalization point $\mu$ only. Large values of $N$ correspond to large values of $\mu$ – and to weak coupling. For large $N$ this implies that,

$$\lambda(\mu N) \sim \frac{24\pi^2}{11 \ln\frac{\mu N}{\Lambda_{RS}}}. \tag{22}$$

Eq. (22) suggests that the confining phase can be explored perturbatively at sufficiently large $N$. This weak-coupling confinement regime was first noticed by Polchinski [10] while exploring the analogy between string theory and large-N gauge theory. However, the background of Eq. (4) is expected to be unstable for temperatures $T > T_d$. Setting the renormalization mass $\mu \sim T_d$ in Eq. (22), the unstable regime corresponds to couplings $\lambda < \lambda_d$ with

$$\lambda_d(N) \sim \frac{24\pi^2}{11 \ln(\frac{NT_d}{\Lambda_{RS}})}. \tag{23}$$

For any finite value of $N$, the phase transition occurs at a (perhaps small) but nevertheless finite value of the coupling. An asymptotic perturbative expansion thus is not possible in the center-symmetric phase for any fixed value of $N$. Because of Eq. (23), a perturbative evaluation of the (leading) $\mathcal{O}(N^0)$ contribution to the free energy could nevertheless be reasonably accurate.

Since the usual $1/N$-expansion of $SU(N)$ is algebraic in $1/N$, the logarithmic dependence of the coupling on $N$ in Eq. (23) is somewhat unexpected. However, the center-symmetric orbit of Eq. (3) is described by a connection that is itself $N$-dependent. This leads to a nontrivial $N$-dependence of the momentum scale in planar diagrams. The usual ultraviolet behavior of the model then results in a logarithmic dependence on $N$ of the effective coupling. The perturbative analysis of $U(N)$ gauge theory in the confining phase in this sense becomes self-consistent at large $N$.

## V. OTHER CONTRIBUTIONS TO THE FREE ENERGY OF CENTER-SYMMETRIC $U(N)$ GAUGE THEORY

We saw that the contribution from planar 't Hooft diagrams to the free energy in the center-symmetric phase is of order $1/N^2$ only. 'T Hooft diagrams with the topology of a torus may superficially contribute to the free energy density in order $N^0$. To conclude that the free energy of $U(N)$ indeed is of order $N^0$ in a center-symmetric $1/N$ expansion we have to consider some remaining possibilities.

### A. No contributions to the free energy of $\mathcal{O}(N)$

Fields in the fundamental representation of the group explicitly break the center symmetry and superficially could give rise to contributions to the free energy that are of order $N$. There in fact are no such contributions in an expansion about the center-symmetric background of Eq. (4). The argument is rather similar to the one employed in the gluonic case. Vacuum diagrams that superficially are of order $N$ are planar diagrams with one fundamental color loop only. [A sphere with a hole, topologically a disc $D_2$.] A typical 't Hooft diagram of this kind is shown in Fig. 2(b). We now have that $L_p = L_c$ and can augment to composite loop momenta as before. The difference to planar gluonic vacuum diagrams is the absence of an extra perimeter loop over color only. This suppresses such contributions by a factor of $N$ compared to the planar gluonic ones of Fig. 2(a). The sums over the time-components of composite loop momenta now extend over *half-integer* multiples of the fundamental frequency $2\pi T/N$. The previous scaling argument shows that such vacuum diagrams contribute to the free energy density in order $1/N^3$:

$$F_{D_2}(T, g^2, N) = Tf_{D_2}(T/N, \lambda) = \frac{T^4}{N^3}\tilde{f}_{D_2}(\lambda(T/\Lambda)) \tag{24}$$

Below we explicitly find that this is also true for contributions of order $\lambda^0$. There thus are no contributions of order $N^2$ or $N$ in the expansion of the free energy density of a $U(N)$ gauge theory about the center-symmetric background of Eq. (4). Since planar contributions to the free energy from adjoint and fundamental fields vanish in the limit of large $N$, the center-symmetric *planar* $U(N)$ model approaches a topological theory without dynamical degrees of freedom.

### B. The free energy of center-symmetric $U(N)$ gauge theory to order $\lambda^0$

The previous diagrammatic analysis does not extend to the 1-loop contribution to the free energy density. We explicitly compute it for an $U(N)$ gauge theory with $N_F$ Dirac fermions in the fundamental representation. The relevant quadratic part of the Lagrangian is,

$$\begin{aligned}\mathcal{L}_0 = \sum_{a,b=1}^{N}\Big[&\frac{1}{4}(\bar{\mathcal{D}}_\mu V^a_{\nu b} - \bar{\mathcal{D}}_\nu V^a_{\mu b})(\bar{\mathcal{D}}_\mu V^b_{\nu a} - \bar{\mathcal{D}}_\nu V^b_{\mu a})\\ &+ \frac{1}{2\alpha}(\bar{\mathcal{D}}_\mu V^a_{\mu b})(\bar{\mathcal{D}}_\nu V^b_{\nu a}) + (\bar{\mathcal{D}}_\mu \bar{C}^a_b)(\bar{\mathcal{D}}_\mu C^b_a)\Big]\\ &+ \sum_{j=1}^{N_F}\sum_{a=1}^{N}[\bar{\Psi}^j_a \gamma_\mu \bar{\mathcal{D}}_\mu \Psi^a_j + im_j \bar{\Psi}^j_a \Psi^a_j]. \end{aligned} \tag{25}$$

In Eq. (25) the $\gamma_\mu$ are the hermitian Euclidean Dirac matrices that satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \mathbf{1}\delta_{\mu\nu}$. The time component of the background covariant derivative $\bar{\mathcal{D}}_4$ for the fundamental and adjoint representation is given in Eq. (8) and (7), respectively, ($\bar{\mathcal{D}}_\mu = \partial_\mu$ for $\mu \neq 4$). The gluon- ($V_\mu$) and ghost- ($C, \bar{C}$) fields satisfy periodic boundary conditions in temporal direction whereas the fermions ($\Psi_j, \bar{\Psi}_j$) are antiperiodic.

Since the free energy does not depend on the gauge parameter, we may for simplicity choose the Feynman-like gauge $\alpha = 1$ to compute it. For the constant background of Eq. (4), the eigenvalues of the operator $\bar{\mathcal{D}}_\mu \bar{\mathcal{D}}_\mu$ are readily obtained and the functional integral over quadratic fluctuations about this background can be formally performed. For $D = 3 - \varepsilon$ spatial dimensions, the regulated contribution to $\partial_T^2 F_0(T, N; \mu, \varepsilon)$, of the noninteracting model is:

$$\partial_T^2 F_0(T, N; \mu, \varepsilon) = \partial_T^2 \frac{T}{2} \sum_{n=-\infty}^{\infty} \sum_{a=1}^{N} \int \frac{d^D k \mu^\varepsilon}{(2\pi)^D} \left\{ \sum_{b=1}^{N} 2 \ln\left[\frac{k^2 + (2\pi T/N)^2 (nN + a - b)^2}{\mu^2}\right] - 4 \sum_{j=1}^{N_F} \ln\left[\frac{k^2 + (2\pi T/N)^2 (nN + a - 1/2)^2 + m_j^2}{\mu^2}\right]\right\}. \tag{26}$$

Noting that $nN + a$ ranges over all integers, Eq. (26) simplifies to,

$$\partial_T^2 F_0(T, N; \mu, \varepsilon) = \partial_T^2 \frac{T}{2N} \sum_{n=-\infty}^{\infty} \int \frac{d^D k \mu^\varepsilon}{(2\pi)^D} \left\{ 2N^2 \ln\left[\frac{k^2 + (2\pi T/N)^2 n^2}{\mu^2}\right] - 4N \sum_{j=1}^{N_F} \ln\left[\frac{k^2 + (2\pi T/N)^2 (n - 1/2)^2 + m_j^2}{\mu^2}\right]\right\}. \tag{27}$$

This expression converges for $D < 1$ spatial dimensions and thus is at most quadratically divergent. Scale invariance of the noninteracting model defined by Eq. (25) implies the absence of quadratic divergences in the massless case [26].

The contribution to the free energy of a noninteracting massless bosonic degree of freedom at temperature $T$ is finite and for $D = 3$ spatial dimensions is [27],

$$F_{\text{boson}}(T, m = 0) = -\frac{T^4 \pi^2}{90}. \tag{28}$$

That from a noninteracting massive fermionic degree of freedom is finite as well [27],

$$F_{\text{fermion}}(T, m) = \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(nm/T). \tag{29}$$

In Eq. (29) $K_2(z)$ is the K-Bessel function normalized so that for small arguments $K_2(|z| \sim 0) = 2/z^2$. [Note that $\frac{7}{8} F_{\text{boson}}(T, m = 0) = F_{\text{fermion}}(T, m = 0) \leq F_{\text{fermion}}(T, m) \leq 0$. The last inequality results because $z^2 K_2(z)$ is a monotonically decreasing function of its argument on the positive real axis, with $z^2 K_2(z) \leq 2$ for all $z \geq 0$. The contribution of massive fermions to the free energy density is exponentially small for $T \ll m$.]

With the integration conditions that the free energy density and the specific entropy vanish at zero temperature, ($F_0(0, N) = \partial_T F_0(0, N) = 0$), the specific heat completely specifies the free energy density. One can read off the 1-loop contribution to the free energy density from Eq. (27): to lowest order in the coupling, the free energy density of center-symmetric $U(N)$ gauge theory at temperature $T$ is that of $2N^2$ noninteracting bosonic degrees of freedom and $4NN_F$ (massive) fermionic degrees of freedom *but at a temperature of* $T/N$. Using Eq. (28) and (29) one has,

$$\begin{aligned} F_0(T, N) &= 2N^2 F_{\text{boson}}(T/N, m = 0) + 4N \sum_{j=1}^{N_F} F_{\text{fermion}}(T/N, m_j) \\ &= -\frac{T^4 \pi^2}{45 N^2} + \frac{2T^4}{\pi^2 N^3} \sum_{j=1}^{N_F} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \times \left(\frac{nNm_j}{T}\right)^2 K_2(nNm_j/T). \end{aligned} \tag{30}$$

We thus find the same behavior in $N$ for the 1-loop contribution to the free energy density as for planar diagrams. We in fact used the same arguments, combining color- and momentum- flow to a single composite momentum $l_4(n, a) = (2\pi T/N)(nN + a)$. Although the center-symmetric background effectively leads to an $N$-dependent rescaling of the temperature, it is perhaps more appealing to view this as an (for sufficiently large $N$) almost complete cancellation of individual contributions to the free energy (see Appendix). Since the background of Eq. (4) essentially shifts frequencies by a fraction of the fundamental frequency $2\pi T$ a partial cancellation can occur: for noninteracting fields this phase shift effectively amounts to a change in boundary conditions and the free energy density is sensitive to this change[4]. However, it is remarkable that this cancellation is almost complete in planar $U(N)$ at large $N$.

[4]The contribution to the free energy density of a bosonic degree of freedom satisfying antiperiodic boundary conditions (corresponding to a shift of the frequency by $\pi T$) for instance is positive.

### C. Contributions to the free energy of order $N^0$

Center-symmetric *planar* $U(N)$ in the limit $N \to \infty$ is devoid of physical degrees of freedom and a topological model. Fortunately, the scaling arguments we used to show the absence of contributions to the free energy of orders $N^2$ and $N$ break down when there are *more* independent momentum- than color- loops, that is when $L_p > L_c$. One then has at least one loop momentum that cannot be augmented to a composite momentum that includes the color flow. Such contributions to the free energy density do not scale with $N$. The corresponding 't Hooft diagrams either include more than one fermion loop or are nonplanar. The leading contribution of $\mathcal{O}(N^0)$ is given by diagrams that topologically either are a torus or a sphere with two holes [i.e. a disk with a hole]. These two classes of 't Hooft diagrams correspond to contributions to the free energy of order $N^0$ from noninteracting, colorless, asymptotic glueball- and meson-states, respectively. These are the stable asymptotic states [3] at large $N$.

The fact that the free energy is $\mathcal{O}(N^0)$ for sufficiently large $N$ implies that in the center-symmetric background of Eq. (4) *all* asymptotic states are color singlets. No higher dimensional representation of the global color group contributes. The center-symmetric expansion "confines" color in this sense. [Note that this definition of confinement is somewhat stronger than Wilson's screening criterion for static color charges in the fundamental representation. The latter cannot be applied in the presence of light meson states.]

To conclude that $U(N)$ gauge theory confines color charge at sufficiently large $N$ one would have to show that the background is *stable* against fluctuations in some (low) temperature regime. The fact that the strong coupling expansion of lattice gauge theory confines and is center-symmetric suggests that this might be the case at sufficiently large effective coupling $\lambda(T/\Lambda) > \lambda_d$. To further conclude that the more realistic $SU(3)$-model confines color charge at low temperatures one in addition has to show that there is no (deconfining) phase transition at some finite $N > 3$. Neither of these issues will be discussed further here. Let us instead look at some interesting aspects of the previous analysis.

## VI. INFRARED-FINITE PERTURBATION THEORY AT FINITE TEMPERATURE

Color momentum is essential in suppressing planar contributions to the free energy. For a background like Eq. (5) that breaks the center symmetry (and corresponds to vanishing color momentum $\xi$), Linde observed [15] that the perturbation series of a non-Abelian gauge theory is infrared divergent at any finite temperature. The most infrared divergent vacuum diagrams are all *planar* and superficially are of order $N^2$. The center-symmetric background of Eq. (4) provides an effective infrared cutoff of order $2\pi T/N$ for all the coset excitations. Since it appears via the covariant derivative, this infrared cutoff is *not* entirely equivalent to an effective gluon mass. Unlike an effective mass, it does not regulate the Abelian sector of the model in the infrared. The gauge bosons of an Abelian $U(1)^N$-model on the other hand do not interact directly, and the infrared behavior of such models is regular when all charged fields are massive [28]. Even though the option of massive off-diagonal fields is not available for a $U(N)$ gauge theory, the center-symmetric background of Eq. (4) provides an infrared cutoff that works rather similarly: it shifts the infrared singularity of any coset field propagator from $\mathbf{k^2} = \mathbf{0}$ to $\mathbf{k}^2 = -4\pi^2 T^2 j^2/N^2$ for some integer $N/2 > j > 0$. Note that although some of these "masses" are rather small at large $N$, they *do not* depend on the coupling $\lambda$.

In the center-symmetric background of Eq. (4) the perturbation series thus is free of infrared divergences without resummation. Of course, when the effective coupling is sufficiently small (at high temperatures) this background presumably is not stable [14] (see below). The center-symmetric background of Eq. (4) cures the pervasive infrared problem of the perturbative expansion at low temperatures only. Although the perturbation series may not converge in this regime, its mere existence to all orders does define the model formally. The regularization of perturbative infrared divergences by the center-symmetric background, however, reshuffles contributions to the $1/N$ expansion of the free energy. It does so in a manner that is consistent with confinement in this phase.

## VII. STABILITY AND (VENEZIANO'S) VECTOR GHOSTS

The result that planar $U(N)$ gauge theory practically has no degrees of freedom at large $N$, implies that center-symmetric *planar* $SU(N)$, although devoid of colored asymptotic states, is *not* a thermodynamically stable model. Center-symmetric $SU(N)$ gauge theory nevertheless can be a perfectly good physical model because the subset of planar diagrams does *not* give the *leading* contribution in $1/N$. There then is no reason why this subset of diagrams should define a thermodynamically viable physical model. Planar diagrams are generated by the Cuntz algebra [29] rather than by a bosonic or fermionic one. There is no proof that such a field theory is thermodynamically stable.

The instability of center-symmetric *planar* $SU(N)$ follows immediately from the previous result for the planar $U(N)$ model without fundamental fields. The color-singlet "photon" decouples in this case and the free energy density of $U(N)$ is just that of the corresponding $SU(N)$ model and of a free photon. Equation (28) together with the previous result for $U(N)$ implies that the free energy density of center-symmetric planar $SU(N)$ is,

$$F_{S_2}^{SU(N)}(T) = \frac{T^4\pi^2}{45} + \mathcal{O}(1/N^2). \quad (31)$$

The *positive* contribution to the free energy of center-symmetric $SU(N)$ is of $\mathcal{O}(N^0)$ and can be interpreted as due to a massless, color-singlet *vector ghost* that compensates the degrees of freedom of the massless, color-singlet photon of center-symmetric planar $U(N)$.

Veneziano [30] has shown that a massless color-singlet vector ghost in planar gluonic $SU(N)$ could saturate the axial Ward Identities and solve the $U_A(1)$-problem at large $N$. Equation (31) is evidence for the existence of a vector-ghost in the confining phase of the planar model. Whether this vector-ghost couples to the axial current in the manner Veneziano suggests, cannot be determined from the free energy. To have a viable confining phase, the vector-ghost of the planar model would have to either decouple by itself (as all states in the planar truncation do) or be part of a BRST-quartet [31] that does not contribute to the free energy.

As discussed in the previous sections, there are additional contributions to the free energy density of $\mathcal{O}(N^0)$ in the center-symmetric phase that are described by nonplanar diagrams. These nonplanar contributions to the free energy density depend on the effective coupling $\lambda$. It is at least conceivable that massless bound states form when $\lambda > \lambda_d$ that complete the BRST quartet and compensate the contribution to the free energy of the vector ghost. Since the vanishing of ghost contributions to the free energy is necessary for the stability of a center-symmetric phase, the critical coupling at which this occurs is a lower bound for $\lambda_d$.

The fact that the noninteracting (Casimir) part of the free energy density of $SU(N)$ is positive (from Eq. (30) with Eq. (28)) for all $N \geq 2$, implies that the center-symmetric phase is not stable at small effective coupling $\lambda(T/\Lambda) \sim 0$. This is consistent with the expectation that the center symmetry is broken for $T > T_d > 0$.

When nonplanar contributions to the free energy of $\mathcal{O}(N^0)$ are included, $SU(N)$ could be thermodynamically stable in the center-symmetric phase at sufficiently large effective coupling $\lambda(T/\Lambda) > \lambda_d$.

## VIII. DISCUSSION AND CONCLUSION

The configuration of Eq. (4) is an absolute minimum of the classical action of unbroken $U(N)$ gauge theory. It is invariant under the discrete global $Z_N$ center-symmetry of the Yang-Mills action. The center-symmetric orbit of vanishing curvature is unique in the infinite volume limit. It is unique even at finite volume if the spatial topology is that of a three-sphere, since the only noncontractible Wilson loops in this case are the Polyakov loops in temporal direction.

Although the background of Eq. (4) is an absolute minimum of the classical Yang-Mills action, it is not an absolute minimum of the free energy density at all temperatures. Because of its symmetries, this orbit is always an extremum of the free energy, but to lowest order of perturbation theory this extremum is a maximum. To the extent that higher order perturbative corrections are negligible, the center symmetry is broken for sufficiently small effective coupling, that is at sufficiently high temperatures [14]. The strong-coupling expansion of lattice gauge theory suggests that a center-symmetric phase is thermodynamically preferred at low temperatures when the effective coupling is sufficiently strong.

An expansion in $1/N$ could be an appealing alternative to the strong-coupling expansion in this nonperturbative regime. The center-symmetry of the gluonic sector is preserved in every order of the perturbative expansion about the background configuration of Eq. (4). Certain qualitative conclusions about the $1/N$-expansion of the free energy can be obtained by examining this series. There are no contributions to the free energy of order $N^2$ or $N$ at any perturbative order. For large $N$, a center-symmetric planar truncation of $U(N)$ approaches a topological field theory without dynamical degrees of freedom. This confirms the result of Gocksch and Neri [4] that the free energy of lattice gauge theory in the planar limit does not depend on the temperature (and therefore vanishes) at large $N$. The leading contribution is of order $N^0$ and due to vacuum diagrams that represent the free energy of color-singlet quark-antiquark mesons and glueballs.

This is as one expects for the confining phase of a $U(N)$ gauge theory. Perhaps more significant is that perturbative calculations at finite temperature in principle are feasible in the background of Eq. (4). The severe infrared divergences of ordinary perturbation theory observed by Linde [15] do not occur in this center-symmetric expansion. The reduced coupling $\lambda$ furthermore becomes weak in the confining phase for sufficiently large $N$ (see Sec. IV B). Some aspects of confinement therefore may be accessible in a perturbative framework [10]. The perturbative analysis of the model in the vicinity of the center-symmetric minimal action orbit in this sense is (self)consistent. However, the existence of a phase transition at a perhaps very small but nevertheless finite value of the coupling restricts the accuracy of a perturbative analysis in the confining phase at any fixed value of $N$.

Planar $SU(N)$ turns out to be thermodynamically unstable at sufficiently large $N$ due to a massless color-singlet vector-ghost. This is a consequence of the fact that the $U(1)$-photon decouples and the free energy of planar $U(N)$ in the confining phase vanishes as $N \rightarrow \infty$. We speculate that the vector ghost couples to the axial current in the manner conjectured by Veneziano [30]. It then should be part of a BRST-quartet [31] that, as a whole, does not contribute to the free energy. This could be the case at temperatures $T < T_d$ when other contributions to the free energy density of $\mathcal{O}(N^0)$ are included.

Although the center symmetry of the action is explicitly broken by fields in the fundamental representation, they do not contribute in order $N$ to the free energy in an expansion about the center-symmetric background of Eq. (4). Spontaneous chiral symmetry breaking perhaps can be investigated in this background: to lowest order in the coupling and for large $N$ several fermionic degrees of freedom are almost zero modes in the background of Eq. (4). Whether this is sufficient to trigger spontaneous chiral symmetry breaking has not been explored. If so, the thermodynamic instability of the background for $T > T_d$ would imply that chiral- and deconfinement-transition temperatures coincide.

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## APPENDIX: CASIMIR CONTRIBUTIONS TO THE FREE ENERGY OF CENTER-SYMMETRIC $U(N)$

We here calculate contributions to the free energy density in the center-symmetric background of Eq. (4) to zeroth order in perturbation theory without recourse to the scaling argument. The calculation explicitly shows that contributions of individual degrees of freedom cancel.

Consider the regularized expression[5] for the free energy density given by Eq. (26). The free energy density to lowest order in the coupling can be decomposed,

$$F_0(T,N) = 2\sum_{a,b=1}^{N} I(T,(a-b)/N;0) - 4\sum_{j=1}^{N_F}\sum_{a=1}^{N} I(T,(a-1/2)/N;m_j), \tag{A1}$$

into individual contributions $I(T,\delta;m)$ that depend on the phase $\delta$ and mass $m$ associated with a particular degree of freedom. $I(T,\delta;m)$ is formally given by,

$$I(T,\delta;m) = \lim_{D\to 3^-}\frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^Dk\mu^{3-D}}{(2\pi)^D}\ln(k^2+m^2 + 4\pi^2T^2(n+\delta)^2) \tag{A2}$$

The expression of Eq. (A2) is not well defined. However, making use of the fact that the free energy density $F$ *and* the specific entropy $-\partial_T F$ vanish at $T=0$, it suffices to obtain the second derivative $\partial_T^2 I$. Explicitly taking the derivatives in Eq. (26), the expression for $\partial_T^2 I(T,\delta;m)$ may be written,

$$\begin{aligned}\partial_T^2 I(T,\delta;m) &= \lim_{D\to 3^-}\sum_{n=-\infty}^{\infty}\int\frac{d^Dk\mu^{3-D}}{(2\pi)^D} \times \frac{4\pi^2T(n+\delta)^2}{k^2+m^2+4\pi^2T^2(n+\delta)^2} \times\left[1+\frac{2(k^2+m^2)}{k^2+m^2+4\pi^2T^2(n+\delta)^2}\right] \\ &= -\lim_{D\to 3^-}\partial_T^2 T\int_0^\infty\frac{d\lambda}{2\lambda}\sum_{n=-\infty}^{\infty}\int\frac{d^Dk\mu^{3-D}}{(2\pi)^D} \times e^{-\lambda(k^2+m^2+4\pi^2T^2(n+\delta)^2)}\end{aligned} \tag{A3}$$

[Note that the final (finite) result does not depend on the renormalization point $\mu$. The latter was introduced to have a free energy density with the canonical dimension.] We next evaluate the momentum integrals in the last expression of Eq. (A3),

$$\partial_T^2 I(T,\delta;m) = -\lim_{D\to 3^-}\partial_T^2 T\int_0^\infty\frac{d\lambda}{2\lambda} \times\frac{\mu^3 e^{-\lambda m^2}}{(4\pi\lambda\mu^2)^{D/2}}\sum_n e^{-4\pi^2\lambda T^2(n+\delta)^2}. \tag{A4}$$

To separate the summation over the integers $n$ from the dependence on $\delta$ it is convenient to use the Fourier-dependence representation of the Gaussian :

$$\begin{aligned}\sum_n e^{-4\pi^2\lambda T^2(n+\delta)^2} &= \int_{-\infty}^{\infty}\frac{dp}{T\sqrt{4\pi\lambda}}e^{-p^2/(4T^2\lambda)}e^{2\pi ip\delta}\sum_n e^{2\pi ipn} \\ &= \sum_n\frac{e^{2\pi in\delta}}{T\sqrt{4\pi\lambda}}e^{-n^2/(4T^2\lambda)} \\ &= \frac{1}{T\sqrt{4\pi\lambda}}\left\{1+2\sum_{n=1}^{\infty}\cos(2\pi n\delta) \times e^{-n^2/(4T^2\lambda)}\right\}.\end{aligned} \tag{A5}$$

[Note that the second expression for the sum is dual to the original one in the sense that the "radius" of the temporal direction has been inverted $4\lambda T^2 \to 1/(4\pi^2\lambda T^2)$.] Substituting the last expression for the sum into Eq. (A4) and noting that the constant term in the braces of Eq. (A5) does not survive differentiation, one finds that in $D=3$ spatial dimensions,

[5] For simplicity and to easily include fermions, the following computation uses dimensional regularization. The gluonic contribution to the free energy of $SU(N)$ has also been computed in lattice regularization [13]. Evaluating Neuberger's lattice result confirms that this gluonic contribution to the free energy is $O(1/N^2)$ at the center-symmetric background and subleading for sufficiently large $N$.

$$I(T,\delta;m) = -\sum_{n=1}^{\infty}\cos(2\pi n\delta)\int_0^{\infty}\frac{d\lambda}{\lambda}\frac{e^{-\lambda m^2-n^2/(4T^2\lambda)}}{(4\pi\lambda)^2} = -\sum_{n=1}^{\infty}\frac{\cos(2\pi n\delta)}{2\pi^2}\left(\frac{Tm}{n}\right)^2 K_2(nm/T). \quad \text{(A6)}$$

The integration constants in Eq. (A6) have been determined so that the free energy density and the specific entropy vanish at $T = 0$.

The result of Eq. (A6) can be checked in various limits: the free energy density of a noninteracting bosonic degree of freedom satisfying periodic boundary conditions is obtained with $\delta = 0$; $\delta = 1/2$ corresponds to the (positive) free energy of a noninteracting bosonic degree of freedom satisfying antiperiodic boundary conditions, etc. One evidently can achieve some cancellation in the total free energy by mixing several degrees of freedom satisfying different boundary conditions. In the center-symmetric background of Eq. (4), the gluons effectively satisfy different boundary conditions. If the fundamental and antifundamental color indices of the gluon (in the background gauge) are $a$ and $b$, the corresponding shift of the Matsubara frequency is $2\pi T(a-b)/N$. Note that diagonal Abelian degrees of freedom (with $a = b$) do not suffer a phase shift and that all bosonic degrees of freedom are $N$-periodic.

The gluonic part of the free energy density in Eq. (A1) is found by evaluating,

$$2\sum_{a,b=1}^{N}\cos(2\pi n(a-b)/N) = \sum_{a,b=1}^{N}e^{2\pi i n(a-b)/N} + c.c. = 2\left|\sum_{a=1}^{N}e^{2\pi i a n/N}\right|^2 = 2N^2\sum_k \delta_{n,kN}. \quad \text{(A7)}$$

Here $\delta_{n,kN}$ is the Kronecker symbol (one when $n$ is an integer multiple of $N$ and zero otherwise). Semiclassically, contributions to the free energy density from gluonic paths with $(n \bmod N) \neq 0$ windings thus cancel completely. Using Eq. (A6) (for $m \to 0$) and Eq. (A7) the gluonic contribution in Eq. (A1) is,

$$2\sum_{a,b=1}^{N} I(T,(a-b)/N;0) = -2T^4\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{\cos(2\pi n(a-b)/N)}{\pi^2 n^4} = -T^4\sum_{k=1}^{\infty}\frac{2N^2}{\pi^2(kN)^4} = -\frac{T^4\pi^2}{45N^2}. \quad \text{(A8)}$$

This verifies the scaling argument for the first term of Eq. (30). The fermionic contribution in Eq. (30) is similarily obtained from Eq. (A6) and (A1). We have that,

$$\sum_{a=1}^{N}\cos(2\pi n(a-1/2)/N) = \frac{1}{2}\sum_{a=1}^{N}e^{2\pi i n(a-1/2)/N} + c.c. = N\sum_k(-1)^k\delta_{n,kN}. \quad \text{(A9)}$$

Using Eq. (A9) and (A6) in Eq. (A1) gives the second term of Eq. (30).


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