Quantized gauge-affine gravity in the superfiber bundle approach

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The quantization of gauge-affine gravity within the superfiber bundle formalism is proposed. By introducing an even pseudotensorial 1-superform over a principal superfiber bundle with superconnection, we obtain the geometrical Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST transformations of the fields occurring in such a theory. Reducing the four-dimensional general affine group double-covering $\overline{GA}(4, \mathbb{R})$ to the Poincaré group double-covering $\overline{ISO}(1, 3)$ we also find the BRST and anti-BRST transformations of the fields present in Einstein's gravity. Furthermore, we give a prescription leading to the construction of both BRST-invariant gauge-fixing action for gauge-affine gravity and Einstein's gravity.

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I. INTRODUCTION

One of the most outstanding problems in modern theoretical physics is to construct a consistent theory of quantum Einstein gravity. Several models have been proposed (for a review see e.g. Ref. [1] and references therein), however none of these models, both renormalizable and unitary, has been found. This is basically due to the dimensionful nature of the gravitational coupling constant [2] which destroys the predictivity of quantum Einstein gravity, i.e. it is impossible to have a renormalizable theory.

On the other hand, a serious progress has been achieved by Ne'eman and Šijački [3] for solving this problem. They proposed a model for quantum gravity that reproduces Einstein's gravity at low energy with a fair possibility to be renormalizable and unitary. However, it has been proved by Stelle in [4] that a theory containing Einstein's action with term quadratic in the appropriate curvatures was renormalizable, but violated unitarity. The failure of unitarity in that model arises through the Riemannian condition which relates the connection to the metric (i.e. torsion free and metric compatible). To avoid this difficulty one has attempted to consider spacetime with torsion and thus guaranteeing the independence of metric and connection fields. In this context, the Poincaré gauge theory (PGT) has been developed as a gravitational theory based on the double-covering $\overline{ISO}(1, 3) = SL(2, \mathbb{C}) \otimes \mathbb{R}^4$ of the Poincaré group $ISO(1, 3) = SO(1, 3) \otimes \mathbb{R}^4$ [5,6]. We note that the connection is not an independent variable, since the metricity condition is also preserved in this model [5–7]. However, it has been confirmed that no PGT model can be renormalizable if one imposes unitarity [8].

Another possibility for doing away with the Riemannian condition consists to have gravitational gauge model in which the Poincaré group acting on the local frames is extended to a larger gauge group for frames, namely $\overline{GA}(4, \mathbb{R})$. The resulting gravitational model is a metric-affine gauge theory of gravity (MAG) which has been suggested in Ref. [3]. The theory has a metric-affine spacetime with torsion and nonmetricity and incorporates gravitational models like Einstein's gravity. The model is based on gauging the four-dimensional general affine group $GA(4, \mathbb{R}) = GL(4, \mathbb{R}) \otimes \mathbb{R}^4$ [9–11], or its double-covering $\overline{GA}(4, \mathbb{R}) = \overline{GL}(4, \mathbb{R}) \otimes \mathbb{R}^4$ [1,12,13]. The existence of a double-covering$\overline{GL}(4, \mathbb{R})$ of the general linear group $GL(4, \mathbb{R})$ has been realized in Ref. [14]. Here, the spinorial double-covering exists only in infinite matrix representations and the corresponding infinite-component fields, the so-called manifields [14,15]. The renormalizability of MAG model has been proved [16,17], but unitarity has not been properly checked to date.

Recently, in [18] the algebraic structure of Becchi-Rouet-Stora-Tyutin (BRST) transformations [19] of a metric-affine gauge gravity based on the Hamiltonian formalism has been analyzed. This approach leads to the same BRST transformations obtained in [20,21] in the context of the Batalin and Vilkovisky formalism [22]. Here, the authors generalize the work developed by Okubo [23] where a new type of BRST operator has been constructed only for spacetimes with teleparallelism. They follow the rather transparent exposition of van Holten [24] which departs from the Hamiltonian formalism and replaces the Lagrange multipliers for the first class constraints by ghost operators.

Moreover, BRST transformations equivalent to those given in [18,20,21] can also be obtained geometrically. Indeed, as shown in Ref. [25], we have used a superspace formalism to determine geometrically the BRST and anti-BRST algebra for gauge-affine gravity. Our method was based on the introduction of $\overline{GA}(4, \mathbb{R})$-superconnection over a (4, 2)-dimensional superspace obtained by extending a metric-affine space with two anticommuting coordinates. This superconnection represents the gauge fields and their associated ghost and antighost fields occurring in gauge-affine gravity. In particular, the introduction of the

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coordinate ghost and antighost fields leads to the construction of a basis, where the local expression of the superconnection becomes more natural. By using this basis, we have determined the BRST and anti-BRST transformations from the structure equations by imposing horizontality conditions on the supercurvature.

In the present paper, we discuss the quantization of gauge-affine gravity theory by using the superfiber bundle formalism, see e.g. Ref. [26] and references therein, in analogy to what is realized for the case of super-Yang-Mills theory [27,28] and the four-dimensional non-Abelian topological antisymmetric tensor gauge theory, the so-called BF theory [29]. In section II, we show how the various fields of gauge-affine gravity and their geometrical BRST and anti-BRST transformations can be determined via a principal superfiber bundle endowed with a superconnection and an even pseudotensorial 1-superform in the adjoint representation. Reducing the four-dimensional general affine group double covering to the Poincaré group double covering we also find the BRST and anti-BRST transformations of the fields present in quantum Einstein gravity. The obtained geometrical BRST and anti-BRST transformations are nilpotent. In section III we first build a gauge-fixing superaction for gauge-affine gravity as a natural generalization of the one corresponding to the usual Yang-Mills theory. Then, a gauge-fixing action is obtained as the lowest component of the gauge-fixing superaction. However, we also work in the same spirit of construction as in [27–29] for building the gauge-fixing action for quantum Einstein gravity. Section IV is devoted to concluding remarks.

II. GEOMETRICAL BRST AND ANTI-BRST ALGEBRA

Let $P(M, G_s)$ be a principal superfiber bundle with superconnection ϕ. The base space M is the four-dimensional metric-affine spacetime and the structure group G_s is the direct product of the general linear group double-covering $\overline{GL}(4, \mathbb{R})$ with the general affine group double-covering $\overline{GA}(4, \mathbb{R})$ as well the two-dimensional old translation group $S^{0,2}$. We consider P as being globally trivial with respect to $S^{0,2}$. This will be related to the fact that the BRST and anti-BRST transformations are defined globally.

The Lie superalgebra g_s of the structural Lie supergroup G_s is given by

$$g_s = \overline{gl}(4, \mathbb{R}) \oplus \overline{ga}(4, \mathbb{R}) \oplus s^{0,2}. \tag{1}$$

Let $(T^\rho_\sigma)_{\{\rho,\sigma=1,\ldots,4\}}$, $(T^a_b)_{\{a,b=1,\ldots,4\}}$, $(P_b)_{\{b=1,\ldots,4\}}$, and $(F_\alpha)_{\{\alpha=1,2\}}$ be the generators of $\overline{GL}(4, \mathbb{R})$, $\overline{GA}(4, \mathbb{R})$, and $S^{0,2}$, respectively. They satisfy the following commutation relations

$$\begin{aligned}
&[T^\tau_\varepsilon, T^\rho_\sigma] = (\delta^\tau_\sigma \delta^\rho_\mu \delta^\nu_\varepsilon - \delta^\rho_\varepsilon \delta^\nu_\sigma \delta^\tau_\mu) T^\mu_\nu,\\
&[T^a_b, T^c_d] = (\delta^a_d \delta^c_e \delta^f_b - \delta^c_b \delta^f_d \delta^a_e) T^e_f,\\
&[T^a_b, P_c] = \delta^a_c \delta^d_b P_d,\\
&[P_a, P_b] = [T^a_b, T^\rho_\sigma] = [P_a, F_\alpha] = 0,\\
&[T^\rho_\sigma, P_b] = [F_\alpha, T^\rho_\sigma] = [T^a_b, F_\alpha] = [F_\alpha, F_\beta] = 0.
\end{aligned} \tag{2}$$

Let Ω be an even 2-superform associated to the superconnection ϕ and ϑ an even pseudotensorial 1-superform of the type (ad, g_s).[1] The introduction in $P(M, G_s)$ besides the usual superconnection ϕ an even 1-superform ϑ will be related, as we will see later, to the fact that the imposed constraints on the supercurvature Ω can be obtained by the fact that the covariant differentiation of a pseudotensorial 1-superform ϑ is tensorial.

In order to realize supercurvature constraints, we need to introduce an even 1-superform generalized superconnection λ such that

$$\lambda = \phi - \vartheta. \tag{3}$$

At this point, let us mention that the introduction of the generalized superconnection is related, on the one hand, to the fact that the double-covering group $\overline{\mathrm{Diff}}(4, \mathbb{R})$ of the group of general coordinate transformations (GCT) (i.e., the group of diffeomorphisms) is realized through the direct product of the general linear group double-covering $\overline{GL}(4, \mathbb{R})$ with the translation group $\mathbb{R}^4$ as well a simply connected Lie subgroup [15,30,31] and, on the other hand, as we will see later, to the fact that the gauge-fixing action for quantum Einstein gravity can be deduced from gauge-fixing action for gauge-affine gravity by reducing the linear connection to the symmetric Levi-Civita connection.

Acting the exterior covariant superdifferential D on λ we define then the generalized supercurvature Λ (even 2-superform) given by

$$\Lambda = D\lambda = \Omega - \Theta, \tag{4}$$

where the associated supercurvature Ω and Θ to ϕ and ϑ are defined by $\Omega = D\phi$ and $\Theta = D\vartheta$, respectively. They satisfy the structure equations

$$\Omega = d\phi + \frac{1}{2}[\phi, \phi], \tag{5}$$

$$\Theta = d\vartheta + [\phi, \vartheta], \tag{6}$$

where d is the exterior superdifferential and $[,]$ the graded Lie bracket.

Let $z = (z^M) = (x^\mu, \theta^\alpha)$ be a local coordinates system on P, where $(x^\mu)_{\mu=1,\ldots,4}$ are the coordinates of the metric-affine spacetime M and $(\theta^\alpha)_{\alpha=1,2}$ are ordinary anticommuting variables. Upon expressing the generalized super-

[1] Here (ad) means adjoint representation.

connection λ and the generalized supercurvature Λ as

$$\lambda = dz^M \lambda_M = dz^M(\phi_M - \vartheta_M),$$
$$\Lambda = \frac{1}{2} dz^N \wedge dz^M \Lambda_{MN} = \frac{1}{2} dz^N \wedge dz^M (\Omega_{MN} - \Theta_{MN}), \tag{7}$$

we have

$$\Omega_{MN} = \partial_M \phi_N - (-1)^{mn} \partial_N \phi_M + [\phi_M, \phi_N], \tag{8a}$$
$$\Theta_{MN} = \partial_M \vartheta_N - (-1)^{mn} \partial_N \vartheta_M - [\phi_N, \vartheta_M] + (-1)^{mn}[\phi_M, \vartheta_N], \tag{8b}$$

where $m = | z^M |$ is the Grassmann degree of z^M. Note that the Grassmann degrees of the superfield components ϕ_M, ϑ_M, λ_M, Ω_{MN}, Θ_{MN}, and Λ_{MN} are given by

$$| \phi_M | = | \vartheta_M | = | \lambda_M | = m,$$
$$| \Omega_{MN} | = | \Theta_{MN} | = | \Lambda_{MN} | = m + n \ (\mathrm{mod}2),$$

since ϕ, ϑ, λ, Ω, Θ, and Λ are even superforms.

Moreover, the generalized supercurvature is a tensorial 2-superform, in particular we have $i(X)\Lambda = 0$, where i denotes the contraction of vectors with forms and X is a vertical superfield in P. Using the fact that $\partial_\alpha = \partial/\partial\theta^\alpha$ is vertical, we obtain the following supercurvature equations

$$\Lambda_{\alpha\beta} = 0, \tag{9a}$$
$$\Lambda_{\alpha\mu} = 0, \tag{9b}$$

i.e.

$$\Omega_{\alpha\beta} = \Theta_{\alpha\beta}, \tag{10a}$$
$$\Omega_{\alpha\mu} = \Theta_{\alpha\mu}. \tag{10b}$$

Furthermore, the g_s-valued component superfields ϕ_M, ϑ_M, Ω_{MN}, and Θ_{MN} are given by

$$\phi_M = \phi^a_{bM} T^b_a + \phi^\mu_{\nu M} T^\nu_\mu + \phi^a_M P_a + \phi^\alpha_M F_\alpha,$$
$$\vartheta_M = \vartheta^a_{bM} T^b_a + \vartheta^\mu_{\nu M} T^\nu_\mu + \vartheta^a_M P_a + \vartheta^\alpha_M F_\alpha,$$
$$\Omega_{MN} = \Omega^a_{bMN} T^b_a + \Omega^\mu_{\nu MN} T^\nu_\mu + \Omega^a_{MN} P_a + \Omega^\alpha_{MN} F_\alpha,$$
$$\Theta_{MN} = \Theta^a_{bMN} T^b_a + \Theta^\mu_{\nu MN} T^\nu_\mu + \Theta^a_{MN} P_a + \Theta^\alpha_{MN} F_\alpha. \tag{11}$$

According to Eqs. (10) and (11) we find

$$\Omega^a_{b\mu\alpha} = \Theta^a_{b\mu\alpha}, \tag{12a}$$
$$\Omega^a_{b\alpha\beta} = \Theta^a_{b\alpha\beta}, \tag{12b}$$
$$\Omega^\rho_{\sigma\mu\alpha} = \Theta^\rho_{\sigma\mu\alpha}, \tag{12c}$$
$$\Omega^\rho_{\sigma\alpha\beta} = \Theta^\rho_{\sigma\alpha\beta}, \tag{12d}$$
$$\Omega^a_{\mu\beta} = \Theta^a_{\mu\beta}, \tag{12e}$$
$$\Omega^a_{\alpha\beta} = \Theta^a_{\alpha\beta}, \tag{12f}$$
$$\Omega^\alpha_{\mu\beta} = \Theta^\alpha_{\mu\beta}, \tag{12g}$$
$$\Omega^\alpha_{\gamma\beta} = \Theta^\alpha_{\gamma\beta}. \tag{12h}$$

The components Λ^α_{MN} associated to F_α give the $S^{0,2}$-generalized supertorsion

$$\Lambda^\alpha_{MN} = \Omega^\alpha_{MN} - \Theta^\alpha_{MN}. \tag{13}$$

According to (12g) and (12h) supplemented with the constraint

$$\Lambda^\alpha_{\mu\nu} = 0,$$

we find then that the $S^{0,2}$-generalized supertorsion vanishes. Moreover, the potentials ϕ^α_M being pure gauge, we can then impose the following supercurvature constraint

$$\Omega^\alpha_{MN} = \partial_M \phi^\alpha_N - (-1)^{mn} \partial_N \phi^\alpha_M = 0, \tag{14}$$

and therefore we also have

$$\Theta^\alpha_{MN} = \partial_M \vartheta^\alpha_N - (-1)^{mn} \partial_N \vartheta^\alpha_M = 0. \tag{15}$$

In addition, we impose that the geometrical structure of the principal superfiber bundle $P(M, G_s)$ should incorporate the metric-affine structure of the spacetime M such that for $\theta^\alpha = 0$, the components of the superfields $\Lambda_{\mu\nu}$ permit us to find the standard results concerning the torsion and the curvature of metric-affine spacetime. This allows us to have

$$\Lambda_{\mu\nu} = \Omega_{\mu\nu}, \tag{16}$$

and therefore

$$\Theta^\rho_{\tau\mu\nu} = \Theta^a_{b\mu\nu} = \Theta^a_{\mu\nu} = \Theta^\alpha_{\mu\nu} = 0. \tag{17}$$

Since the potentials ϕ^α_M being pure gauge, we consider, hereafter, that the components ϕ_M are $\overline{gl}(4, \mathbb{R}) \oplus \overline{ga}(4, \mathbb{R})$-valued superfields and can be written as

$$\phi = dz^M \phi_M = dx^\mu \phi_\mu + d\theta^\alpha \eta_\alpha, \tag{18}$$

where

$$\phi_\mu = \phi^a_{b\mu} T^b_a + \phi^\rho_{\tau\mu} T^\tau_\rho + \phi^a_\mu P_a, \tag{19a}$$
$$\eta_\alpha = \eta^a_{b\alpha} T^b_a + \eta^\rho_{\tau\alpha} T^\tau_\rho + \eta^a_\alpha P_a. \tag{19b}$$

Now, in order to derive the BRST structure of gauge-affine gravity it is necessary to give the geometrical description of the fields present in such theory. To this purpose, we assign to the anticommuting coordinates θ^1 and θ^2 the ghost numbers (-1) and $(+1)$, respectively, and ghost number zero for an even quantity: either a coordinate, a superform, or a generator. These rules permit us to determine the ghost numbers of the superfields $(\phi^\rho_{\tau\mu}, \phi^a_{b\mu}, \phi^a_\mu, \eta^a_{b1}, \eta^a_{b2})$ which are given by $(0, 0, 0, 1, -1)$. So, the lowest components $\phi^\rho_{\tau\mu|}$, $\phi^a_{b\mu|}$, $\phi^a_{\mu|}$, $\eta^a_{b1|}$, and $\eta^a_{b2|}$ can be identified with the linear connection $\Gamma^\rho_{\tau\mu}$, the affine connection $\omega^a_{b\mu}$, the vierbein e^a_μ, the $\overline{GA}(4, \mathbb{R})$ ghost c^a_b, and its antighost $\bar{c}^a_b$, respectively. The symbol "|" indicates that the superfield is evaluated at $\theta^\alpha = 0$. Moreover, we introduce the coordinate (diffeomorphism) ghost and antighost superfields η^{μ}_{α} by the following replacement

$$\eta^{\mu}_{\tau\alpha} = \partial_{\tau}\eta^{\mu}_{\alpha} + \phi^{\mu}_{\tau\rho}\eta^{\rho}_{\alpha}, \tag{20a}$$

$$\eta^{a}_{\alpha} = 0. \tag{20b}$$

This permits us, on the one hand, to introduce the coordinate ghost $c^{\mu} = \eta^{\mu}_{1|}$ and its antighost $\bar{c}^{\mu} = \eta^{\mu}_{2|}$ and, on the other hand, to justify the introduction of general linear group double-covering $\overline{GL}(4, \mathbb{R})$ with generators (T^{ρ}_{σ}) in the structure group G_s of the principal superfiber bundle $P(M, G_s)$. Furthermore, knowing the expression of the components of Θ we can determine the components of ϑ by using the relation (8b). Some components of Θ have been determined from Eqs. (15) and (17). As trivial solutions we have

$$\vartheta^{\alpha}_{\beta} = \vartheta^{\alpha}_{\mu} = \vartheta^{a}_{\mu} = \vartheta^{a}_{b\mu} = \vartheta^{\rho}_{\tau\mu} = 0. \tag{21}$$

The determination of the other components of ϑ such $\vartheta^{a}_{b\alpha}$, $\vartheta^{\tau}_{\rho\alpha}$, and ϑ^{a}_{β} can be easily obtained from the remaining Θ components. The latter, after straightforward calculations, acquire the form

$$\Theta^{a}_{b\mu\alpha} = \eta^{\rho}_{\alpha}\Omega^{a}_{b\rho\mu} + D_{\mu}\{\phi^{a}_{b\rho}\eta^{\rho}_{\alpha}\}, \tag{22a}$$

$$\Theta^{\rho}_{\tau\alpha\beta} = -\frac{1}{2}[\eta^{\sigma}_{\alpha}, \eta^{\nu}_{\beta}]\Omega^{\rho}_{\tau\sigma\nu}, \tag{22b}$$

$$\Theta^{a}_{b\alpha\beta} = \begin{cases} -2\eta^{\rho}_{\alpha}\partial_{\rho}\eta^{a}_{b\beta} & \text{if} \quad \alpha = \beta, \\ 0 \quad \text{if} & \alpha \neq \beta, \end{cases} \tag{22c}$$

$$\Theta^{a}_{\mu\alpha} = \eta^{\rho}_{\alpha}\Omega^{a}_{\rho\mu} + D_{\mu}\{\eta^{\rho}_{\alpha}\phi^{a}_{\rho}\}, \tag{22d}$$

$$\Theta^{a}_{\alpha\beta} = 0, \tag{22e}$$

where $D_{\mu} = \partial_{\mu} + [\phi_{\mu},]$ is the $\overline{ga}(4, \mathbb{R})$-valued covariant superderivative. It is worth noting that the anholonomic and holonomic components of the superconnection $\phi^{\sigma}_{\mu\nu}$ and $\phi^{a}_{b\mu}$ are related by the supervierbein ϕ^{a}_{μ} as follows

$$\phi^{\sigma}_{\mu\nu} = \phi^{\sigma}_{a}(\partial_{\nu}\phi^{a}_{\mu} + \phi^{b}_{\mu}\phi^{a}_{b\nu}). \tag{23}$$

Therefore, the components $\Theta^{\rho}_{\tau\mu\alpha}$ can be derived from the components $\Theta^{a}_{b\mu\alpha}$ and $\Theta^{a}_{\mu\alpha}$ as follows

$$\Theta^{\rho}_{\tau\mu\alpha} = \eta^{\nu}_{\alpha}\Omega^{\rho}_{\tau\mu\nu}. \tag{24}$$

However, the operational representation for an infinitesimal $S^{0,2}$-motion in P is given by

$$r(\theta^{\alpha}) = 1 + \theta^{\alpha}Q_{\alpha}, \tag{25}$$

where $(Q_{\alpha})_{\alpha=1,2}$ are the differential operators representing the $S^{0,2}$-generators (F_{α}). According to the fact that the superconnection is a pseudotensorial 1-superform in the adjoint representation, we have

$$\phi^{A}_{M}(x^{\mu}, \zeta^{\alpha} + \theta^{\alpha}) = r(\theta^{\alpha})\phi^{A}_{M}(x^{\mu}, \zeta^{\alpha})r^{-1}(\theta^{\alpha}). \tag{26}$$

It is straightforward to compute (26), and we find

$$\phi^{A}_{M}(x^{\mu}, \zeta^{\alpha} + \theta^{\alpha}) = \phi^{A}_{M}(x^{\mu}, \zeta^{\alpha}) + \theta^{\alpha}[Q_{\alpha}, \phi^{A}_{M}(x^{\mu}, \zeta^{\alpha})] + \frac{1}{2}\theta^{\alpha}\theta^{\beta}[Q_{\beta,}[Q_{\alpha}, \phi^{A}_{M}(x^{\mu}, \zeta^{\alpha})]]. \tag{27}$$

By expanding $\phi^{A}_{M}(x^{\mu}, \theta^{\alpha})$ in power series of θ^{α}, we have

$$\phi^{\sigma}_{\tau\mu} = \Gamma^{\sigma}_{\tau\mu} + \theta^{\alpha}A^{\sigma}_{\tau\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}B^{\sigma}_{\tau\mu\beta\alpha}, \tag{28a}$$

$$\phi^{a}_{b\mu} = \omega^{a}_{b\mu} + \theta^{\alpha}M^{a}_{b\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}N^{a}_{b\mu\beta\alpha}, \tag{28b}$$

$$\phi^{a}_{\mu} = e^{a}_{\mu} + \theta^{\alpha}K^{a}_{\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}L^{a}_{\mu\beta\alpha}, \tag{28c}$$

$$\eta^{a}_{b\delta} = c^{a}_{b\delta} + \theta^{\alpha}R^{a}_{b\delta\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}S^{a}_{b\delta\beta\alpha}, \tag{28d}$$

$$\eta^{\mu}_{\delta} = c^{\mu}_{\delta} + \theta^{\alpha}V^{\mu}_{\delta\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}W^{\mu}_{\delta\beta\alpha}, \tag{28e}$$

where $B^{\sigma}_{\tau\mu\beta\alpha}$, $N^{a}_{b\mu\beta\alpha}$, $L^{a}_{\mu\beta\alpha}$, $S^{a}_{b\delta\beta\alpha}$, and $W^{\mu}_{\delta\beta\alpha}$ are antisymmetric with respect to the indices α and β. Evaluating (27) at $\zeta^{\alpha} = 0$ and in view of Eq. (28), we obtain

$$A^{\sigma}_{\tau\mu\alpha} = [Q_{\alpha}, \Gamma^{\sigma}_{\tau\mu}] = \partial_{\alpha}\phi^{\sigma}_{\tau\mu|}, \tag{29a}$$

$$M^{a}_{b\mu\alpha} = [Q_{\alpha}, \omega^{a}_{b\mu}] = \partial_{\alpha}\phi^{a}_{b\mu|}, \tag{29b}$$

$$K^{a}_{\mu\alpha} = [Q_{\alpha}, e^{a}_{\mu}] = \partial_{\alpha}\phi^{a}_{\mu|}, \tag{29c}$$

$$R^{a}_{b\delta\alpha} = [Q_{\alpha}, c^{a}_{b\delta}] = \partial_{\alpha}\eta^{a}_{b\delta|}, \tag{29d}$$

$$V^{\mu}_{\delta\alpha} = [Q_{\alpha}, c^{\mu}_{\delta}] = \partial_{\alpha}\eta^{\mu}_{\delta|}. \tag{29e}$$

We also obtain similar relations for the other field components

$$B^{\sigma}_{\tau\mu\beta\alpha} = [Q_{\beta,}[Q_{\alpha}, \Gamma^{\sigma}_{\tau\mu}]] = \partial_{\beta}\partial_{\alpha}\phi^{\sigma}_{\tau\mu|}, \tag{30a}$$

$$N^{a}_{b\mu\beta\alpha} = [Q_{\beta,}[Q_{\alpha}, \omega^{a}_{b\mu}]] = \partial_{\beta}\partial_{\alpha}\phi^{a}_{b\mu|}, \tag{30b}$$

$$L^{a}_{\mu\beta\alpha} = [Q_{\beta,}[Q_{\alpha}, e^{a}_{\mu}]] = \partial_{\beta}\partial_{\alpha}\phi^{a}_{\mu|}, \tag{30c}$$

$$S^{a}_{b\delta\beta\alpha} = [Q_{\beta,}[Q_{\alpha}, c^{a}_{b\delta}]] = \partial_{\beta}\partial_{\alpha}\eta^{a}_{b\delta|}, \tag{30d}$$

$$W^{\mu}_{\delta\beta\alpha} = [Q_{\beta,}[Q_{\alpha}, c^{\mu}_{\delta}]] = \partial_{\beta}\partial_{\alpha}\eta^{\mu}_{\delta|}. \tag{30e}$$

In analogy with the Yang-Mills case [28], we remark that the operators Q_1 and Q_2 represent the BRST and anti-BRST operators Q and $\overline{Q}$, respectively.

Evaluating (8a) at $\theta^{\alpha} = 0$ and using (12), (20), (22), (28), and (29) we obtain the following geometrical BRST transformations

$$
\begin{aligned}
[Q, \Gamma^{\sigma}_{\tau\mu}] &= \Gamma^{\rho}_{\tau\mu}\partial_\rho c^\sigma - \partial_\mu\partial_\tau c^\sigma - \Gamma^{\sigma}_{\tau\rho}\partial_\mu c^\rho - \Gamma^{\sigma}_{\rho\mu}\partial_\tau c^\rho - c^\rho\partial_\rho\Gamma^{\sigma}_{\tau\mu},\\
[Q, \omega^a_{b\mu}] &= \partial_\mu c^a_b + c^d_b\omega^a_{d\mu} - c^a_d\omega^d_{b\mu} + \omega^a_{b\sigma}\partial_\mu c^\sigma + c^\rho\partial_\rho\omega^a_{b\mu},\\
[Q, e^a_\mu] &= e^a_\sigma\partial_\mu c^\sigma + c^\rho\partial_\rho e^a_\mu - e^b_\mu c^a_b,\\
[Q, c^a_b] &= c^\rho\partial_\rho c^a_b - c^a_f c^f_b,\\
[Q, c^\sigma] &= c^\rho\partial_\rho c^\sigma,\\
[Q, \bar{c}^a_b] &= B^a_b,\\
[Q, \bar{c}^\sigma] &= B^\sigma,\\
[Q, B^a_b] &= 0,\\
[Q, B^\sigma] &= 0,
\end{aligned}
\tag{31}
$$

and also the geometrical anti-BRST transformations, which can be derived from (31) by the following rules: $X \longrightarrow X$, if $X = \Gamma^{\sigma}_{\tau\mu}, \omega^a_{b\mu}, e^a_\mu$; $X \longrightarrow \bar{X}$, if $X = Q, c^\mu, c^a_b, B^\mu, B^a_b$, and $X = \bar{\bar{X}}$, where

$$
\begin{aligned}
B^\mu + \bar{B}^\mu &= c^\rho\partial_\rho\bar{c}^\mu + \bar{c}^\rho\partial_\rho c^\mu,\\
B^a_b + \bar{B}^a_b &= c^\rho\partial_\rho\bar{c}^a_b + \bar{c}^\rho\partial_\rho c^a_b - c^a_d\bar{c}^d_b - \bar{c}^a_d c^d_b.
\end{aligned}
\tag{32}
$$

Let us note that the obtained BRST and anti-BRST transformations are nilpotent, i.e.

$$Q^2 = \bar{Q}^2 = [Q, \bar{Q}] = 0. \tag{33}$$

Now, we apply the same geometrical framework to find the BRST and anti-BRST transformations of the fields occurring in quantum Einstein gravity. To this end, we must reduce the general affine group double-covering $\overline{GA}(4, \mathbb{R})$ to the Poincaré double-covering $\overline{ISO}(1, 3)$. The BRST transformations of the fields associated to Poincaré double-covering have already be given in [32]. Reducing $\overline{GA}(4, \mathbb{R})$ to $\overline{ISO}(1, 3)$ leads us to keep from (12) only

$$\Omega^{\rho}_{\sigma\alpha\beta} = \Theta^{\rho}_{\sigma\alpha\beta}. \tag{34}$$

On the other hand, Einstein's theory is Riemannian, i.e. it precludes the propagation of either torsion or nonmetricity. Only the coordinate metric field $g_{\mu\nu}$ propagates. Here the coordinate metric field $g_{\mu\nu}$ is related to the Minkowski metric η_{ab} through the vierbein e^a_μ as follows

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu, \tag{35}$$

and can be written, in view of (28c), as a lowest component of a superfield $G_{\mu\nu}$ which can be put in the form

$$G_{\mu\nu} = g_{\mu\nu} + \theta^\alpha H_{\mu\nu\alpha} + \frac{1}{2}\theta^\alpha\theta^\beta E_{\mu\nu\beta\alpha},$$

where $H_{\mu\nu\alpha}$ and $E_{\mu\nu\beta\alpha}$ follow from (28c). This remark permits us to find the BRST transformation of the coordinate metric field $g_{\mu\nu}$ through the BRST transformation of the vierbein e^a_μ

$$[Q, g_{\mu\nu}] = \eta_{ab}[Q, e^a_\mu]e^b_\nu + \eta_{ab}e^a_\mu[Q, e^b_\nu].$$

The latter becomes

$$[Q, g_{\mu\nu}] = g_{\mu\sigma}\partial_\nu c^\sigma + c^\rho\partial_\rho g_{\mu\nu} + g_{\sigma\nu}\partial_\mu c^\sigma, \tag{36}$$

by using (31) and the fact that $c_{bd} = -c_{db}$.

Moreover, according to (22b), (28e), and (34), we obtain

$$V^{\mu}_{\delta\alpha} = c^\rho_\delta\partial_\rho c^\mu_\alpha, \qquad V^\tau_{12} + V^\tau_{21} = c^\rho_1\partial_\rho c^\tau_2 + c^\rho_2\partial_\rho c^\tau_1. \tag{37}$$

Therefore, making use of (29e) and keeping the same identifications, we find the following BRST transformations [32]

$$[Q, c^\sigma] = c^\rho\partial_\rho c^\sigma, \qquad [Q, \bar{c}^\sigma] = B^\sigma, \qquad [Q, B^\sigma] = 0. \tag{38}$$

We also obtain the geometrical anti-BRST transformations, which can be derived from (36) and (38) by the following mirror symmetry of the ghost numbers: $X \longrightarrow X$, if $X = g_{\mu\nu}$, $X \longrightarrow \bar{X}$, if $X = Q, c^\mu, B^\mu$, and $\bar{\bar{X}} \longrightarrow X$.

III. GAUGE-FIXING QUANTUM ACTION

In the present section, we show how to construct a BRST-invariant gauge-fixing quantum action for gauge-affine gravity as the lowest component of a gauge-fixing superaction. To this purpose, we propose starting with a gauge-fixing superaction similar to that obtained in the case of super-Yang-Mills theory as given in [27,28]

$$
\begin{aligned}
S_{sgf} &= \int d^4x L_{sgf},\\
L_{sgf} &= (\partial_1\phi_2)(\partial^\mu\phi_\mu) + (\partial^\mu\phi_2)(\partial_1\phi_\mu) + (\partial_1\phi_2)(\partial_1\phi_2).
\end{aligned}
\tag{39}
$$

We note first that it is the superconnection ϕ which is $\overline{gl}(4, \mathbb{R}) \oplus \overline{ga}(4, \mathbb{R})$-valued and represents the fields occurring in quantized gauge-affine gravity. This allows us to write the superaction S_{sgf} as follows

$$S_{sgf} = S^t_{sgf} + S^d_{sgf}, \tag{40}$$

where S^t_{sgf} and S^d_{sgf} are associated to the tangent and spacetime indices, respectively.

According to the fact that

$$\partial_1 \phi^a_{b2|} = B^a_b = [Q, \bar{c}^a_b], \qquad \partial_1 \phi^a_{b\mu|} = [Q, \omega^a_{b\mu}]$$

and in view of (39), we can write the gauge-fixing action associated to the tangent indices, $S^t_{gf} = S^t_{sgf|}$, in the following form

$$S^t_{gf} = \int d^4x \{B^a_b \partial^\mu \omega^b_{a\mu} + \partial^\mu \bar{c}^a_b [Q, \omega^b_{a\mu}] + B^a_b B^b_a\}. \quad (41)$$

Concerning the superaction S^d_{sgf} we note that the antighost $\bar{c}^\rho$ of the general coordinate transformations and the auxiliary field B^ρ are introduced through the relations (20a) and (28e). Thus, the superaction S^d_{sgf} can be obtained from the prescription (39) by substituting the component superfield $\phi^\rho_{\tau 2}$ with the antighost superfield (η^ρ_2) and using the necessary contraction of the components $\phi^\rho_{\tau\mu}$. This gives

$$S^d_{sgf} = \int d^4x (\partial_1 \eta^\rho_2 \partial^\mu \phi^\rho_{\rho\mu} + \partial^\mu \eta^\rho_2 \partial_1 \phi^\rho_{\rho\mu} + \partial_1 \eta^\rho_2 \partial_1 \eta^\rho_2). \quad (42)$$

Using the fact that

$$\partial_1 \eta^\rho_{2|} = B^\rho = [Q, \bar{c}^\rho]$$

and

$$\partial_1 \phi^\rho_{\rho\mu|} = [Q, \Gamma^\rho_{\rho\mu}],$$

we obtain

$$S^d_{sgf|} = S^d_{gf} = \int d^4x (B^\rho \partial^\mu \Gamma^\rho_{\rho\mu} + \partial^\mu \bar{c}^\rho [Q, \Gamma^\rho_{\rho\mu}] + B^\rho B_\rho). \quad (43)$$

Then, it is quite easy to show that the gauge-fixing action,

$$S_{gf} = S^t_{gf} + S^d_{gf}, \quad (44)$$

is invariant with respect to the geometrical BRST transformations. In fact, we have

$$[Q, S^t_{gf}] = \int d^4x (B^a_b [Q, \partial^\mu \omega^b_{a\mu}] + [Q, \partial^\mu \bar{c}^a_b][Q, \omega^b_{a\mu}]),$$
$$[Q, S^d_{gf}] = \int d^4x (B^\rho [Q, \partial^\mu \Gamma^\rho_{\rho\mu}] + [Q, \partial^\mu \bar{c}^\rho][Q, \Gamma^\rho_{\rho\mu}]), \quad (45)$$

and using the fact that the geometrical BRST operator Q commutes with the differential operator we get

$$[Q, S_{gf}] = \int d^4x \{\partial^\mu (B^a_b [Q, \omega^b_{a\mu}] + B^\rho [Q, \Gamma^\rho_{\rho\mu}])\}. \quad (46)$$

From this, it follows that the Q invariance of S_{gf} is guaranteed modulo a total divergence. So we have constructed the Q-invariant gauge-fixing action for gauge-affine gravity theory [17,20].

Furthermore, it is also interesting to construct the gauge-fixing action for quantum Einstein gravity in analogy with what is realized in super-Yang-Mills theory [27,28]. Let us first remark that the expression (41) corresponds to the gauge-fixing action S^t_{gf} associated to the tangent $\overline{ISO}(1, 3)$ group where we should consider the BRST transformation $[Q, \omega^b_{a\mu}]$ as in (31), see also Ref. [32]. To determine the gauge-fixing action S^d_{gf} associated to the diffeomorphisms group, we proceed as below but we should substitute in (39) the superfield components of the superconnection by the dynamic fields occurring in quantum Einstein gravity, namely, the antighost $\bar{c}^\mu$ and the metric $g_{\rho\mu}$ which is introduced by $\tilde{g}_{\rho\mu} = \sqrt{-g} g_{\rho\mu}$, where g is the determinant of $g_{\rho\mu}$, we obtain

$$S^d_{gf} = \int d^4x (B^\rho \partial^\mu \tilde{g}_{\rho\mu} + \partial^\mu \bar{c}^\rho [Q, \tilde{g}_{\rho\mu}] + B^\rho B_\rho), \quad (47)$$

where the BRST transformation of the field $\tilde{g}_{\rho\mu}$ is given by

$$[Q, \tilde{g}_{\rho\mu}] = \partial_\sigma (c^\sigma \tilde{g}_{\rho\mu}) + \tilde{g}_{\sigma\mu} \partial_\rho c^\sigma + \tilde{g}_{\sigma\rho} \partial_\mu c^\sigma. \quad (48)$$

In fact, knowing that $\tilde{g}_{\rho\mu} = \sqrt{-g} g_{\rho\mu}$, we have

$$[Q, \tilde{g}_{\rho\mu}] = [Q, \sqrt{-g}] g_{\rho\mu} + \sqrt{-g}[Q, g_{\rho\mu}].$$

Then by using the fact that

$$[Q, \sqrt{-g}] = \frac{-1}{2\sqrt{-g}} [Q, g],$$
$$[Q, g] = g g^{\mu\nu} [Q, g_{\mu\nu}] = 2g \partial_\sigma c^\sigma + c^\sigma \partial_\sigma g,$$

we have

$$[Q, \sqrt{-g}] = \partial_\sigma (c^\sigma \sqrt{-g}),$$

and so we can easily derive the relation (48). Finally the BRST-invariant gauge-fixing action S^d_{gf} associated to the diffeomorphisms group can be written as follows [32]

$$S^d_{gf} = \int d^4x \{B^\rho \partial^\mu \tilde{g}_{\rho\mu} + \partial^\mu \bar{c}^\rho (\partial_\sigma (c^\sigma \tilde{g}_{\rho\mu}) + \tilde{g}_{\sigma\mu} \partial_\rho c^\sigma + \tilde{g}_{\sigma\rho} \partial_\mu c^\sigma) + B^\rho B_\rho\}. \quad (49)$$

IV. CONCLUSION

In the present paper a geometric formulation of quantized gauge-affine gravity has been provided using a superfiber bundle formalism with base space simply the metric-affine spacetime and a structure group the direct product of the general linear group double-covering $\overline{GL}(4, \mathbb{R})$ with the general affine group double-covering $\overline{GA}(4, \mathbb{R})$ as well the two-dimensional old translation group $S^{0,2}$. In this geometrical framework, the gauge fields and their associated ghost and antighost fields occurring in quantized gauge-affine gravity have been described through a $\overline{GL}(4, \mathbb{R}) \otimes \overline{GA}(4, \mathbb{R})$-superconnection. Furthermore, in order to realize supercurvature constraints we introduce over a principal superfiber bundle an even pseudotensorial 1-superform which permits us to introduce a generalized superconnection, and by applying the exterior covariant superdifferential this gives the generalized supercurvature. Then the supercurvature constraints are determined by the fact that the generalized supercurvature is an even tensorial 2-superform which leads to the determination of the gauge-affine gravity BRST and anti-BRST transformations. The obtained BRST transformations are nilpotent and equivalent to those given in [18,20,25]. Reducing the four-dimensional general affine group double-covering $\overline{GA}(4, \mathbb{R})$ to the Poincaré group double-covering $\overline{ISO}(1, 3)$ we have also found the BRST and anti-BRST transformations of the fields present in quantum Einstein gravity. Moreover, we have shown how to construct the gauge-fixing superaction for gauge-affine gravity in analogy with what is realized in super-Yang-Mills theory [27,28]. Its lowest component represents the gauge-fixing action and is invariant under the geometrical BRST transformations. By using the fact that the dynamic field occurring in Einstein's gravity is represented by the tensor metric $g_{\mu\nu}$ and following the same spirit of construction of the superaction as in [27,28] we have found the gauge-fixing action for quantum Einstein gravity recovering then the standard results [32].

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