

# Physical process version of the first law of thermodynamics for black holes in higher dimensional gravity

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The problem of *physical process* version of the first law of black hole thermodynamics for charged rotating black hole in  $n$ -dimensional gravity is elaborated. The formulas for the first order variations of mass, angular momentum and canonical energy in Einstein  $(n - 2)$ -gauge form field theory are derived. These variations are expressed by means of the perturbed matter energy momentum tensor and charge matter current density.

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## I. INTRODUCTION

In their seminal paper Bardeen, Carter and Hawking [1], considering linear perturbations of a stationary electrovac black hole to another stationary black hole, found the first law of black hole thermodynamics. Contrary to the derivation presented in Ref. [1] Sudarsky *et al.* [2] using the Arnowitt-Deser-Misner (ADM) formalism derived the first law of black hole thermodynamics for Einstein-Yang-Mills (EYM) theory valid for arbitrary asymptotically flat perturbations of a stationary black hole. In Ref. [3] this method, was used to study the first law of black hole thermodynamics in Einstein-Maxwell axion-dilaton gravity (EMAD) being the low energy limit of the heterotic string theory. In Ref. [4] the first law of black hole mechanics in  $n$ -dimensional gravity was established.

The first law of black hole thermodynamics for an arbitrary diffeomorphism invariant Lagrangians with metric and matter fields possessing stationary and axisymmetric black hole solutions was widely studied in Refs. [5]. The case of a charged and rotating black hole where fields were not smooth through the event horizon was considered in Ref. [6]. The case of the higher curvature terms and higher derivative terms in the metric was considered in [7], while a generalized theory of gravity subject to the Lagrangian being arbitrary function of metric, Ricci tensor and a scalar field was treated in Ref. [8]. One can also think about a *physical process* version of the first law of black hole thermodynamics obtained by changing a stationary black hole by some infinitesimal physical process, e.g., when matter is thrown into black hole. Assuming that the black hole eventually settle down to a stationary state and calculating the changes of black hole's parameters one can find this law. If the resulting relation fails comparing to the known version of the first law of black hole thermodynamics it will provide inconsistency with the assumption that the black hole settles down to a final stationary state. This fact will give a strong evidence against cosmic censorship.

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The *physical process* version of the first law of black hole thermodynamics in Einstein theory was proved in [9]. Then, it was generalized for Einstein-Maxwell (EM) black holes in Ref. [10] and for EMAD gravity black holes in [11]. In our paper we shall find a *physical process* version of the first black hole mechanics in Einstein  $(n - 2)$ -gauge form field theory. The convention will follow Ref. [12].

## II. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK HOLE MECHANICS

We begin with the Lagrangian of generalized Maxwell  $(n - 2)$ -gauge form field in  $n$ -dimensional spacetime as follows:

$$\mathbf{L} = \epsilon^{(n)} R - F_{(n-2)}^2, \quad (1)$$

where by  $\epsilon$  we denote the volume element,  $F_{(n-2)} = dA_{(n-3)}$  is  $(n - 2)$ -gauge form field. One can remark that using in  $n$ -dimensional spacetime the generalized Maxwell field  $F_{j_1 \dots j_{n-2}}$  enables one to treat both *magnetic* and *electric* components of it.

Consider, next, the first order variation of the conserved quantities in this theory. Our main task will be to obtain the explicit formulae for the variation of mass and angular momentum. The result of the variations yields

$$\delta \mathbf{L} = \epsilon (G_{\mu\nu} - T_{\mu\nu}(F_{(n-2)})) \delta g^{\mu\nu} + 2(n-2)! (\nabla_{j_1} F^{j_1 \dots j_{n-2}}) \delta A_{j_2 \dots j_{n-2}} + d\Theta, \quad (2)$$

where the energy momentum tensor for  $(n - 2)$ -gauge form field is given by the expression

$$T_{\mu\nu}(F_{(n-2)}) = (n-2) F_{\mu j_2 \dots j_{n-2}} F_{\nu}^{j_2 \dots j_{n-2}} - \frac{1}{2} g_{\mu\nu} F_{j_1 \dots j_{n-2}} F^{j_1 \dots j_{n-2}}. \quad (3)$$

The totally divergent term in Eq. (2) is a functional of the field variables  $A_{j_1 \dots j_{n-3}}$  and their variations  $\delta A_{j_1 \dots j_{n-3}}$  which for simplicity we have denoted, respectively, by  $\psi_\alpha$  and  $\delta\psi_\alpha$ . Inspection of relation (2) reveals the following form of the symplectic  $(n - 1)$ -form  $\Theta_{j_1 \dots j_{n-1}}[\psi_\alpha, \delta\psi_\alpha]$ , namely

$$\Theta_{j_1 \dots j_{n-1}}[\psi_\alpha, \delta\psi_\alpha] = \epsilon_{\mu j_1 \dots j_{n-1}}[\omega^\mu - 2(n-2)!F^{\mu j_2 \dots j_{n-2}} \times \delta A_{j_2 \dots j_{n-2}}] \quad (4)$$

where  $\omega_\mu$  implies

$$\omega_\mu = \nabla^\alpha \delta g_{\alpha\mu} - \nabla_\mu \delta g_\beta^\beta. \quad (5)$$

By virtue of Eq. (2) one enables to read off the source-free Einstein  $(n-2)$ -gauge form fields equations of motion

$$G_{\mu\nu} - T_{\mu\nu}(F_{(n-2)}) = 0, \quad (6)$$

$$\nabla_{j_1} F^{j_1 \dots j_{n-2}} = 0. \quad (7)$$

One identifies the variations of the fields  $\delta\psi_\alpha$  with a general coordinate transformations  $\mathcal{L}_\xi \psi_\alpha$  induced by an arbitrary Killing vector  $\xi_\alpha$ . The Noether  $(n-1)$ -form, with respect to this Killing vector  $\xi_\alpha$ , implies [5]

$$J_{j_1 \dots j_{n-1}} = \epsilon_{mj_1 \dots j_{n-1}} J^m[\psi_\alpha, \mathcal{L}_\xi \psi_\alpha], \quad (8)$$

where the vector field  $J^m[\psi_\alpha, \mathcal{L}_\xi \psi_\alpha]$  is given by

$$J^\delta[\psi_\alpha, \mathcal{L}_\xi \psi_\alpha] = \Theta^\delta[\psi_\alpha, \mathcal{L}_\xi \psi_\alpha] - \xi^\delta L. \quad (9)$$

Using the above relation (9) we get the resultant expression for the Noether current  $(n-1)$ -form with respect to the Killing vector field  $\xi_\alpha$ . It yields

$$J_{j_1 \dots j_{n-1}} = dQ_{j_1 \dots j_{n-1}}^{GR} + 2\epsilon_{\delta j_1 \dots j_{n-1}}(G_\eta^\delta - T_\eta^\delta(F_{(n-2)}))\xi^\eta \quad (10)$$

$$\begin{aligned} & -2(n-2)!(n-3)\epsilon_{mj_1 \dots j_{n-1}} \nabla_{\alpha_2}(\xi^d A_{d\alpha_3 \dots \alpha_{n-2}} F^{m\alpha_2 \dots \alpha_{n-2}}) \\ & + 2(n-2)!(n-3)\epsilon_{mj_1 \dots j_{n-1}} \xi^k A_{k\alpha_3 \dots \alpha_{n-2}} \\ & \times \nabla_{\alpha_2}(F^{m\alpha_2 \dots \alpha_{n-2}}), \end{aligned}$$

where  $Q_{j_1 \dots j_{n-2}}^{GR}$  implies

$$Q_{j_1 \dots j_{n-2}}^{GR} = -\epsilon_{j_1 \dots j_{n-2} ab} \nabla^a \xi^b. \quad (11)$$

Quite nontrivial calculations reveal that the relation (10) can be written in the following form:

$$\begin{aligned} J_{j_1 \dots j_{n-1}} &= dQ_{j_1 \dots j_{n-1}} + 2\epsilon_{\delta j_1 \dots j_{n-1}}(G_\eta^\delta - T_\eta^\delta(F_{(n-2)}))\xi^\eta \\ &+ 2(n-2)!(n-3)\epsilon_{mj_1 \dots j_{n-1}} \nabla_{\alpha_2} F^{m\alpha_2 \dots \alpha_{n-2}} \\ &\times \xi^k A_{k\alpha_3 \dots \alpha_{n-2}}. \end{aligned} \quad (12)$$

Having in mind that  $J[\xi] = dQ[\xi] + \xi^\alpha \mathbf{C}_\alpha$ , where  $\mathbf{C}_\alpha$  is an  $(n-1)$  form locally constructed from the dynamical fields we may identify  $Q_{\alpha\beta}$  as the Noether charge. Thus, the Noether charge is subject to the expression

$$Q_{j_1 \dots j_{n-2}} = Q_{j_1 \dots j_{n-2}}^{GR} + Q_{j_1 \dots j_{n-2}}^F, \quad (13)$$

where

$$Q_{j_1 \dots j_{n-2}}^F = 2(n-3)\epsilon_{mkj_1 \dots j_{n-2}} F^{mk\alpha_3 \dots \alpha_{n-2}} \xi^d A_{d\alpha_3 \dots \alpha_{n-2}}. \quad (14)$$

On the other hand, the quantity  $C_{aj_1 \dots j_{n-1}}$  implies the following:

$$\begin{aligned} C_{aj_1 \dots j_{n-1}} &= 2\epsilon_{mj_1 \dots j_{n-1}}[G_a^m - T_a^m(F_{(n-2)})] \\ &+ 2(n-2)!(n-3)\epsilon_{mj_1 \dots j_{n-1}} \nabla_{\alpha_2} F^{m\alpha_2 \dots \alpha_{n-2}} \\ &\times A_{a\alpha_3 \dots \alpha_{n-2}}. \end{aligned} \quad (15)$$

If  $\mathbf{C}_\alpha$  is equal to zero one has the source-free equations fulfilled, when it does not hold we obtain

$$G_{\mu\nu} - T_{\mu\nu}(F_{(n-2)}) = T_{\mu\nu}(\text{matter}), \quad (16)$$

$$\nabla_{j_1} F^{j_1 \dots j_{n-2}} = j^{j_2 \dots j_{n-2}}(\text{matter}). \quad (17)$$

Let us suppose that  $(g_{\mu\nu}, A_{\alpha_1 \dots \alpha_{n-3}})$  be the solution of the source-free equations of motion with  $(n-2)$ -gauge form field. Let us assume further that  $(\delta g_{\mu\nu}, \delta A_{\alpha_1 \dots \alpha_{n-3}})$  be linearized perturbations fulfilling equations of motion with sources  $\delta T_{\mu\nu}(\text{matter})$  and  $\delta j_{j_2 \dots j_{n-2}}(\text{matter})$ . Thus, for a perturbed  $\delta C_{aj_1 \dots j_{n-1}}$  quantity we have the following relation:

$$\begin{aligned} \delta C_{aj_1 \dots j_{n-1}} &= 2\epsilon_{mj_1 \dots j_{n-1}}[\delta T_a^m(\text{matter}) + (n-2)!(n-3) \\ &\times A_{a\alpha_3 \dots \alpha_{n-2}} \delta j^{m\alpha_3 \dots \alpha_{n-2}}(\text{matter})]. \end{aligned} \quad (18)$$

As was shown in Ref. [10] when  $\xi_\alpha$  is a Killing vector field of a background spacetime and it also describes a symmetry of background matter fields. Then, one can write the explicit formula for a conserved quantity  $\delta H_\xi$  connected with the aforementioned Killing vector field, namely

$$\delta H_\xi = - \int_\Sigma \xi^\alpha \delta \mathbf{C}_\alpha + \int_{\partial\Sigma} (\delta Q(\xi) - \xi \cdot \Theta). \quad (19)$$

In our case  $\delta H_\xi$  has the form as follows:

$$\begin{aligned} \delta H_\xi &= -2 \int_\Sigma \epsilon_{mj_1 \dots j_{n-1}}[\delta T_a^m(\text{matter})\xi^a + (n-2)!(n-3) \\ &\times \xi^a A_{a\alpha_3 \dots \alpha_{n-2}} \delta j^{m\alpha_3 \dots \alpha_{n-2}}(\text{matter})] \\ &+ \int_{\partial\Sigma} [\delta Q(\xi) - \xi \cdot \Theta]. \end{aligned} \quad (20)$$

By virtue of choosing  $\xi^\alpha$  to be an asymptotic time translation  $t^\alpha$  one can conclude that  $M = H_t$ . Thus, we finally obtain the variation of the ADM mass

$$\begin{aligned} \alpha \delta M &= -2 \int_\Sigma \epsilon_{mj_1 \dots j_{n-1}}[\delta T_a^m(\text{matter})t^a + (n-2)!(n-3) \\ &\times t^a A_{a\alpha_3 \dots \alpha_{n-2}} \delta j^{m\alpha_3 \dots \alpha_{n-2}}(\text{matter})] \\ &+ \int_{\partial\Sigma} [\delta Q(t) - t \cdot \Theta], \end{aligned} \quad (21)$$

where  $\alpha = \frac{n-3}{n-2}$ . On the other hand, taking the Killing vector fields  $\phi_{(i)}$  which are responsible for the rotation in the adequate directions, we arrive at the relations for angular momenta

$$\begin{aligned} \delta J_{(i)} &= 2 \int_{\Sigma} \epsilon_{mj_1 \dots j_{n-1}} [\delta T_a^m(\text{matter}) \phi_{(i)}^a + (n-2)!(n-3) \\ &\quad \times \phi_{(i)}^a A_{a\alpha_3 \dots \alpha_{n-2}} \delta j^{m\alpha_3 \dots \alpha_{n-2}}(\text{matter})] \\ &\quad - \int_{\partial\Sigma} [\delta Q(\phi_{(i)} - \phi_{(i)} \cdot \Theta)]. \end{aligned} \quad (22)$$

In order to consider the *physical process* version of the first law of black hole thermodynamics let us suppose that one has a classical, stationary black hole solution to the equations of motion of Einstein  $(n-2)$ -gauge form field system (6) and (7). Then, one perturbs the considered black hole by dropping charged matter. Assuming that the black hole will be not destroyed in the process of this phenomenon and settles down to a stationary final state, one can calculate the change of black hole's parameters. To proceed to the *physical process* version of the first law of black hole thermodynamics let us assume moreover that  $(\delta g_{\mu\nu}, \delta A_{\alpha_1 \dots \alpha_{n-3}})$  are solutions to the source-free Einstein equations with  $(n-2)$  form field. Furthermore, suppose that the event horizon the Killing vector field  $\chi^\mu$  is of the form as

$$\chi^\mu = t^\mu + \sum_i \Omega_{(i)} \phi^{\mu(i)} \quad (23)$$

Let us assume further that  $\Sigma_0$  is an asymptotically flat hypersurface which terminates on the event horizon and take into account the initial data on  $\Sigma_0$  for a linearized perturbations  $(\delta g_{\mu\nu}, \delta A_{\alpha_1 \dots \alpha_{n-3}})$  with  $\delta T_{\mu\nu}(\text{matter})$  and  $\delta j^{\alpha_2 \dots \alpha_{n-2}}(\text{matter})$ . We require that  $\delta T_{\mu\nu}(\text{matter})$  and  $\delta j^{\alpha_2 \dots \alpha_{n-2}}(\text{matter})$  disappear at infinity and the initial data for  $(\delta g_{\mu\nu}, \delta A_{\alpha_1 \dots \alpha_{n-3}})$  vanish in the vicinity of the black hole horizon  $\mathcal{H}$  on the hypersurface  $\Sigma_0$ . It envisages the fact that for the initial time  $\Sigma_0$ , the considered black hole is unperturbed. Consequently, taking into account Eqs. (21) and (22) and having in mind that the perturbations vanish near the internal boundary  $\partial\Sigma_0$  of the initial surface, we can write the following:

$$\begin{aligned} \alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} &= -2 \int_{\Sigma_0} \epsilon_{mj_1 \dots j_{n-1}} [\delta T_a^m(\text{matter}) \phi_{(i)}^a \\ &\quad + (n-2)!(n-3) \phi_{(i)}^a A_{a\alpha_3 \dots \alpha_{n-2}} \\ &\quad \times \delta j^{m\alpha_3 \dots \alpha_{n-2}}(\text{matter})] \\ &= \int_{\mathcal{H}} \gamma^\alpha k_\alpha \bar{\epsilon}_{j_1 \dots j_{n-1}}, \end{aligned} \quad (24)$$

where  $\bar{\epsilon}_{j_1 \dots j_{n-1}} = n^\delta \epsilon_{\delta j_1 \dots j_{n-1}}$  and  $n^\delta$  is the future directed unit normal to the hypersurface  $\Sigma_0$ . In the last line of Eq. (24) we replace  $n^\delta$  by the vector  $k^\delta$  tangent to the affinely parametrized null generators of the black hole event horizon  $\mathcal{H}$ . It can be done due to the fact of the conservation of current  $\gamma^\alpha$  and the assumption that all of the matter falls into the considered black hole. Of course, one should also integrate over the event horizon  $\mathcal{H}$ .

Now, it is easy to see that we have left with the following:

$$\alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} + \delta \mathcal{E}_F = 2 \int_{\mathcal{H}} \delta T_{\mu}^{\nu} \xi^{\mu} k_{\nu}, \quad (25)$$

where by  $\delta \mathcal{E}_F$  is the canonical energy of  $(n-2)$ -gauge form fields [4].

Our next task will be to find the change in the area of black hole horizon. Having in mind that the null generators of the event horizon of the perturbed black hole coincide with the null generators of the unperturbed stationary black hole [10] (when we use the diffeomorphism freedom in identifying the perturbed spacetime with the background one) the result of this gauge choice is that the perturbation in the location of the event horizon disappears and we obtain that  $\delta k_\mu \propto k_\mu$ .

Consider, next the Raychauduri equation of the form as follows:

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{(n-2)} - \sigma_{ij} \sigma^{ij} - R_{\mu\nu} \xi^\mu \xi^\nu. \quad (26)$$

Because of the fact that the expansion  $\theta$  and shear  $\sigma_{\mu\nu}$  vanish in the stationary background, one gets the perturbed Raychauduri's equation written as

$$\begin{aligned} \frac{d(\delta\theta)}{d\lambda} &= -\delta(T_{\mu\nu}(\text{total})k^\mu k^\nu) |_{\mathcal{H}} \\ &= -\delta(T_{\mu\nu}(\text{matter}))k^\mu k^\nu |_{\mathcal{H}} \\ &\quad - \delta(T_{\mu\nu}(F_{(n-2)}))k^\mu k^\nu |_{\mathcal{H}}, \end{aligned} \quad (27)$$

where we exploit the fact that  $T(F_{(n-2)})_{\mu\nu} k^\mu k^\nu |_{\mathcal{H}} = 0$  and  $\delta k_\mu \propto k_\mu$  to eliminate terms in the form  $T(F_{(n-2)})_{\mu\nu} k^\mu \delta k^\nu$ . One finds that the remaining term in Eq. (27) reads

$$\begin{aligned} \delta T_{\mu\nu}(F_{(n-2)})k^\mu k^\nu |_{\mathcal{H}} &= [2(n-2)\delta F_{\mu j_2 \dots j_{n-2}} F_{\nu}^{j_2 \dots j_{n-2}} \\ &\quad - \frac{1}{2} \delta g_{\mu\nu} F_{j_1 \dots j_{n-2}} F^{j_1 \dots j_{n-2}} \\ &\quad - g_{\mu\nu} \delta F_{j_1 \dots j_{n-2}} F^{j_1 \dots j_{n-2}}] k^\mu k^\nu. \end{aligned} \quad (28)$$

The last two expressions in the above equation are equal to zero, because of the fact that the vector  $k_\mu$  is a null vector both in the perturbed as well as in unperturbed case. One also finds that  $F_{\nu j_2 \dots j_{n-2}} k^\nu \propto k_{j_2} \dots k_{j_{n-2}}$ . For since we take into account the antisymmetry of  $\delta F_{\mu j_2 \dots j_{n-1}}$  the first term in the related equation also vanishes. Hence, one concludes that

$$\frac{d(\delta\theta)}{d\lambda} = -\delta T_{\mu\nu}(\text{matter})k^\mu k^\nu |_{\mathcal{H}}, \quad (29)$$

Calculations of the right-hand side of Eq. (29) are identical

as in Ref. [9]. For the readers' convenience we quote the main steps. Namely, it is possible to substitute for the Killing vector  $k_\mu$  the following expression

$$k_\mu = \left( \frac{\partial}{\partial V} \right)_\mu = \frac{1}{\kappa V} (t^\mu + \sum_i \Omega_{(i)} \phi^{\mu(i)}), \quad (30)$$

where  $\kappa$  is the surface gravity. On the other hand,  $V$  is an affine parameter along the null geodesics tangent to  $\xi_\beta$  generating the adequate Killing horizon. One can introduce the function  $v$  (it is called *Killing parameter time*) on the portion of Killing horizon. It satisfies the relation  $\xi^\beta \nabla_\beta v = 1$  and it is related with  $V$  by the expression  $V = \exp(\kappa v)$ . Then, we multiply both sides of the resulting equation by  $\kappa V$  and integrate over the event horizon.

One should also recall that expansion  $\theta$  measures the local rate of change of the cross-sectional area as the observer moves up the null geodesics. It can be parameterize in the following way:  $\theta = \frac{1}{A} \frac{dA}{d\lambda}$ , where  $\lambda$  is an affine parameter which parametrized null geodesics generators of

the horizon. The left-hand side of Eq. (29) is evaluated by integration by parts, having in mind that  $V = 0$  at the lower limit and  $\theta$  has to vanish faster than  $1/V$  as  $V$  tends to infinity when the considered black hole settled down to a stationary final state after throwing some charged matter into it. The consequence of the above establishes the result

$$\kappa \delta A = \int_{\mathcal{H}} \delta T_\nu^\mu (\text{matter}) \xi^\nu k_\mu. \quad (31)$$

In the light of what has been shown we obtained the *physical process* version of the first law of black hole mechanics in Einstein ( $n - 2$ )-gauge form fields gravity of the same form as known from Ref. [4], namely

$$\alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} + \delta \mathcal{E}_F = \kappa \delta \mathcal{A}. \quad (32)$$

Like in four-dimensional case a proof of the *physical process* version of the first law of thermodynamics for  $n$ -dimensional black hole also provides support for cosmic censorship.

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- [1] J. M. Bardeen, B. Carter, and S. W. Hawking, *Commun. Math. Phys.* **31**, 161 (1973).  
 [2] D. Sudarsky and R. M. Wald, *Phys. Rev. D* **46**, 1453 (1992).  
 [3] M. Rogatko, *Phys. Rev. D* **58**, 044011 (1998).  
 [4] M. Rogatko, *Phys. Rev. D* **71**, 024031 (2005).  
 [5] R. M. Wald, *Phys. Rev. D* **48**, R3427 (1993), V. Iyer and R. M. Wald, *Phys. Rev. D* **50**, 846 (1994), V. Iyer and R. M. Wald, *Phys. Rev. D* **52**, 4430 (1995); V. Iyer, *Phys. Rev. D* **55**, 3411 (1997).  
 [6] S. Gao, *Phys. Rev. D* **68**, 044016 (2003).  
 [7] T. Jacobsen, G. Kang, and R. C. Myers, *Phys. Rev. D* **49**, 6587 (1994); T. Jacobsen, G. Kang, and R. C. Myers, *Phys. Rev. D* **52**, 3518 (1995).  
 [8] J. Koga and K. Maeda, *Phys. Rev. D* **58**, 064020 (1998).  
 [9] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).  
 [10] S. Gao and R. M. Wald, *Phys. Rev. D* **64**, 084020 (2001).  
 [11] M. Rogatko, *Class. Quant. Grav.* **19**, 3821 (2002).  
 [12] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).