Spherically symmetric, static spacetimes in a tensor-vector-scalar theory

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Recently, a relativistic gravitation theory has been proposed [J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004)] that gives the modified Newtonian dynamics in the weak acceleration regime. The theory is based on three dynamic gravitational fields and succeeds in explaining a large part of extragalactic and gravitational lensing phenomenology without invoking dark matter. In this work, I consider the strong gravity regime of TeVeS. I study spherically symmetric, static, and vacuum spacetimes relevant for a nonrotating black hole or the exterior of a star. Two branches of solutions are identified: in the first, the vector field is aligned with the time direction, while in the second, the vector field has a nonvanishing radial component. I show that in the first branch of solutions the β and γ parametrized post-Newtonian (PPN) coefficients in TeVeS are identical to these of general relativity, while in the second the β PPN coefficient differs from unity, violating observational determinations of it (for the choice of the free function F of the theory made in Bekenstein's paper). For the first branch of solutions, I derive analytic expressions for the physical metric and discuss their implications. Applying these solutions to the case of black holes, it is shown that they violate causality (since they allow for superluminal propagation of metric, vector, and scalar waves) in the vicinity of the event horizon and/or that they are characterized by negative energy density carried by the fields.

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I. INTRODUCTION

On cosmological scales, Newtonian gravitational theory underpredicts the acceleration of stars and gas. Furthermore, galaxies and clusters of galaxies show anomalously large gravitational lensing when only baryonic matter is taken into account. A natural "cure" for these discrepancies is to assume the existence of dark matter, which dominates over visible matter [1–5]. Such dark matter might solve the "missing mass" problem within the standard theory of gravity (i.e., general relativity or GR). For this picture to be complete, however, the origin of dark matter needs to be identified.

A second approach to the acceleration discrepancy and lensing anomaly is to look for alternative theories of gravity that modify GR on large scales. Among many other attempts, the modified Newtonian dynamics (MOND) paradigm has been proposed [6,7]. It is characterized by an acceleration scale a_0 so that

$$\tilde{\mu}(|\vec{a}|/a_0)\vec{a} = -\nabla\Phi_{\rm N}, \tag{1}$$

where $\Phi_{\rm N}$ is to be understood as the Newtonian potential, $\tilde{\mu}(x) = x$ for $x \ll 1$, while $\tilde{\mu}(x) = 1$ for $x \gg 1$.

This empirical law has been very successful in explaining the rotation curves in a large number of (spiral, low surface brightness, and elliptical) galaxies using the observed distributions of gas and stars as input [8–10]. MOND can also explain the observed correlation between the infrared luminosity of a disk galaxy L_K and the asymptotic rotational velocity v_a (i.e., the Tully-Fisher law: $L_K \propto v_a^4$ [11]), on the assumption (suggested by population synthesis models) that the M/L_K ratio is constant.

However, MOND is merely a prescription for gravity and not a self-consistent theory. It violates, for example, conservation of momentum and angular momentum and does not provide the formulation to describe light deflection or to build a cosmological model. A theory of gravity is needed that has the MOND characteristics in the weak acceleration limit but also has full predictive power.

In this context, a new relativistic theory of gravity has been introduced by Bekenstein [12]. It consists of three dynamical gravitation fields: a tensor field ($g_{\alpha\beta}$), a vector field (U_α), and a scalar field (ϕ) leading to the acronym TeVeS. The theory involves a free function F, a length scale ℓ (that can be related to a_0), and two positive dimensionless constants κ, K.

TeVeS gives MOND in the weak acceleration limit (and therefore inherits the successes of MOND on a large scale), makes similar prediction on gravitational lensing as GR (with dark matter), and provides a formulation for constructing cosmological models. As a drawback, however, one can mention that TeVeS still appears to need dark matter to address the cosmological matter problem, i.e., the fact that observations require that the source term of Friedmann's equations is a factor of ∼6 the baryonic matter density.

A self-consistent theory must be causal (i.e., not to allow for superluminal propagation of any measurable field or energy) and in TeVeS this is the case provided that $\phi > 0$ (see Ref. [12], Sec. VIII). It can be shown that, for a range of initial conditions, Friedmann-Robertson-Walker cosmological models with flat spaces in TeVeS expand forever with $0 < \phi \ll 1$ throughout. Moreover, in the vicinity of a star embedded in this cosmological background ϕ is still positive. So, in a wide range of environments, TeVeS has been shown to be causal.

The predictions of the theory have thus been explored to some extent for a range of strengths of the gravitational field: from the MOND limit to the post-Newtonian corrections in the inner solar system. The strong gravity regime (e.g., in the vicinity of a black hole or a neutron star) of TeVeS has not been studied. This regime is a topic of this work. The motivation for this work is twofold. First, it is interesting to see how the physics of compact objects (i.e., black holes, neutron stars) differ in TeVeS with respect to GR and what constraints (if any) observations can put on its free parameters. Second, one can check the consistency of the theory (e.g., its causality, positivity of energy carried by the fields) in these extreme conditions.

In Sec. II, I summarize the fundamentals of TeVeS and in Sec. III I consider its strong gravity limit. I limit myself to static, spherically symmetric, and vacuum spacetimes relevant for a nonrotating black hole or the exterior of a star. Two branches of solutions are identified: in the first the vector field is aligned with the time direction, while in the (not previously explored) second branch the vector field has a nonvanishing radial component. I show that the β and γ parametrized post-Newtonian (PPN) coefficients in TeVeS are identical to those of GR in the first branch of solutions, while the β PPN coefficient differs in the second. For the choice of the free function F made in Ref. [12], I find that TeVeS predicts a value for β that is in conflict with recent observational determinations of it. In Sec. IV, I consider the first branch of solutions and derive exact solutions for the metric for arbitrary values of the parameters of the theory. The observational properties of the black holes in TeVeS are discussed in Sec. V along with the issue of superluminal propagation of waves in the black hole vicinity. Conclusions are given in Sec. VI.

II. THE BASIC EQUATIONS OF TEVES

TeVeS is based on three dynamical gravitational fields: a tensor field (the Einstein metric $g_{\alpha\beta}$), a 4-vector field U_α, and a scalar field ϕ with an additional nondynamical scalar field σ. The physical metric $\tilde{g}_{\alpha\beta}$ in TeVeS is connected to these fields through the expression

$$\tilde{g}_{\alpha\beta} \equiv e^{-2\phi} g_{\alpha\beta} - 2U_\alpha U_\beta \sinh(2\phi). \tag{2}$$

The total action in TeVeS is the sum of four terms S_g, S_s, S_U, and S_m (see Ref. [12]), where S_g is identical to the Hilbert-Einstein action and is the part that corresponds to the tensor field, while S_s, S_U, S_m are the actions of the two scalar fields, the vector field, and the matter, respectively. The basic equations of TeVeS are derived by varying the total action S with respect to $g^{\alpha\beta}$, ϕ, σ, U_α.

Doing so for $g^{\alpha\beta}$, one arrives at the metric equations

$$G_{\alpha\beta} = 8\pi G[\tilde{T}_{\alpha\beta} + (1 - e^{4\phi})U^\mu \tilde{T}_{\mu(\alpha}U_{\beta)} + \tau_{\alpha\beta}] + \Theta_{\alpha\beta}, \tag{3}$$

where a pair of indices surrounded by parentheses stands for symmetrization, i.e., $A_{(\alpha}B_{\beta)} = A_\alpha B_\beta + A_\beta B_\alpha$, the $G_{\alpha\beta}$ denotes the Einstein tensor for $g_{\alpha\beta}$, $\tilde{T}_{\alpha\beta}$ is the energy momentum tensor, and

$$\tau_{\alpha\beta} \equiv \sigma^2[\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - U^\mu\phi_{,\mu}(U_{(\alpha}\phi_{,\beta)} - \tfrac{1}{2}U^\nu\phi_{,\nu}g_{\alpha\beta})] - \tfrac{1}{4}G\ell^{-2}\sigma^4 F(kG\sigma^2)g_{\alpha\beta}, \tag{4}$$

$$\Theta_{\alpha\beta} \equiv K(g^{\mu\nu}U_{[\mu,\alpha]}U_{[\nu,\beta]} - \tfrac{1}{4}g^{\sigma\tau}g^{\mu\nu}U_{[\sigma,\mu]}U_{[\tau,\nu]}g_{\alpha\beta}) - \lambda U_\alpha U_\beta, \tag{5}$$

where a pair of indices surrounded by brackets stands for antisymmetrization, i.e., $A_{[\alpha}B_{\beta]} = A_\alpha B_\beta - A_\beta B_\alpha$.

Similarly, one derives a scalar equation that can be brought into the form

$$[\mu(k\ell^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu})h^{\alpha\beta}\phi_{,\alpha}]_{;\beta} = kG[g^{\alpha\beta} + (1 + e^{-4\phi})U^\alpha U^\beta]\tilde{T}_{\alpha\beta}, \tag{6}$$

where $h^{\alpha\beta} \equiv g^{\alpha\beta} - U^\alpha U^\beta$ and $\mu(y)$ is defined by

$$-\mu F(\mu) - \tfrac{1}{2}\mu^2 \dot{F}(\mu) = y, \tag{7}$$

where $\dot{F} \equiv dF/d\mu$. The scalar field σ is given by

$$kG\sigma^2 = \mu(k\ell^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}). \tag{8}$$

Note that the form of the function $F(\mu)$ [or, equivalently, $y(\mu)$] is not predicted by the theory and is essentially a free function. In next section I give the form used in Bekenstein's paper. As it will turn out, however, the results derived here are essentially independent of the exact choice of $F(\mu)$ and quite general. On the other hand, our final conclusions do depend on the choice of F, since it influences the way in which observations put constraints on the parameters of the theory (see, for example, Sec.).

Finally, the vector equation is derived through variation of S with respect to U_α

$$KU^{[\alpha;\beta]}{}_{;\beta} + \lambda U^\alpha + 8\pi G\sigma^2 U^\beta \phi_{,\beta} g^{\alpha\gamma}\phi_{,\gamma} = 8\pi G(1 - e^{-4\phi})g^{\alpha\mu}U^\beta \tilde{T}_{\mu\beta}, \tag{9}$$

where $U^\alpha \equiv g^{\alpha\beta}U_\beta$ and λ is a Lagrange multiplier. These four equations determine λ and three of the components of U^α, with the fourth being determined by the normalization of the vector field

$$g^{\alpha\beta}U_\alpha U_\beta = -1. \tag{10}$$

A. The function F

The function $F(\mu)$ [or, equivalently, of $y(\mu)$] is a free function since there is no theory for it. One has large freedom in choosing the form of F, each to be checked on implications for cosmological models, galactic rotation curves, and constraints from measurements in the outer solar system. Bekenstein in Ref. [12] made the following choice:

$$y(\mu) = \frac{3}{4}\frac{\mu^2(\mu-2)^2}{1-\mu} \tag{11}$$

which, using Eq. (7), leads to

$$F(\mu) = \frac{3}{8}\frac{\mu(4+2\mu-4\mu^2+\mu^3)+4\ln(1-\mu)}{\mu^2}. \tag{12}$$

It can be shown that the range $0<\mu<1$ (i.e., $y>0$) is relevant for quasistationary systems and $2<\mu<\infty$ (i.e., $y<0$) for cosmology. For this specific choice of $F(\mu)$, one can put a lower limit on the value of the κ parameter of the theory so that it is not in conflict with the measured motions of planets of the outer solar system (see Ref. [12], Sec. IV). On the other hand, small values of κ are relevant for cosmological models. Together, these constraints indicate a value of κ around ~ 0.03. It should be stressed that it depends on the specific choice of the form of $F(\mu)$.

The Newtonian limit of a spherically symmetric system has been explored in Sec. IV C of Ref. [12], where it is shown that for gravitational accelerations $|\vec{a}|/a_0 \gg 8\pi^2/\kappa^2$ the quantity $y \to \infty$ and, consequently, $\mu \to 1$. As an arithmetic example, at Earth's and Mercury's orbit μ differs from unity by about 2×10^{-6} and 5×10^{-8}, respectively, for the specific choice (12) of function F and $\kappa = 0.03$. Since in this study we focus on the strong gravity limit, we can safely take $\mu = 1$ and, therefore, [see Eq. (8)]

$$\sigma^2 = \frac{1}{\kappa G}. \tag{13}$$

Strictly speaking, μ has been shown to be of order unity in the Newtonian limit but not necessary in the relativistic limit. However, using the analytic solutions derived in Sec. IV of this work, I have checked that taking $\mu = 1$ is an excellent approximation also in this limit.

III. SPHERICAL SYMMETRIC, STATIC SPACETIMES IN TEVES

From this point on, we focus on the strong gravity limit of TeVeS and explore the spacetime in the vicinity of a spherically symmetric mass. The isotropic form of a spherical symmetric, static metric is

$$g_{\alpha\beta}dx^\alpha dx^\beta = -e^\nu dt^2 + e^\zeta(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2), \tag{14}$$

where both ν and ζ are only functions of r. For the static system of our case, the vector field has two nonvanishing components

$$U^\alpha = (U^t, U^r, 0, 0), \tag{15}$$

where U^t and U^r are functions of the radial coordinate.

Taking $\phi = \phi(r)$, the scalar-field equation (6) in vacuum may be written as

$$\frac{e^{-(\nu+3\zeta)/2}}{r^2}[r^2 e^{(\nu+3\zeta)/2}\phi'(e^{-\zeta} - (U^r)^2)]' = 0, \tag{16}$$

where a prime stands for ordinary derivative with respect to r.

Equation (16) can be integrated once to give

$$\phi' = C\frac{e^{-(\nu+3\zeta)/2}}{r^2[e^{-\zeta} - (U^r)^2]}, \tag{17}$$

where C is an integration constant. To determine this constant, one must consider the source of the gravitational field. An analysis relevant for an extended source (e.g., a star) is given in Bekenstein [12] Sec. V, where it is shown that one can define the "scalar" mass m_s as a (non-negative) particular integral over $\tilde{\rho}$ and $\tilde{P}$ (defined as the proper energy density and pressure expressed in the physical metric) of the star's matter so that the integration constant is given by

$$C \equiv \kappa G m_s/(4\pi). \tag{18}$$

The r component of the vector equation (9) (the θ and ϕ components vanish because of the symmetry of the problem under consideration) can be brought into the form

$$U^r\left(\lambda + \frac{\kappa(Gm_s)^2}{2\pi}\frac{e^{-(\nu+4\zeta)}}{r^4[e^{-\zeta} - (U^r)^2]^2}\right) = 0. \tag{19}$$

This equation shows that there are two cases: either U^r vanishes or one has a constraint on the Lagrange multiplier λ. In the former case, the vector field is aligned with the time direction, while in the latter, there is a nonvanishing radial component of the vector field. Since the mathematical analysis of the two cases is rather different, I examine them separately.

A. Case I: The vector field is aligned to the time direction

When U^r vanishes, the vector field is determined by the normalization expression (10), which yields

$$U^\alpha = (e^{-\nu/2}, 0, 0, 0). \tag{20}$$

The physical metric is then given by Eq. (2), which reduces to $\tilde{g}_{tt} = g_{tt}e^{2\phi}$, $\tilde{g}_{ii} = g_{ii}e^{-2\phi}$. To determine $\tilde{g}^{\alpha\beta}$, one needs to solve for ν and ζ and, through them, for the metric $g_{\alpha\beta}$ and the scalar ϕ. To this end, the differential equations resulting from the tt, rr, and $\theta\theta$ components of the metric equation (3) must be solved. This is equivalent to the procedure one follows to arrive at the GR solutions.

Since we are looking for vacuum spacetimes, the terms that include the matter energy density in Eq. (3) are zero. So, we are left with the $\tau_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ terms. The $\tau_{\alpha\beta}$ [see

Eq. (4)] contains $\phi_{,\alpha}$ terms, while the last term depends on the function F. In the strong acceleration limit, it is possible to show (see Sec. V in Ref. [12]) that the last term is completely negligible in comparison to the other terms. This is exactly the limit we are interested in here and so we will neglect the F term. Using Eqs. (4), (17), and (20), we then find

$$\tau_{tt} = \frac{\kappa G m_s^2}{32\pi^2}\frac{e^{-2\zeta}}{r^4}, \tag{21}$$

$$\tau_{rr} = \frac{\kappa G m_s^2}{32\pi^2}\frac{e^{-(\zeta+\nu)}}{r^4}, \tag{22}$$

$$\tau_{\theta\theta} = -\frac{\kappa G m_s^2}{32\pi^2}\frac{e^{-(\zeta+\nu)}}{r^2}. \tag{23}$$

To proceed with the calculation of the $\Theta_{\alpha\beta}$ terms [defined by Eq. (5)], one first needs to compute the Lagrange multiplier $\lambda_{\rm I}$ (where the index I is used to show that it corresponds to case I) from the t component of Eq. (9) (the other components vanish because of the symmetry of the problem and because we study the case where $U^r = 0$). Using Eq. (20) and that $U^\beta \phi_{,\beta} = 0$ and $\tilde{T}_{\alpha\beta} = 0$, we have

$$\lambda_{\rm I} = -Ke^{-\zeta}\left(\frac{\nu''}{2} + \frac{\nu'\zeta'}{4} + \frac{\nu'}{r}\right). \tag{24}$$

Substituting Eqs. (20) and (24) in (5), we get

$$\Theta_{tt} = Ke^{\nu-\zeta}\left(\frac{(\nu')^2}{8} + \frac{\nu''}{2} + \frac{\nu'\zeta'}{4} + \frac{\nu'}{r}\right), \tag{25}$$

$$\Theta_{rr} = -\frac{K}{8}(\nu')^2, \tag{26}$$

$$\Theta_{\theta\theta} = \frac{K}{8}(r\nu')^2. \tag{27}$$

We can now use the tt and rr components of the metric equation [Eq. (3)] to derive a system of ordinary differential equations for ζ and ν. Using Eqs. (21), (22), (25), and (26) in (3), and after some rearrangement, one finds that

$$\zeta'' + \frac{(\zeta')^2}{4} + \frac{2\zeta'}{r} = -\frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4} - K\left(\frac{(\nu')^2}{8} + \frac{\nu''}{2} + \frac{\nu'\zeta'}{4} + \frac{\nu'}{r}\right) \tag{28}$$

and

$$\frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{2} + \frac{\zeta' + \nu'}{r} = \frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4} - K\frac{(\nu')^2}{8}. \tag{29}$$

These two equations are, in principle, enough to solve the metric. However, it turns out that it is useful to make use also of the $\theta\theta$ component of the metric equation

$$\frac{\zeta'' + \nu''}{2} + \frac{(\nu')^2}{4} + \frac{\zeta' + \nu'}{2r} = -\frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4} + K\frac{(\nu')^2}{8}. \tag{30}$$

The study of the properties of these equations for the appropriate boundary conditions constitutes most of the rest of this work.

B. Case II: The vector field has a nonvanishing r component

In the nonaligned case, $U^r \neq 0$, the Lagrange multiplier $\lambda_{\rm II}$ is given by [see Eq. (19)]

$$\lambda_{\rm II} = -\frac{\kappa(Gm_s)^2}{2\pi}\frac{e^{-(\nu+4\zeta)}}{r^4[e^{-\zeta} - (U^r)^2]^2}. \tag{31}$$

The components of the vector field are connected to the functions ν and ζ through the normalization equation (10)

$$e^\nu(U^t)^2 - e^\zeta(U^r)^2 = 1, \tag{32}$$

and the t component of the vector equation (9) (given in a compact form) yields

$$KU^{[t;\beta]}{}_{;\beta} - \frac{\kappa(Gm_s)^2}{2\pi}\frac{e^{-(\nu+4\zeta)}}{r^4[e^{-\zeta} - (U^r)^2]^2}U^t = 0, \tag{33}$$

where the $U^{[t;\beta]}{}_{;\beta}$ term involves derivatives of the four unknown functions ν, ζ, U^r, U^t.

The last two expressions can be combined with the tt and rr components of the metric equation (3) to arrive at a closed system of four differential equations with four unknown functions. To this end, one has to calculate the relevant $\tau_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ terms. Equation (4) yields

$$\tau_{tt} = \frac{\kappa G m_s^2}{32\pi^2}\frac{e^{-3\zeta}}{r^4[e^{-\zeta} - (U^r)^2]}, \tag{34}$$

$$\tau_{rr} = \frac{\kappa G m_s^2}{32\pi^2}\frac{e^{-(\nu+3\zeta)}}{r^4[e^{-\zeta} - (U^r)^2]^2}(1 - 3(U^r)^2). \tag{35}$$

For Θ_{tt} and Θ_{rr} we have [see Eq. (5)]

$$\Theta_{tt} = \frac{K}{2}e^{2\nu-\zeta}[\nu' U^t + (U^t)']^2 + \frac{\kappa(Gm_s)^2}{2\pi} \times \frac{(U^t)^2 e^{\nu-4\zeta}}{r^4[e^{-\zeta} - (U^r)^2]^2}, \tag{36}$$

$$\Theta_{rr} = -\frac{K}{2}e^{\nu}[\nu' U^t + (U^t)']^2 + \frac{\kappa(Gm_s)^2}{2\pi} \times \frac{(U^r)^2 e^{-(\nu+2\zeta)}}{r^4[e^{-\zeta} - (U^r)^2]^2}. \tag{37}$$

The task in the next subsection is to study the asymptotic behavior of the physical metric far from the source in the two cases I and II and derive the post-Newtonian corrections predicted by TeVeS.

C. Asymptotic behavior of the metric far from the source

Far from the source (but not too far, so that the MOND corrections can be safely neglected), the metric can be taken to be asymptotically flat. Expanding the e^{ζ}, e^{ν} to powers of r/r_g (where r_g is a length scale to be determined), we have

$$e^{\nu} = 1 - r_g/r + a_2(r_g/r)^2 + \cdots \tag{38}$$

and

$$e^{\zeta} = 1 + b_1 r_g/r + b_2(r_g/r)^2 + \cdots, \tag{39}$$

where the proportionality constant of the second term in the expansion (38) has been absorbed by r_g. We now proceed to calculate the coefficients a_i and b_i of the metric and equivalent coefficients of the physical metric $\tilde{g}_{\alpha\beta}$ for the two cases I and II (defined in the previous section).

1. Case I: $U^r = 0$

If the vector field is aligned with the time direction, one can substitute the expansions (38) and (39) into the metric equations (28) and (29), match coefficients of like powers if $1/r$, and solve for the coefficients a_i, b_i. Doing so to the order of $(1/r)^3$, the metric has the form

$$e^{\nu} = 1 - \frac{r_g}{r} + \frac{1}{2}\frac{r_g^2}{r^2} - \frac{1}{96}\left[18 + \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - K\right]\frac{r_g^3}{r^3} \tag{40}$$

and

$$e^{\zeta} = 1 + \frac{r_g}{r} + \frac{1}{16}\left[6 - \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 + K\right]\frac{r_g^2}{r^2} + \frac{1}{96}\left[6 - \frac{10\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 + 5K\right]\frac{r_g^3}{r^3}. \tag{41}$$

In this expansion, one can see that the first corrections introduced by TeVeS with respect to the Schwarzschild metric appear in the $(r_g/r)^2$ term in e^{ζ} and in the $(r_g/r)^3$ term in e^{ν}.

Actually, these asymptotic expansions differ from expressions given in Ref. [12] [compare Eqs. (40) and (41) of this work with Eqs. (89)–(91) in Sec. V of [12]]. The reason for this difference is a sign error in the β_1 term of the Lagrange multiplier in Bekenstein's Eq. (82) (see also the erratum of Ref. [12]). Because of this discrepancy, we need to rederive the post-Newtonian corrections predicted by TeVeS. The physical metric $\tilde{g}_{\alpha\beta}$ is given by the expressions $\tilde{g}_{tt} = g_{tt}e^{2\phi}$, $\tilde{g}_{ii} = g_{ii}e^{-2\phi}$; so we still need the asymptotic behavior of ϕ. Integrating (17) and using Eqs. (40) and (41), we have

$$\phi(r) = \phi_c - \frac{\kappa G m_s}{4\pi r} - \frac{\kappa G m_s}{192\pi}\left[1 + \frac{\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - \frac{K}{2}\right]\frac{r_g^2}{r^3} + \mathcal{O}(r^{-5}), \tag{42}$$

where ϕ_c is the cosmological value of ϕ at a specific epoch, which can be absorbed by rescaling of the t and r coordinates: $t' = te^{\phi_c}$ and $r' = re^{-\phi_c}$. Doing so and dropping the primes for simplicity in the notation, the physical metric is

$$\tilde{g}_{tt} = -1 + \left(\frac{\kappa G m_s}{2\pi} + r_g\right)\frac{1}{r} - \frac{1}{8}\left(2r_g + \frac{\kappa m_s}{\pi}\right)^2\frac{1}{r^2} + \frac{1}{192}\left(2r_g + \frac{\kappa G m_s}{\pi}\right)\left[4\left(\frac{\kappa G m_s}{\pi r_g} + 2\right)^2 + 2 + \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - K\right]\frac{r_g^2}{r^3} + \cdots \tag{43}$$

and

$$\tilde{g}_{rr} = 1 + \left(\frac{\kappa G m_s}{2\pi} + r_g\right)\frac{1}{r} + \frac{1}{16}\left[2\left(\frac{\kappa G m_s}{\pi r_g} + 2\right)^2 - 2 - \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 + K\right]\frac{r_g^2}{r^2} + \cdots. \tag{44}$$

Identifying the $1/r$ term of the tt component with $2G_N m/r$ (where G_N is Newton's constant), the physical metric can be brought into the form

$$\tilde{g}_{tt} = -1 + 2G_N m/r - 2G_N^2 m^2/r^2 + \cdots \tag{45}$$

and

$$\tilde{g}_{rr} = 1 + 2G_N m/r + \cdots, \tag{46}$$

which is identical to GR up to order of post-Newtonian corrections. This means that one has to go to higher order terms in TeVeS to obtain the corrections to GR. No constraints can be set to the parameters of TeVeS κ and K from measurements of the standard post-Newtonian coefficients, if the radial component of the vector field vanishes.

At this point, one more comment is in order. By inspection of Eqs. (40) and (41) one notices that the quantity $\frac{\kappa}{\pi} \times (Gm_s/r_g)^2 - K/2$ (times some factor) appears in all the corrections introduced by TeVeS with respect to equivalent of the general relativistic solution. This quantity will also appear in the analytic solutions derived in Sec. IV of this work.

Case II: $U^r \neq 0$

In the nonaligned case, one needs to consider the asymptotic expansion of the vector field components. For $r_g/r \ll 1$ the vector field relaxes to its cosmological value, i.e., $U^t \to 1$ and $U^r \to 0$ (since there is no preferred spatial direction). So, expanding to powers of r_g/r, we have

$$U^t = 1 + c_1 r_g/r + c_2(r_g/r)^2 + \cdots \tag{47}$$

and

$$U^r = d_1 r_g/r + d_2(r_g/r)^2 + \cdots. \quad (48)$$

From this point on, the method we follow to calculate the post-Newtonian corrections is similar to that of the previous subsection. Substituting these expansions and Eqs. (38) and (39) into Eqs. (32) and (33) and the tt and rr components of the metric equation (3) and matching coefficients of like powers if $1/r$, we derive the coefficients a_i, b_i, c_i, and d_i. This analysis is carried out down to the order necessary to calculate the post-Newtonian coefficients and gives for $K \ll 1$, $\kappa \ll 1$

$$a_2 = \frac{1}{2} + \frac{\kappa(Gm_s)^2}{4\pi r_g^2} + \frac{K}{8}, \quad (49)$$

$$b_1 = 1, \quad (50)$$

$$b_2 = \frac{3}{8} - \frac{3}{8}\frac{\kappa(Gm_s)^2}{\pi r_g^2} - \frac{K}{16}, \quad (51)$$

$$c_1 = \frac{1}{2}, \quad (52)$$

$$c_2 = \frac{1}{16}\left(5 + \frac{4\kappa(Gm_s)^2}{K\pi r_g^2} \pm \sqrt{\frac{8\kappa(Gm_s)^2}{K\pi r_g^2} + 5}\right), \quad (53)$$

and

$$d_1 = \frac{1}{4}\left(1 \pm \sqrt{\frac{8\kappa(Gm_s)^2}{K\pi r_g^2} + 5}\right). \quad (54)$$

The $\pm$ sign in the last two coefficients comes from the fact that the normalization expression (32) contains squares of the vector components. The $+$ sign corresponds to $U^r > 0$ and vice versa.

The asymptotic behavior of the scalar field is found after expanding and integrating Eq. (17)

$$\phi(r) = \phi_c - \frac{\kappa G m_s}{4\pi r} + \mathcal{O}(r^{-3}). \quad (55)$$

The physical metric is given by Eq. (2) and a rescaling of the r, t coordinates by $t' = te^{\phi_c}$ and $r' = re^{-\phi_c}$ is needed so that it can asymptote to the Minkowskian form. Notice, however, that ϕ_c is not absorbed by this rescaling unlike the $U^r = 0$ case, because of the more complicated connection of the physical metric with the fields of TeVeS in this case. Assuming again $K \ll 1$, $\kappa \ll 1$, and furthermore that $\phi_c \ll 1$, we have for the standard β, γ post-Newtonian coefficients, as predicted by TeVeS for the case that $U^r \neq 0$,

$$\beta = 1 + \frac{\kappa}{8\pi} + \frac{K}{4} + \phi_c\left(3 + \frac{\kappa}{\pi K} \pm \sqrt{\frac{2\kappa}{\pi K} + 5}\right) \quad (56)$$

and

$$\gamma = 1. \quad (57)$$

Here again the $\pm$ sign in the expression for β is determined by the sign of U^r [see Eq. (54)].

While the γ coefficient coincides with the GR prediction, the β differs from unity. The best determination of β comes from lunar laser ranging tests (see, for example, Ref. [13]), which in combination with the value for γ measured by the Cassini experiment [14] yields $\beta - 1 \lesssim 10^{-4}$ [15]. How does this result compare with the prediction of Eq. (56)? It is important to note that the term multiplied with the (positive) ϕ_c in Eq. (56) is positive for any value of $\kappa/K > 0$ and choice of the $\pm$ sign, so one has the inequality $\beta - 1 \geq \kappa/(8\pi) + K/4$. For the choice of the function F made in Ref. [12], κ is constrained to be $\simeq 0.03$, which results in $\beta - 1 \gtrsim 2.5 \times 10^{-3}$ (taking the K term much smaller). This is in conflict with observations.

Summarizing, in this section I have shown that if the vector field is aligned with the time direction, the standard post-Newtonian coefficients derived by TeVeS are identical to those of GR, while if $U^r \neq 0$, the PPN correction for the β coefficient is in conflict with best determinations of β. This means that either one has to assume that $U^r = 0$ or a different choice of the function F than that of Ref. [12] has to be made so that TeVeS is in accordance with solar system phenomenology.

IV. ANALYTIC SOLUTIONS WHEN U^r VANISHES

Until now, I have kept the study of spherical symmetric spacetimes in TeVeS quite general. From this point on, I focus on the branch of solutions for which $U^r = 0$; i.e., the vector field is aligned to the time direction. As it turns out, exact analytic solutions are possible in this case.

A. Solutions in the $K \to 0$ limit

The system of Eqs. (28)–(30) is rather complicated. Here, we first consider some special cases and then use the intuition we gain to derive the general solution. In the simplest case where both $\kappa = 0$ and $K = 0$, the metric equations in TeVeS coincide with these in GR and their right-hand side is zero (i.e., no source terms appear). In this limit the integration of Eqs. (28) and (29) is straightforward, leading to the familiar GR solution

$$e^{\nu} = \left(\frac{1 - r_g/4r}{1 + r_g/4r}\right)^2, \quad (58)$$

$$e^{\zeta} = (1 + r_g/4r)^4, \quad (59)$$

where the boundary conditions (40) and (41) have been used. In this case, one can show [see Eq. (17)] that ϕ is constant at its cosmological value ϕ_c and that $g_{\alpha\beta}$ coincides with that predicted by GR. The physical metric is given by $\tilde{g}_{tt} = g_{tt}e^{2\phi_c}$, $\tilde{g}_{ii} = g_{ii}e^{-2\phi_c}$. The factors $e^{\pm 2\phi_c}$

can be absorbed by an appropriate rescaling of the t and r coordinates, resulting in a physical metric equivalent to that of GR.

As a next step toward the most general solution, we take the limit $K \to 0$ but allow κ to be arbitrary. In this limit we essentially decouple the theory from the vector field, and the metric equations become

$$\zeta'' + \frac{(\zeta')^2}{4} + \frac{2\zeta'}{r} = -\frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4}, \tag{60}$$

$$\frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{2} + \frac{\zeta'+\nu'}{r} = \frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4}, \tag{61}$$

and

$$\frac{\zeta''+\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\zeta'+\nu'}{2r} = -\frac{\kappa(Gm_s)^2}{4\pi}\frac{e^{-(\zeta+\nu)}}{r^4}. \tag{62}$$

It turns out that Eqs. (60)–(62) are equivalent to spherical symmetric spacetimes in metric-massless scalar theories of gravity. The exact solution was originally written down by Buchdahl in Ref. [16] (see also Ref. [17]). Here, I will briefly repeat the derivation.

From the addition of Eqs. (61) and (62), we find

$$2(\nu''+\zeta'') + (\nu'+\zeta')^2 + 6\frac{\nu'+\zeta'}{r} = 0, \tag{63}$$

which can be integrated once to give

$$\nu' + \zeta' = \frac{4r_c^2}{r(r^2-r_c^2)}, \tag{64}$$

where we have introduced the integration constant r_c^2. This constant can be evaluated by expanding Eq. (64) to powers of $1/r$ and comparing with the expansions (40) and (41). After some algebra, we find

$$r_c = \frac{r_g}{4}\sqrt{1+\frac{\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2}. \tag{65}$$

Equation (64) can be integrated again to yield

$$\nu + \zeta = 2\ln\left(\frac{r^2-r_c^2}{r^2}\right), \tag{66}$$

where the second integration constant has been set to unity so that the asymptotic form of the solution is a flat spacetime (i.e., $e^{\nu+\zeta} \to 1$ for $r \to \infty$).

One verifies that, after setting

$$\zeta' = \frac{4r_c^2}{r(r^2-r_c^2)} - \frac{r_g}{r^2-r_c^2} \tag{67}$$

and using Eq. (66) to derive ν', the metric equations are all satisfied. After integrating for ν and ζ, one has the exact solution for the metric components

$$e^{\nu} = \left(\frac{r-r_c}{r+r_c}\right)^{r_g/2r_c} \tag{68}$$

and

$$e^{\zeta} = \frac{(r^2-r_c^2)^2}{r^4}\left(\frac{r-r_c}{r+r_c}\right)^{-r_g/2r_c}, \tag{69}$$

where r_c is given by Eq. (65). It is straightforward to check that, in the limit where $\kappa m_s \to 0$, one derives the well known general relativistic expressions.

Having solved for the metric components, one can integrate Eq. (17) to derive the r dependence of the scalar field and then the physical metric through Eq. (2). However, the results derived in this section are of limited generality since they correspond to the $K = 0$ case, where the effect of the vector field to the metric equations is ignored. The generalization of the solutions to the case where $K \neq 0$ is the task of the next section.

B. Spherically symmetric, vacuum solution for the metric for arbitrary κ, K

We turn to the general case where both κ and K are nonzero. While at first sight the metric equations look quite complicated in this case, it turns out that one can repeat the procedure of the previous section to derive more general spherical symmetric, vacuum solutions for the metric which are identical to (68) and (69), provided that one makes the substitution

$$\frac{\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 \to \frac{\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - \frac{K}{2} \tag{70}$$

in the definition of r_c [Eq. (65)], i.e.,

$$r_c = \frac{r_g}{4}\sqrt{1+\frac{\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - \frac{K}{2}}. \tag{71}$$

In the κ, K parameter plane, the line defined by

$$K = \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 \tag{72}$$

has a special significance. With Eq. (72) $r_c = r_g$ and the metric is identical to the general relativistic one [18]. This is a result of the fact that the energy density contributed by the scalar is exactly canceled out by the *negative* energy density of the Θ_{tt} in the right-hand side of the tt component of Eq. (3), i.e., $8\pi G\tau_{tt} + \Theta_{tt} = 0$. Actually, when

$$K > \frac{2\kappa}{\pi}\left(\frac{Gm_s}{r_g}\right)^2, \tag{73}$$

the total energy density of vacuum contributed by the fields is negative in the whole spacetime. This can have important consequences for the theory since it may lead to instability of the vacuum from the quantum point of view.

The behavior of the scalar field can be followed by integrating Eq. (17) and the use of Eq. (66)

$$\phi(r) = \phi_c + \frac{\kappa G m_s}{8\pi r_c}\ln\left(\frac{r-r_c}{r+r_c}\right), \tag{74}$$

where ϕ_c stands for the cosmological value of the scalar field at a specific epoch. Just as in metric-scalar theories, one can see that, unless $\kappa m_s = 0$, the scalar field diverges logarithmically at $r = r_c$ and that there is always a radius $r_1 > r_c$ where $\phi(r_1) = 0$ and becomes negative further in. I will return to this point in the next section where I discuss how black holes look in TeVeS and how much they differ from the ones predicted by GR.

The components of physical metric are related to ν, ζ, and ϕ through the expressions (2), (68), and (69) that yield

$$\tilde{g}_{tt} = -\left(\frac{r - r_c}{r + r_c}\right)^a \tag{75}$$

and

$$\tilde{g}_{rr} = \frac{(r^2 - r_c^2)^2}{r^4}\left(\frac{r - r_c}{r + r_c}\right)^{-a}, \tag{76}$$

where $a \equiv (r_g/2r_c) + (\kappa G m_s/4\pi r_c)$.

The expressions (75) and (76) describe spherically symmetric, vacuum spacetimes; i.e., they describe the spacetime down to the surface of a star. The two dimensionless parameters κ and K of the theory provide the parameter space that is to be explored. In addition to these, I have kept the scalar mass m_s and the gravitational radius r_g as free parameters so that the derived results are quite general and applicable to the case of both a black hole and the exterior of a star. In Appendix D of Ref. [12], a detailed description of the procedure to calculate m_s and r_g in terms of its gravitational mass m_g of the star is given. Unfortunately, this method is not applicable to the case of a black hole, and a different approach is needed to determine m_s and r_g.

V. HOW DO BLACK HOLES LOOK IN TEVES?

The characteristic radius of the physical metric described by Eqs. (75) and (76) is r_c. At r_c the tt component of the metric vanishes and the question that arises is whether and under which conditions r_c is the location of the horizon of a black hole. A first step toward answering this question is to calculate the surface area at this radius, which turns out to be proportional to $\tilde{g}_{rr}(r_c)$. A black hole must have a finite surface area at r_c which [in view of Eq. (76)] constrains a to be ≤ 2.

A second constraint on a [19] comes from the demand that there is no essential singularity at r_c. For our solution, the Ricci scalar R is

$$R = \frac{2(a^2 - 4)r_c^2 r^4 (r - r_c)^{a-4}}{(r + r_c)^{a+4}}. \tag{77}$$

From this expression, one can see that the Ricci scalar is finite when $a = 2$ or $a \geq 4$. Considered together, the two constraints (i.e., of finite surface area and Ricci scalar at r_c) imply that only the value $a = 2$ describes a black hole. Using the definition of a, we have that for $a = 2$

$$r_c = \frac{r_g}{4} + \frac{\kappa G m_s}{8\pi}, \tag{78}$$

and the physical metric has the form

$$\tilde{g}_{tt} = -\left(\frac{r - r_c}{r + r_c}\right)^2 \tag{79}$$

and

$$\tilde{g}_{rr} = \left(1 + \frac{r_c}{r}\right)^4. \tag{80}$$

This is exactly the GR solution after setting $r_c = G_N m/2$. So, the physical metric in TeVeS is identical to that of GR for a nonrotating black hole.

Furthermore, one can use the definition of r_c [see Eq. (71)] in Eq. (78) to solve for m_s and finds

$$\frac{G m_s}{r_g} = \frac{1 + \sqrt{1 + (2 - \frac{\kappa}{2\pi})\frac{\pi K}{\kappa}}}{2 - \frac{\kappa}{2\pi}}. \tag{81}$$

I have already shown in the previous section that when $m_s \neq 0$, there is always a region close to r_c where the scalar field turns negative. Bekenstein in Ref. [12], on the other hand, has shown that TeVeS becomes acausal (i.e., it suffers from superluminal propagation of metric, vector, and scalar-field disturbances) when $\phi < 0$. As a result, the theory appears to behave in an unphysical way in the vicinity of our black hole solution. On the other hand, our solutions have been derived under the assumption that $U^r = 0$. Perhaps the causality problem can be overcome by allowing for $U^r \neq 0$.

VI. CONCLUSIONS

Bekenstein's recent relativistic gravitational theory (TeVeS) that leads to MOND in the relevant limit has been proposed as a modification to GR. TeVeS has several attractive features; for example, it predicts the right amount of gravitational lensing when only the observed mass is used and provides a covariant formulation to construct cosmological models.

The free parameters in TeVeS can be constrained by the large extragalactic phenomenology. In this work, instead, we have looked at TeVeS in the strong gravity limit. Two branches of solutions are identified: the first is characterized by the vector field being aligned with the time direction, while in the (not previously explored) second branch the vector filed has a nonvanishing radial component. I have shown that the β and γ PPN coefficients in TeVeS are identical to these of GR in the first branch of solutions, while the β PPN coefficient differs in the two theories in the second. Despite the fact that the results derived here are essentially independent of the exact choice of the free function F of the theory, our final conclusions do depend on it, since the choice of F influences the way in which observations put constraints on the parameters of the theory. For the second branch of solutions and for the choice of the free function F made in Ref. [12], TeVeS predicts β that is in conflict with recent observational determinations of it.

For the first branch of solutions, I derive analytic expression for the physical metric. These solutions are an extension of those that describe spherical symmetric spacetimes in tensor-massless scalar theories and depend on the values of the two dimensionless parameters κ, K of TeVeS and the ratio Gm_s/r_g. One of the findings of this work is that the energy density contributed by the vector field is negative and, when $K > \frac{2\kappa}{\pi}(Gm_s/r_g)^2$, the total energy density of vacuum also becomes negative, possibly turning it unstable from the quantum point of view.

In the case of a black hole, our solutions for the metric are identical to the Schwarzschild solution in GR. On the other hand, these solutions are shown to be acausal in the vicinity of the black hole. Possibly, the issues of the negative energy density contributed by the vector field and of causality close to a black hole do not appear in the case where $U^r \neq 0$. In this case, however, a different choice of the free function F will be needed so that TeVeS is not in conflict with solar system phenomenology.

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