Effects of cosmological magnetic helicity on the cosmic microwave background

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Cosmological magnetic fields induce temperature and polarization fluctuations in the cosmic microwave background (CMB) radiation. A cosmological magnetic field with current amplitude of order 10^{-9} G is detectable via observations of CMB anisotropies. This magnetic field (with or without helicity) generates vector perturbations through vortical motions of the primordial plasma. This paper shows that magnetic field helicity induces parity-odd cross correlations between CMB temperature and *B*-polarization fluctuations and between *E*- and *B*-polarization fluctuations, correlations which are zero for fields with no helicity (or for any parity-invariant source). Helical fields also contribute to parity-even temperature and polarization anisotropies, canceling part of the contribution from the symmetric component of the magnetic field. We give analytic approximations for all CMB temperature and polarization anisotropy vector power spectra due to helical magnetic fields. These power spectra offer a method for detecting cosmological helical magnetic fields, particularly when combined with Faraday rotation measurements which are insensitive to helicity.

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I. INTRODUCTION

The linear theory for the evolution of small inhomogeneities in the standard spatially homogeneous and isotropic cosmological model, developed by Lifshitz and others [1], shows that vector perturbations[1] decay as the Universe expands.[2] This is why vector perturbations are usually neglected when computing cosmic microwave background (CMB) fluctuations. On the other hand, the presence of a cosmological magnetic field [3–8] alters this situation. In this paper we focus on vector perturbations induced by a cosmological magnetic field, existing since radiation-matter equality or earlier (for a specific inflation model see [9]). Both an homogeneous magnetic field, as well as the more realistic stochastic one, induce transverse MHD (Alfvén) waves [3]. Nondecaying cosmological Alfvén waves result in CMB temperature and polarization anisotropies [4–7,10,11].[3]

A cosmological magnetic field could have helicity [15]. Magnetic field helicity plays an important role in the MHD dynamo, used in some models of galactic magnetic field amplification [16]. Unlike gravitational waves which are damped on scales below the Hubble radius at decoupling (the corresponding damping multipole number $l \sim 100$), Alfvén waves survive down to smaller damping scales ($l \sim 2000$) [7], thus a cosmological magnetic field can affect small-scale CMB fluctuations. From this point of view, the vector mode is more relevant for constraining a cosmological magnetic field from CMB fluctuation measurements.

In this paper we present analytic expressions for all CMB fluctuation vector power spectra that arise from a helical cosmological magnetic field. In particular, we also compute how magnetic helicity affects parity-even CMB fluctuation power spectra. We propose a scheme to constrain cosmological magnetic helicity from CMB temperature and polarization anisotropy observations. The symmetric part of the magnetic field spectrum can be reconstructed from measurements of the rotation of the CMB polarization plane as a consequence of the Faraday effect [17]; this is because magnetic helicity does not contribute to the Faraday rotation effect [18–20]. On the other hand, the helical part of the magnetic field spectrum induces parity-odd cross correlations between temperature and *B*-polarization anisotropies, and between *E*- and *B*-polarization anisotropies [10,14]; such cross correlations are not induced by the Faraday effect [20].[4]

For our computations we use the formalism of Ref. [6], extending it to account for magnetic field helicity. To compute CMB temperature and polarization anisotropy power spectra we use the total angular momentum method of Ref. [22]. Our results are obtained using analytic approximations and are valid for $l < 500$. We present results in terms of a ratio between CMB fluctuation contributions from the symmetric and helical parts of the magnetic field power spectrum.

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[1]Vector perturbations are also known as transverse peculiar velocity or vorticity perturbations.

[2]This is for adiabatic initial conditions, which are consistent with observations. However, a nondecaying vector perturbation exists if there is an initial large-scale photon-baryon fluid vorticity compensated by a neutrino vorticity such that the total large-scale vorticity in relativistic components vanishes [2]. Such a mode can be excited only after neutrino decoupling [2].

[3]In addition to the vector mode, a cosmological magnetic field generates CMB fluctuations through the scalar and tensor modes as described in Refs. [3,6,12–14].

[4]This is not true for an homogeneous magnetic field [21].

The rest of the paper is organized as follows. In the next section we model the helical magnetic field source term for vorticity (Alfvén) waves. In Secs. III and IV we present analytic approximations for the parity-even and parity-odd CMB fluctuation power spectra contributions induced by magnetic helicity. In Sec. V we discuss our results and conclude. In Appendixes A and B we present some details of the computation including analytic approximations used for some of the integrations.

II. MAGNETIC-FIELD-INDUCED VECTOR PERTURBATIONS

A. Magnetic field source term

We assume the existence of a cosmological magnetic field generated during or prior to the radiation-dominated epoch, with the energy density of the field a first-order perturbation to the standard Friedmann-Lemaître-Robertson-Walker homogeneous cosmological spacetime model. Neglecting fluid backreaction onto the magnetic field, the spatial and temporal dependence of the field separates, $\mathbf{B}(t,\mathbf{x}) = \mathbf{B}(\mathbf{x})/a^2$; here a is the cosmological scale factor. As a phenomenological normalization of the magnetic field, we smooth the field on a comoving length λ with a Gaussian smoothing kernel $\propto \exp[-x^2/\lambda^2]$ to obtain the smoothed magnetic field with average value of squared magnetic field ${B_\lambda}^2 \equiv \langle \mathbf{B}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})\rangle|_\lambda$ and magnetic helicity ${H_\lambda}^2 \equiv \lambda|\langle \mathbf{B}(\mathbf{x})\cdot[\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{x})]\rangle|_\lambda$. See Ref. [20] for a more detailed discussion.

We also assume that the primordial plasma is a perfect conductor on all scales larger than the Silk damping wavelength λ_S (the thickness of the last scattering surface) set by photon and neutrino diffusion. We model magnetic field damping by an ultraviolet cutoff wave number $k_D = 2\pi/\lambda_D$ [6,13],

$$\left(\frac{k_D}{\mathrm{Mpc}^{-1}}\right)^{n_B+5} \approx 2.9\times 10^4 \left(\frac{B_\lambda}{10^{-9}\ \mathrm{G}}\right)^{-2}\left(\frac{k_\lambda}{\mathrm{Mpc}^{-1}}\right)^{n_B+3} h. \tag{1}$$

Here n_B is the spectral index of the symmetric part of the magnetic field power spectrum [see Eq. (4) below], h is the Hubble constant in units of 100 km sec^{-1} Mpc^{-1}, $k_\lambda = 2\pi/\lambda$ is the smoothing wave number, and $\lambda_D \ll \lambda_S$. This assumes that magnetic field damping is due to the damping of Alfvén waves from photon viscosity.

Assuming that the stochastic magnetic field is Gaussianly distributed, and accounting for the possible helicity of the field, the magnetic field spectrum in wave number space is [10],

$$\langle B_i^\star(\mathbf{k}) B_j(\mathbf{k}')\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}')[P_{ij}(\hat{\mathbf{k}})P_B(k) + i\epsilon_{ijl}\hat{k}_l P_H(k)]. \tag{2}$$

Here $P_{ij}(\hat{\mathbf{k}}) \equiv \delta_{ij} - \hat{k}_i\hat{k}_j$ is the transverse plane projector with unit wave number components $\hat{k}_i = k_i/k$, ϵ_{ijl} is the antisymmetric tensor, and $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ is the Dirac delta function. We use

$$B_j(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} B_j(\mathbf{x}), \qquad B_j(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} B_j(\mathbf{k}), \tag{3}$$

when Fourier transforming between real and wave number spaces; we assume flat spatial hypersurfaces. $P_B(k)$ and $P_H(k)$ are the symmetric and helical parts of the magnetic field power spectrum, assumed to be simple power laws on large scales,

$$P_B(k) \equiv P_{B0}k^{n_B} = \frac{2\pi^2\lambda^3 B_\lambda^2}{\Gamma(n_B/2+3/2)}(\lambda k)^{n_B}, \qquad P_H(k) \equiv P_{H0}k^{n_H} = \frac{2\pi^2\lambda^3 H_\lambda^2}{\Gamma(n_H/2+2)}(\lambda k)^{n_H}, \qquad k < k_D, \tag{4}$$

and vanishing on small scales when $k > k_D$. Here Γ is the Euler gamma function. These power spectra are generically constrained by $P_B(k) \ge |P_H(k)|$ [16,23], which implies $n_H > n_B$ [14,23]. In addition, finiteness of the magnetic field energy density requires $n_B > -3$ (to prevent an infrared divergence of magnetic field energy density). Finiteness of the magnetic field average helicity requires $n_H > -4$; this is automatically satisfied as a consequence of $n_H > n_B > -3$.

To obtain the magnetic field source term in the transverse peculiar velocity perturbation equation of motion we need to extract the transverse vector part of the magnetic field stress-energy tensor $\tau_{ij}(\mathbf{k})$. This is done through $\Pi_{ij}(\mathbf{k}) = (P_{ib}(\hat{\mathbf{k}})\hat{k}_j + P_{jb}(\hat{\mathbf{k}})\hat{k}_i)\hat{k}_a\tau_{ab}(\mathbf{k})$, and the Π_{ij} tensor is related to the vector (divergenceless and transverse) part of the Lorentz force $L_i^{(V)}(\mathbf{k}) = k_j\Pi_{ij}(\mathbf{k}) = P_{ib}(\hat{\mathbf{k}})k_a\tau_{ab}(\mathbf{k})$ [Eq. (2.16) of Ref. [6]]. For the normalized Lorentz force vector $\Pi_i \equiv L_i^{(V)}/k$, the general spectrum in wave number space is similar to Eq. (2); that is,[5]

$$\langle \Pi_i^\star(\mathbf{k})\Pi_j(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')[P_{ij}f(k) + i\epsilon_{ijq}\hat{k}_q g(k)], \tag{5}$$

where $f(k)$ and $g(k)$ represent the symmetric and helical parts of the vector Lorentz force power spectrum. These functions are related to the magnetic field stress-energy tensor spectrum through

[5]We thank Lewis (private communication, 2004, and Ref. [7]) for pointing out a missing $(2\pi)^3$ factor in the expression for $\langle \Pi_i^*(\mathbf{k})\Pi_j(\mathbf{k}')\rangle$ given in Ref. [6]. In what follows we use other results from Ref. [6] corrected for a similar missing factor.

$$(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')f(k) = \frac{1}{4}[P_{aj}(\hat{\mathbf{k}})\hat{k}_i P_{am}(\hat{\mathbf{k}}')\hat{k}'_l + P_{ai}(\hat{\mathbf{k}})\hat{k}_j P_{al}(\hat{\mathbf{k}}')\hat{k}'_m] \times \langle \tau^\star_{ij}(\mathbf{k})\tau_{lm}(\mathbf{k}')\rangle, \quad (6)$$

$$(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')g(k) = -\frac{i}{4}\hat{k}_q[\epsilon_{jmq}\hat{k}_i\hat{k}'_l + \epsilon_{ilq}\hat{k}_j\hat{k}'_m] \times \langle \tau^\star_{ij}(\mathbf{k})\tau_{lm}(\mathbf{k}')\rangle. \quad (7)$$

The functions $f(k)$ and $g(k)$ are evaluated in Appendix A,

$$f(k) \simeq \mathcal{F}_B(\lambda k_D)^{2n_B+3}\left[1+\frac{n_B}{n_B+3}\left(\frac{k}{k_D}\right)^{2n_B+3}\right] - \mathcal{F}_H(\lambda k_D)^{2n_H+3}\left[1+\frac{n_H-1}{n_H+4}\left(\frac{k}{k_D}\right)^{2n_H+3}\right], \quad (8)$$

$$g(k) \simeq \mathcal{G}\lambda k(\lambda k_D)^{n_B+n_H+2}\left[1+\frac{n_H-1}{n_B+3}\left(\frac{k}{k_D}\right)^{n_B+n_H+2}\right], \quad (9)$$

for $k < k_D$; $f(k)$ and $g(k)$ vanish for $k > k_D$. Here $\mathcal{F}_B$, $\mathcal{F}_H$ and $\mathcal{G}$ are constants that depend on the magnetic field power spectrum indices n_B and n_H, and the power spectrum normalization,

$$\mathcal{F}_B = \frac{\lambda^3 B_\lambda^4}{16(2n_B+3)\Gamma^2(n_B/2+3/2)}, \quad \mathcal{F}_H = \frac{\lambda^3 H_\lambda^4}{24(2n_H+3)\Gamma^2(n_H/2+2)}, \quad \mathcal{G} = \frac{\lambda^3 B_\lambda^2 H_\lambda^2}{24(n_B+n_H+2)\Gamma(n_B/2+3/2)\Gamma(n_H/2+2)}. \quad (10)$$

The contribution of magnetic field helicity to the symmetric source $f(k)$ is negative (as in the case for tensor perturbations [14]). The magnetic source terms in Eqs. (8) and (9) vanish on scales smaller than the cutoff scale $\lambda < \lambda_D$ because of magnetic field damping. The singularities at $n_B = -3/2$, $n_H = -3/2$ and $n_B + n_H = -2$ in Eqs. (10) are removable [6,14]. For $n_B > -3/2$ the terms proportional to $k_D^{2n_B+3}$ and $k_D^{2n_H+3}$ dominate in the expression for $f(k)$ in Eq. (8), and the symmetric source term depends on the cutoff wave number k_D but not on k, and so is a white noise spectrum [23].

B. Vorticity perturbations

A cosmological magnetic field contributes, via the linearized Einstein equations, to all three kinds of perturbations, scalar, vector, and tensor modes (for a recent review see [7]). Here we focus on the effects a stochastic magnetic field with helicity has on vector perturbations.[6] The vector metric perturbation may be described in terms of two gauge-invariant divergenceless three-dimensional vector fields, the vector potential $\mathbf{V}$ (which is a vector perturbation of the extrinsic curvature), and a vector parametrizing the transverse peculiar velocity of the plasma, the vorticity $\boldsymbol{\Omega} = \mathbf{v} - \mathbf{V}$, where $\mathbf{v}$ is the spatial part of the four-velocity perturbation of a stationary fluid element [4]. In the absence of a source a vector perturbation decays with time and so can be ignored.

Since electromagnetism is conformally invariant it is possible to rescale fields by appropriate powers of the scale factor and simply obtain Maxwell's equations in the expanding Universe from the Minkowski spacetime Maxwell equations. Since the fluid velocity is small the displacement current in Ampère's law may be neglected; this implies the current $\mathbf{J} = \nabla \times \mathbf{B}/(4\pi)$. The residual ionization is large enough to ensure that magnetic field lines are frozen into the plasma, so the induction law takes the form $(\partial/\partial t)\mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B})$. As a result the baryon Euler equation for $\mathbf{v}$ has a Lorentz force $\mathbf{L}(\mathbf{x}) \simeq -\{\mathbf{B}(\mathbf{x}) \times [\nabla \times \mathbf{B}(\mathbf{x})]\}/(4\pi)$ as a source term. The photons are neutral so the photon Euler equation does not have a Lorentz force source term. The Euler equations for photons and baryons are [6,22]

$$\dot{\Omega}_\gamma + \dot{\tau}(\mathbf{v}_\gamma - \mathbf{v}_b) = 0, \quad (11)$$

$$\dot{\Omega}_b + \frac{\dot{a}}{a}\Omega_b - \frac{\dot{\tau}}{R}(\mathbf{v}_\gamma - \mathbf{v}_b) = \frac{\mathbf{L}^{(V)}(\mathbf{k})}{a^4(\rho_b + p_b)}, \quad (12)$$

where an overdot represents a derivative with respect to conformal time η, and $\Omega_\gamma = \mathbf{v}_\gamma - \mathbf{V}$ and $\Omega_b = \mathbf{v}_b - \mathbf{V}$ are the vorticities of the photon and baryon fluids. Here $\dot{\tau} = n_e\sigma_T a$ is the differential optical depth, with n_e the free electron density and σ_T the Thomson cross section, $R \equiv (\rho_b + p_b)/(\rho_\gamma + p_\gamma) \simeq 3\rho_b/4\rho_\gamma$ is the momentum density ratio between baryons and photons, and $L_i^{(V)}$ is the transverse vector (divergenceless) part of the Lorentz force.

The average Lorentz force $\langle \mathbf{L}(\mathbf{x})\rangle = -\langle \mathbf{B} \times [\nabla \times \mathbf{B}]\rangle/(4\pi)$ vanishes, while the rms Lorentz force $\langle \mathbf{L}(\mathbf{x}) \cdot \mathbf{L}(\mathbf{x})\rangle^{1/2}$ is nonzero and acts as a source in the vector perturbation equation. The magnetic helicity spectrum $P_H(k)$ contributes to the symmetric part of the Lorentz force spectrum, see the expression for $f(k)$ in Eq. (8). As a result magnetic helicity affects the symmetric vorticity perturbation spectrum via the Euler equation, and so contributes to parity-even CMB fluctuations. The helical part of the Lorentz force spectrum is completely determined by $g(k)$, Eq. (9), and acts as a source for the helical part of the vorticity perturbation spectrum. Solving the Euler equations in the tight-coupling limit when $\mathbf{v}_\gamma \simeq \mathbf{v}_b$, we have [5,6],[7]

[6]See Ref. [10] for a study of helical vorticity fields. They did not account for magnetic field helicity acting as a source in the helical vorticity field perturbation equation of motion.

[7]We use the "helicity" basis of Sec. 1.1.3 of Ref. [24] and decompose a vector $\mathbf{A} = \sum_{\mu=-1}^{1}\mathbf{e}_{(\mu)}A^{(\mu)}$, where $\mathbf{e}^{(\pm 1)}$ and $\mathbf{e}^{(0)}$ are the unit basis vectors. The unit vector $\mathbf{e}^{(0)}$ is chosen to be in the direction of wave propagation $\mathbf{e}^{(0)} = \hat{\mathbf{k}}$ and $\mathbf{e}^{(\pm 1)}(\mathbf{k}) = -i(\mathbf{e}_1 \pm i\mathbf{e}_2)/\sqrt{2}$.

$$\Omega^{(\pm 1)}(\eta, \mathbf{k}) \simeq \frac{k\Pi^{(\pm 1)}(\mathbf{k})\eta}{(1+R)(\rho_{\gamma 0}+p_{\gamma 0})}. \quad (13)$$

Here $p_{\gamma 0}$ and $\rho_{\gamma 0}$ are the photon pressure and energy density today, and the $\Omega^{(0)}$ component vanishes due to the transversality condition. This result can also be obtained from the Einstein equation [6].

The expression in Eq. (13) is valid on scales λ larger than the comoving Silk scale λ_S. On smaller scales the photon viscosity becomes comparable to the effects of the magnetic field and so must be accounted for in the Euler equation. On these smaller scales with $k > k_S$ [5],

$$\Omega^{(\pm 1)}(\eta, \mathbf{k}) \simeq \frac{\Pi^{(\pm 1)}(\mathbf{k})}{(kL_\gamma/5)(\rho_{\gamma 0}+p_{\gamma 0})}, \quad (14)$$

where L_γ is the photon mean free path length.

We define the CMB anisotropies in terms of the vorticity perturbation power spectrum,[8]

$$\langle \Omega_i^*(\mathbf{k})\Omega_j(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')[P_{ij}|\Omega|^2(k) + i\hat{k}_q\epsilon_{ijq}\omega(k)]. \quad (15)$$

Here $|\Omega|^2(k)$ is the symmetric part of the vorticity power spectrum, directly related to the symmetric magnetic field source spectrum $f(k) = |\Pi|^2(k)$, while $\omega(\eta, k)$ is the helical part of the vorticity spectrum and is determined by the helical magnetic field source term $g(k)$,

$$\Omega(\eta, k) \simeq \begin{cases} \frac{k\eta}{(1+R)(\rho_{\gamma 0}+p_{\gamma 0})}\Pi(k), & k < k_S, \\ \frac{1}{(kL_\gamma/5)(\rho_{\gamma 0}+p_{\gamma 0})}\Pi(k), & k > k_S, \end{cases}$$
$$\omega(\eta, k) \simeq \begin{cases} [\frac{k\eta}{(1+R)(\rho_{\gamma 0}+p_{\gamma 0})}]^2 g(k), & k < k_S, \\ [\frac{1}{(kL_\gamma/5)(\rho_{\gamma 0}+p_{\gamma 0})}]^2 g(k), & k > k_S. \end{cases} \quad (16)$$

Here the factor $1 + R$ for $k < k_S$ reflects the suppression of the vorticity field due to the tight coupling between baryons and photons, because photons, being neutral, are not influenced by the magnetic field Lorentz force source.

III. PARITY-EVEN CMB FLUCTUATIONS FROM MAGNETIC HELICITY

Cosmological magnetic field vector and tensor mode contributions to large angular scale ($l < 100$) CMB fluctuations are of the same order of magnitude [6], while small angular scale ($l > 100$) CMB fluctuations are dominated by the vector mode contribution [14]. Stochastic nonhelical magnetic field effects on the CMB temperature and polarization anisotropies are discussed in detail in Refs. [5–7]. Here we compute the CMB vector mode fluctuation arising from the helical part of the magnetic field power spectrum (also see Ref. [10]).

To compute the CMB temperature and polarization anisotropy power spectra we use the total angular momentum representation [22]. Given CMB temperature and polarization anisotropy integral solutions of the Boltzmann temperature equation, the CMB fluctuation power spectra are [22]

$$C_l^{\mathcal{X}\mathcal{X}'} = \frac{2}{\pi}\int dk k^2 \sum_m \frac{\mathcal{X}_{l(m)}(\eta_0, k)}{2l+1}\frac{\mathcal{X}'_{l(m)}(\eta_0, k)}{2l+1}, \quad (17)$$

where η_0 is the conformal time now and $\mathcal{X}$ is either Θ, E, or B, which represent, respectively, the temperature, E-polarization, and B-polarization anisotropies. For the vector mode the sum includes only terms with $m = \pm 1$. In what follows we use results from Ref. [6] for the CMB fluctuation power spectra $C_{(S)l}^{\mathcal{X}\mathcal{X}'}$ induced by the symmetric (nonhelical) part of magnetic field power spectrum and proportional to $f_B(k) \sim \int d^3p P_B(p) P_B(|\mathbf{k}-\mathbf{p}|)$.

The complete parity-even CMB fluctuation power spectra may be expressed as

$$C_l^{\mathcal{X}\mathcal{X}'} = C_{(S)l}^{\mathcal{X}\mathcal{X}'} - C_{(A)l}^{\mathcal{X}\mathcal{X}'}, \quad (18)$$

where the $C_{(A)l}^{\mathcal{X}\mathcal{X}'}$ are the power spectra induced by magnetic helicity, i.e., proportional to $f_H(k)$. The minus sign reflects the negative contribution of magnetic helicity to the total parity-even CMB fluctuation power spectra.[9] This result also holds for the tensor mode case [14]. The fractional difference $\kappa_l^{\mathcal{X}\mathcal{X}'} \equiv 1 - C_{(A)l}^{\mathcal{X}\mathcal{X}'}/C_{(S)l}^{\mathcal{X}\mathcal{X}'}$, where $0 < \kappa_l^{\mathcal{X}\mathcal{X}'} < 1$, can be used to characterize the reduction of the parity-even CMB fluctuation power spectra amplitudes as a consequence of nonzero magnetic helicity. The ratio $C_{(A)l}^{\mathcal{X}\mathcal{X}'}/C_{(S)l}^{\mathcal{X}\mathcal{X}'}$ may be expressed in terms of $g(k)/f(k)$, i.e., in terms of P_{0H}/P_{0B} and spectral indexes n_H and n_B. We find that for any parity-even CMB fluctuation power spectrum

$$\kappa_l^{\mathcal{X}\mathcal{X}'} = 1 - \frac{2(2n_B+3)}{3(2n_H+3)}\left(\frac{P_{H0}k_D^{n_H-n_B}}{P_{B0}}\right)^2 \mathcal{R}_a^{\mathcal{X}\mathcal{X}}(n_B, n_H, l). \quad (19)$$

Here $\mathcal{R}_a^{\mathcal{X}\mathcal{X}'}(n_B, n_H, l)$ are dimensionless functions of l and spectral indexes n_B and n_H, and the index $a = 1$ and $a = 2$ corresponds to $n_H > -3/2$ and $n_H < -3/2$, respectively. In this section we present explicit forms for $\mathcal{R}_a^{\mathcal{X}\mathcal{X}'}$. Parity-odd CMB fluctuation power spectra, such as $C_l^{\Theta B}$ and C_l^{EB}, receive a contribution only from the helical part of the magnetic field source power spectrum, $g(k)$, i.e., from

[8] Both the vorticity Ω and the vector potential $\mathbf{V}$ appear in the equations for the CMB temperature and polarization fluctuations. The set of the equations governing the vector perturbation dynamics is given in Eqs. (35)–(38) of Ref. [7]. These show that the contribution from Ω to the CMB anisotropies dominates over that from $\mathbf{V}$, justifying neglect of the $\mathbf{V}$ contribution to CMB temperature and polarization anisotropies [4–6].

[9] Reference [10] ignores the magnetic helicity contribution to the symmetric vorticity power spectrum. That is, in their computation they ignore all terms proportional to $\int d^3p P_H(p) P_H(|\mathbf{k}-\mathbf{p}|)$.

terms proportional to $\int d^3pP_B(p)P_H(|\mathbf{k}-\mathbf{p}|)$ and $\int d^3pP_H(p)P_B(|\mathbf{k}-\mathbf{p}|)$.

A. CMB temperature anisotropies

Vector perturbations induce CMB temperature anisotropies via the Doppler and integrated Sachs-Wolfe effects. Neglecting a possible dipole contribution from the velocity perturbation $\mathbf{v}$ today, and using the Boltzmann temperature transport equation solutions Θ_l for vector perturbations [see Eqs. (5.2), (5.5), and (5.6) of Ref. [6]], we get[10]

$$C^{\Theta\Theta}_{(A)l} = l(l+1)\frac{(2\pi)^{2n_H+8}v^4_{H\lambda}}{6(2n_H+3)\Gamma^2(n_H/2+3/2)}\frac{(k_D\eta_0)^{2n_H+3}}{(k_\lambda\eta_0)^{2n_H+6}} \times\left[\frac{\eta^2_{\rm dec}}{(1+R_{\rm dec})^2}\int_0^{k_S}dkk+\frac{25}{L^2_{\gamma{\rm dec}}}\int_{k_S}^{k_D}\frac{dk}{k^3}\right] \times\left\{1+\frac{n_H-1}{n_H+4}\left(\frac{k}{k_D}\right)^{2n_H+3}\right\}J^2_{l+1/2}(k\eta_0). \quad (20)$$

Here $J_{l+1/2}(x)$ is a Bessel function, and we have used the analogy with Alfvén velocity, $v_{A\lambda}\equiv B_\lambda/\sqrt{4\pi(\rho_{\gamma0}+p_{\gamma0})}$, to introduce the helicity velocity $v_{H\lambda}\equiv H_\lambda/\sqrt{4\pi(\rho_{\gamma0}+p_{\gamma0})}$. For $n_H>-3/2$ the integral expressing the CMB temperature anisotropy in Eq. (20) is dominated by the first term (1) in the curly brackets, while for $n_H<-3/2$ the second term $\propto (n_H-1)/(n_H+4)$ in the curly brackets dominates.

The integral in Eq. (20) is split into two parts. The first integral is evaluated using the solution for vorticity that does not account for the effects of viscosity, Eq. (13), while the second integral makes use of the solution in Eq. (14) which accounts for the effects of viscosity. However, numerical integration shows that the first integral $\int_0^{k_S}$ dominates, since the second integral with vorticity damped by photon viscosity (when $k_S<k<k_D$) contributes less than of order 1% for $l\leq 500$, for all values of the spectral indexes n_B and n_H [6]. Thus the contribution of the second integral may be safely neglected. The first integral is evaluated using Eqs. (5.9)–(5.11) and (5.15) of Ref. [6], and Appendix B of this paper.

Retaining only the leading-order terms for $l\leq 500$, for $n_H>-3/2$ we find

$$\mathcal{R}_1^{\Theta\Theta}\simeq\begin{cases}1, & n_B>-3/2,\\ \frac{2(n_B+3)(n_B+2)}{n_B}\left(\frac{k_D}{k_S}\right)^{2n_B+3}, & -2<n_B<-3/2,\\ \frac{2^{2n_B+6}(n_B+3)\Gamma(-2n_B-4)}{n_B\Gamma^2(-n_B-2)}\left(\frac{k_S\eta_0}{l}\right)\left(\frac{k_D\eta_0}{l}\right)^{2n_B+3}, & -3<n_B<-2.\end{cases} \quad (21)$$

Even for a magnetic field with maximal helicity [when $|P_H(k)|=P_B(k)$], for $n_B\simeq n_H>-3/2$ the ratio $C^{\Theta\Theta}_{(A)l}/C^{\Theta\Theta}_{(S)l}=2/3$ ($\kappa_l^{\Theta\Theta}=1/3$) and the CMB temperature anisotropy power spectrum $C_l^{\Theta\Theta}>0$. For $n_B>-2$ the function $\mathcal{R}_1^{\Theta\Theta}(n_B,n_H,l)$ is independent of l.[11] For $-3<n_B<-2$, $\mathcal{R}_1^{\Theta\Theta}(n_B,n_H,l)$ is a growing function of l, scaling as l^{-2n_B-4}. The maximum growth rate of l^2 occurs for $n_B=-3$.

For $n_H<-3/2$ we distinguish two different regions, $-2<n_H<-3/2$ and $-3<n_H<-2$. Using Eqs. (B1)–(B8) [also see Eqs. (5.10)–(5.12) and (5.15) of Ref. [6]], and $k_S\eta_0\gg l$, we find

$$\mathcal{R}_2^{\Theta\Theta}\simeq\begin{cases}\frac{(n_H-1)(n_B+3)(n_B+2)}{n_B(n_H+4)(n_H+2)}\left(\frac{k_D}{k_S}\right)^{2(n_B-n_H)}, & -2<n_B<-3/2,\\ \frac{2^{2n_B+6}(n_H-1)(n_B+3)\Gamma(-2n_B-4)}{n_B(n_H+4)(n_H+2)\Gamma^2(-n_B-2)}\left(\frac{k_D}{k_S}\right)^{2(n_B-n_H)}\left(\frac{k_S\eta_0}{l}\right)^{2n_B+4}, & n_B<-2<n_H<-3/2,\\ \frac{2^{2n_B-2n_H-1}(n_H-1)(n_B+3)\Gamma(-2n_B-4)\Gamma^2(-n_H-2)}{n_B(n_H+4)\Gamma(-2n_H-4)\Gamma^2(-n_B-2)}\left(\frac{k_D\eta_0}{l}\right)^{2(n_B-n_H)}, & n_B\leq n_H<-2.\end{cases} \quad (22)$$

The function $\mathcal{R}_2^{\Theta\Theta}$ is independent of l for $-2<n_B\leq n_H<-3/2$, while $\mathcal{R}_2^{\Theta\Theta}$ grows as l^{-2n_B-4} for $-3<n_B<-2\leq n_H<-3/2$, which coincides with the growth rate of $\mathcal{R}_1^{\Theta\Theta}(n_B,n_H,l)$ for $-3<n_B<-2$ and $n_H>-3/2$. Thus if $n_B<-2$, independent of whether $n_H>-3/2$ or $-2<n_H<-3/2$, $\mathcal{R}_a^{\Theta\Theta}(n_B,n_H,l)\propto l^{-2n_B-4}$. For $-3<n_B\leq n_H<-2$ the function $\mathcal{R}_2^{\Theta\Theta}(n_B,n_H,l)$ monotonically increases with l if $n_H>n_B$, while it is independent of l for $n_H=n_B$.

The approximate expressions above are accurate to better than 15% for $n_H>-2$, to better than 30% for $-2.5\leq n_H\lesssim -2$, and to better than a few percent for $n_H<-2.5$ [6]. We do not reproduce the explicit forms of the power spectra for the cases $n_H=-3/2$ or $n_H=-2$, and $n_B=-3/2$ or $n_B=-2$, since the corresponding results can be easily derived via a straightforward extension of the computation presented in Ref. [6]; see Eq. (5.16) there.

B. CMB polarization anisotropies

Vector perturbations generate both E and B CMB polarization anisotropies [25,26]. Since scalar perturbations do not induce a magnetic (B) CMB polarization anisotropy, the future detection of a B polarization signal will indicate the presence of a vector and/or a tensor perturbation mode. In this subsection we consider the CMB polarization an-

[10] For $C^{\Theta\Theta}_{(S)l}$ see Eq. (5.7) of Ref. [6].

[11] For $n_B=-2$ there is a weak dependence on l, $\mathcal{R}_1^{\Theta\Theta}(n_B,n_H,l)\sim 1/\ln(k_S\eta_0/l)$.

isotropies that result from vector perturbations induced by a helical cosmological magnetic field.

1. *E* polarization

Using the integral solution for the CMB electric (E) polarization E_l, see Eq. (6.6) of Ref. [6], the CMB E-polarization power spectrum from magnetic helicity is[12]

$$C^{EE}_{(A)l} = (l-1)(l+2)\frac{(2\pi)^{2n_H+8}v^4_{H\lambda}}{54(2n_H+3)\Gamma^2(n_H/2+2)} \times \frac{(k_D\eta_0)^{2n_H+3}}{(k_\lambda\eta_0)^{2n_H+6}}L^2_{\gamma\mathrm{dec}}\left(\frac{\eta_{\mathrm{dec}}\eta_0}{1+R_{\mathrm{dec}}}\right)^2 \times \int_0^{k_S} dk k^5\left[1+\frac{n_H-1}{n_H+4}\left(\frac{k}{k_D}\right)^{2n_H+3}\right] \times \left[(l+1)\frac{J_{l+1/2}(k\eta_0)}{(k\eta_0)^2}-\frac{J_{l+3/2}(k\eta_0)}{k\eta_0}\right]^2. \quad (23)$$

The contribution from the integral $\int_{k_S}^{k_D}$ (i.e., the contribution from the region where vorticity is damped by photons) is negligible. When evaluating the integral in Eq. (23) we retain only the dominant term, and this differs depending on whether $n_H > -3/2$ or $n_H < -3/2$. We also use Eq. (B3) to approximate the cross term $J_{l+1/2}(k\eta_0)J_{l+3/2}(k\eta_0)$. To evaluate $\mathcal{R}^{EE}_a(n_B, n_H, l)$ we consider the following three regions: (i) $n_B > -3/2$ and $n_H > -3/2$; (ii) $-3 < n_B < -3/2$ and $n_H > -3/2$; and (iii) $-3 < n_B < -3/2$ and $n_H < -3/2$.

For $n_H > -3/2$, using Eqs. (B2)–(B5) and (B8) and retaining only the leading terms in the limit where $k_S\eta_0 \gg l$, we find

$$\mathcal{R}^{EE}_1 \simeq \begin{cases} 1, & n_B > -3/2, \\ \frac{2(n_B+3)^2}{n_B}\left(\frac{k_D}{k_S}\right)^{2n_B+3}, & -3 < n_B < -3/2. \end{cases} \quad (24)$$

The function $\mathcal{R}^{EE}_1 = \mathcal{R}^{\Theta\Theta}_1$ for $n_B > -3/2$, and $\mathcal{R}^{EE}_1 = (n_B+3)\mathcal{R}^{\Theta\Theta}_1$ for $-2 < n_B < -3/2$. The expression for $\mathcal{R}^{EE}_1$ is the same for all values of n_B in the range $-3 < n_B < -3/2$, while $\mathcal{R}^{\Theta\Theta}_1$ differs depending on whether $-2 < n_B < -3/2$ or $-3 < n_B < -2$, see Eq. (21).

For $n_H < -3/2$ we need to consider the ranges $-2 < n_H < -3/2$ and $n_H < -2$ separately [6]. Using Eqs. (B2)–(B5) and (B8), we find that for $n_H < -2$ the expression for $C^{EE}_{(A)l}$ is identical to the one in the region $-2 < n_H < -3/2$ to within 15% accuracy. This simplifies the computation, and for $-3 < n_B \le n_H < -3/2$ we find

$$\mathcal{R}^{EE}_2 \simeq \frac{(n_H-1)(n_B+3)^2}{n_B(n_H+4)(n_H+3)}\left(\frac{k_S}{k_D}\right)^{2(n_H-n_B)}, \quad (25)$$

while for $-2 < n_B \le n_H < -3/2$ we have $\mathcal{R}^{EE}_2(n_B, n_H, l) = (n_B+3)(n_H+2)/((n_B+2)(n_H+3))\mathcal{R}^{\Theta\Theta}_2(n_B, n_H, l)$, where $\mathcal{R}^{\Theta\Theta}_2$ is given in Eq. (22).

[12]Just as for the temperature integral solution Θ_l, E_l is also expressed in terms of Ω. To compute C^{EE}_l we use Eqs. (16) and (17) along with Eq. (8). An expression for $C^{EE}_{(S)l}$ is given in Eqs. (6.7) and (6.9)–(6.12) of Ref. [6].

2. *B* polarization

To compute the contribution from cosmological magnetic helicity to the CMB B-polarization power spectrum we use the integral solution for B_l, Eq. (6.16) of Ref. [6]. The CMB B-polarization power spectrum is

$$C^{BB}_{(A)l} = (l-1)(l+2)\frac{(2\pi)^{2n_H+8}v^4_{H\lambda}}{54(2n_H+3)\Gamma^2(n_H/2+2)} \times \frac{(k_D\eta_0)^{2n_H+3}}{(k_\lambda\eta_0)^{2n_H+6}}L^2_{\gamma\mathrm{dec}}\left(\frac{\eta_{\mathrm{dec}}\eta_0}{1+R_{\mathrm{dec}}}\right)^2 \times \int_0^{k_S} dk k^5\left[1+\frac{n_H-1}{n_H+4}\left(\frac{k}{k_D}\right)^{2n_H+3}\right]\frac{J^2_{l+1/2}(k\eta_0)}{(k\eta_0)^2}. \quad (26)$$

Here again the contribution from the region where viscous effects are important is negligibly small. An expression for $C^{BB}_{(S)l}$ is given in Eqs. (6.18) and (6.19) of Ref. [6].

Using Eq. (B2) and retaining the leading terms we get

$$\mathcal{R}^{BB}_1 \simeq \mathcal{R}^{EE}_1, \qquad \mathcal{R}^{BB}_2 \simeq \mathcal{R}^{EE}_2, \quad (27)$$

to better than 20% accuracy [6].

Both functions R^{EE}_a and R^{BB}_a (i.e., the coefficients κ^{EE}_l and κ^{BB}_l) are independent of l. This means that cosmological magnetic helicity reduces the CMB polarization power spectrum amplitudes by the same scale factor for the E and the B polarizations. The ratio between contributions to the E- or B-polarization signal from the helical and the symmetric parts of the magnetic field are independent of l for the entire range of spectral indices in the case of the vector mode, while for the tensor mode case the ratios depend on l for $n_H < -2$ [14].

3. Temperature-*E*-polarization cross correlation

We may obtain the CMB temperature-E-polarization cross-correlation power spectrum $C^{\Theta E}_l$ from the integral solutions for the temperature and E-polarization anisotropies. Like $C^{\Theta\Theta}_l$ this power spectrum is also parity even and only the symmetric part of the magnetic field source $f_B(k)$ [which also contains a contribution from the helical part of the magnetic field power spectrum $P_H(k)$] contributes to it. As discussed in Ref. [22] (see Fig. 5 there), the vector dipole temperature anisotropy radial function $j^{(1V)}_l = j_l\sqrt{l(l+1)/2}/x$ does not correlate well with its E-polarization anisotropy radial function $\epsilon^{(V)}_l = [j_l/x^2 + j'_l/x]\sqrt{(l-1)(l+2)}/2$ [here $j'_l(x)$ is the partial derivative with respect to x of the Bessel function of fractional order j_l], while the vector quadrupole temperature anisotropy radial function $j^{(2V)}_l = \sqrt{3l(l+1)/2}(j_l/x)'$ does. To compute the vector mode CMB temperature-E-polarization cross-correlation power spectrum we therefore have to retain the term proportional to $j_l^{(2V)}$ in the vector temperature integral solution Θ_l which was neglected previously[13] in the derivation of $C_l^{\Theta\Theta}$ [see Eq. (20)]. With the $j_l^{(2V)}$ term the CMB temperature anisotropy integral solution is [22]

$$\frac{\Theta_l^{(V)}(\eta_0,k)}{2l+1} \simeq \sqrt{\frac{l(l+1)}{2}}\Omega(\eta_{\rm dec},k)\left[\frac{j_l(k\eta_0)}{k\eta_0}+\frac{kL_{\gamma{\rm dec}}}{3}\left\{(l-1)\frac{j_l(k\eta_0)}{(k\eta_0)^2}-\frac{j_{l+1}(k\eta_0)}{k\eta_0}\right\}\right]. \quad (28)$$

$C_{(S)l}^{\Theta E}$ is given in Eq. (7.3) of Ref. [6], and for $C_{(A)l}^{\Theta E}$ we get

$$C_{(A)l}^{\Theta E} = -\sqrt{l(l-1)(l+1)(l+2)}\frac{(2\pi)^{2n_H+8}v_{H\lambda}^4}{18(2n_H+3)\Gamma^2(n_H/2+2)}\frac{(k_D\eta_0)^{2n_H+3}}{(k_\lambda\eta_0)^{2n_H+6}}\left(\frac{\eta_{\rm dec}\eta_0}{1+R_{\rm dec}}\right)^2 L_{\gamma{\rm dec}}$$
$$\times\int_0^{k_S} dkk^4\left[1+\frac{n_H-1}{n_H+4}\left(\frac{k}{k_D}\right)^{2n_H+3}\right]\left[(l+1)\frac{J_{l+1/2}^2(k\eta_0)}{(k\eta_0)^3}-\frac{J_{l+1/2}(k\eta_0)J_{l+3/2}(k\eta_0)}{(k\eta_0)^2}+\frac{kL_{\gamma{\rm dec}}}{3}\right.$$
$$\left.\times\left\{(l^2-1)\frac{J_{l+1/2}^2(k\eta_0)}{(k\eta_0)^4}-2l\frac{J_{l+1/2}(k\eta_0)J_{l+3/2}(k\eta_0)}{(k\eta_0)^3}+\frac{J_{l+3/2}^2(k\eta_0)}{(k\eta_0)^2}\right\}\right]. \quad (29)$$

The first two terms in the second pair of square brackets in this integral arise from the correlation of $j_l^{(1V)}$ with $\epsilon_l^{(V)}$. A numerical evaluation of the integral [6] shows that these two terms roughly cancel each other as a consequence of the low correlation between $j_l^{(1V)}$ and $\epsilon_l^{(V)}$ [22]. This may also be seen by using Eqs. (B6) and (B7),

$$\frac{l+1}{x}J_{l+1/2}^2(x) - J_{l+1/2}(x)J_{l+3/2}(x)$$
$$\simeq \frac{1}{2}[J_{l-1/2}(x)J_{l+1/2}(x) - J_{l+1/2}(x)J_{l+3/2}(x)]$$
$$\simeq \sin(2x-l\pi) - \sin(2x-l\pi-\pi) \simeq 0. \quad (30)$$

In what follows we neglect these two terms. We use Eqs. (B2)–(B5) to evaluate the three terms in the curly brackets of Eq. (29). The terms from the correlation between $j_l^{(2V)}$ and $\epsilon_l^{(V)}$ are suppressed by an additional factor of $kL_{\gamma{\rm dec}}$, relative to the two terms from the correlation between $j_l^{(1V)}$ and $\epsilon_l^{(V)}$. In the limit $l \gg 1$, these terms and the squared sum of the two Bessel function terms in the expression for C_l^{EE} in Eq. (23) (the last factor inside the integral of this equation) are almost identical. Thus apart from an overall minus sign, the $C_l^{\Theta E}$ are approximately equal to the corresponding C_l^{EE} [6]. Here our approximation might not be as accurate because, accounting for the suppression factor $kL_{\gamma{\rm dec}}$, the neglected contribution from the correlation between $j_l^{(1V)}$ and $\epsilon_l^{(V)}$ could be comparable to the retained contribution from the correlation between $j_l^{(2V)}$ and $\epsilon_l^{(V)}$.

IV. PARITY-ODD CMB FLUCTUATIONS FROM MAGNETIC HELICITY

Magnetic helicity induces parity-odd cross correlations between the E- and B-polarization anisotropies, as well as between temperature and B-polarization anisotropies [10,14]. Such off-diagonal parity-odd cross correlations also occur in the case of an homogeneous magnetic field from the Faraday rotation effect [21], but not in the case of a stochastic magnetic field, even one with nonzero helicity [20]. Faraday rotation measurements cannot be used to detect magnetic helicity [18–20]. A possible way of detecting magnetic helicity directly from CMB fluctuation data is to detect the above parity-odd CMB correlations or to detect the effects magnetic helicity has on parity-even CMB fluctuations. In this section we study the parity-odd CMB cross correlations generated from vorticity perturbations. The corresponding tensor mode contributions are derived in Sec. VI of Ref. [14].

A. Temperature-B-polarization cross correlation

To compute the cross correlation between the CMB temperature and E-polarization anisotropies we use the integral solutions for Θ_l and B_l given in Eqs. (5.6) and (6.16) of Ref. [6] and find

$$C_l^{\Theta B} = -\sqrt{(l-1)l(l+1)(l+2)}$$
$$\times\frac{(2\pi)^{n_B+n_H+8}v_{A\lambda}^2v_{H\lambda}^2}{18(n_B+n_H+2)\Gamma(n_B/2+3/2)\Gamma(n_H/2+2)}$$
$$\times\frac{(k_D\eta_0)^{n_B+n_H+2}}{(k_\lambda\eta_0)^{n_B+n_H+6}}\left[\frac{\eta_{\rm dec}\eta_0}{1+R_{dec}}\right]^2\frac{L_{\gamma{\rm dec}}}{\eta_0}$$
$$\times\int_0^{k_S} dkk^3\left[1+\frac{n_H-1}{n_B+3}\left(\frac{k}{k_D}\right)^{n_B+n_H+2}\right]J_{l+1/2}^2(k\eta_0). \quad (31)$$

To evaluate this integral it is helpful to consider separately the index ranges (i) $n_B + n_H > -2$ and (ii) $n_H + n_B < -2$.

When $n_B + n_H > -2$ the integral on the right-hand side (r.h.s.) of Eq. (31) is dominated by the first term in the

[13]Since it is suppressed relative to the term proportional to $j_l^{(1V)}$.

square brackets. Using Eq. (B2) with $p = 3$ we find

$$C_l^{\Theta B} = -l^2 \frac{(2\pi)^{n_B+n_H+7} v_{A\lambda}^2 v_{H\lambda}^2}{27(n_B + n_H + 2)\Gamma(n_B/2 + 3/2)\Gamma(n_H/2 + 2)} \times \frac{(k_D\eta_0)^{n_B+n_H+2}}{(k_\lambda\eta_0)^{n_B+n_H+6}} \left[\frac{\eta_{\rm dec}}{(1 + R_{\rm dec})\eta_0}\right]^2 \frac{L_{\gamma \rm dec}}{\eta_0} (k_S\eta_0)^3. \quad (32)$$

When $n_H + n_B < -2$ the integral on the r.h.s. of Eq. (31) is dominated by the second term ($\propto k^{n_B+n_H+2}$) in the square brackets. For $n_B + n_H < -5$ the integral in Eq. (31) [$\int_0^{k_S} dk k^{n_B+n_H+5} J_{l+1/2}^2(k\eta_0)$] converges for large k_S which can then be extended to ∞, and so the integral may be evaluated using Eq. (B1). We find for $-6 < n_B + n_H < -5$,

$$C_l^{\Theta B} = -l^{n_B+n_H+7} \frac{(2\pi)^{n_B+n_H+8} 2^{n_B+n_H+4} v_{A\lambda}^2 v_{H\lambda}^2}{9(n_B + n_H + 2)\Gamma(n_B/2 + 3/2)\Gamma(n_H/2 + 2)} \frac{1}{(k_\lambda\eta_0)^{n_B+n_H+6}} \left[\frac{\eta_{\rm dec}}{(1 + R_{\rm dec})\eta_0}\right]^2 \frac{L_{\gamma \rm dec}}{\eta_0} \left(\frac{n_H - 1}{n_B + 3}\right) \times \frac{\Gamma(-n_B - n_H - 5)}{\Gamma^2(-n_B/2 - n_H/2 - 2)}. \quad (33)$$

For $n_B + n_H > -5$ (but still $n_B + n_H < -2$) the integral in Eq. (31) diverges at large k_S and so the upper limit cannot be replaced by ∞, and the integral cannot be evaluated by using Eq. (B1). Instead we approximate it by using Eq. (B2). We find for $-5 < n_B + n_H < -2$,

$$C_l^{\Theta B} = -l^2 \frac{(2\pi)^{n_B+n_H+7} v_{A\lambda}^2 v_{H\lambda}^2}{9(n_B + n_H + 2)(n_B + n_H + 5)\Gamma(n_B/2 + 3/2)\Gamma(n_H/2 + 2)} \frac{(k_S\eta_0)^{n_B+n_H+5}}{(k_\lambda\eta_0)^{n_B+n_H+6}} \left[\frac{\eta_{\rm dec}}{(1 + R_{\rm dec})\eta_0}\right]^2 \frac{L_{\gamma \rm dec}}{\eta_0} \left(\frac{n_H - 1}{n_B + 3}\right). \quad (34)$$

When $n_B + n_H = -5$ the integration can be done by using Eq. (B2) with $p = 0$.

At large angular scales ($l < 100$) where the contribution from the tensor mode is significant, for $n_B + n_H > -2$ the vector mode $C_l^{\Theta B(V)}$ and the tensor mode $C_l^{\Theta B(T)}$ [see Eq. (98) of Ref. [14]] have the same l dependence $\propto l^2$. For all other values of spectral indexes n_B and n_H, the growth rate (with l) of $C_l^{\Theta B(V)}$ is faster than $C_l^{\Theta B(T)}$. In particular, when the integral in Eq. (31) converges at large k_S (for $-6 < n_B + n_H < -5$), $C_l^{\Theta B(V)}/C_l^{\Theta B(T)} \propto l^3$. When $-5 < n_B + n_H < -2$, the integral for the tensor mode temperature-B-polarization cross-correlation power spectrum converges at large k [see Eq. (97) of Ref. [14]], while it diverges for vorticity perturbations, Eq. (31), resulting in $C_l^{\Theta B(V)}/C_l^{\Theta B(T)} \propto l^{-n_B-n_H-2}$. The ratio between temperature-B-polarization signals from vector and tensor modes is independent of the amplitudes of the average magnetic field (B_λ) and average magnetic helicity (H_λ).

For maximally helical magnetic fields with $n_H \simeq n_B$, due to the suppression factor $L_{\gamma,\rm dec}/\eta_0$ the temperature-E-polarization cross-correlation power spectrum $C_l^{\Theta E}$ is smaller the temperature-B-polarization cross-correlation power spectrum $C_l^{\Theta B}$,[14] $C_l^{\Theta E} \ll C_l^{\Theta B}$, but both are $\propto l^2$, if $n_B + n_H > -5$. The same suppression factor makes $C_l^{\Theta B}$ smaller than $C_l^{\Theta\Theta}$. For an arbitrary helical field $C_l^{\Theta B}/C_l^{\Theta E}$ depends on the ratio $(P_{H0}/P_{B0})k_D^{n_H-n_B}$ and order unity prefactors that depend on n_B and n_H. A dependence on l appears only if $n_B + n_H < -5$ when the ratio $C_l^{\Theta B}/C_l^{\Theta E}$ decreases as $\propto l^{n_B+n_H+5}$.

B. E- and B-polarization cross correlation

To compute cross correlations between E- and B-polarization anisotropies we use the integral solutions for E_l and B_l given in Eqs. (6.14) and (6.16) of Ref. [6]. We find

$$C_l^{EB} = -(l-1)(l+2) \frac{(2\pi)^{n_B+n_H+8} v_{H\lambda}^2 v_{A\lambda}^2 \eta_0^5}{54(n_B + n_H + 2)\Gamma(n_B/2 + 3/2)\Gamma(n_H/2 + 2)} \frac{(k_D\eta_0)^{n_B+n_H+2}}{(k_\lambda\eta_0)^{n_B+n_H+6}} \left[\frac{\eta_{\rm dec}\eta_0}{1 + R_{\rm dec}}\right]^2 L_{\gamma \rm dec}^2 \times \int_0^{k_S} dk k^5 \left[1 + \frac{n_H - 1}{n_B + 3}\left(\frac{k}{k_D}\right)^{n_B+n_H+2}\right] \left[(l+1)\frac{J_{l+1/2}^2(k\eta_0)}{(k\eta_0)^3} - \frac{J_{l+1/2}(k\eta_0)J_{l+3/2}(k\eta_0)}{(k\eta_0)^2}\right]. \quad (35)$$

The combination of Bessel functions in the second set of square brackets in this integral is identical to the first two terms in the second set of square brackets in Eq. (29), which is negligibly small according to Eq. (30). Also the E- and B-polarization anisotropy cross-correlation power spectrum has an additional suppression factor of $kL_{\gamma,\rm dec}$ relative to the expression in Eq. (29). This implies $C_l^{EB} \ll C_l^{\Theta B}$. Note that this is consistent with the result of Ref. [22] that $j_l^{(1V)}$ does not correlate well with $\epsilon_l^{(V)}$. The corresponding C_l^{EB} amplitudes in the tensor mode case (for $l < 100$)

[14] If $n_H \simeq n_B > -3/2$, $C_l^{\Theta E}/C_l^{\Theta B} \simeq L_{\gamma,\rm dec}/(2\eta_0)$.

are not suppressed and are of the same order of magnitude as the tensor mode temperature-B-polarization anisotropy cross-correlation power spectrum [14].

V. CONCLUSION

In this paper we consider how cosmological magnetic helicity affects CMB fluctuations. Even for a cosmological magnetic field with maximal helicity, such effects may be detectable only if the current magnetic field amplitude is at least 10^{-10} or 10^{-9} G on Mpc scales. Our analytical expressions for CMB fluctuation power spectra are valid (to the accuracy of our approximations) for $n_B > -3$.[15]

A cosmological magnetic field generates a B-polarization signal via induced vector and/or tensor modes, so a detection of such a signal may indicate the presence of a cosmological magnetic field.[16] However, it has to be emphasized that a B-polarization anisotropy signal can also arise in other ways, such as from primordial tensor perturbations [25], gravitational lensing [29], or Faraday rotation of the CMB anisotropy polarization plane [17,19,20]. The B-polarization anisotropy power spectrum $l^2 C_l^{BB}$ peak position may help identify the B-polarization source. For example, cosmological-magnetic-field-induced tensor perturbations only contribute on large angular scales $l < 100$, while B-polarization anisotropy from gravitational lensing has a peak amplitude $l^2 C_l^{BB} \sim 10^{-14}$ at $l \sim 1000$ [29]. The Faraday rotation B-polarization anisotropy signal from a field with $B_\lambda = 10^{-9}$ G (at $\lambda =$ 1 Mpc) and spectral index $n_B = -2$ peaks at a substantially smaller scale $l \sim 10^4$ with a frequency-dependent peak amplitude $l^2 C_l^{BB} \sim 10^{-12}$ (at 10 GHz) and $l^2 C_l^{BB} \sim 10^{-14}$ (at 30 GHz) [20]. A nonhelical cosmological magnetic field with $B_\lambda = 10^{-9}$ G at $\lambda = 1$ Mpc induces a B-polarization anisotropy signal via the vector perturbation mode with a peak amplitude $l^2 C_l^{BB} \sim 10^{-13}$ at $l \sim 1000$ [7]. We have shown that a magnetic field with maximal helicity results in the reduction of the B-polarization anisotropy signal on all scales by a factor of $1/3$ for $-3/2 < n_B \simeq n_H$, relative to the nonhelical magnetic field case.

We have presented analytical expressions for all CMB fluctuation power spectra affected by cosmological magnetic helicity. Our results show that cosmological magnetic helicity can affect CMB anisotropies, in addition to the effects it has on MHD dynamo amplification and processes in the early Universe [15,16]. It would be useful to incorporate our analytical expressions for $l^2 C_l^{XX'}$ into a numerical code (e.g., that of Lewis [7]) to compute CMB temperature and polarization anisotropies generated by a general cosmological magnetic source.

To set an observational limit on cosmological magnetic helicity H_λ [or $P_H(k)$] one may proceed as follows. The first step is to determine the average magnetic field B_λ [or $P_B(k)$] using measurements of the Faraday rotation of the CMB polarization plane [17,19,20]. Then one may use measurements of the parity-odd temperature-B-polarization anisotropies cross-correlation power spectrum $C_l^{\Theta B}$ for $l > 100$ (to insure that the tensor mode does not contribute).[17] According to Eq. (31), if B_λ and n_B are known, $C_l^{\Theta B}$ is determined by H_λ and n_H (i.e., the helical part of the magnetic field spectrum). A future detection of $\Theta - B$ cross correlations may be used to constrain H_λ. It should be emphasized that on scales $l > 100$ magnetic-field-induced cross correlations between E and B polarization are negligibly small, $C_l^{EB} \ll C_l^{\Theta B}$, which can be used as a cross-check of the source of B-polarization anisotropy. To bound the range of n_H for a given n_B, the l dependence of the ratio $C_{(A)l}^{\Theta\Theta}/C_{(S)l}^{\Theta\Theta}$ can be used. In particular, if, for $-3 < n_B < -2$, $C_{(A)l}^{\Theta\Theta}/C_{(S)l}^{\Theta\Theta}$ is an increasing function of l growing as l^{-2n_B-4}, then $n_H > -2$, while if it scales as $l^{2(n_H - n_B)}$ then $n_B \leq n_H < -2$. If, for $n_B < -2$, $C_{(A)l}^{\Theta\Theta}/C_{(S)l}^{\Theta\Theta}$ is l independent, then $n_H \simeq n_B$. If $n_B > -2$, $C_{(A)l}^{\Theta\Theta}/C_{(S)l}^{\Theta\Theta}$ is l independent for any allowed $n_H \geq n_B > -2$.

It is possible that there are other ways to detect magnetic helicity. On the other hand, Ref. [18] argues that a detection of magnetic helicity, even for cluster magnetic fields, is a very difficult task.

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[15]However, a cosmological magnetic field with spectral index $n_B \simeq 2$ (as might be generated by an MHD cascade in the early Universe [16]) has significant power on small (galaxy cluster) scales and so measurements of Faraday rotation in clusters imply an upper limit on the smoothed amplitude $B_\lambda < 10^{-12}$ G on Mpc scales [16,27]. Such a "blue" cosmological magnetic field cannot significantly affect CMB anisotropies. A strong constraint on magnetic field amplitude on Mpc scales for $n_B > -2$ arises from gravitational waves production via a magnetic source, if the magnetic field is generated with a power law spectrum $P_B(k) = (k/k_{\max})^{n_B} P_B(k_{\max})$ (where $k_{\max} = \eta_{\rm gen}^{-1}$ and $\eta_{\rm gen}$ is the moment of magnetic field generation) that can be extrapolated, without damping, down to the Hubble radius when the magnetic field is generated, e.g., $\approx 10^{-4}$ Mpc if the magnetic field is generated at the electroweak phase transition (or even smaller for a magnetic field generated during inflation) [28].

[16]See Ref. [26] for CMB polarization anisotropy measurements.

[17]Magnetic helicity is the only known source of diagonal temperature-B-polarization anisotropies cross correlations. A homogeneous magnetic field induces off-diagonal $\Theta - B$ cross correlations [21], but not diagonal correlations. Faraday rotation does not induce diagonal or off-diagonal $\Theta - B$ cross correlations [20].

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APPENDIX A: DERIVATION OF MAGNETIC FIELD SOURCE TERMS FOR VORTICITY PERTURBATIONS

The derivation of the symmetric magnetic field source term for the vorticity perturbation equation of motion is given in [6]. To obtain the complete magnetic field source term for the vector metric perturbation we use the Fourier transformed magnetic field energy-momentum tensor,

$$\tau_{ij}(\mathbf{k}) = \frac{1}{2(2\pi)^4}\int d^3p\Big[B_i(\mathbf{p})B_j(\mathbf{k}-\mathbf{p}) - \frac{1}{2}B_l(\mathbf{p})B_l(\mathbf{k}-\mathbf{p})\delta_{ij}\Big], \tag{A1}$$

and its wave number space power spectrum $\langle\tau^\star_{ij}(\mathbf{k})\tau_{lm}(\mathbf{k}')\rangle$ [6]. The symmetric part of the magnetic source power spectrum $f(k)$ and the helical part $g(k)$ of the magnetic source power spectrum can be obtained via Eqs. (6) and (7), see Eqs. (A4)–(A7) of [6] and Eq. (A1) of Ref. [14]. The part of $f(k) = f_B(k) - f_H(k)$ that depends on the symmetric part of the magnetic field power spectrum, $f_B(k) \sim \int d^3pP_B(p)P_B(|\mathbf{k}-\mathbf{p}|)$, is [6]

$$f_B(k) \simeq \frac{\lambda^3B_\lambda^4}{16(2n_B+3)\Gamma^2(n_B+3/2)} \times\Big[(\lambda k_D)^{2n_B+3} + \frac{n_B}{n_B+3}(\lambda k)^{2n_B+3}\Big]. \tag{A2}$$

The magnetic helicity contribution, $f_H(k) \sim \int d^3pP_H(p)P_H(|\mathbf{k}-\mathbf{p}|)$, to $f(k)$ is

$$f_H(k) = \frac{1}{8(2\pi)^5}\int d^3pP_H(p)P_H(|\mathbf{k}-\mathbf{p}|) \times\frac{p(1-\gamma^2)}{\sqrt{k^2-2kp\gamma+p^2}}, \tag{A3}$$

where $\gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}$. The function $f_H(k)$ is positive but it contributes to $f(k)$ with a minus sign, $f(k) = f_B(k) - f_H(k)$, and so it decreases the overall symmetric magnetic field source term. This result contradicts that of Ref. [10]. Reality requires $P_B(k) \geq |P_H(k)|$, and the total symmetric source term is always positive, $f(k) > 0$. The mixed terms $\int d^3pP_B(p)P_H(|\mathbf{k}-\mathbf{p}|)$ and $\int d^3pP_H(p)P_B(|\mathbf{k}-\mathbf{p}|)$ do not contribute to $f(k)$, while the helical spectrum $g(k)$ is completely defined by these terms,

$$g(k) = \frac{1}{16(2\pi)^5}\int d^3pP_B(p)P_H(|\mathbf{k}-\mathbf{p}|) \times\frac{(k-2p\gamma)(1-\gamma^2)}{\sqrt{k^2-2kp\gamma+p^2}}. \tag{A4}$$

The helical spectrum $g(k)$ does not receive a contribution from the diagonal terms $\int d^3pP_B(p)P_B(|\mathbf{k}-\mathbf{p}|)$ and $\int d^3pP_H(p)P_H(|\mathbf{k}-\mathbf{p}|)$.

To evaluate the expressions for $f_H(k)$ and $g(k)$, we first integrate over γ and then integrate over p. The integration over γ can be done analytically, see the appendix of Ref. [6]. To integrate over p we approximate the result of the γ integration by using the binomial expansion $(1+x)^n = 1 + nx + n(n-1)x^2/2 + \mathcal{O}(x^3)$. For the vector case the dominant terms for $x \ll 1$ are those of quadratic order. Additionally, the integration is split into two parts, $\int_0^{k_D}dp = \int_0^k dp + \int_k^{k_D}dp$, and the binomial expansion for $k > p$ used for $\int_0^k dp$, while the second integral $\int_k^{k_D}dp$ is evaluated using the binomial expansion for $k < p$ [6,14]. The result is the vector perturbation source power spectrum symmetric $[f(k)]$ and helical $[g(k)]$ terms in Eqs. (8) and (9) above.

APPENDIX B: BESSEL FUNCTIONS INTEGRALS

We need to evaluate integrals of the form $\int_0^{x_S} dxJ_p(ax)J_q(ax)x^{-b}$, which contain products of Bessel functions. For $b > 0$ when the integral converges and is dominated by $x \ll x_S$, the upper limit x_S can be replaced by ∞ (with an accuracy of a few percent for $b > 1$, and 15%–30% for $0 < b < 1$, depending on the value of $p - q$). We can then use Eq. (6.574.2) of Ref. [30],

$$\int_0^\infty dxJ_p(ax)J_q(ax)x^{-b} = \frac{a^{b-1}\Gamma(b)\Gamma((p+q-b+1)/2)}{2^b\Gamma((-p+q+b+1)/2)\Gamma((p+q+b+1)/2)\Gamma((p-q+b+1)/2)}, \tag{B1}$$

which is valid for $\mathrm{Re}(p+q+1) > \mathrm{Re}b > 0$, and $a > 0$.

To evaluate the integral $\int_0^{x_S} dxx^pJ^2_{l+1/2}(x)$ with $p > 0$ and $x_S \gg l$, we use the asymptotic expansion of $J_p(x)$ for large arguments, Eq. (9.2.1) of Ref. [31], $J_{l+1/2}(x) \simeq \sqrt{2/(\pi x)}\cos[x-(l+1)\pi/2] \simeq \sqrt{2/(\pi x)}\cos[x-(l+1)\pi/2]$. Replacing the oscillatory function $\cos^2$ by its rms value of $1/2$, we obtain [6,14],

$$\pi\int_0^{x_S} dxx^pJ^2_{l+1/2}(x) \simeq \pi\int_l^{x_S} dxx^pJ^2_{l+1/2}(x) \simeq \begin{cases} x_S^p/p, & p>0, \\ \ln(x_S/(l+1/2)), & p=0. \end{cases} \tag{B2}$$

For the integral $\int_0^{x_S} dx x^p J_{l+1/2}(x) J_{l+3/2}(x)$ we also use the large argument ($x \gg l$) approximation for the Bessel functions, and find (see Ref. [14] for a numerical check)

$$\pi x J_{l+1/2}(x) J_{l+3/2}(x) \simeq 2\cos\left(x-(l+1)\frac{\pi}{2}\right)\cos\left(x-(l+2)\frac{\pi}{2}\right) = 2\cos\left(x-(l+1)\frac{\pi}{2}\right)\sin\left(x-(l+1)\frac{\pi}{2}\right) = \sin(2x-(l+1)\pi) = (-1)^{l+1}\sin(2x). \tag{B3}$$

So for $p > 0$,

$$\pi \int_0^{x_S} dx x^p J_{l+1/2}(x) J_{l+3/2}(x) \simeq (-1)^{l+1} \int_l^{x_S} dx x^{p-1}\sin(2x) \simeq \frac{(-1)^l}{2}(x_S^{p-1}\sin(2x_D) - (l+1/2)^{p-1}\sin(2l)). \tag{B4}$$

This approximation tends to underestimate; it is good to a few percent for $p > 1$ and is within 30% for $0 < p \leq 1$. For $p = 0$ the integral $\int_0^{x_S} dx J_{l+1/2}(x) J_{l+3/2}(x)$ may be evaluated using Eq. (11.4.42) of Ref. [31],

$$\int_0^{x_S} dx J_{l+1/2}(x) J_{l+3/2}(x) = \frac{1}{2}. \tag{B5}$$

For the integral $\int_0^{x_S} dx x^p [(l+1)J_{l+1/2}(x)/x - J_{l+3/2}(x)]^2$ appearing in Eq. (23), for large enough l we have the approximation,

$$\left(\frac{l+1}{x}J_{l+1/2}(x) - J_{l+3/2}(x)\right)^2 \approx [J'_{l+1/2}(x)]^2 = \frac{1}{4}[J_{l-1/2}(x) - J_{l+3/2}(x)]^2. \tag{B6}$$

We now approximate the cross term $J_{l-1/2}(x) J_{l+3/2}(x)$ in the limit $x \gg l$ by using

$$\pi x J_{l-1/2}(x) J_{l+3/2}(x) \simeq 2\cos\left(x - l\frac{\pi}{2}\right)\cos\left(x-(l+2)\frac{\pi}{2}\right) = -2\cos^2\left(x - l\frac{\pi}{2}\right). \tag{B7}$$

As in the computation of Eq. (B2) we may replace the $\cos^2$ by $1/2$, and so find for $p \geq 0$,

$$\pi\int_0^{x_S} dx x^p \left(\frac{l+1}{x}J_{l+1/2}(x) - J_{l+3/2}(x)\right)^2 \simeq \frac{\pi}{4}\int_0^{x_S} dx x^p (J_{l-1/2}^2 - 2J_{l-1/2}J_{l+3/2} + J_{l+3/2}^2) \simeq \pi\int_0^{x_S} dx x^p J_l^2 \simeq \begin{cases} x_S^p/p, & p > 0, \\ \ln(x_S/(l+1/2)), & p = 0. \end{cases} \tag{B8}$$

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