# Bounds on cubic Lorentz-violating terms in the fermionic dispersion relation

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We study the recently proposed Lorentz-violating dispersion relation for fermions and show that it leads to two distinct cubic operators in the momentum. We compute the leading order terms that modify the nonrelativistic equations of motion and use experimental results for the hyperfine transition in the ground state of the $^9\mathrm{B}e^+$ ion to bound the values of the Lorentz-violating parameters $\eta_1$ and $\eta_2$ for neutrons. The resulting bounds depend on the value of the Lorenz-violating background four-vector in the laboratory frame.

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## I. INTRODUCTION

The possibility of violation of the Lorentz symmetry has been widely discussed in the recent literature (see e.g. [1]). Indeed, the spontaneous breaking of this fundamental symmetry may arise in the context of string/M-theory due to existence of nontrivial solutions in string field theory [2], in loop quantum gravity [3,4], in noncommutative field theories [5][1], in quantum gravity inspired spacetime foam scenarios [7] or through the spacetime variation of fundamental coupling constants [8]. This breaking could be tested, for instance, in ultrahigh energy cosmic rays [9].

Recently, it has been proposed a method of introducing cubic modifications into dispersion relations by means of dimension five operators for fermions [10]. The upper bounds for the parameters that characterize these modifications are based on low-energy experiments, being $|\xi| \lesssim 10^{-6}$ for the electromagnetic sector, $|\eta_{Q,u,d}| \lesssim 10^{-6}$ for first quark generation and $|\eta^e_{L,R}| \lesssim 10^{-5}$ for electrons [10].

In this paper, we shall consider cubic Lorentz-violating terms for fermions in the nonrelativistic limit and obtain new upper bounds for neutrons, based on spectroscopical results for the $^9\mathrm{B}e^+$ ground state, as discussed by Bollinger *et al.* [11].

## II. THE MODEL

We consider terms in the Lagrangian density which describes a Dirac spinor field, correspondig to dimension five operators which break the Lorentz symmetry by means of a background four-vector $n^\mu$ [10]. These terms have the following features: (i) have one more derivative than the usual kynetic term, (ii) are gauge invariant, (iii) are Lorentz invariant, apart from $n^\mu$, (iv) are irreducible to lower dimension operators by means of the equations of motion and (v) do not correspond to a total derivative and are suppressed by a single power of the Planck mass, $M_P$.

Under these conditions, the two possible operators can be combined in the following form [10]:

$$\mathcal{L}_f = \frac{1}{M_P}\bar{\psi}(\eta_1 \not{n} + \eta_2 \not{n}\gamma_5)(n\cdot\partial)^2\psi. \tag{1}$$

The parameters $\eta_1$ and $\eta_2$ can, for instance, in the case of string theory, be regarded as vacuum expectation values of tensor operators arising from the spontaneous symmetry breaking mechanism [2].

First, it should be pointed out that the Lagrangian density Eq. (1) is not symmetric in what respects the fields $\psi$ and $\bar{\psi}$ and, thus, one should include its hermitian conjugate. The complete fermionic Lagrangian density is, hence, given by

$$\mathcal{L}_f = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{M_P}\bar{\psi}(\eta_1\not{n} + \eta_2\not{n}\gamma_5)(n\cdot\partial)^2\psi + \mathrm{h.c.}, \tag{2}$$

which must satisfy the following Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\varphi)}\right) = 0, \tag{3}$$

where $\varphi$ denotes a generic field of the Lagrangian density. For $\varphi = \bar{\psi}$, Eq. (3) leads to the modified Dirac equation:

$$\left[i\not{\partial} - m + \frac{1}{M_P}(\eta_1\not{n} + \eta_2\not{n}\gamma_5)(n\cdot\partial)^2\right]\psi = 0. \tag{4}$$

For $\varphi = \psi$, we obtain, as expected, the hermitian conjugated equation.

In order to obtain the correspondent dispersion relation, we operate Eq. (4) with $(i\partial + m + \frac{1}{M_P}(\eta_1\not{n} + \eta_2\not{n}\gamma_5)\times(n\cdot\partial)^2)$, and after neglecting terms of order $M_P^{-2}$ we obtain, by using

$$\{\not{\partial}, \not{n}\} = 2(n\cdot\partial), \tag{5}$$

$$\{\not{\partial}, \not{n}\gamma_5\} = [\not{\partial}, \not{n}]\gamma_5 = -2i\gamma_5\sigma^{\mu\nu}n_\nu\partial_\mu, \tag{6}$$

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[1] Notice however, that in a model where a scalar field is coupled to gravity, Lorentz invariance may hold, at least at first nontrivial order in perturbation theory of the noncommutative parameter [6].

 

that

$$(\Box + m^2)\psi = \frac{2i}{M_P}(\eta_1 (n\cdot\partial)^3 - i\eta_2\gamma_5\sigma^{\mu\nu}n_\nu\partial_\mu(n\cdot\partial)^2)\psi. \tag{7}$$

Finally, in the frame where $n^\mu = (1, 0, 0, 0)$, we find the dispersion relation

$$E^2 - |\vec{p}|^2 - m^2 - \frac{2}{M_P}(\eta_1 E^3 + i\eta_2\gamma_5\sigma^{0\mu}p_\mu E^2) = 0. \tag{8}$$

Thus, we conclude that the terms in $\eta_1$ and $\eta_2$ yield two different cubic modifications in the momentum operator of the fermionic dispersion relation. The first one is similar to the one of Ref. [10], while the second is a new term identified here for the first time.

## III. THE NON-RELATIVISTIC LIMIT AND THE $^9$Be$^+$ ION ENERGY SPECTRUM

Let us now determine how the Lorentz-violating terms in Eq. (4) affect the equations of motion in the nonrelativistic limit. For this, we can write the four component spinor $\psi$ in the form

$$\psi = \begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix}, \tag{9}$$

where $\hat{\varphi}$ and $\hat{\chi}$ are two component spinors. Equation (4) can be, thus, written as a system of two equations:

$$i\partial_0\hat{\varphi} + i(\vec{\sigma}\cdot\vec{\nabla})\hat{\chi} - m\hat{\varphi} = -\frac{1}{M_P}[A(n\cdot\partial)^2\hat{\varphi} + B(n\cdot\partial)^2\hat{\chi}], \tag{10}$$

$$i\partial_0\hat{\chi} + i(\vec{\sigma}\cdot\vec{\nabla})\hat{\varphi} + m\hat{\chi} = -\frac{1}{M_P}[A(n\cdot\partial)^2\hat{\chi} + B(n\cdot\partial)^2\hat{\varphi}], \tag{11}$$

where $A \equiv \eta_1 n_0 - \eta_2(\vec{n}\cdot\vec{\sigma})$ and $B \equiv \eta_2 n_0 - \eta_1(\vec{n}\cdot\vec{\sigma})$. In the low-energy limit, $E - m \ll m$, and we can separate the slowly and the rapidly time-varying parts of spinors $\hat{\varphi}$ and $\hat{\chi}$ in the following way:

$$\begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix} = e^{-\mathrm{imt}}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{12}$$

Hence, Eqs. (10) and (11) become:

$$i\partial_0\varphi + i(\vec{\sigma}\cdot\vec{\nabla})\chi = -\frac{1}{M_P}[A(F\varphi) + B(F\chi)], \tag{13}$$

$$i\partial_0\chi + i(\vec{\sigma}\cdot\vec{\nabla})\varphi + 2m\chi = -\frac{1}{M_P}[A(F\chi) + B(F\varphi)], \tag{14}$$

where the operator $F$ is given by

$$F = n_0^2(\partial_0^2 - 2im\partial_0 - m^2) + 2n_0(-im + \partial_0)(\vec{n}\cdot\vec{\nabla}) + (\vec{n}\cdot\vec{\nabla})^2. \tag{15}$$

As we are looking for the leading order terms for Lorentz violation in the nonrelativistic limit, we can neglect terms of order $M_P^{-1}$ in Eq. (14) in order to obtain a zeroth-order relation between the spinors $\varphi$ and $\chi$. As $\chi$ varies slowly in time, we can also neglect its time derivative, and so

$$\chi \approx \frac{-i(\vec{\sigma}\cdot\vec{\nabla})}{2m}\varphi = \frac{(\vec{\sigma}\cdot\vec{p})}{2m}\varphi \ll \varphi. \tag{16}$$

Substituting this result into Eq. (13) and neglecting terms of order $m/M_P$ and $m^2/M_P$, as well as those terms which include time derivatives of the spinors that are suppressed by the Planck mass $M_P$, we obtain

$$i\partial_0\varphi = \frac{1}{2m}\nabla^2\varphi - \frac{1}{M_P}\left[A(\vec{n}\cdot\vec{\nabla})^2 - \frac{i}{2m}B(\vec{n}\cdot\vec{\nabla})^2(\vec{\sigma}\cdot\vec{\nabla})\right]\varphi. \tag{17}$$

We have then found the two leading order terms that modify the kynetic term of the Schrödinger equation for the positive energy spinor $\varphi$. In general, these terms will modify the Hamiltonian for a system of $N$ particles through a Lorentz-violating potential given by:

$$\hat{V} = -\frac{1}{M_P}\sum_{k=1}^{N}\left[(\eta_1 n_0 - \eta_2(\vec{n}\cdot\vec{\sigma}))(\vec{n}\cdot\vec{\nabla}_k)^2 - \frac{i}{2m_k}(\eta_2 n_0 - \eta_1(\vec{n}\cdot\vec{\sigma}))(\vec{n}\cdot\vec{\nabla}_k)^2(\vec{\sigma}\cdot\vec{\nabla}_k)\right] \tag{18}$$

where $\vec{\nabla}_k \equiv \partial/\partial\vec{r}_k$, and $\vec{r}_k$, $k = 1, \ldots, N$, is the position vector of the *k-th* particle with mass $m_k$, respectively.

In a 1989 paper, Steven Weinberg proposed the use of a hyperfine transition in the ground state of the $^9$B$e^+$ ion to test a nonlinear generalization of quantum mechanics [12]. Although we are looking for the effects of linear Lorentz-violating operators in the Schrödinger equation, Weinberg's method can be easily adapted to our purposes.

Consider a system in a coherent superposition of two quantum states, $\psi_1$ and $\psi_2$, whose energy eigenvalues in the absence of Lorentz violation are $E_1$ and $E_2$, respectively. This system is described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{V}$ can be treated as a perturbative potential compared to the system's Lorentz invariant Hamiltonian $\hat{H}_0$, as we expect the effects of the Lorentz invariance violation to be small at this energy scale. To first order in perturbation theory, the Schrödinger time-dependent equation for state $\psi_k$, $k = 1, 2$, takes the form

$$i\hbar\frac{\partial\psi_k}{\partial t} = (E_k + \langle\hat{V}\rangle_k)\psi_k = \hbar\omega_k\psi_k, \tag{19}$$

where $\langle\hat{V}\rangle_k \equiv \langle\psi_k|\hat{V}|\psi_k\rangle$, and has the general solution $\psi_k = c_k e^{-i\omega_k t}$.

The constants $c_k$ can be parametrized as $c_1 = \sin(\frac{\theta}{2})$ and $c_2 = \cos(\frac{\theta}{2})$ [11]. The relative phase of the two states, correspondent to the time dependence of $\psi_2^\dagger\psi_1$, is given by

$$\omega_p \equiv \omega_1 - \omega_2 = \omega_0 + \frac{\langle \hat{V} \rangle_1 - \langle \hat{V} \rangle_2}{\hbar}, \tag{20}$$

where $\omega_0 \equiv (E_1 - E_2)/\hbar$ is the frequency of the transition between the unperturbed states. The perturbative terms will, thus, depend on the parameter $\theta$ and, hence, measuring the $\theta$ dependence of $\omega_p$ allows for determining the effects of the Lorentz invariance violation on the system.

A two level system is mathematically equivalent to a spin $1/2$ system which undergoes precession about an external uniform magnetic field, with $\theta$ being the angle between the spin and magnetic field vectors and $\omega_p$ the precession frequency. Bollinger *et al.* have used this idea to search for a $\theta$ dependence of the precession frequency of the hyperfine transition $|m_I, m_J\rangle = |-\frac{1}{2}, \frac{1}{2}\rangle \rightarrow |-\frac{3}{2}, \frac{1}{2}\rangle$ in the ground state of the $^9\mathrm{B}e^+$ ion [11].

In their discussion it has been assumed that the $^9\mathrm{B}e^+$ nuclear spin was decoupled from the valence electron's spin, so that $\psi_1 \equiv |-\frac{3}{2}, \frac{1}{2}\rangle$ and $\psi_2 \equiv |-\frac{1}{2}, \frac{1}{2}\rangle$ are pure $|m_I, m_J\rangle$ states. With this hypothesis, they obtained the upper bound

$$\left|\frac{\omega_p(\theta_B) - \omega_p(\theta_A)}{2\pi}\right| \leq 12.1 \mu\mathrm{Hz} \tag{21}$$

for $\theta_A = 1.02$ rad and $\theta_B = 2.12$ rad.

To determine how the breaking of the Lorentz symmetry produces a $\theta$ dependence in $\omega_p$, we have to compute the expectation value of the perturbative potential on states $\psi_1$ and $\psi_2$. We first point out that

$$(\vec{n} \cdot \vec{\nabla})^2 (\vec{\sigma} \cdot \vec{\nabla}) = -i n^i n^j \sigma^k p^i p^j p^k, \tag{22}$$

where $p^i$ is the *i-th* component of the vector momentum. As, for bound states like $\psi_1$ and $\psi_2$, any odd power of the momentum operator has a zero expectation value, the term in B will not affect the perturbative potential's expectation value [13].

The $^9\mathrm{B}e^+$ ion is a system composed by three electrons, two of which in a closed $1s$ shell, and a nucleus with five neutrons and four protons. As, in the considered transition, $\Delta m_J = 0$, we expect the perturbative potential to alter both states energy eigenvalues in the same way, not affecting the transition frequency. In the ion's nucleus, the *pairing interaction* induces nucleons to group up into pairs of neutrons and pairs of protons with zero angular momentum [14]. Hence, the ion's nuclear spin is entirely carried by one of its neutrons.

In this way, $\psi_1$ and $\psi_2$ can be treated as states of a particle with spin $I = 3/2$ and projections on the quantization axis, which is usually defined as the external magnetic field's direction, $m_I = -3/2$ and $m_I = -1/2$, respectively. If $\hat{e}_3$ defines the direction of the quantization axis,

$$\langle I, m_I | \sigma^k | I, m_I \rangle = 2 m_I \delta_{k3}, \tag{23}$$

and therefore

$$\langle \hat{V} \rangle_1 = \frac{|c_1|^2}{M_P} [\eta_1 n_0 + 3\eta_2 n^z] n^i n^j \langle p^i p^j \rangle_1, \tag{24}$$

$$\langle \hat{V} \rangle_2 = \frac{|c_2|^2}{M_P} [\eta_1 n_0 + \eta_2 n^z] n^i n^j \langle p^i p^j \rangle_2. \tag{25}$$

Hence, we find (inserting back the missing $h$ factors)

$$\omega_p(\theta) = \omega_0 - \frac{n^i n^j \langle p^i p^j \rangle}{M_P \hbar} [\eta_1 n_0 (\cos^2(\theta/2) - \sin^2(\theta/2)) + \eta_2 n^z (\cos^2(\theta/2) - 3\sin^2(\theta/2))], \tag{26}$$

where we have assumed that $\langle p^i p^j \rangle \equiv \langle p^i p^j \rangle_1 \approx \langle p^i p^j \rangle_2$. Finally, we obtain

$$\frac{\omega_p(\theta_B) - \omega_p(\theta_A)}{2\pi} = \frac{n^i n^j \langle p^i p^j \rangle}{h M_P} [a \eta_1 n_0 + b \eta_2 n^z], \tag{27}$$

where the constants $a$ and $b$ are defined as

$$a \equiv \cos(\theta_A) - \cos(\theta_B) \simeq 1.045, \tag{28}$$

$$b \equiv -\cos^2\left(\frac{\theta_B}{2}\right) + 3\sin^2\left(\frac{\theta_B}{2}\right) + \cos^2\left(\frac{\theta_A}{2}\right) - 3\sin^2\left(\frac{\theta_A}{2}\right) \simeq 2.091. \tag{29}$$

As for a neutron, $\langle p^2 \rangle / m_n^2 \sim 10^{-2}$ [13], and assuming that the Lorentz symmetry breaking does not privilege any spatial direction, $n^x = n^y = n^z \equiv n$, we obtain:

$$\frac{n^i n^j \langle p^i p^j \rangle}{h M_P} \sim \frac{9 n^2 \langle p^2 \rangle}{h M_P} \sim (2 \times 10^3) n^2 \text{ Hz}. \tag{30}$$

## IV. RESULTS

As presently there is no way of determining the form of the background four-vector $n^\mu$, we can only estimate bounds on the values of the parameters $\eta_1$ and $\eta_2$.

First, we consider the case where $n^\mu$ is a timelike four-vector in some cosmic frame ($n \cdot n = 1$). Thus, in the laboratory frame, $n_0 \sim 1$ and the typical size of the spatial components will be of order $n \sim 10^{-3}$ due to the relative motion of our galaxy, the Solar System and the Earth [10,15]. Hence,

$$\frac{\omega_p(\theta_B) - \omega_p(\theta_A)}{2\pi} \simeq (2 \times 10^{-3} \eta_1 + 4.5 \times 10^{-6} \eta_2) \text{ Hz}. \tag{31}$$

Using Bollinger *et al.* result Eq. (21), we obtain the following upper bounds for the Lorentz-violating parameters:

$$|\eta_1| \lesssim 6 \times 10^{-3}, \qquad |\eta_2| \lesssim 3, \tag{32}$$

where we have assumed $\eta_1(\eta_2) = 0$ to obtain a bound for $\eta_2(\eta_1)$.

If $n^\mu$ is spacelike in some cosmic frame ($n \cdot n$= -1), we will have, in the laboratory frame, $n_0 \sim 10^{-3}$ and $n \sim \sqrt{3}/3$. Thus,

$$\frac{\omega_p(\theta_B) - \omega_p(\theta_A)}{2\pi} \simeq (0.76\eta_1 + 8.7 \times 10^2 \eta_2)\ \mathrm{Hz}, \quad (33)$$

and, in this case, we obtain the upper bounds

$$|\eta_1| \lesssim 2 \times 10^{-5}, \qquad |\eta_2| \lesssim 1 \times 10^{-8}. \quad (34)$$

Finally, considering the case where $n^\mu$ is a lightlike four-vector in the laboratory frame ($n \cdot n = 0$), with $n_0 \sim 1$ and $n \sim \sqrt{3}/3$, we get

$$\frac{\omega_p(\theta_B) - \omega_p(\theta_A)}{2\pi} \sim (7.5 \times 10^2 \eta_1 + 8.7 \times 10^2 \eta_2)\ \mathrm{Hz}, \quad (35)$$

and the correspondent upper bounds

$$|\eta_1| \lesssim 2 \times 10^{-8}, \qquad |\eta_2| \lesssim 1 \times 10^{-8}. \quad (36)$$

## V. CONCLUSIONS

In this paper, we have considered the introduction of cubic Lorentz-violating terms in the fermionic dispersion relation. We have concluded that the two possible Lorentz-violating parameters yield different terms in the fermionic dispersion relation, both cubic in the momentum operator components. In the nonrelativistic limit, we have found the two leading order terms altering the equations of motion for fermions and determined the effect of these terms in the $^9Be^+$ ion's energy spectrum. Using the method developed by Weinberg and the experimental result of Bollinger *et al.*, we have obtained new bounds on the value of the parameters $\eta_1$ and $\eta_2$ for neutrons. We have determined $|\eta_1| \lesssim 6 \times 10^{-3}$ and $|\eta_2| \lesssim 3$ for a timelike background Lorentz-violating four-vector, $|\eta_1| \lesssim 2 \times 10^{-5}$ and $|\eta_2| \lesssim 1 \times 10^{-8}$ for a spacelike four-vector, and $|\eta_1| \lesssim 2 \times 10^{-8}$ and $|\eta_2| \lesssim 1 \times 10^{-8}$ for a lightlike four-vector.

The values of the Lorentz-violating parameters $\eta_1$ and $\eta_2$ are, hence, highly dependent on the form of the background four-vector, particularly on its spatial components. Bollinger *et al.* experimental results are consistent with high values for these parameters, especially $|\eta_2|$, in the case where the spatial components of $n^\mu$ have small values in the laboratory frame, $n \sim 10^{-3}$ (a timelike background four-vector). On the other hand, this experiment yields quite strong constraints when $n \sim 1$ (a spacelike or lightlike background four-vector).

In general, $n^\mu$ may have different spatial components in the laboratory frame due to the motion of the Earth with respect to the cosmic frame where the background four-vector has a simple form. If some of these components are further suppressed, the upper bounds on the values of the Lorentz-violating parameters will be larger than the ones presented above.

In any case, it is somewhat striking that 15 yr-old experiments like the one considered in this paper can lead to relevant upper bounds for these parameters and shed some light on the physics of very high energy scales.

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