Renormalized broken-symmetry Schwinger-Dyson equations and the two-particle irreducible $1/N$ expansion for the $O(N)$ model

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We derive the renormalized Schwinger-Dyson equations for the one- and two-point functions in the auxiliary field formulation of Coleman, Jackiw, and Politzer [S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D **10**, 2491 (1974).] for $\lambda\phi^4$ field theory, to order $1/N$, in the 2PI-$1/N$ expansion. We show that the renormalization of the broken-symmetry theory depends only on the counter terms of the symmetric theory with $\phi = 0$, as discussed in our previous paper [F. Cooper, B. Mihaila, and J. F.Dawson, Phys. Rev. D **70**, 105008 (2004).]. We find that the 2PI-$1/N$ expansion violates the Goldstone theorem at order $1/N$. In using the $O(4)$ model as a low energy effective field theory of pions to study the time evolution of disoriented chiral condensates one has to *explicitly* break the $O(4)$ symmetry to give the physical pions a nonzero mass. In this effective theory, we expect that the additional small contribution to the pion mass due to the violation of the Goldstone theorem in the 2PI-$1/N$ equations to be unimportant for an adequate description of the phenomenology.

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I. INTRODUCTION

Lately there has been interest in using two-particle irreducible (2PI)-$1/N$ methods to investigate various aspects of quantum field theory [1,2]. In a previous work [3] we showed how to renormalize the Schwinger-Dyson (SD) equations for the symmetric phase of the $O(N)$ model in the auxiliary field formalism, to order $1/N$. This was done by first using the multiplicative renormalization approach [4] to find the exact renormalized SD equations, and then to realize that to leading order in $1/N$ one only needs to set the renormalized vertex function for $\phi\phi\chi$ to one ($\Gamma_R = 1$), in order to consistently truncate the infinite hierarchy of renormalized Green's functions. Here, χ is the auxiliary field related to ϕ^2.

In order to carry out dynamical simulations with nonzero values of $\langle\phi\rangle$, which occurs, for example, when chiral condensates are produced [5], it is important to extend that result to the case of broken symmetry, $\phi \neq 0$. In this paper we show that by extending the multiplicative renormalization scheme used for the symmetric phase, we can obtain finite renormalized SD equations for the broken-symmetry phase. Since the 2PI-$1/N$ approach is a resummation of the ordinary $1/N$ expansion, it is important to ask to what extent the Ward-Takahashi identities preserved in the original perturbative $1/N$ approach [4] are preserved in the 2PI-$1/N$ approach. One of these identities leads to the Goldstone theorem [6]. Goldstone's theorem states that if continuous symmetry is broken, and there is a residual symmetry, there should be massless particles in the theory corresponding to the number of symmetries left unbroken.

As has been previously pointed out for the 2PI-$1/N$, the Goldstone theorem is formally satisfied if one determines the masses from the inverse propagators derived from the one-particle irreducible (1PI) generating functional for the ϕ fields [2,7]. However one expects (and we find) that the inverse propagators obtained directly from the 2PI-$1/N$ generating functional for the would be massless particles do *not* vanish as p^2 as $p^2 \to 0$ in violation of Goldstone's theorem. What we explicitly find is that the condition for the spontaneous symmetry breakdown found from the renormalized equation for the expectation value of $\langle\phi_i\rangle$ leads to a mass for the would be Goldstone bosons at order $1/N$. The evolution equations obtained from the 2PI-$1/N$ effective action are energy preserving.

If we modify by hand these equations to enforce the Goldstone theorem, we would then violate energy conservation at order $1/N$. This violation of the Goldstone theorem can be more satisfactorily remedied by constructing an improved effective action functional as discussed in Van Hees and Knoll [7]. However, this then leads to a

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much more complicated set of equations which includes (in addition to solving for the one- and two-point function equations) solving simultaneously the Bethe-Salpeter equations for the vertex function. Given present computational power, this would be not feasible for $3+1$ dimensional calculations at this time. The improved effective action is not entirely satisfactory in that the propagators on internal lines still do not obey the Goldstone theorem.

One can hope, however, from a phenomenological point of view, that the violation of the Goldstone theorem by this approximation is not very serious. In making a realistic phenomenological model of pions using the $O(4)$ model one has to *explicitly* break the $O(4)$ symmetry if we want the pion to have the correct physical mass. This is done by introducing an external source coupling to the field with nonvanishing expectation value (i.e. the σ particle). One then determines the magnitude of this external source by using the partially conserved axial current equation (PCAC). This was discussed in a previous paper on disoriented chiral condensates [5]. As long as the mass generated by the breakdown of the Goldstone theorem is small compared to the mass generated by the *explicit* violation of the symmetry then this breakdown should not be important for phenomenological applications. Our renormalization procedure will lean heavily on our previous result for obtaining renormalized SD equations in the symmetric vacuum [3].

Before continuing with our approach we will discuss some previous approaches to Goldstone theory problems. Firstly, in a direct $1/N$ expansion to order $1/N^2$, Binoth *et al.* [8] have performed all renormalizations. They found no inconsistencies with the Goldstone theorem, and the residual $O(N-1)$ symmetry is preserved. This is a very comforting result, but as we have discussed previously [9], a direct $1/N$ expansion leads to secularity problems in the dynamics which is our main interest here. In an important paper, Arrizabalaga *et al.* [10] realized that one can avoid the problems with the Goldstone theorem discussed here, by breaking the symmetry and then taking the limit of zero symmetry breaking. This is valid if one is interested in $O(N)$-invariant initial conditions, but also having Goldstone particles. We will discuss this approach later. Finally, Ivanov *et al.* [11] have discussed how to preserve the Goldstone theorem in the simpler Hartree approximation, by adding terms to the 2PI generating functional which vanish when the symmetry is restored, but which explicitly enforce the Goldstone condition. This is a promising approach, which needs to be explored further.

II. THE $O(N)$ SCALAR FIELD THEORY

In the auxiliary field formulation of Coleman *et al.* [12], the $O(N)$ model can be described by an action written in terms of the auxiliary field χ

$$S[\phi_i,\chi]=\int \mathrm{d}^d x\Big\{-\frac{1}{2}\phi_i(x)[\Box+\chi(x)]\phi_i(x)+\frac{\chi^2(x)}{2g}+\frac{\mu^2}{g}\chi(x)\Big\}. \tag{2.1}$$

Here and in what follows we let $g=\lambda/N$. To treat the $N+1$ fields on equal footing we introduce the notation

$$\begin{aligned}\phi_a(x)&=[\chi(x),\phi_1(x),\phi_2(x),\ldots,\phi_N(x)],\\ j_a(x)&=[j_0(x),j_1(x),j_2(x),\ldots,j_N(x)],\end{aligned} \tag{2.2}$$

with $a=0,i=1,\ldots,N$. Using this notation, the complete action for the $O(N)$ model is given by:

$$S[\phi;j]=-\frac{1}{2}\int \mathrm{d}^d x\int \mathrm{d}^d x'\phi_a(x)\Delta^{-1}_{ab}(x,x')\phi_b(x')+\int \mathrm{d}^d x\Big\{-\frac{1}{6}\gamma_{abc}\phi_a(x)\phi_b(x)\phi_c(x)+\phi_a(x)j_a(x)\Big\}, \tag{2.3}$$

where $\Delta^{-1}_{ab}(x,x')=\Delta^{-1}_{ab}(x)\delta(x,x')$ with

$$\Delta^{-1}_{ab}(x)=\begin{pmatrix}-1/g & 0\\ 0 & \Box\delta_{ij}\end{pmatrix}, \tag{2.4}$$

and where $\gamma_{abc}=\delta_{a0}\delta_{ij}+$ cyclic permutations. Here we have put $j_0(x)=J(x)+\mu^2/g$. The coupling constant $g=\lambda/N$ is of order $1/N$. For the dynamics the integrals and delta functions $\delta_C(x,x')$ are defined on the closed time path (CTP) contour, which incorporates the initial value boundary condition [13].

The generating functional $Z[j]$ and connected Green's function generator $W[j]$ are defined by a path integral:

$$Z[j]=e^{\mathrm{i}W[j]}=\prod_{a=0}^{N}\int \mathrm{d}\phi_a e^{\mathrm{i}S[\phi;j]}. \tag{2.5}$$

We define one-point functions by:

$$\phi_a(x)=\frac{\delta W[j]}{\delta j_a(x)}, \tag{2.6}$$

which satisfy the equations:

$$\Delta^{-1}_{ab}(x)\phi_b(x)+\frac{1}{2}\gamma_{abc}\{\phi_b(x)\phi_c(x)+G_{bc}(x,x)/\mathrm{i}\}=j_a(x), \tag{2.7}$$

where $G_{ab}(x,x')$ is the two-point Green's function, defined by:

$$G_{ab}(x,x')=\frac{\delta\phi_a(x)}{\delta j_b(x')}=\frac{\delta^2 W[j]}{\delta j_a(x)\delta j_b(x')}=\begin{pmatrix}D(x,x') & K_j(x,x')\\ \bar{K}_i(x,x') & G_{ij}(x,x')\end{pmatrix}. \tag{2.8}$$

We also define the generating functional $\Gamma[\phi]$ of 1PI vertices by a Legendre transformation:

$$\Gamma[\phi] = W[j] - \int \mathrm{d}^d x \phi_a(x) j_a(x), \tag{2.9}$$

and one-point vertex functions by:

$$\Gamma^{(1)}_a(x) = -\frac{\delta\Gamma[\phi]}{\delta\phi_a(x)} = j_a(x), \tag{2.10}$$

so that from (2.7), we have:

$$\Gamma^{(1)}_a(x) = \Delta^{-1}_{ab}(x)\phi_b(x) + \frac{1}{2}\gamma_{abc}\{\phi_b(x)\phi_c(x) + G_{bc}(x,x)/i\}, \tag{2.11}$$

which gives the familiar equations of motion

$$\begin{aligned}&[\Box + \chi(x)]\phi_i(x) + K_i(x,x)/i = j_i(x),\\ &\chi(x) = -\mu^2 - g j_0(x) + \frac{g}{2}\sum_i [\phi_i^2(x) + G_{ii}(x,x)/\mathrm{i}].\end{aligned} \tag{2.12}$$

The two-point vertex functions are defined by:

$$\Gamma^{(2)}_{ab}(x,x') = -\frac{\delta^2\Gamma[\phi]}{\delta\phi_a(x)\delta\phi_b(x')} = \frac{\delta j_a(x)}{\delta\phi_b(x')}, \tag{2.13}$$

so that by differentiating (2.11), we find:

$$\Gamma^{(2)}_{ab}(x,x') = G^{-1}_{0ab}(x,x') + \Sigma_{ab}(x,x'), \tag{2.14}$$

where

$$\begin{aligned}G^{-1}_{0ab}(x,x') &= [\Delta^{-1}_{ab}(x) + \gamma_{abc}\phi_c(x)]\delta(x,x')\\ &= \begin{pmatrix} D_0^{-1}(x,x') & \bar{K}^{-1}_{0j}(x,x') \\ K^{-1}_{0i}(x,x') & G^{-1}_{0ij}(x,x')\end{pmatrix},\end{aligned} \tag{2.15}$$

with

$$\begin{aligned}D_0^{-1}(x,x') &= -g\delta(x,x'),\\ G^{-1}_{0ij}[\chi](x,x') &= [\Box + \chi(x)]\delta_{ij}\delta(x,x'),\\ K^{-1}_{0i}[\phi](x,x') &= \bar{K}^{-1}_{0i}[\phi](x,x') = \phi_i(x)\delta(x,x').\end{aligned}$$

and

$$\Sigma_{ab}(x,x') = \frac{1}{2i}\gamma_{abc}\frac{\delta G_{bc}(x,x)}{\delta\phi_b(x')} = \begin{pmatrix}\Pi(x,x') & \Omega_j(x,x') \\ \bar{\Omega}_i(x,x') & \Sigma_{ij}(x,x')\end{pmatrix}. \tag{2.16}$$

The two-point vertex and Green's functions are inverses of each other:

$$\int \mathrm{d}^d x' \Gamma^{(2)}_{ab}(x,x') G_{bc}(x',x'') = \delta_{ac}\delta(x,x''), \tag{2.17}$$

from which we find schematically that

$$\frac{\delta G_{ab}}{\delta\phi_c} = -G_{aa_1}G_{bb_1}\Gamma^{(3)}_{a_1,b_1,c}, \tag{2.18}$$

where

$$\Gamma^{(3)}_{abc}(x,x',x'') = -\frac{\delta^3\Gamma[\phi]}{\delta\phi_a(x)\delta\phi_b(x')\delta\phi_c(x'')} \tag{2.19}$$

is the three-point vertex function. So the self-energy $\Sigma_{ab}(x,x')$ can be written as:

$$\Sigma_{ab}(x,x') = \frac{\mathrm{i}}{2}\gamma_{aa_1b_1}G_{a_1a_2}G_{b_1b_2}\Gamma_{a_2,b_2,b}. \tag{2.20}$$

Differentiating Eq. (2.14) again with respect to $\phi_c(x'')$ gives an equation for the three-point vertex function:

$$\Gamma^{(3)}_{abc}(x,x',x") = \gamma_{abc}\delta(x,x')\delta(x,x") + \Delta\Gamma^{(3)}_{abc}(x,x',x"), \tag{2.21}$$

where

$$\Delta\Gamma^{(3)}_{abc}(x,x',x'') = \frac{\delta\Sigma_{ab}(x,x')}{\delta\phi_c(x'')} \sim O(1/N). \tag{2.22}$$

For the purpose of renormalization it is useful to think of Eq. (2.21) as an identity

$$\gamma = \Gamma - \Delta\Gamma \equiv \bar{\Gamma}, \tag{2.23}$$

since we have showed that both Γ and $\Delta\Gamma$ renormalize the same way in our previous paper [3].

For the exact equations it is convenient to introduce the notations

$$\Gamma^{(2)}_{ab}(x,x') = \begin{pmatrix} D_2^-1(x,x') & \Xi_j(x,x') \\ \bar{\Xi}_i(x,x') & G^{-1}_{2,ij}(x,x')\end{pmatrix}, \tag{2.24}$$

such that

$$\begin{aligned}\Gamma^{(2)}_{00} &\equiv D_2^{-1}(x,x') = D_0^{-1}(x,x') + \Pi(x,x'),\\ \Gamma^{(2)}_{ij} &\equiv G^{-1}_{2,ij}(x,x') = G^{-1}_{0,ij}(x,x') + \Sigma_{ij}(x,x'),\\ \Gamma^{(2)}_{0j} &\equiv \Xi_j(x,x') = K^{-1}_{0,j}(x,x') + \Omega_j(x,x'),\\ \Gamma^{(2)}_{i0} &\equiv \bar{\Xi}_i(x,x') = K^{-1}_{0,i}(x,x') + \bar{\Omega}_i(x,x').\end{aligned}$$

In the homogeneous vacuum we can invert these equations in momentum space to obtain schematically

$$\begin{aligned}&D_2^{-1}D + \Xi_m\bar{K}_m = \delta_C, \qquad \bar{\Xi}_i D + G^{-1}_{2,im}\bar{K}_m = 0,\\ &D_2^{-1}K_j + \Xi_m G_{mj} = 0, \qquad \bar{\Xi}_i K_j + G^{-1}_{2,im}G_{mj} = \delta_{ij}\delta_C.\end{aligned} \tag{2.26}$$

We find:

$$K_i = \bar{K}_i = -D_2\Xi_m G_{mi} = -G_{2,im}\bar{\Xi}_m D, \tag{2.27}$$

$$D = -g + g\bar{\Pi}D, \tag{2.28}$$

$$G_{ij} = G_0\delta_{ij} - G_0\bar{\Sigma}_{in}G_{nj}, \tag{2.29}$$

where we have introduced the notations

$$\bar{\Sigma}_{ij} = \Sigma_{ij} - \bar{\Xi}_i D_2 \Xi_j, \tag{2.30}$$

$$\bar{\Pi} = \Pi - \Xi_m G_{2,mn}\bar{\Xi}_n. \tag{2.31}$$

The above equations are, in principle, exact. In practice, however, the exact SD hierarchy of equations needs to be truncated. Two approximation schemes have been developed in the past few years: the bare vertex approximation

(BVA) [14], where the resulting dynamics is based on ignoring vertex corrections (i.e. $\Gamma \equiv \gamma$), and the 2PI-$1/N$ expansion [1], where one further ignores terms of order $1/N^2$.

In this paper, it is useful to define renormalization at $p^2 = 0$, for the vacuum sector. As shown in our previous paper [3], $\Sigma(p^2)$ is quadratically divergent and requires two subtractions. Expanding about $p^2 = 0$, we have

$$\Sigma(p^2) = \Sigma(0) + \Sigma_1 p^2 + \Sigma^{[\mathrm{sub2}]}(p^2), \tag{2.32}$$

where $\Sigma_1 = \frac{d\Sigma}{dp^2}|_{p^2=0}$, and $\Sigma^{[\mathrm{sub2}]} \propto p^4$ as $p^2 \to 0$. Then, the wave function renormalization constant is introduced as

$$Z_2^{-1} = -\frac{dG^{-1}(p^2)}{dp^2}\bigg|_{p^2=0} = 1 - \Sigma_1. \tag{2.33}$$

The vacuum renormalized mass parameter is defined as

$$M^2 = Z_2[\chi + \Sigma(0)]. \tag{2.34}$$

The vertex renormalization constant Z_1 is equal to Z_2 by a Ward-like identity and is defined by

$$Z_1^{-1} = \Gamma(p, p)|_{p^2=0} = 1 + \frac{\partial\Sigma(p^2)}{\partial\chi}\bigg|_{p^2=0}, \tag{2.35}$$

and $\Gamma_R(p, q) = Z_1\Gamma(p, q)$. We have shown in the vacuum sector, that

$$G_R^{-1}(p^2) = Z_2 G^{-1}(p^2) = p^2 + M^2 + \Sigma_R^{[\mathrm{sub2}]}(p^2), \tag{2.36}$$

where $\Sigma_R^{[\mathrm{sub2}]}(p^2)$ is explicitly finite and written only in terms of renormalized Green's functions and renormalized vertex functions.

Also, in 3 + 1 dimensions, coupling constant renormalization is needed. Since the renormalized coupling constant g_R is the negative of the inverse χ propagator at $p^2 = 0$, and is a renormalization group invariant, one can obtain a finite equation for D^{-1} with the following single subtraction

$$D^{-1}(p^2) = -\frac{1}{g_r} + \Pi^{[\mathrm{sub1}]}(p^2), \tag{2.37}$$

with

$$\Pi^{[\mathrm{sub1}]}(p^2) = \Pi(p^2) - \Pi(0). \tag{2.38}$$

What we showed in our previous paper [3] is that one can write:

$$\Gamma_R(p, q) = 1 + \Delta\Gamma_R^{[sub1]}(p, q), \tag{2.39}$$

where the second term is finite, renormalized, and of order $1/N$.

III. 2PI-$1/N$ EXPANSION

Next we want to compare these exact results with the SD equations coming from the 2PI-$1/N$ approximation. Now we have that the generating functional is given by:

$$\Gamma[\phi_a, G] = S_{\mathrm{cl}}[\phi_a] + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln[G^{-1}] + \frac{\mathrm{i}}{2}\mathrm{Tr}[G_0^{-1}G] + \Gamma_2[G], \tag{3.1}$$

where $\Gamma_2[G]$ is the generating functional of the 2PI graphs [15], and $S_{\mathrm{cl}}[\phi_a]$ is the classical action in Minkowski space. The approximations we are studying include only the two-loop contributions to Γ_2 (see Fig. 1).

The exact equations following from the effective action Eq. (3.1), are the same as Eqs. (2.12) and (2.14) listed above, with the Green's function $G_{0ab}^{-1}[\phi](x, x')$ defined as

$$G_{0ab}^{-1}[\phi](x, x') = -\frac{\delta^2 S_{\mathrm{cl}}}{\delta\phi_a(x)\delta\phi_b(x')}, \tag{3.2}$$

and the self-energy

$$\Sigma_{ab}(x, x') = \frac{2}{\mathrm{i}}\frac{\delta\Gamma_2[G]}{\delta G_{ab}(x, x')}. \tag{3.3}$$

In the 2PI-$1/N$ we keep in $\Gamma_2[G]$ only the first of the two graphs shown in Fig. 1, which is explicitly

$$\Gamma_2[G] = -\frac{1}{4}\iint \mathrm{d}^d x \mathrm{d}^d y G_{ij}(x, y)G_{ji}(y, x)D(x, y). \tag{3.4}$$

The self-energy, given in Eq. (3.3), then reduces to:

$$\begin{aligned} &\Pi(x, x') = \frac{\mathrm{i}}{2}G_{mn}(x, x')G_{mn}(x, x'), \quad \Omega_i(x, x') = 0, \\ &\bar{\Omega}_i(x, x') = 0, \quad \Sigma_{ij}(x, x') = \mathrm{i}G_{ij}(x, x')D(x, x'). \end{aligned} \tag{3.5}$$

In the homogeneous case we will use the $O(N)$ symmetry to choose the symmetry-breaking direction to be in the direction N. In that case only $\langle\phi_N\rangle \neq 0$. This means that G_{ij} is diagonal in general, and only the fields χ and $\phi_N \equiv \sigma$ mix. To determine the σ mass one has to just diagonalize a 2×2 matrix. There is no mixing between χ and the ϕ_i where $i < N$. Thus K_N is the only nonzero entry to the mixed propagator K_i and $\Xi_j(x, x') = \delta_{jN}\phi_N\delta(x, x')$. Let us look at the momentum space equations. The integral equations for D and G_{ij} are now

$$D = -g + g\bar{\Pi}D, \tag{3.6}$$

$$G_{ij} = G_0\delta_{ij} - G_0\bar{\Sigma}_{in}G_{nj}, \tag{3.7}$$

with

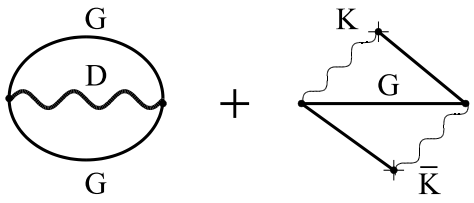

FIG. 1. Graphs included in the 2PI effective action $\Gamma_2[G]$.

$$\bar{\Sigma}_{ij} = \Sigma_{ij} - \delta_{iN}\phi_N \bar{D}_2 \delta_{jN}\phi_N, \tag{3.8}$$

$$\bar{\Pi} = \Pi - \delta_{mN}\phi_N G_{2,mn}\delta_{nN}\phi_N. \tag{3.9}$$

Iterating the equation for G shows it is diagonal and only G_{NN} is different from G_{2NN}. For the self-energy only the NN component is modified from the unbroken case. We also have that

$$D^{-1} = D_2^{-1} - \phi_N G_{2,NN}\phi_N. \tag{3.10}$$

This difference will be important when we discuss Goldstone's theorem. It follows immediately that, in momentum space, we have

$$D^{-1}(p^2) = -\frac{1}{g} + \bar{\Pi}(p^2), \tag{3.11}$$

$$G_{ij}^{-1}(p^2) = (-p^2 + \chi)\delta_{ij} + \bar{\Sigma}_{ij}(p^2), \tag{3.12}$$

and

$$K_i(p^2) = \bar{K}_i(p^2) = -\delta_{iN}\phi_N D_2 G_{NN}. \tag{3.13}$$

Now the correction to $\Pi(p^2)$ goes like $1/p^2$ so this is irrelevant at high momentum. The correction to $\Sigma(p^2)$ goes like $1/\ln p^2$ so this is also negligible compared to $\ln p^2$. This implies that the renormalizability is not changed by symmetry breaking. Also, since G and ϕ^2 renormalize the same way the multiplicative renormalization does not change. Introducing the notations $\phi_N = \sigma$ and $\phi_{i(i\neq N)} = \pi_i$ and letting $G_{ij} = G\delta_{ij}$ we have that the inverse propagator for the π mesons is

$$G_{\pi\pi}^{-1}(p^2) = -p^2 + \chi + \Sigma(p^2), \tag{3.14}$$

and for the σ meson we have instead

$$G_{\sigma\sigma}^{-1}(p^2) = -p^2 + \chi + \Sigma(p^2) - |\phi|^2 D_2(p^2). \tag{3.15}$$

IV. GOLDSTONE THEOREM

The one-point function equation in an external source is

$$[\Box + \chi(x)]\phi_i(x) + K_i(x,x)/i = j_i(x). \tag{4.1}$$

This is to be interpreted as

$$G_{3ij}^{-1}(x,x')\phi_j(x') = j_i(x), \tag{4.2}$$

where

$$G_{3ij}^{-1}(x,x') = [\Box + \chi(x)]\delta_{ij}\delta(x-x') + \Sigma_{3ij}(x,x'). \tag{4.3}$$

Now since

$$K_i(x,x') = -D_2(x,x')\phi_j(x')G_{ji}(x,x'), \tag{4.4}$$

we see that

$$\Sigma_{3ij}(x,x') = \mathrm{i}G_{ij}(x,x')D_2(x,x'). \tag{4.5}$$

Thus apart from $D \to D_2$, this is exactly the self-energy $\Sigma_{ij}(x,x')$. Thus G_3 is made finite by exactly the same 2 subtractions of wave function renormalization and mass renormalization as the full G. The renormalized one-point function equation is then

$$G_{3R}^{-1}(x,x')\phi_R(x') = 0. \tag{4.6}$$

In momentum space, in the vacuum, we have

$$G_{3R}^{-1}(p^2) = -p^2 + M_3^2(0) + \Sigma_{3R}^{\mathrm{sub2}}(p^2), \tag{4.7}$$

where the self-energy is subtracted twice at $p^2 = 0$. We also have

$$M_3^2 = Z_2[\chi + \Sigma_3(0)]. \tag{4.8}$$

The condition for broken symmetry is that

$$(\chi\delta_{ij} + \Sigma_{3ij})\phi_i = 0. \tag{4.9}$$

Choosing the direction of the expectation value $\langle\vec{\phi}\rangle$ to define the $i = N$ direction we have

$$\chi + \Sigma_{3NN} = 0, \tag{4.10}$$

for spontaneous symmetry breakdown. We need to ask whether this insures N-1 Goldstone bosons (see also previous discussions on this topic in [2,7,16,17]).

Now the N-1 would be Goldstone bosons come from the inverse propagator $G_{\pi\pi}^{-1}$, which after renormalization at $p^2 = 0$ gives

$$G_R^{-1}(p^2) = -p^2 + M^2(0) + \Sigma_R^{\mathrm{sub2}}(p^2), \tag{4.11}$$

with Σ_R^{sub2} (the twice subtracted at $p^2 = 0$ renormalized self-energy) proportional to p^4 at small p^2. The condition for a Goldstone theorem is that $G_R^{-1} = ap^2$ for small p^2 so that there is a zero mass pole in the propagator. This requires

$$M^2(0) = Z_2[\chi + \Sigma(0)] = 0. \tag{4.12}$$

However the condition for broken symmetry is that

$$M_3^2 = Z_2[\chi + \Sigma_3(0)] = 0. \tag{4.13}$$

The difference between Σ and Σ_3 is of order $1/N$, and is proportional to $\langle\phi\rangle^2$.

If we want to preserve the Goldstone theorem in our dynamical simulations we could use D_2 and not the full D in our update equations for the self-energy. However this would then violate energy conservation (by terms of order $1/N$) previously guaranteed by the use of the effective action. Note, however, that if were only interested in $O(4)$ symmetric initial condition, but having Goldstone particles, the strategy of Arizabalaga *et al.* works perfectly. By first choosing $\langle\phi\rangle$ small, but not zero, Eq. (4.10) must be satisfied. Taking the limit $\langle\phi\rangle$ goes to zero later, the difference between Σ and Σ_3 vanishes, and we have no conceptual problem. The difficulty only arises when one is interested in non $O(N)$ symmetric initial conditions for the expectation value of ϕ.

In leading order $1/N$, the self-energy Σ is zero and the condition for symmetry breakdown is then $\chi = 0$ which automatically leads to N-1 Goldstone particles, and there is

no problem with the Goldstone theorem. (This fact has been verified to order $1/N^2$ in a direct $1/N$ expansion by Binoth *et al.* [8].) As for the mass of the σ meson one has that

$$m_\sigma^2(0) - m_\pi^2(0) = -Z_2\phi^2 D_2(0) \equiv g_R\phi_R^2/2. \quad (4.14)$$

This is the renormalized version of what happens in the classical theory. To make a realistic model of pions, one has to *explicitly break* the $O(4)$ symmetry by setting $j_0 = H$ as in Ref. [5]. Doing this the quantum field equation for the σ field becomes

$$[\Box + \chi]\sigma = H. \quad (4.15)$$

Therefore, H is renormalized the same way as σ, and the renormalized PCAC equation coming from

$$A_\mu^i = \pi^i\partial_\mu\sigma - \sigma\partial_\mu\pi^i, \qquad (i = 1\ldots N-1), \quad (4.16)$$

becomes

$$\partial^\mu A_{R\mu}^i(x) = H_R\pi_R^i(x), \quad (4.17)$$

with $H_R = f_\pi m_\pi^2$. As long as the pion mass generated from the breakdown of the Goldstone theorem is small compared to the mass coming from the explicit symmetry breakdown, the violation of the Goldstone theorem by this approximation will not be important in dynamical simulations of an effective theory of disoriented chiral condensates.

One way to ''solve'' the Goldstone problem is to introduce an ''improved'' action [2,7]

$$\Gamma^*[\phi] = \Gamma[\phi, \chi[\phi], G[\phi]]. \quad (4.18)$$

The second derivative of this action is guaranteed to satisfy the Goldstone theorem by construction. Because of the $O(N)$ symmetry Γ^* is only a function of $\phi\cdot\phi \equiv \phi^2$. Thus the condition for a minimum is

$$\frac{\partial\Gamma^*}{\partial\phi_i} = 2\frac{\partial\Gamma^*}{\partial\phi^2}\phi_i = 0. \quad (4.19)$$

So that for $\langle\phi_i\rangle \neq 0$, we have

$$\Gamma^{*\prime} = \frac{\partial\Gamma^*}{\partial\phi^2} = 0, \quad (4.20)$$

at the minimum. The inverse propagator is now

$$G_{ij}^{-1} = \frac{\partial^2\Gamma^*}{\partial\phi_i\partial\phi_j} = 2\Gamma^{*\prime}[\delta_{ij} - \phi_i\phi_j/\phi^2] + (2\Gamma^{*\prime} + 4\Gamma^{*\prime\prime})\phi_i\phi_j/\phi^2. \quad (4.21)$$

From this equation one infers that the transverse degrees of freedom are massless and the longitudinal ones are not. The construction of Γ^* though feasible in $3+1$ dimensions in static cases, is not at present numerically feasible in the dynamical case where one has to solve further Bethe-Salpeter equations for the three-point vertex functions. The details of the construction of Γ^* are found in [7].

Before closing, let us remark that in a recent paper, Ivanov *et al.* [11] have proposed a new way of circumventing the violation of Goldstone's theorem, at leading order, in the simpler Hartree approximation. Specifically, these authors have outlined a modified self-consistent Hartree approximation, which preserves features present in the ϕ-derivable approach, such as energy conservation and thermodynamic consistency. By adding terms to the 2PI generating functional which vanish when the symmetry is restored, their approach explicitly enforces the Goldstone theorem. This may be a promising approach, and it will be interesting to see if the same strategy can be pursued at next-to-leading order in $1/N$.

V. CONCLUSIONS

In what is a follow-up to our previous paper [3], in which we have discussed the renormalization of the symmetric $O(N)$ model, $\phi = 0$, to next-to-leading order in $1/N$, in the SD framework, in this paper we have shown that the 2PI-$1/N$ expansion of the $O(N)$ model in the homogeneous broken-symmetry vacuum is also renormalizable to order $1/N$. We have derived finite equations for the renormalized Green's functions, and shown that Goldstone's theorem is violated. We have briefly discussed some current ideas about how to circumvent this problem. Our major interest here was to obtain finite renormalized equations for numerical simulations of $O(4)$ model dynamics. To make a realistic model of the time evolution of the chiral phase transition with physical π mesons requires introduction of explicit symmetry breakdown [5], which will make the violation of the Goldstone theorem unimportant in phenomenological applications, when compared with the mass generated by the explicit breaking of the $O(4)$ symmetry. The renormalization presented here is easily generalized to the time-dependent equations and we are in the process of reinvestigating the problem of disoriented chiral condensates using the $O(4)$ model in the 2PI-$1/N$ expansion.

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