On the infrared freezing of perturbative QCD in the Minkowskian region

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The infrared freezing of observables is known to hold at fixed orders of perturbative QCD if the Minkowskian quantities are defined through the analytic continuation from the Euclidean region. In a recent paper [D. M. Howe and C. J. Maxwell, Phys. Rev. D **70**, 014002 (2004)] it is claimed that infrared freezing can be proved also for Borel-resummed all-orders quantities in perturbative QCD. In the present paper we obtain the Minkowskian quantities by the analytic continuation of the all-orders Euclidean amplitudes expressed in terms of the inverse Mellin transform of the corresponding Borel functions [I. Caprini and M. Neubert, J. High Energy Phys. 03 (1999) 007]. Our result shows that if the principle of analytic continuation is preserved in Borel-type resummations, the Minkowskian quantities exhibit a divergent increase in the infrared regime, which contradicts the claim made in [D. M. Howe and C. J. Maxwell, Phys. Rev. D **70**, 014002 (2004)]. We discuss the arguments given in this paper and show that the special redefinition of Borel summation at low energies adopted there does not reproduce the lowest order result obtained by analytic continuation.

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I. INTRODUCTION

Since the advent of QCD it was realized that the application of the renormalization-group improved perturbation theory is natural in the deep Euclidean region, where the running coupling is small and the physical hadronic thresholds are absent. The application of perturbative QCD for physical observables defined as Minkowskian quantities requires the analytic continuation from the spacelike to the timelike axis of the complex momentum plane. At high energies, the analytic continuation of the strong running coupling $a(-s)$ from the Euclidean region $s < 0$ to the Minkowskian region $s > 0$ can be expanded in powers of $1/\ln(s/\Lambda^2)$. So, in the asymptotic region the expansion parameter is the same on the spacelike axis and the timelike one. At lower energies, however, one must take into account the finite terms appearing from the analytic continuation of $\ln(-s/\Lambda^2) \to \ln(s/\Lambda^2) - i\pi$. The problem was investigated in the early 1980s by several authors [1,2], who tried to identify the most natural parameter for the perturbative QCD expansions of timelike observables. In [1] the authors compare the expansion parameters $a(s)$, $|a(-s)|$ and $\mathrm{Re}a(-s)$ for $s > 0$ and notice that $|a(-s)|$ seems suitable since it remains finite in the Landau region $s < \Lambda^2$. However, the choice of the modulus $|a(-s)|$ as expansion parameter does not absorb all the π^2 factors which arise from the analytic continuation, as shown by Radyushkin [2], who derived explicit formulae for the timelike observables to every finite order of perturbative series. The analytic continuation was subsequently applied in the perturbative calculation of Minkowskian quantities [3] and in phenomenological analyses of inclusive observables like the rates of the processes $e^+e^- \to$ hadrons and $\tau \to$ hadrons, using either low orders of perturbation theory [4–6] or resummations based on the Borel method [7–9].

While for a long time the applications of perturbative QCD in the region $0 < s < \Lambda^2$ were not considered reliable, the interest in the low energies increased when it was realized that some Minkowskian quantities, obtained in a consistent way by analytic continuation, remain finite in the timelike infrared limit $s \to 0$. This property, called "infrared freezing," was shown to hold in every finite order of perturbation theory [10,11], and is actually put on the basis of the so-called "analytic perturbation theory." In this approach [10,11], the perturbative expansions of the Minkowskian observables are defined with a regular effective coupling, and the Euclidean quantities are obtained thereof by means of dispersion relations known to be valid in QCD [12] under plausible assumptions.

One may ask whether the infrared freezing is only a feature of the finite order QCD expansions or it survives beyond finite orders. This is a nontrivial question, especially since the QCD perturbative series is known to be divergent. In [13], using the Borel summation of the QCD perturbative series in the leading-β_0 approximation, the authors conclude that the infrared finite limit of the Minkowskian observables is valid also at all orders in perturbative QCD. Since the perturbative series of QCD is ambiguous, it is not impossible, in principle, to implement a desired property by a suitable summation prescription. It is however natural to require that the procedure respects the principle applied to finite orders, which in the

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present case is the analytic continuation. In the arguments given in [13] this principle is abandoned at some stage. The reason is that the authors use a Borel representation expressed as an infinite series of renormalons in the large-β_0 approximation, which does not display the dependence on the momentum in a transparent way. So the question of what is the infrared limit of the Minkowskian quantities when defined in a consistent way by analytic continuation from the deep Euclidean region, as is done in the case of fixed orders, remains open. In the present paper we address this question.

To this end, we choose an alternative representation of the Borel-summed Euclidean quantities, derived in [14], which is more convenient for the analytic continuation since it explicitly displays momentum dependence. A remarkable merit of this approach is that we do not need to represent the Minkowskian quantity in terms of any expansion parameter $|a(s)|$, $a(|s|)$ or $\mathrm{Re}\,a(s)$ (as in [1]), assuming only that the quantity admits certain integral representations as discussed below in Sec. III. Note that the same technique was applied in [15] for the analytic continuation in the coupling plane, leading to results consistent with those obtained in [16].

As in [13], we choose as Euclidean quantity the Adler function in massless QCD and as Minkowskian quantity the spectral function of the polarization function. In the next section we briefly review the analytic continuation from the Euclidean to the Minkowskian region of fixed-order perturbative expansions in QCD, stressing upon the fact that a consistent analytic continuation is free of ambiguities. In Sec. III we perform the similar analytic continuation of the whole Borel-resummed Adler function, written in a compact form in [14], which displays the energy dependence in an explicit way. In this section we treat in detail the one-loop coupling. The situation beyond one-loop is discussed briefly in Sec. IV, where we show that our conclusion about the infrared behavior of the Minkowskian observables remains valid also in this case. We use the analytic expression of the two-loop coupling derived recently in [17,18], working in the assumption, true in the real-world QCD, that the Euclidian coupling is not causal. In Sec. V we review the Borel summation presented in [13] and show that it does not reproduce correctly the lowest order result obtained by analytic continuation in the infrared limit.

II. ANALYTIC CONTINUATION OF FIXED-ORDER PERTURBATIVE EXPANSIONS

We consider the Adler function in massless QCD defined as

$$\mathcal{D}(s) = -s\frac{\mathrm{d}\Pi(s)}{\mathrm{d}s} - 1, \tag{1}$$

where $\Pi(s)$ is the correlation function of two vector currents. The function $\Pi(s)$ can be obtained from $\mathcal{D}(s)$ by logarithmic integration:

$$\Pi(s) = k - \ln(-s) - \int^{s} \mathrm{d}\ln(-s')\mathcal{D}(s'), \tag{2}$$

where k is a constant and the integration is along a contour in the complex plane which starts at a fixed point and ends at s, without crossing the singularities of the integrand. This definition is consistent with asymptotic freedom and the general properties of the QCD Green functions. Causality and unitarity imply that $\Pi(s)$ and $\mathcal{D}(s)$ are real analytic functions in the complex s plane [i.e. $\Pi(s^*) = \Pi^*(s)$ and $\mathcal{D}(s^*) = \mathcal{D}^*(s)$], cut along the positive real axis from the threshold for hadron production at $s = 4m_\pi^2$ to infinity. Along the cut, the Minkowskian quantity of interest is related to the spectral function $\mathrm{Im}\Pi(s + i\epsilon)$ by

$$\mathcal{R}(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s + i\epsilon) - 1. \tag{3}$$

Following [13], we consider the renormalization group improved truncated expansion of the Adler function in perturbative QCD

$$\mathcal{D}^{(N)}(s) = a(-s) + \sum_{n\geq 1}^{N} d_n a^{n+1}(-s), \tag{4}$$

with the one-loop coupling defined as

$$a(s) = \frac{\alpha_s(s)}{\pi} = \frac{1}{\beta_0 \ln(s/\Lambda^2)}, \tag{5}$$

where Λ is the QCD scale parameter and $\beta_0 = (11N_c - 2n_f)/12$ is the first coefficient of the β function (we follow in general the notations in [13], except for using $\beta_0 = b/2$ instead of b). In our analysis we shall assume that β_0 is positive, which means that infrared freezing does not hold for the Euclidian quantities like $D(s)$ for $s < 0$. The first coefficients in (4), d_n, $n \leq 3$, were calculated in [3].

Using (2) we obtain the polarization amplitude as:

$$\Pi^{(N)}(s) = k - \ln(-s) - \frac{\ln\ln(-s/\Lambda^2)}{\beta_0} + \sum_{n\geq 1}^{N} d_n\left(\frac{1}{\beta_0}\right)^{n+1}\frac{1}{n\ln^n(-s/\Lambda^2)}. \tag{6}$$

We recall that the above expressions are derived in the deep Euclidean region $s < -\Lambda^2$ or, more generally, for complex values of s, with $|s| > \Lambda^2$. In this region the expressions are consistent with the general properties derived from field theory, which require that $\Pi(s)$ and $\mathcal{D}(s)$ must be real for $s < 0$. The analytic continuations of (4) and (6) to low values of $|s|$ contain however unphysical singularities on the spacelike axis, which are absent from the exact amplitudes: $\mathcal{D}^{(N)}(s)$ has a Landau pole at $s = -\Lambda^2$, and $\Pi^{(N)}(s)$ has a Landau cut along the interval $-\Lambda^2 < s < 0$. Here we are interested in the imaginary part of $\Pi^{(N)}(s)$ on the upper edge of the timelike axis $s > 0$. Using (6), the spectral function (3) is obtained, at finite orders, as

$$\mathcal{R}^{(N)}(s) = A_1(s) + \sum_{n\geq 1}^{N} d_n A_{n+1}(s), \tag{7}$$

where [2,10,13],

$$A_1(s) = \frac{1}{\pi\beta_0}[\arctan(\pi\beta_0 a(s)) + \pi\theta(\Lambda^2 - s)]$$
$$A_n(s) = \frac{1}{\pi\beta_0}\frac{a^{n-1}(s)}{n-1}\mathrm{Im}[(1 - i\pi\beta_0 a(s))^{1-n}], \quad n > 1, \tag{8}$$

with $a(s)$ defined in (5). We note that in the first Eq. (8) arctan denotes the standard function defined in the interval $(-\pi/2, +\pi/2)$, with $\arctan(0) = 0$, and the term $\theta(\Lambda^2 - s)$ accounts for the fact that the real part of $\ln(s/\Lambda^2)$ becomes negative when $s < \Lambda^2$. Indeed, writing

$$\ln(-s/\Lambda^2) = \ln(s/\Lambda^2) - i\pi = \sqrt{\ln^2(s/\Lambda^2) + \pi^2}\exp[i\phi] \tag{9}$$

for s positive and above the cut, one can see that the phase ϕ is continuous at $s = \Lambda^2$, where it passes to the second quadrant [as shown in [11], the first Eq. (8) may be written also as $A_1(s) = 1/\pi\beta_0 \arccos[L/\sqrt{L^2 + \pi^2}]$, where $L = \ln s/\Lambda^2 = 1/(\beta_0 a(s))$].

We mention that in some papers the Minkowskian quantity $\mathcal{R}(s)$ for $s > \Lambda^2$ is defined in terms of the Adler function through an integral along an open contour which ends at $s \pm i\epsilon$. Usually, this contour is chosen as the circle of radius $|s|$ centered at the origin, since in this case the integrals of the finite order expansions can be done analytically [2,9]. While this procedure is suited for s much larger than Λ^2, for points close to Λ^2 the result is sensitive to small deformations of the integration contour, which may or may not include the Landau pole. In [13] this ambiguity is solved by an ad-hoc choice of the branch of the arctan function which appears after integration, so as to lead to infrared freezing for finite order expansions. We stress that the procedure of calculating the discontinuity of the polarization function applied in the above Eqs. (3)–(6) is free of such ambiguities.

As was mentioned in the introduction, in applications at large energies one expands the functions $A_n(s)$ in powers of the small coupling $a(s)$ defined in (5). This gives for $\mathcal{R}(s)$ [3]

$$\mathcal{R}(s) \sim a(s) + d_1 a^2(s) + \left(d_3 - d_1\frac{\pi^2\beta_0^2}{3}\right)a^3(s)\ldots. \tag{10}$$

The approximate expansion of $\mathcal{R}$ thus obtained can in no way be used at low energies, since $a(s)$ becomes infinite at $s = \Lambda^2$. On the other hand, as Eqs. (7) and (8) imply, the $\mathcal{R}^{(N)}(s)$ are regular for all s, including $s = \Lambda^2$, and have a finite, universal infrared limit

$$\mathcal{R}^{(N)}(0) = \frac{1}{\beta_0}, \tag{11}$$

for N any positive integer.

III. ANALYTIC CONTINUATION OF THE BOREL-SUMMED AMPLITUDE

The perturbation expansion (4) of $\mathcal{D}(s)$ in powers of the renormalized coupling $a(-s)$ is known to be neither convergent nor Borel summable. We consider the Borel transform $B_D(u)$ defined in the standard way in terms of the perturbative coefficients d_n of $\mathcal{D}$:

$$B_D(u) = \sum_{n=0}^{\infty}\frac{d_n}{n!}\left(\frac{u}{\beta_0}\right)^n, \quad d_0 = 1. \tag{12}$$

From the $n!$ large order growth of d_n it is known that $B_D(u)$ has singularities (ultraviolet and infrared renormalons) on the real axis of the u-plane [19]. For the Adler function, the ultraviolet renormalons are placed along the range $u \leq u_1$, $u_1 = -1$ and the infrared renormalons along $u \geq u_2$, $u_2 = 2$. Because of the infrared renormalons, the usual Borel-Laplace integral is not well-defined and requires an integration prescription. Defining

$$\mathcal{D}^{(\pm)}(s) = \frac{1}{\beta_0}\int_{C_\pm} \mathrm{e}^{-u/(\beta_0 a(-s))}B_D(u)\mathrm{d}u = \frac{1}{\beta_0}\lim_{\epsilon\to 0}\int_{0\pm i\epsilon}^{\infty\pm i\epsilon}\mathrm{e}^{-u/(\beta_0 a(-s))}B_D(u)\mathrm{d}u, \tag{13}$$

one can adopt as prescription, for each value of $a(-s)$ with $\mathrm{Re}\,a(-s) > 0$, either $\mathcal{D}^{(+)}(s)$ or $\mathcal{D}^{(-)}(s)$, or a linear combination of them, with coefficients ξ and $1 - \xi$ such as correctly to reproduce the known perturbative (asymptotic) expansion (4) of $\mathcal{D}(s)$ (obtained by truncating the Taylor expansion (12) at a finite order N). Note that all these (and many other) integration prescriptions have, according to a theorem by Watson [20], the same asymptotic series in powers of $a(-s)$ and therefore possess the same, original perturbative expansion.

Once a prescription is adopted, one has a well-defined function, different prescriptions yielding different functions with different properties. In the present work we use, as in [13], the principal value (PV) prescription

$$\mathcal{D}(s) = \frac{1}{2}[\mathcal{D}^{(+)}(s) + \mathcal{D}^{(-)}(s)]. \tag{14}$$

As shown in [14], this prescription gives real values along the spacelike axis outside the Landau region, which is consistent with the general analyticity requirements imposed by causality and unitarity. Moreover, we work in the V-scheme, where all the exponential dependence in the Laplace integrals (13) is absorbed in the running coupling, and denote by Λ_V^2 the corresponding QCD scale parameter.

Our purpose is to obtain the Minkowskian qaauntity $\mathcal{R}$ by analytically continuing the Euclidean Borel-summed

Adler function (14). To this end it is convenient to use a representation of the Borel function B_D in terms of its inverse Mellin transform $\hat{w}_D$ defined as [21]

$$\hat{w}_D(\tau) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} \mathrm{d}u B_D(u)\tau^{u-1}. \tag{15}$$

The inverse relation

$$B_D(u) = \int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\tau^{-u}, \tag{16}$$

defines the function $B_D(u)$ in a strip parallel to the imaginary axis with $u_1 < \mathrm{Re}u < u_2$. The relations (15) and (16) are valid if the following L^2 condition holds [22]:

$$\frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} \mathrm{d}u|B_D(u)|^2 < \infty, \tag{17}$$

where $u_0 \in [u_1, u_2]$. The function $\hat{w}_D(\tau)$ was calculated in [21] in the large-β_0 approximation [23,24], where it has different analytic expressions, which we denote by $\hat{w}_D^{(<)}$ and $\hat{w}_D^{(>)}$, depending on whether τ is less or greater than 1, respectively:

$$\begin{aligned} \hat{w}_D(\tau) &= \hat{w}_D^{(<)}(\tau), \quad 0 < \tau < 1 \\ \hat{w}_D(\tau) &= \hat{w}_D^{(>)}(\tau), \quad \tau > 1. \end{aligned} \tag{18}$$

As discussed in [14], one expects the inverse Mellin transform $\hat{w}_D$ to have different expressions for $\tau < 1$ and $\tau > 1$ in general, also beyond the leading β_0-approximation. Indeed, $\hat{w}_D^{(<)}$ is given by a sum over the residua of the infrared renormalons, while $\hat{w}_D^{(>)}$ is calculated in terms of the residua of the ultraviolet renormalons, and there are no reasons to expect these two contributions to be equal. In the large-β_0 approximation, the expressions of the functions $\hat{w}_D^{(<)}$ and $\hat{w}_D^{(>)}$ are [21]:

$$\hat{w}_D^{(<)}(\tau) = \frac{8}{3}\left\{\tau\left(\frac{7}{4} - \ln\tau\right) + (1+\tau)[L_2(-\tau) + \ln\tau\ln(1+\tau)]\right\}, \tag{19}$$

$$\hat{w}_D^{(>)}(\tau) = \frac{8}{3}\left\{1 + \ln\tau + \left(\frac{3}{4} + \frac{1}{2}\ln\tau\right)\frac{1}{\tau} + (1+\tau) \times [L_2(-\tau^{-1}) - \ln\tau\ln(1+\tau^{-1})]\right\},$$

where $L_2(x) = -\int_0^x \frac{\mathrm{d}t}{t}\ln(1-t)$ is the Euler dilogarithm.

As noticed in [21], the function $\hat{w}_D(\tau)$ defined in (18) is continuous together with its first three derivatives, and satisfies the normalization condition:

$$\int_0^\infty \hat{w}_D(\tau)\mathrm{d}\tau = 1. \tag{20}$$

On the other hand, Eqs. (19) define two independent functions, $\hat{w}_D^{(<)}$ and $\hat{w}_D^{(>)}$, which are analytic in the whole τ complex plane except for logarithmic branch-points. These functions are not bounded everywhere: the function $\hat{w}_D^{(<)}(\tau)$ is unbounded for $\tau > 1$ (growing at infinity like $\tau\ln^2\tau$), while $\hat{w}_D^{(>)}(\tau)$ grows like $\ln\tau/\tau$ for $\tau \to 0$. This behavior is seen in Fig. 1, where we represent the function $\hat{w}_D(\tau)$ defined by (18), together with $\hat{w}_D^{(<)}(\tau)$ for $\tau > 1$ and $\hat{w}_D^{(>)}(\tau)$ for $\tau < 1$. The same figure shows also the real parts of the functions $\hat{w}_D^{(<)}(\tau)$ and $\hat{w}_D^{(>)}(\tau)$ for $\tau < 0$, where they become complex. As we will see below, the growth of $\mathrm{Re}\hat{w}_D^{(<)}(\tau)$ for $\tau \to -\infty$ will have important consequences for the problem investigated in the present work.

Following the technique described in detail in [14], we shall express the function $\mathcal{D}(s)$ for complex values of s, with $\mathrm{Re}a(-s) > 0$, in terms of the inverse Mellin transform $\hat{w}_D$. Using Eqs. (13) and (14) as starting points, we rotate the integration contours $C_\pm$ in the complex u-plane up to a line parallel to the imaginary axis where the representation (16) of B_D holds and can be inserted into the Borel integral. If the integrals are convergent, we can reverse the order of integration upon u and τ, and perform first the integral upon the variable u, which can be done exactly. As explained in [14], when s is in the upper half of the complex plane, the contour C_+ can be rotated towards the positive imaginary axis in the u-plane since the integrals remain convergent, while for the integral along the contour C_- it is necessary to first cross the real positive axis of the u-plane, picking up contribution of the residua of the corresponding singularities, i.e. the infrared renormalons. When s is in the lower half of the complex plane, convergence is achieved if the contours are rotated towards the negative imaginary axis in the u-plane, and the roles of the contours C_+ and C_- are reversed. This gives different expressions for $\mathcal{D}(s)$ in the upper/lower semiplanes of the s plane:

$$\begin{aligned} \mathcal{D}(s) &= \frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \frac{\hat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)} - \frac{i\pi}{\beta_0} \times \left(-\frac{\Lambda_V^2}{s}\right)\hat{w}_D^{(<)}(-\Lambda_V^2/s), \quad \mathrm{Im}s > 0 \\ \mathcal{D}(s) &= \frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \frac{\hat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)} + \frac{i\pi}{\beta_0} \times \left(-\frac{\Lambda_V^2}{s}\right)\hat{w}_D^{(<)}(-\Lambda_V^2/s), \quad \mathrm{Im}s < 0, \end{aligned} \tag{21}$$

where the first terms are given by the integration with respect to u, and the last terms are produced by the residua of the infrared renormalons picked up by crossing of the positive axis. We recall that the expressions (21) were obtained by using the Principal Value prescription (14).

The corresponding expression for the polarization function $\Pi(s)$ can be obtained by inserting the above result into the definition (2). This gives

$$\Pi(s) = k - \ln\left(\frac{-s}{\Lambda_V^2}\right) - \frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\ln\ln\left(-\frac{\tau s}{\Lambda_V^2}\right) \pm \frac{i\pi}{\beta_0}\int^s \mathrm{d}\ln(-s)\left(-\frac{\Lambda_V^2}{s}\right)\hat{w}_D^{(<)}(-\Lambda_V^2/s), \quad (22)$$

with the contour in the last integral specified below Eq. (2) and the $+/-$ sign corresponding to $\mathrm{Im}s > 0/\mathrm{Im}s < 0$, respectively. The Borel-summed expression (22), obtained for $|s| > \Lambda_V^2$, can be analytically continued into the whole complex plane. We note that $\Pi(s)$ is holomorphic for complex values of s and satisfies the reality condition $\Pi(s^*) = \Pi^*(s)$. On the real axis, this function can have singularities manifested as discontinuities of the imaginary part. The unphysical singularities in the spacelike region $-\Lambda_V^2 < s < 0$ were discussed in detail in [14]. Here we are interested in the spectral function for $s > 0$. A straightforward calculation gives:

$$\mathrm{Im}\Pi(s + i\epsilon) = \pi + \frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\arctan\left(\frac{\pi}{\ln(\tau s/\Lambda_V^2)}\right) + \frac{\pi}{\beta_0}\int_0^{\Lambda_V^2/s}\mathrm{d}\tau \hat{w}_D(\tau) + \frac{\pi}{\beta_0}\mathrm{Re}\int_{-\Lambda_V^2/s}^0 \mathrm{d}\tau \hat{w}_D^{(<)}(\tau). \quad (23)$$

We note that the term in the second line was obtained by means of the relation

$$\mathrm{Im}\left[\ln\ln\left(-\frac{\tau s}{\Lambda_V^2}\right)\right] = \arctan\left(\frac{\pi}{\ln(\tau s/\Lambda_V^2)}\right) + \pi\theta(\Lambda_V^2 - \tau s), \quad (24)$$

already applied (for $\tau = 1$) in deriving the first relation (8), and the last term in (23) is produced to the last term in (22). Using (3) we write also $\mathcal{R}$ as

$$\mathcal{R}(s) = \frac{1}{\pi\beta_0}\int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\arctan\left(\frac{\pi}{\ln(\tau s/\Lambda_V^2)}\right) + \frac{1}{\beta_0} \times \int_0^{\Lambda_V^2/s}\mathrm{d}\tau \hat{w}_D(\tau) + \frac{1}{\beta_0}\mathrm{Re}\int_{-\Lambda_V^2/s}^0 \mathrm{d}\tau \hat{w}_D^{(<)}(\tau). \quad (25)$$

It is easy to check that this expression is continuous for all $s > 0$, including the point $s = \Lambda_V^2$. We consider now the limit of this expression for $s \to 0$, i.e. $\Lambda_V^2/s \to \infty$. Since the first integral tends to zero (recall the comment below Eq. (8) about arctan) and $\hat{w}_D(\tau)$ satisfies the normalization relation (20), we obtain:

$$\mathcal{R}(0) = \frac{1}{\beta_0} + \lim_{s\to 0}\mathrm{Re}\int_{-\Lambda_V^2/s}^0 \mathrm{d}\tau \hat{w}_D^{(<)}(\tau). \quad (26)$$

The first term coincides with the result (11), but we have now an additional term which involves the values of $\hat{w}_D^{(<)}(\tau)$ for negative τ. Moreover, for small values of s, the integral involves arguments $\tau \to -\infty$, where $\mathrm{Re}\hat{w}_D^{(<)}(\tau)$ is unbounded (see Fig. 1). Using (19) (which is valid in the large β_0 limit) it is easy to check that the last integral in (26) diverges like $\ln^2 s/s^2$ for $s \to 0$. This result disproves the statement made in [13]: the infrared limit of the Borel-summed Minkowskian observable $\mathcal{R}(s)$ obtained by analytic continuation from the Euclidian region does not reproduce the infrared freezing observed in the finite orders, and moreover displays an unphysical divergence. This behavior is crucially determined by the summation of an infinity of terms of the series.

In order to understand the transition from the fixed orders to the resummed quantity, it is useful to apply the above formalism to the truncated perturbative expansion, when the Borel transform $B_D(u)$ defined in (12) reduces to a polynomial $B_D^{(N)}(u)$:

$$B_D^{(N)}(u) = 1 + d_1\frac{u}{\beta_0} + \frac{d_2}{2!}\frac{u^2}{\beta_0^2} + \ldots + \frac{d_N}{N!}\frac{u^N}{\beta_0^N}. \quad (27)$$

In this case the Laplace-Borel transform is well defined on the cuts. A straightforward calculation gives

$$\frac{1}{\beta_0}\int_0^\infty \mathrm{e}^{-u/(\beta_0 a(-s))}B_D^{(N)}(u)\mathrm{d}u = \sum_{n=0}^N d_n a^{n+1}(-s), \quad d_0 = 1, \quad (28)$$

i.e. the finite order expansion (4). We want to check

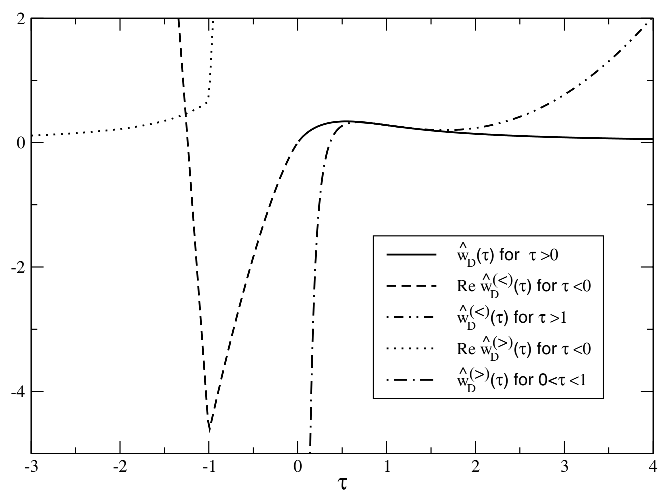

FIG. 1. The function $\hat{w}_D(\tau)$ defined by (18) in the large β_0 approximation [i.e., with $\hat{w}_D^{(<)}(\tau)$ and $\hat{w}_D^{(>)}(\tau)$ given by (19)] is represented by a solid line. We separately display also the function $\hat{w}_D^{(<)}(\tau)$ for $\tau > 1$ (dash-dot-dotted line) and for $\tau < 0$ (dashed line), and the function $\hat{w}_D^{(>)}(\tau)$ for $\tau < 1$ (dash-dotted line and dotted line). For negative values of τ the curves represent the real parts of the corresponding functions. The dashed line is unbounded for τ tending to $-\infty$, while the dash-dot-dotted line grows unboundedly for τ increasing. The dotted and dash-dotted lines are unbounded for τ approaching zero from either side.

whether this expansion, along with the expansion (7) of $\mathcal{R}$, are reproduced in the inverse Mellin formalism.

Clearly, when B_D is a polynomial the condition (17) is not satisfied, so we expect the function $\hat{w}_D$ to be a generalized function (a distribution). In order to calculate it, we consider the alternative distribution function $\hat{W}_D(\tau)$, introduced in [8]:

$$\hat{W}_D(\tau) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} \mathrm{d}u \frac{B_D(u)}{\sin\pi u}\tau^{u-1}. \tag{29}$$

When $B_D(u)$ is a polynomial, the ratio $B_D(u)/\sin\pi u$ satisfies the condition (17), which ensures the existence of the function $\hat{W}_D(\tau)$. For instance, for $B_D(u) = 1$ a straightforward calculation gives

$$\hat{W}_D(\tau) = \frac{1}{\pi(1+\tau)}. \tag{30}$$

On the other hand, as shown in [8,25] (see also Eq. (A.10) of [14]), the connection between the Mellin transforms $\hat{W}_D(\tau)$ and $\hat{w}_D(\tau)$ is

$$\hat{W}_D(\tau) = \frac{1}{\pi}\int_0^\infty \mathrm{d}x \frac{\hat{w}_D(x)}{x+\tau}, \tag{31}$$

where τ can take arbitrary values, except for real negatives. By comparing (30) and (31) it follows that, for $B_D^{(0)}(u) = 1$,

$$\hat{w}_D^{(0)}(x) = \delta(1-x). \tag{32}$$

A straightforward calculation shows that at each finite order the function $\hat{w}_D(x)$ is represented in terms of the distribution $\delta(1-x)$ and its derivatives. For instance, the inverse Mellin transform of $B_D^{(N)}$ defined (27), for $N = 3$, is

$$\begin{aligned}\hat{w}_D^{(3)}(x) = {}& \delta(1-x) - \frac{d_1}{\beta_0}\delta'(1-x) \\ & + \frac{d_2}{2!\beta_0^2}[\delta''(1-x) + \delta'(1-x)] \\ & - \frac{d_3}{3!\beta_0^3}[\delta'''(1-x) + 3\delta''(1-x) + \delta'(1-x)].\end{aligned} \tag{33}$$

Such a representation is not unique: except for the first two terms which remain the same, the higher terms can be written equivalently as the product of the nth derivative of $\delta(1-x)$ with a polynomial of degree $n-1$. For instance, $\hat{w}_D^{(3)}(x)$ in (33) can be expressed in the form

$$\begin{aligned}\hat{w}_D^{(3)}(x) = {}& \delta(1-x) - \frac{d_1}{\beta_0}\delta'(1-x) \\ & + \frac{d_2}{2!\beta_0^2}[(x+1)/2\delta''(1-x)] \\ & - \frac{d_3}{3!\beta_0^3}[(x^2+4x+1)/6\delta'''(1-x)].\end{aligned} \tag{34}$$

It is easy to check that the different expressions (33) and (34) give the same result for the quantities of interest $\mathcal{D}$ and $\mathcal{R}$. An immediate consequence of these expressions is that, for finite orders, $\hat{w}_D^{(N)<}(\tau) = 0$. Inserting $\hat{w}_D^{(3)}$ into the relation (21), which expresses the Adler function in terms of the inverse Mellin transform, we obtain by a straightforward calculation

$$\begin{aligned}\mathcal{D}^{(3)}(s) = {}& \frac{1}{\beta_0}\frac{1}{\ln(-s/\Lambda_V^2)} + d_1\left(\frac{1}{\beta_0}\frac{1}{\ln(-s/\Lambda_V^2)}\right)^2 \\ & + d_2\left(\frac{1}{\beta_0}\frac{1}{\ln(-s/\Lambda_V^2)}\right)^3 + d_3\left(\frac{1}{\beta_0}\frac{1}{\ln(-s/\Lambda_V^2)}\right)^4,\end{aligned} \tag{35}$$

which coincides with the first terms in the expansion (4).

Let us insert also (33) into the resummed expression (25) of $\mathcal{R}$. It is easy to see that the first term $\hat{w}_D^{(0)}(\tau) = \delta(1-\tau)$ contributes both to the first and the second integrals in (25), giving the result

$$\begin{aligned}\mathcal{R}^{(0)}(s) = {}& \frac{1}{\pi\beta_0}\left[\int_0^\infty \delta(1-x)\arctan(\pi/(\ln x + \ln(s/\Lambda^2)))\mathrm{d}x \right.\\ & \left. + \pi\int_0^{\Lambda^2/s}\delta(1-x)\mathrm{d}x\right] \\ = {}& \frac{1}{\pi\beta_0}[\arctan(\pi\beta_0 a(s)) + \pi\theta(\Lambda^2 - s)],\end{aligned} \tag{36}$$

which coincides with the function $A_1(s)$ defined in (8) and satisfies the property of infrared freezing. The higher terms in (33) contribute only to the first integral in (25), reproducing the terms in expression (25). For instance, inserting the second term of (33) in (25) one has

$$\begin{aligned}& -\frac{d_1}{\pi\beta_0^2}\left[\int_0^\infty \delta'(1-x)\arctan(\pi/(\ln x + \ln(s/\Lambda^2)))\mathrm{d}x\right] \\ & \qquad = d_1\frac{a^2}{1 + a^2\beta_0^2\pi^2} = d_1 A_2(s),\end{aligned} \tag{37}$$

with $A_2(s)$ defined in (8). So the formalism of inverse Mellin transform reproduces the finite order expansions which are consistent with the property of infrared freezing (11). But the summation of the whole series leads to a different result. The discussion in this section reveals the difference between the finite orders and the summed expression: it resides in the function $\hat{w}_D^{<}(\tau)$, which is zero at each finite order but is nonvanishing when an infinity of terms are summed and the infrared renormalons show up.

IV. BEYOND THE ONE-LOOP COUPLING

Up to now we restricted the discussion to the one-loop coupling (5). We show now that the same conclusion is valid beyond this approximation. If the one-loop coupling is not inserted into (13), it is easy to see that the two integrals in (22) write in general

$$\Pi(s) \sim -\frac{1}{\beta_0}\int^s \mathrm{d}\ln(-s)\int_0^\infty \mathrm{d}\tau \frac{\hat{w}_D(\tau)}{\ln\tau + \frac{1}{\beta_0 a(-s)}} \pm \frac{i\pi}{\beta_0} \times \int^s \mathrm{d}\ln(-s) e^{-1/(\beta_0 a(-s))} \hat{w}_D^{(<)}(e^{-1/(\beta_0 a(-s))}). \quad (38)$$

As shown recently [17,18], the solution of the two-loop β-function equation can be written analytically in closed form as

$$a(s) = -\frac{1}{c[1 + W(z(s))]}, \qquad z(s) = -\frac{1}{e}\left(\frac{s}{\Lambda^2}\right)^{-\beta_0/c}, \quad (39)$$

where $c = (153 - 19n_f)/24\beta_0$ is the second universal beta-function coefficient and $W(z)$ is the Lambert function defined implicitly by $W(z)e^{W(z)} = z$ [26]. We work in the condition $c > 0$, valid in real-world QCD. Then, as shown in [17], the physical branch of the Lambert function in (39) is W_{-1}, and the coupling does not freeze in the spacelike region.

Denoting $W(z) = W_{-1}(z(-s))$ we obtain from (39):

$$\left(\frac{s}{\Lambda^2}\right)^{-\beta_0/c} = -W(z)e^{W(z)+1}$$
$$\ln\left(\frac{s}{\Lambda^2}\right) = -\frac{c}{\beta_0}(\ln(-1) + \ln z(s) + 1) \quad (40)$$

and

$$\mathrm{d}\ln(-s) = -\frac{c}{\beta_0}\mathrm{d}\ln z(-s) = -\frac{c}{\beta_0}(\mathrm{d}\ln W + \mathrm{d}W) = -\frac{c}{\beta_0}\frac{1+W}{W}\mathrm{d}W. \quad (41)$$

We now insert the two-loop coupling (39) into (38), and notice that the first integral can be performed by making the change of variable (41). Using the relation $W'(z) = W(z)/z(1 + W(z))$ (to be obtained from $We^W = z$ by differentiating the logarithm) we write after a straightforward calculation the first integral in (38) as

$$\Pi \sim -\frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\left[\ln\left(1 + W(z) - \frac{\beta_0}{c}\ln\tau\right) + \frac{1}{1 - \frac{\beta_0}{c}\ln\tau}\ln\frac{W(z)}{1 + W(z) - \frac{\beta_0}{c}\ln\tau}\right]. \quad (42)$$

In order to evaluate the limit $s \to 0$, we use the asymptotic expansion [26]

$$W(z) \sim \ln z - \ln\ln z \quad (43)$$

valid for both $z \to 0$ and $z \to \infty$. Then we obtain from the first logarithm in (42) the limit

$$\lim_{s\to 0}\Pi(s) \sim -\frac{1}{\beta_0}\int_0^\infty \mathrm{d}\tau \hat{w}_D(\tau)\ln\ln\left(\frac{-s\tau}{\Lambda^2}\right), \quad (44)$$

while the second term in (42) vanishes. The expression (44) coincides with the first integral in (22), whose imaginary part, as discussed in Eqs. (23)–(26), has the infrared limit $1/\beta_0$. This result is consistent with the finite order calculations with the two-loop coupling in [13,27].

We turn now to the second integral in (38). It is easy to see that the leading term in the asymptotic behavior (43) gives, for $s \to 0$, an expression identical to the second term in (26). The next to leading term in (43) introduces logarithmic corrections which do not change the singular behavior of the integrand for $s \to 0$, i.e. $\tau \to -\infty$. Thus, although the two-loop coupling leads to a different behavior of the spectral function at nonvanishing s, the singularity in the infrared limit is dominated by the one-loop coupling.

V. COMMENTS ON THE PROOF OF INFRARED FREEZING IN [13]

In Ref. [13] the authors discuss the infrared freezing beyond fixed-order expansions by using a representation of the Minkowskian quantity $\mathcal{R}$ in terms of the Borel transform B_D of the Adler function. This representation is derived by inserting the Borel representation of the Adler function into the expression (6) of the polarization function, which is then used in (3) to compute the quantity $\mathcal{R}$. Adopting the Principal Value prescription (14), a straightforward calculation gives [7,13]:

$$\mathcal{R}(s) = \frac{1}{\beta_0}\mathrm{PV}\int_0^\infty \mathrm{e}^{-u/(\beta_0 a(s))}\frac{\sin\pi u}{\pi u}B_D(u)\mathrm{d}u, \quad (45)$$

with the one-loop coupling $a(s)$ defined in (5).

The integrals in (45) converge only for $s > \Lambda^2$, when $a(s) > 0$. For small s, of interest for the infrared behavior, $a(s)$ is negative and the standard Laplace-Borel integral, along the positive real axis of the u-plane, diverges. In [13] the authors notice that for $a(s) < 0$ a convergent integral is obtained by choosing as integration line the negative axis in the u-plane, instead of the positive one. Therefore, they define the Borel-summed $\mathcal{R}(s)$ for $s < \Lambda^2$ as:

$$\mathcal{R}(s) = \frac{1}{\beta_0}\mathrm{PV}\int_0^{-\infty} \mathrm{e}^{-u/(\beta_0 a(s))}\frac{\sin\pi u}{\pi u}B_D(u)\mathrm{d}u. \quad (46)$$

The Principal Value prescription now regulates the ultraviolet renormalons along the negative real axis, while in the standard definition (45) the prescription regularizes the infrared renormalons. The authors of [13] claim that by this redefinition of the Minkowskian quantity for $s < \Lambda^2$, one recovers infrared freezing (11) also beyond fixed-order perturbation expansions.

In principle, a redefinition of the Borel integral as in (46) is not illegitimate: since we deal with functions which are not Borel summable, the choice of the prescription for the Borel integral is to a large extent arbitrary; as mentioned in Sec. III, there are many different functions that have the same perturbative, divergent asymptotic expansion in powers of the coupling constant. Such a redefinition should however be motivated physically, not just by the fact that the original definition has lost mathematical sense in a region.

Furthermore, when introducing a redefinition of the Borel integral, we must ensure that the fixed-order expansions are reproduced when the Borel transform is expanded in a Taylor series. In [13] the authors claim this requirement to be fulfilled if one expands in powers of u not the whole integrand in (45), but only the Borel transform $B_D(u)$. However, it is easy to see that the prescription (48) fails to reproduce correctly the infrared freezing (11) for finite order expansions.

Indeed, let us insert in (46) the truncated expansion (27) of $B_D(u)$ in powers of u. Then the Borel integral is regular, no prescription is required and we obtain from (46), for $s < \Lambda^2$:

$$\mathcal{R}^{(N)}(s) = \frac{1}{\beta_0}\int_0^{-\infty} \mathrm{e}^{-u/(\beta_0 a(s))}\frac{\sin\pi u}{\pi u} B_D^{(N)}(u)\mathrm{d}u = -\frac{1}{\beta_0}\int_0^{\infty} \mathrm{e}^{-u/(\beta_0(-a(s))}\frac{\sin\pi u}{\pi u} B_D^{(N)}(-u)\mathrm{d}u, \quad (47)$$

where a change of variable was performed in the last step. In the last integral the quantity $-a(s)$ is positive for $s < \Lambda^2$ (recall the definition (5) of the coupling), and we can easily perform the integration for each term of $B_D^{(N)}(-u)$ from (27). A straightforward calculation shows that, for all the terms except the first one, the infrared limit of the integral (47) is zero, in agreement with the behavior of the functions $A_n(s)$ of (8), with $n > 1$. For the first term $B_D(u) = 1$ (which was responsible for the infrared freezing of the truncated expansion in Section II) we apply the identity (41) of [13] which gives

$$\mathcal{R}^{(0)}(s) = -\frac{1}{\beta_0}\int_0^{\infty} \mathrm{e}^{-u/(\beta_0(-a(s)))}\frac{\sin\pi u}{\pi u}\mathrm{d}u = -\frac{1}{\pi\beta_0}\arctan[\pi\beta_0(-a(s)]. \quad (48)$$

In the infrared limit $s \to 0$, Eq. (5) implies $-a(s) \to 0$ through positive values, and from (48) we obtain $\mathcal{R}^{(0)}(0) = 0$, which is not consistent with the infrared limit of the function $A_1(s)$ defined in (8), and with the relation (11). Therefore, the prescription adopted in [13] for the Minkowskian quantity at $s < \Lambda^2$ fails to reproduce correctly the infrared freezing (11) of the truncated expansion.

VI. CONCLUSIONS

In a recent paper [13] it is claimed that by using analytic continuation in the energy plane it is possible to prove the infrared freezing of Minkowskian quantities beyond finite orders in perturbative QCD. In the present work, we applied the technique of the inverse Mellin transform of the Borel function, developed in [14], which gives compact expressions of the QCD amplitudes in the complex plane. By using the analytic continuation of these expressions into the Landau region, we calculated explicitly the spectral functions, as in the fixed-order expansion. As in [13] we adopted the Principal Value prescription, and considered as Euclidean quantity the Adler function in massless QCD. Our result, expressed in Eq. (26), contradicts the conclusion reached in [13]: the summation of higher orders in QCD leads to a divergent increase of the Minkowskian quantities in the infrared limit, if these are calculated by analytic continuation from the Euclidean region. The divergent infrared behavior arises explicitly from the summation of the infinite terms and is related to the infrared renormalons. Of course, one expects that in full QCD this divergent behavior will be compensated by a similar growth of remaining terms in the OPE, calculated with the same prescription.

The difference between our results and those in [13] is explained by the fact that the authors of [13] do not apply consistently the principle of analytic continuation, applied at finite orders. Lacking a compact expression of the Adler function, as the one provided by the inverse Mellin transform used by us, the authors make the analytic continuation of the Borel representation itself, which is valid only outside the Landau region. Therefore, the Borel representation (45) of the Minkowskian quantity considered in [13] converges only for $s > \Lambda^2$, and is useless in the infrared limit. To reach this point the authors change the definition of the Borel integral. However, as we showed in Sec. V, this new prescription for the Borel summation of Minkowskian quantities below the Landau point fails to reproduce the infrared freezing of the truncated expansion.

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