Perturbative corrections to the determination of V_{ub} from the P_+ spectrum in $B \to X_u \ell \bar{\nu}$

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(Received 23 February 2005; published 18 May 2005)

We investigate the relation between the E_γ spectrum in $B \to X_s \gamma$ decay and the P_+ spectrum in semileptonic $B \to X_u \ell \bar{\nu}$ decay (P_+ is the hadronic energy minus the absolute value of the hadronic three-momentum), which provides in principle the theoretically simplest determination of $|V_{ub}|$ from any of the "shape function regions" of $B \to X_u \ell \bar{\nu}$ spectra. We calculate analytically the P_+ spectrum to order $\alpha_s^2 \beta_0$, and study its relation to the $B \to X_s \gamma$ photon spectrum to eliminate the leading dependence on non-perturbative effects. We compare the result of fixed order perturbation theory to the next-to-leading log renormalization group improved calculation, and argue that fixed order perturbation theory is likely to be a more appropriate expansion. Implications for the perturbative uncertainties in the determination of $|V_{ub}|$ from the P_+ spectrum are discussed.

DOI: 10.1103/PhysRevD.71.093007 PACS numbers: 12.15.Hh, 13.20.He

I. INTRODUCTION

The determination of the magnitude of the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix element V_{ub} via inclusive decays is theoretically involved, because the experimental cuts required to suppress the $B \to X_c \ell \bar{\nu}$ background tend to restrict the $B \to X_u \ell \bar{\nu}$ phase space in a way that gives rise to theoretical complications [1]. The local operator product expansion (OPE) [2,3] breaks down in the regions $E_\ell > (m_B^2 - m_D^2)/(2m_B)$ and $m_X < m_D$ [4,5] (where E_ℓ is the charged lepton energy and m_X the hadronic invariant mass), because numerically $m_c^2 \sim \Lambda_{\rm QCD} m_b$. In these regions an expansion in $\Lambda_{\rm QCD}/m_b$ in terms of nonlocal matrix elements is still possible [6], and the leading term can be measured in $B \to X_s \gamma$ decay [7]. (The local OPE is valid in the $q^2 > (m_B - m_D)^2$ region [8], q^2 being the lepton invariant mass, but there are other issues for that cut [1].) The kinematic region, in which the final hadronic state has high energy $\sim m_b$ but low invariant mass $\sim \sqrt{\Lambda_{\rm QCD} m_b}$, is typically known as the "shape function region."

It was pointed out in Ref. [9] (see also [10,11]) that separating $b \to u$ from $b \to c$ using another variable, $P_+ \equiv E_X - |\vec{p}_X|$, where E_X and $\vec{p}_X$ are the energy and three-momenta of the hadronic final state, may provide advantages compared to E_ℓ or m_X. At lowest order in $\Lambda_{\rm QCD}/m_b$ $\mathrm{d}\Gamma_{B\to X_u \ell\bar{\nu}}/\mathrm{d}P_+$ is proportional to $\mathrm{d}\Gamma_{B\to X_s\gamma}/\mathrm{d}E_\gamma$ evaluated at $E_\gamma = (m_B - P_+)/2$ (since P_+ in $B \to X_s \gamma$ equals $m_B - 2E_\gamma$). Thus, to predict the $B \to X_u \ell \bar{\nu}$ rate in the $P_+ < m_D^2/m_B$ region that is free from the charm background, we only need to know the $B \to X_s \gamma$ photon spectrum for $E_\gamma > (m_B^2 - m_D^2)/(2m_B)$; i.e., a 330 MeV region near the endpoint, which is already precisely measured [12–14]. This is also what is needed to predict the E_ℓ endpoint spectrum, but it is significantly smaller than what is required to determine the m_X spectrum [15,16].

A convenient way to express the relation between the P_+ spectrum in semileptonic $b \to u$ decay and the photon energy spectrum in $B \to X_s \gamma$ decay is to relate weighted integrals of the two spectra [17] (see also [10]). One can write

$$\int_0^\Delta \mathrm{d}P_+ \frac{\mathrm{d}\Gamma_u}{\mathrm{d}P_+} \propto \frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} \int_0^\Delta \mathrm{d}P_\gamma W(\Delta, P_\gamma) \frac{\mathrm{d}\Gamma_s}{\mathrm{d}P_\gamma}, \qquad (1)$$

where we have defined $P_\gamma \equiv m_B - 2E_\gamma$, and at leading order in $\Lambda_{\rm QCD}/m_b$ the weight function $W(\Delta, P_\gamma)$ is calculable in perturbation theory.

In the shape function region, $\Delta \sim \Lambda_{\rm QCD}$, the perturbative expansion of W contains logarithms of the ratio of scales $\sqrt{m_b \Lambda_{\rm QCD}}/m_b$. For $m_b \gg \Lambda_{\rm QCD}$, these logarithms are large and can spoil the convergence of perturbation theory, so must be resummed using renormalization group techniques. This can be done using traditional perturbative QCD methods [17–20] or by using the Soft-Collinear Effective Theory (SCET) [21]. Currently W can be extracted from known results to leading log (LL) and next-to-leading log (NLL) accuracy. However, for the true value $\sqrt{m_b \Lambda_{\rm QCD}}/m_b \sim 1/3$, it is not clear that the leading log expansion is appropriate, and a fixed order calculation in α_s may give a better approximation to W.[1]

[1]This is reminiscent of the resummation of logs of $m_c/m_b \sim 1/3$ in exclusive $B \to D^* \ell \bar{\nu}$ decay at zero recoil, in which the leading log calculation is a poor approximation to the one- or two-loop results [22,23].

In this paper we address this issue by calculating the order $\alpha_s^2\beta_0$ corrections to the P_+ spectrum, where $\beta_0 = 11 - 2n_f/3$ is the first coefficient of the QCD β-function (we shall call this order BLM, referring to Brodsky, Lepage and Mackenzie [24]), from which we determine the corresponding corrections to the weighting function W. Since $\beta_0 \sim 9$ is a large number, such terms dominate the two-loop corrections to many processes [24,25]. We find that a considerable part of the two-loop expression arises from terms which in the renormalization group improved perturbation theory only occur at next-to-next-to leading log (NNLL) order, and we argue that fixed order perturbation theory is a more appropriate expansion. We discuss the implications of this for the uncertainty in the determination of $|V_{ub}|$.

II. THE SPECTRA AT ORDER $\alpha_s^2\beta_0$

We first present analytical results for the parton level spectra in $B \to X_u \ell \bar{\nu}$ and $B \to X_s\gamma$ to order $\alpha_s^2\beta_0$. Our results for the P_+ spectrum are new, whereas those for the photon spectrum are collected from the existing literature for completeness.

At the parton level, the appropriate variables are

$$\hat{p}_+ \equiv (v - q/m_b)\cdot n, \qquad \bar{x} \equiv 1 - 2E_\gamma/m_b, \tag{2}$$

for $B \to X_u \ell \bar{\nu}$ and for $B \to X_s\gamma$ decay, respectively, where n is a lightlike four-vector in the direction of $-\vec{q}$, v is the four-velocity of the B meson. These variables are simply related to the experimentally observed hadronic variables by

$$\begin{aligned} P_+ &\equiv E_X - |\vec{p}_X| = (m_B v - q)\cdot n = m_b\hat{p}_+ + \Lambda, \\ P_\gamma &\equiv m_B - 2E_\gamma = m_b\bar{x} + \Lambda, \end{aligned} \tag{3}$$

where $\Lambda \equiv m_B - m_b$.

A. The $\hat{p}_+$ spectrum in $B \to X_u \ell \bar{\nu}$

The $\hat{p}_+$ spectrum to order α_s is given by [9]

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma_u}{\mathrm{d}\hat{p}_+} = \delta(\hat{p}_+) - \frac{\alpha_s(m_b)C_F}{4\pi}\bigg[4\bigg(\frac{\ln\hat{p}_+}{\hat{p}_+}\bigg)_* + \frac{26}{3}\bigg(\frac{1}{\hat{p}_+}\bigg)_* + \bigg(\frac{13}{36} + 2\pi^2\bigg)\delta(\hat{p}_+) + 4\hat{p}_+^2(3 - 2\hat{p}_+)\ln^2\hat{p}_+ + \frac{2}{3}(2 - 23\hat{p}_+ - 9\hat{p}_+^2 + 8\hat{p}_+^3)\ln\hat{p}_+ - \frac{316 + 407\hat{p}_+ - 1101\hat{p}_+^2 + 708\hat{p}_+^3 - 200\hat{p}_+^4 + 33\hat{p}_+^5 - 7\hat{p}_+^6}{18}\bigg] + \mathcal{O}(\alpha_s^2), \tag{4}$$

where $\Gamma_0 = G_F^2|V_{ub}|^2 m_b^5/(192\pi^3)$, $C_F = 4/3$, and m_b is the b quark pole mass. The $*$ distributions (for $n \geq 0$ integers) are defined by

$$\int_0^z \mathrm{d}x f(x)\bigg(\frac{\ln^n x}{x}\bigg)_* = f(0)\frac{\ln^{n+1}z}{n+1} + \int_0^z \mathrm{d}x[f(x) - f(0)]\frac{\ln^n x}{x}. \tag{5}$$

Integrating Eq. (4) over $0 < \hat{p}_+ < 1$ reproduces the total rate $1 - (\alpha_s C_F/(4\pi))(2\pi^2 - 25/2)$.

The BLM part of the two-loop result may be obtained from the one-loop calculation with an arbitrary gluon mass using the method of Ref. [26]. We calculated the spectrum for $\hat{p}_+ \neq 0$, for which only bremsstrahlung graphs contribute, and then determined the coefficient of $\delta(\hat{p}_+)$ by comparing with the total rate [27]. The result is

$$\begin{aligned} \frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma_u^{(\mathrm{BLM})}}{\mathrm{d}\hat{p}_+} = \frac{\alpha_s^2(m_b)}{\pi^2}\beta_0\bigg\{&\frac{1}{2}\bigg(\frac{\ln^2\hat{p}_+}{\hat{p}_+}\bigg)_* + \frac{1}{2}\bigg(\frac{\ln\hat{p}_+}{\hat{p}_+}\bigg)_* + \bigg(\frac{\pi^2}{18} - \frac{113}{72}\bigg)\bigg(\frac{1}{\hat{p}_+}\bigg)_* - \bigg(\frac{4}{3}\zeta_3 + \frac{41}{72}\pi^2 - \frac{1333}{1728}\bigg)\delta(\hat{p}_+) \\ &- \frac{2\hat{p}_+^2(3 - 2\hat{p}_+)}{3}\bigg[\mathrm{Li}_3(\hat{p}_+) + \mathrm{Li}_3(\hat{p}_+(2 - \hat{p}_+)) - 2\mathrm{Li}_3(1/(2 - \hat{p}_+)) + \frac{1}{4}\mathrm{Li}_3((1 - \hat{p}_+)^2) \\ &- \mathrm{Li}_2(\hat{p}_+)\ln(\hat{p}_+(2 - \hat{p}_+)^2) - \frac{\pi^2}{6}\ln(\hat{p}_+(2 - \hat{p}_+)) + \frac{1}{2}\ln(1 - \hat{p}_+)\ln^2\hat{p}_+ - \frac{5}{6}\ln^3\hat{p}_+ + \frac{1}{3}\ln^3(2 - \hat{p}_+)\bigg] \\ &+ \frac{6 + 11\hat{p}_+ - 32\hat{p}_+^2 + 18\hat{p}_+^3 - 4\hat{p}_+^4}{18\hat{p}_+}\mathrm{Li}_2((1 - \hat{p}_+)^2) - \frac{\pi^2}{18\hat{p}_+} + (1 - \hat{p}_+)\mathrm{Li}_2(\hat{p}_+ - 1) \\ &+ \frac{53}{216(1 - \hat{p}_+)} - \frac{674 - 1333\hat{p}_+ + 606\hat{p}_+^2}{216(1 - \hat{p}_+)^3}\ln(\hat{p}_+(2 - \hat{p}_+)) + \frac{6 - 69\hat{p}_+ - 123\hat{p}_+^2 + 100\hat{p}_+^3}{36}\ln^2\hat{p}_+ \\ &+ \frac{605 + 184\hat{p}_+ - 237\hat{p}_+^2 - 106\hat{p}_+^3}{108}\ln\hat{p}_+ \\ &+ \frac{2374 + 6219\hat{p}_+ - 15\,589\hat{p}_+^2 + 9890\hat{p}_+^3 - 2352\hat{p}_+^4 + 417\hat{p}_+^5 - 87\hat{p}_+^6}{864} \\ &+ \frac{530 + 137\hat{p}_+ - 1341\hat{p}_+^2 + 840\hat{p}_+^3 - 200\hat{p}_+^4 + 33\hat{p}_+^5 - 7\hat{p}_+^6}{216}\ln\frac{2 - \hat{p}_+}{\hat{p}_+^2}\bigg\}, \end{aligned} \tag{6}$$

where $\mathrm{Li}_2(z) = -\int_0^z \mathrm{d}t\ln(1 - t)/t$ is the dilogarithm, and $\mathrm{Li}_3(z) = \int_0^z \mathrm{d}t\mathrm{Li}_2(t)/t$.

B. The $\bar{x}$ spectrum in $B \to X_s \gamma$

We concentrate on the part of the spectrum that arises from the operator $O_7 = (e/16\pi^2) m_b \bar{s}_L \sigma^{\mu\nu} F_{\mu\nu} b_R$. This gives rise to the dominant part of the photon spectrum, and while other operators also influence the spectrum, the photon spectrum is only known analytically to order $\alpha_s^2 \beta_0$ for this piece (and for O_8, but not for the terms involving the four-quark operators). To study the convergence of the perturbative expansions and assess the theoretical uncertainties in the P_+ spectrum, it is sufficient to consider O_7. However, ultimately, for the actual determination of $|V_{ub}|$ the other contributions should also be included.

The photon spectrum to order α_s is given by [28,29]

$$\frac{1}{\Gamma_\gamma} \frac{\mathrm{d}\Gamma_{77}}{\mathrm{d}\bar{x}} = \delta(\bar{x}) - \frac{\alpha_s(m_b) C_F}{4\pi} \left[4\left(\frac{\ln \bar{x}}{\bar{x}}\right)_* + 7\left(\frac{1}{\bar{x}}\right)_* + \left(5 + \frac{4}{3}\pi^2\right)\delta(\bar{x}) - 6 - 3\bar{x} + 2\bar{x}^2 + 2(2-\bar{x})\ln\bar{x} \right] + \mathcal{O}(\alpha_s^2), \quad (7)$$

where $\Gamma_\gamma = G_F^2 |V_{tb} V_{ts}^*|^2 \alpha_{\rm em} m_b^3 [\overline{m}_b(m_b) C_7^{\rm eff}(m_b)]^2/(32\pi^4)$, and we have set $\mu = m_b$ for convenience. The BLM correction to the photon spectrum due to O_7 may be obtained by combining results given in [30,31]; again, setting $\mu = m_b$ it reads

$$\frac{1}{\Gamma_\gamma} \frac{\mathrm{d}\Gamma_{77}^{(\mathrm{BLM})}}{\mathrm{d}\bar{x}} = \frac{\alpha_s^2(m_b)}{\pi^2} \beta_0 \left[\frac{1}{2}\left(\frac{\ln^2 \bar{x}}{\bar{x}}\right)_* + \frac{13}{36}\left(\frac{\ln \bar{x}}{\bar{x}}\right)_* + \left(\frac{\pi^2}{18} - \frac{85}{72}\right)\left(\frac{1}{\bar{x}}\right)_* - \left(\frac{1}{3}\zeta_3 + \frac{91}{216}\pi^2 + \frac{631}{432}\right)\delta(\bar{x}) \right.$$
$$\left. + \frac{(6 + 6\bar{x} - 3\bar{x}^2)\mathrm{Li}_2(1-\bar{x}) - \pi^2}{18\bar{x}} + \frac{2-\bar{x}}{4}\ln^2\bar{x} - \frac{38 - 33\bar{x} + 7\bar{x}^2 + 6\bar{x}^3}{36(1-\bar{x})}\ln\bar{x} + \frac{66 + 21\bar{x} - 38\bar{x}^2}{72} \right]. \quad (8)$$

III. RELATION BETWEEN THE SPECTRA

One can define a weighting function W that relates, in the shape function regions, weighted integrals of the photon energy spectrum in $B \to X_s \gamma$ to integrals of the P_+ spectrum in $B \to X_u l \nu$:

$$\int_0^\Delta \mathrm{d}P_+ \frac{\mathrm{d}\Gamma_u}{\mathrm{d}P_+} = \frac{|V_{ub}|^2}{|V_{tb} V_{ts}^*|^2} \frac{\pi}{6\alpha_{\rm em} C_7^{\rm eff}(m_b)^2} \frac{m_B^2}{\overline{m}_b(m_b)^2} \int_0^\Delta \mathrm{d}P_\gamma W(\Delta, P_\gamma) \frac{\mathrm{d}\Gamma_{77}}{\mathrm{d}P_\gamma}. \quad (9)$$

The weighting function W can be computed perturbatively at leading order in $\Lambda_{\rm QCD}/m_b$, because the shape function contribution that captures the leading nonperturbative physics at scales of order ΛQCD is identical in the two spectra. The corresponding perturbation series depends on the scales m_b and $\sqrt{m_b \Lambda_{\rm QCD}}$. The function W is also free of logarithms of the form $\alpha_s^n \ln^m(m_b^2/\mu^2)$, $m = n+1, \ldots, 2n$, which are universal in the two spectra and cancel from the relation in Eq. (9).

The origin of the $m_B^2/\overline{m}_b^2$ factor in the definition of W deserves comment. At leading order in $\Lambda_{\rm QCD}/m_b$, the effects of the shape function $f(\omega)$ may be simply included by making the replacement

$$m_b \to m_b^* = m_b + \omega, \quad (10)$$

in the tree level partonic rate, and then convoluting the differential rate with the shape function $f(\omega)$ [7],

$$\mathrm{d}\Gamma = \int \mathrm{d}\Gamma^{\rm parton}|_{m_b \to m_b^*} f(\omega) \mathrm{d}\omega. \quad (11)$$

This prescription also generalizes to higher orders in $\Lambda_{\rm QCD}/m_b$, provided that the replacement (10) is only applied to factors of m_b which arise from kinematics, and not from coefficients of operators in the Hamiltonian [16] and correctly reproduces the class of "kinematic" subleading effects (proportional to the leading order shape function) which do not arise from the expansion of the heavy quark fields.

Applying this procedure to the tree level spectra, we have

$$\frac{\mathrm{d}\Gamma_u}{\mathrm{d}P_+} \propto \int (m_b + \omega)^5 \delta(P_+ - \Lambda + \omega) f(\omega) \mathrm{d}\omega = \int (m_B - \tilde{\omega})^5 \delta(P_+ - \tilde{\omega}) f(\Lambda - \tilde{\omega}) \mathrm{d}\tilde{\omega} \quad (12)$$

for semileptonic decays (where $\tilde{\omega} \equiv \Lambda - \omega$), and

$$\frac{\mathrm{d}\Gamma_{77}}{\mathrm{d}P_\gamma} \propto \overline{m}_b(m_b)^2 \int (m_B - \tilde{\omega})^3 \delta(P_\gamma - \tilde{\omega}) f(\Lambda - \tilde{\omega}) \mathrm{d}\tilde{\omega} \quad (13)$$

for radiative decays, where two powers of $\overline{m}_b(m_b)$ originate from C_7 in the effective Hamiltonian to which the replacement (10) does not apply. This gives

$$\int_0^\Delta \frac{d\Gamma_u}{dP_+} dP_+ \propto \int_0^\Delta \frac{(m_B - P_\gamma)^2}{\overline{m}_b(m_b)^2} \frac{d\Gamma_{77}}{dP_\gamma} dP_\gamma = \frac{m_B^2}{\overline{m}_b(m_b)^2} \int_0^\Delta \left(1 - \frac{2P_\gamma}{m_B} + \dots\right) \frac{d\Gamma_{77}}{dP_\gamma} dP_\gamma. \tag{14}$$

Thus, it is natural to pull the leading factor of $m_B^2/\overline{m}_b(m_b)^2$ out of the definition of W. As we will show, factoring out this term rather than leaving the "partonic" factor $m_b^2/\overline{m}_b(m_b)^2$ in the definition of W dramatically reduces the absolute size of the perturbative corrections to W.

To calculate W, we first expand both parton level spectra in powers of $\hat{p}_+$ and $\bar{x}$ respectively, since in the shape function region both are $\mathcal{O}(\Lambda_{QCD}/m_b)$. At leading order in Λ_{QCD}/m_b, this simply corresponds to keeping the first four terms in Eqs. (6) and (8). At leading order in the Λ_{QCD}/m_b expansion relevant for the shape function regime, the Feynman diagrams which give the coefficient of the shape function give the partonic result with the substitution $\hat{p}_+ \to \hat{p}_+ + \hat{k}_+$ and $\bar{x} \to \bar{x} + \hat{k}_+$, where $\hat{k}_+$ is the light-cone component of the residual momentum of the heavy quark, which is of the same order. (The dimensionless variables $\hat{k}_+$ and $\hat{\omega}$ have support between $-\infty$ and Λ/m_b.) Thus, we have

$$\frac{1}{\Gamma_0} \frac{d\Gamma_u^{(BLM)}}{d\hat{p}_+} = \frac{\alpha_s^2(m_b)}{\pi^2} \beta_0 \delta(\hat{\omega} - \hat{k}_+) \bigg[\frac{1}{2} \left(\frac{\ln^2(\hat{p}_+ + \hat{\omega})}{\hat{p}_+ + \hat{\omega}} \right)_* + \frac{1}{2} \left(\frac{\ln(\hat{p}_+ + \hat{\omega})}{\hat{p}_+ + \hat{\omega}} \right)_* + \left(\frac{\pi^2}{18} - \frac{113}{72} \right) \left(\frac{1}{\hat{p}_+ + \hat{\omega}} \right)_* - \left(\frac{4}{3}\zeta_3 + \frac{41}{72}\pi^2 - \frac{1333}{1728} \right) \delta(\hat{p}_+ + \hat{\omega}) + \dots \bigg], \tag{15}$$

and similarly for $d\Gamma_{77}/d\bar{x}$. This gives, for each spectrum, the matching conditions onto the leading nonlocal operator $O_0(\omega) = \bar{b}\delta(\omega - iD_+)b$ in the OPE. The B meson matrix element of $O_0(\omega)$ gives the leading order b quark light-cone distribution function, or shape function. In principle, to determine the coefficient of $O_0(\omega)$ we must include the radiative corrections to the parton level matrix element of $O_0(\omega)$; however, since these terms are common to both spectra and therefore drop out of W, we do not need to worry about them here.

Including the full one-loop corrections and the BLM two-loop contributions, and setting the scale $\mu = m_b$ for simplicity, the function W is given by

$$W(\Delta, P_\gamma) = 1 + \frac{C_F\alpha_s(m_b)}{4\pi} \left(\frac{5}{3} \ln\frac{m_b}{\Delta - P_\gamma} - \frac{2\pi^2}{3} + \frac{167}{36} \right) + \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2} \beta_0 \left(\frac{5}{6} \ln^2 \frac{m_b}{\Delta - P_\gamma} + \frac{14}{3} \ln\frac{m_b}{\Delta - P_\gamma} - \frac{16\pi^2}{9} + \frac{3857}{144} - 12\zeta_3 \right) + \dots \tag{16}$$

where the ellipses denote non-BLM two-loop terms, higher orders in perturbation theory and nonperturbative corrections suppressed by powers of Λ_{QCD}/m_b. Equation (16) is the main result of this paper.

It is instructive to compare Eq. (16) with the next-to-leading log result in SCET. Using the results of Refs. [10,32] (see also [33]), we find

$$W^{NLL}(\Delta, P_\gamma) = T(a) \left\{ 1 + \frac{C_F\alpha_s(m_b)}{4\pi} H(a) + \frac{C_F\alpha_s(\mu_i)}{4\pi} \left[4f_2(a) \ln\frac{m_b(\Delta - P_\gamma)}{\mu_i^2} - 3f_2(a) + 2f_3(a) \right] \right\}, \tag{17}$$

where

$$T(a) = \frac{2(6-a)}{(4-a)(3-a)},$$
$$H(a) = -\frac{4(486 - 389a + 103a^2 - 9a^3)}{(6-a)(4-a)^2(3-a)^2} - 4\psi'(3-a) - cf_2(a), \tag{18}$$
$$f_2(a) = -\frac{30 - 12a + a^2}{(6-a)(4-a)(3-a)},$$
$$f_3(a) = \frac{2(138 - 90a + 18a^2 - a^3)}{(6-a)(4-a)^2(3-a)^2},$$

ψ' is the derivative of the digamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$, and

$$a = \frac{\Gamma_0}{\beta_0} \ln\frac{\alpha_s(\mu_i)}{\alpha_s(m_b)}, \qquad c = \frac{4}{\beta_0} \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) \left(\frac{\alpha_s(\mu_i)}{\alpha_s(m_b)} - 1 \right). \tag{19}$$

Here $\Gamma_0 = 4C_F$ and $\Gamma_1 = 8C_F(67/6 - \pi^2/2 - 5n_f/9)$ are the first two coefficients of the cusp anomalous dimension, and $\beta_1 = 102 - 38n_f/3$. Setting $\mu_i \sim \Lambda_{QCD}m_b$ sums all leading and subleading logarithms of the form $\alpha_s^n \ln^n(\mu_i/m_b)$ and $\alpha_s^n \ln^{n-1}(\mu_i/m_b)$. Expanding Eq. (17) to order $\alpha_s^2(m_b)$, we find

$$W^{\mathrm{NLL}}(\Delta, P_\gamma) = 1 + \frac{C_F\alpha_s(m_b)}{4\pi}\left(\frac{5}{3}\ln\frac{m_b^2}{\mu_i^2}\right) + \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2}\left(\frac{5\beta_0}{6} + \frac{92}{27}\right)\ln^2\frac{m_b^2}{\mu_i^2} + \frac{C_F\alpha_s(m_b)}{4\pi}\left(\frac{5}{3}\ln\frac{\mu_i^2}{m_b(\Delta - P_\gamma)} - \frac{2\pi^2}{3} + \frac{167}{36}\right) + \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2}\left[\left(\frac{5\beta_0}{3} + \frac{184}{27}\right)\ln\frac{\mu_i^2}{m_b(\Delta - P_\gamma)} + \frac{14\beta_0}{3} + \frac{64}{3}\psi''(3) - \frac{85\pi^2}{27} + \frac{1234}{81}\right]\ln\frac{m_b^2}{\mu_i^2} + \ldots, \tag{20}$$

where the first three terms are the leading log result, and the following ones are subleading logs. Note that the renormalization group equations sum logs of m_b^2/μ_i^2, not $m_b(\Delta - P_\gamma)/\mu_i^2$, since for $\mu_i^2 \sim \Lambda_{\mathrm{QCD}} m_b$ only the first are parametrically enhanced. The result is formally independent of μ_i, as can be seen by reexpanding the logs in Eq. (20),

$$W^{\mathrm{NLL}}(\Delta, P_\gamma) = 1 + \frac{C_F\alpha_s(m_b)}{4\pi}\left(\frac{5}{3}\ln\frac{m_b}{\Delta - P_\gamma} - \frac{2\pi^2}{3} + \frac{167}{36}\right) + \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2}\left[\left(\frac{5\beta_0}{6} + \frac{92}{27}\right)\ln^2\frac{m_b}{\Delta - P_\gamma} + \left(\frac{14\beta_0}{3} + \frac{64}{3}\psi''(3) - \frac{85\pi^2}{27} + \frac{1234}{81}\right)\ln\frac{m_b}{\Delta - P_\gamma} + \ldots\right] + \ldots, \tag{21}$$

where we have dropped terms of order $\alpha_s^2\ln^n[m_b(\Delta - P_\gamma)/\mu_i^2]$, which are next-to-next-to-leading order. This result agrees with the corresponding one-loop and two-loop BLM terms in Eq. (16).

Eqs. (16) and (21) both provide approximations to the full expression for W at two-loops, so it is instructive to compare them. Numerically, the $\mathcal{O}(\alpha_s^2)$ terms are

$$\begin{aligned} W^{(\alpha_s^2)} &= \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2}\left[(0.83\beta_0 + 3.41)\ln^2\frac{m_b}{\Delta - P_\gamma} + (4.67\beta_0 - 19.1)\ln\frac{m_b}{\Delta - P_\gamma} - (5.19\beta_0 + c_0)\right] \\ &= \frac{C_F\alpha_s^2(m_b)}{(4\pi)^2}\left[\left(6.94\frac{\beta_0}{25/3} + 3.41\right)\ln^2\frac{m_b}{\Delta - P_\gamma} + \left(38.9\frac{\beta_0}{25/3} - 19.1\right)\ln\frac{m_b}{\Delta - P_\gamma} - \left(43.2\frac{\beta_0}{25/3} + c_0\right)\right], \end{aligned} \tag{22}$$

where a complete two-loop calculation is required to determine c_0.

For both the double and single log terms, the BLM-enhanced term is about a factor of 2 larger than the non-BLM term, which suggests that it may also dominate the nonlogarithmic term to a similar degree. In contrast, the leading log approximation is clearly poorly behaved at this order: for $\mu_i^2/m_b^2 \sim (\Delta - P_\gamma)/m_b = 1/9$, the double, single and nonlogarithmic terms in Eq. (22) are in the ratio

$$\mathcal{O}(\log^2):\mathcal{O}(\log):\mathcal{O}(\log^0) = 1:0.87:(-0.86 - 0.02c_0). \tag{23}$$

This reflects the fact that the logarithmic enhancement is not sufficient for the double log to dominate over the single or zero log terms, nor for the single log to dominate over the BLM-enhanced piece of the zero log term. The same conclusion is reached by comparing the size of the LL and NLL terms in Eq. (20). This suggests that a fixed order calculation, rather than a leading log calculation, is more appropriate for W. We will discuss the numerical significance of this for the extraction of $|V_{ub}|$, along with the poor convergence of both the leading log and fixed order calculations, in the next section.

Another source of corrections to W comes from terms suppressed by $\Lambda_{\mathrm{QCD}}/m_b$. Some of these modify W without introducing new unknown hadronic matrix elements, while others involve nonperturbative matrix elements of higher-dimension nonlocal operators [6], which cannot presently be computed. The mismatch of the powers of m_b in the two spectra gives rise to the order $\Lambda_{\mathrm{QCD}}/m_b$ correction to W contained in Eq. (14), $-2P_\gamma/m_B$. At the same order, there are also corrections that come from expanding the b quark fields in powers of $\Lambda_{\mathrm{QCD}}/m_b$. These effects are sensitive to the Dirac structure of the current, so do not cancel from W. We can extract the full $\Lambda_{\mathrm{QCD}}/m_b$ correction from Ref. [34], and find

$$W^{(\Lambda_{\mathrm{QCD}}/m)} = -\frac{8P_\gamma - 2\Lambda}{3m_B} + \ldots. \tag{24}$$

The corrections that depend on subleading shape functions are also given in [34] and involve five unknown nonperturbative functions, whose effects we do not attempt to model here. Some of the $\mathcal{O}(\alpha_s)$ corrections to Eq. (24) can be obtained by expanding the results in Sec. II to higher order in $\hat{p}_+$ and $\bar{x}$. We find that these are small compared to the perturbative uncertainties in W at leading order in $\Lambda_{\mathrm{QCD}}/m_b$.

IV. IMPLICATIONS AND DISCUSSIONS

To eliminate the charm background kinematically from semileptonic $B \to X_u \ell \bar{\nu}$ decays, one has to impose a cut $P_+ < m_D^2/m_B \simeq 0.66\mathrm{GeV}$. Then measurements of the P_+ spectrum in $B \to X_u \ell \bar{\nu}$ and the $P\gamma$ spectrum in $B \to X_s\gamma$, together with the theoretical input of $W(\Delta, P_\gamma)$ will allow a determination of $|V_{ub}|$ using Eq. (9).

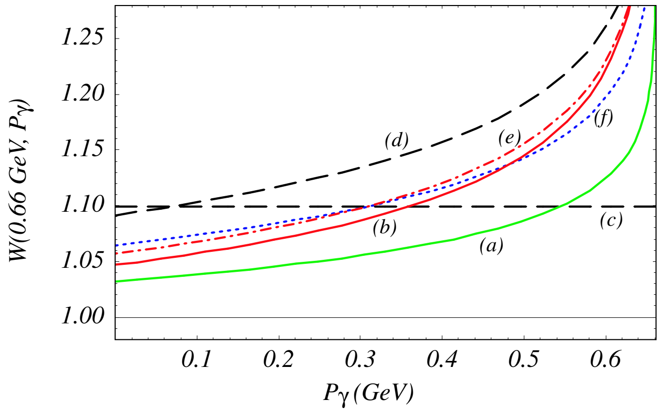

FIG. 1 (color online). $W(\Delta, P_\gamma)$ as a function of P_γ. At tree level $W = 1$, and the curves include different orders in the expansion: (a) the $\mathcal{O}(\alpha_s)$ result; (b) all known $\mathcal{O}(\alpha_s^2)$ terms, Eq. (22) with $c_0 = 0$; (c) the LL RGE resummed result; (d) the NLL RGE resummed result, Eq. (17); (e) the $\mathcal{O}(\alpha_s^2\beta_0)$ result, Eq. (16) and (f) the NLL expression expanded to $\mathcal{O}(\alpha_s^2)$, Eq. (20).

In Fig. 1 we plot $W(\Delta, P_\gamma)$ for $\Delta = 0.66$ GeV, $\alpha_s(m_b) = 0.22$ and $m_b = 4.8$ GeV in different approximations. At tree level, $W(\Delta, P_\gamma) = 1$. Curve 1(a) is the order α_s result, while the result with all known α_s^2 contributions is shown in curve 1(b) [i.e., Eq. (22), the BLM result plus the non-BLM coefficient of the double and single logs]. The dashed black curves show the RGE resummed results for $\mu_i = \sqrt{m_b\Delta} \simeq 1.78$ GeV. The straight line 1(c) is the LL result, whereas the long dashed curve 1(d) is the NLL result in Eq. (17). The LL and NLL results use 1- and 2-loop running for α_s, respectively. Finally, the dot-dashed curve 1(e) contains just the BLM terms at order α_s^2, Eq. (16), while the short dashed curve 1(f) contains NLL expression expanded to order α_s^2, Eq. (20).

The difference between the dashed 1(d) and 1(f) curves shows that the terms in the NLL sum beyond $\mathcal{O}(\alpha_s^2)$ are not negligible. However, the logs that are summed do not dominate over other contributions in the perturbation series; e.g., the difference between Eqs. (20) and (21) is NNLL, but it is comparable to the non-BLM LL and NLL terms at $\mathcal{O}(\alpha_s^2)$. Therefore, we view the fixed order result, curve 1(b), as our best estimate of the perturbation theory prediction of W. The difference of these curves provides an estimate of the uncertainty related to higher order uncalculated terms. In addition, the μ_i-dependence of the NLL resummed result is larger using two-loop than one-loop running for α_s, also indicating that renormalization group improved perturbation theory may not lead to an improved expansion.

To assess the significance of these results for the determination of $|V_{ub}|$ using Eq. (9), we integrated these various expansions of $W(\Delta, P_\gamma)$ against a simple parametrization of the experimentally measured $B \to X_s\gamma$ spectrum [13]. For the optimal cut, $\Delta = 0.66$ GeV, the integral on the right-hand side of Eq. (9) normalized to that integral at tree level (corresponding to $W = 1$) is shown in Table I at order α_s, $\alpha_s^2\beta_0$, using the LL and NLL RGE resummations, and all known $\mathcal{O}(\alpha_s^2)$ terms. The simple parametrization

TABLE I. Weighted integral $\int_0^\Delta \mathrm{d}P_\gamma W(\Delta, P_\gamma)(\mathrm{d}\Gamma_s/\mathrm{d}P_\gamma)$, normalized to $\int_0^\Delta \mathrm{d}P_\gamma(\mathrm{d}\Gamma_s/\mathrm{d}P_\gamma)$ for $\Delta = 0.66$ GeV, taking the simple parametrization (25) of the experimental photon energy spectrum.

Tree	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2\beta_0)$	LL	NLL	all known $\mathcal{O}(\alpha_s^2)$
1	1.10	1.19	1.10	1.22	1.18

$$\frac{\mathrm{d}\Gamma_s}{\mathrm{d}P_\gamma}\bigg|_{\mathrm{exp.}} \propto x^{(a-1)}e^{-ax}, \qquad x \equiv \frac{P_\gamma}{\tilde{\Lambda}}, \tag{25}$$

with $\tilde{\Lambda} \sim 0.9$ GeV and $a \sim 6.1$ provides a crude but, for our purposes, sufficient fit to the data,[2] as the ratios in Table I are quite insensitive to the precise shape of the spectrum. (Using the parameters $\tilde{\Lambda} \sim 0.66$ GeV and $a \sim 3.3$, which is the shape function fit rather than the photon spectrum [35], and so corresponds to a rather different shape, only changes the entries in Table I to 1, 1.08, 1.15, 1.10, 1.18, 1.14, respectively. However, it changes $\int_0^\Delta \mathrm{d}P_\gamma(\mathrm{d}\Gamma_s/\mathrm{d}P_\gamma)$ by about a factor of 2.)

The convergence of the result in Table I going from tree level to $\mathcal{O}(\alpha_s)$ to $\mathcal{O}(\alpha_s^2\beta_0)$ is poor, and that going from tree level to LL to NLL resummation is worse. This poor behavior of the perturbation series may be related to the fact that in the nonlocal OPE there are $\mathcal{O}(\Lambda_{\mathrm{QCD}}/m_b)$ nonperturbative corrections to Eq. (9) arising both from the explicit factor of Λ in Eq. (24) as well as the subleading shape functions, and so there is a renormalon ambiguity in the perturbation series for W at this order.

The subleading twist terms in Eq. (24) give a -0.19 correction to the numbers in Table I. (The "trivial" part of this correction, $-2P_\gamma/m_B$, that is due to the mismatch of the two powers of m_b smeared in the two spectra, gives numerically the same result.) While this is a calculable effect and not an uncertainty, there are incalculable subleading shape functions that enter at the same order, which can only be modeled. Thus, the perturbative and nonperturbative uncertainties in this determination of $|V_{ub}|$ are probably comparable.

Because the experimental photon spectrum is peaked around $P_\gamma \sim 0.8$ GeV, the weighted integral (9) is dominated by small values of $\Delta - P_\gamma$, increasing the importance of the logarithmically enhanced terms in (21).

[2]A parametrization of the physical $B \to X_s\gamma$ spectrum is not available, only the raw spectrum or fits to shape function parameters [35]. However, the shape function fit includes $\mathcal{O}(\alpha_s)$ corrections differently than how they enter W. Thus, the numbers in Table I should be taken only as indicative of the size of the corrections.

However, this numerically large logarithm is not summed by the RGE, since it is not logarithms of $(\Delta - P_\gamma)/m_b$ but rather of μ_i/m_b which are summed.

Had we not factored out $m_B^2/\overline{m}_b^2$ in the definition of W in Eq. (9), there would be an additional perturbative factor of $m_b^2/\overline{m}_b^2$ in W which, when expanded out, would modify the expressions in Eqs. (16), $H(a)$ in (18), (20)–(22), and (24). If we define a weighting function $W'(\Delta, P_\gamma)$ in this way, then instead of Eq. (22) we obtain

$$W'^{(\alpha_s^2)} = \frac{C_F \alpha_s^2(m_b)}{(4\pi)^2}\bigg[(0.83\beta_0 + 3.41)\ln^2\frac{m_b}{\Delta - P_\gamma} + (4.67\beta_0 - 1.35)\ln\frac{m_b}{\Delta - P_\gamma} + (32.3\beta_0 + c_0')\bigg], \tag{26}$$

and the nonlogarithmic BLM term is considerably larger than either of the logarithmic terms. In this case the numerical results in Table I would read 1, 1.29, 1.51, 1.10, 1.43, 1.53, respectively, and we would have to assign a much larger perturbative uncertainty to W' than to W. However, this is due to the bad perturbative behavior of $m_b^2/\overline{m}_b^2$, and not of the spectra themselves. In this case the analog of Eq. (24), $W'^{(\Lambda_{\rm QCD}/m)} = -8(P_\gamma - \Lambda)/(3m_B) + \ldots$, gives a -0.01 correction to the figures in the previous sentence. It is interesting to check the consistency of the results. Combining all known order α_s^2 and $\Lambda_{\rm QCD}/m_b$ terms, we obtain

$$m_B^2 \frac{\int_0^\Delta {\rm d}P_\gamma W(\Delta, P_\gamma)({\rm d}\Gamma_{77}/{\rm d}P_\gamma)}{\int_0^\Delta {\rm d}P_\gamma W'(\Delta, P_\gamma)({\rm d}\Gamma_{77}/{\rm d}P_\gamma)} \simeq (4.27\ {\rm GeV})^2, \tag{27}$$

quite consistently with the physical value of $\overline{m}_b(m_b)$. In comparison, Eq. (27) with the NLL result for W and W' gives $(4.50\ {\rm GeV})^2$. We learn that if we keep all two-loop corrections, the physical result is quite independent of whether we calculate it in terms of W or W', while the same cannot be said about the NLL resummation result.

The measured $B \to X_s\gamma$ photon spectrum together with $W(\Delta, P_\gamma)$ given by the sum of Eqs. (22) and (24) determines $\int_0^\Delta {\rm d}P_\gamma W(\Delta, P_\gamma)({\rm d}\Gamma_s/{\rm d}P_\gamma)$, which in turn determines $|V_{ub}|$ from a measurement of the partially integrated P_+ spectrum in $B \to X_u \ell\bar\nu$ using Eq. (9). The theoretical uncertainty of $|V_{ub}|$ from perturbation theory alone using this method is half the error of the results in Table I. However, because of the poor behavior of the perturbation series, the full two-loop calculation of $W(\Delta, P_\gamma)$ would be desirable. Furthermore, the perturbative series is likely to improve if the unknown matrix elements at $\mathcal{O}(\Lambda_{\rm QCD}/m_b)$ are expressed in terms of physical quantities, so that the leading renormalon ambiguity in W is canceled. In addition, and probably more importantly, effects of operators other than O_7 need to be included, in particular, that of O_2 and O_8 may be important. In the NLL resummed result, including these is straightforward using Eq. (4) in [32]. However, for the BLM result one needs to combine the analytically known virtual contributions [30] with the bremsstrahlung contributions [31], which are only known numerically for the O_2 operator. Work in this direction is in progress.

In summary, we calculated the order $\alpha_s^2\beta_0$ corrections to the P_+ spectrum in $B \to X_u\ell\bar\nu$ decay and studied the uncertainties in extracting $|V_{ub}|$ using a measurement of the P_γ spectrum in $B \to X_s\gamma$. We showed that the factor of $m_b^2/\overline{m}_b^2$ in $W(\Delta, P_\gamma)$ at lowest order naturally becomes $m_B^2/\overline{m}_b^2$ when subleading effects are included, and results in much reduced perturbative corrections. We found that the NLL RGE resummation is of limited use, because the logs that it sums do not dominate over the nonlogarithmic terms. This may have implications for the phenomenological usefulness of other applications of RGE resummations in inclusive heavy to light decays.

ACKNOWLEDGMENTS

We thank Adam Leibovich and Iain Stewart for helpful conversations. Z. L. and M. L. thank the Max Planck Institute for Physics (Werner-Heisenberg-Institute) for its hospitality while some of this work was completed. This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098 (Z. L.); and by the Natural Sciences and Engineering Research Council of Canada (M. L.).

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