Leptonic *CP* violation in a two parameter model

Samina S. Masood*
Department of Earth Sciences, SUNY Oswego, Oswego, New York 13126, USA

Salah Nasri†
Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA

Joseph Schechter‡
Department of Physics, Syracuse University, Syracuse, New York 13244-1130, USA

(Received 23 December 2004; published 10 May 2005)

We further study the "complementary" ansatz, $\mathrm{Tr}(M_\nu) = 0$, for a prediagonal light Majorana type neutrino mass matrix. Previously, this was studied for the *CP* conserving case and the case where the two Majorana type *CP* violating phases were present but the Dirac type *CP* violating phase was neglected. Here we employ a simple geometric algorithm which enables us to "solve" the ansatz including all three *CP* violating phases. Specifically, given the known neutrino oscillation data and an assumed two parameter (the third neutrino mass m_3 and the Dirac *CP* phase δ) family of inputs we predict the neutrino masses and Majorana *CP* phases. Despite the two parameter ambiguity, interesting statements emerge. There is a characteristic pattern of interconnected masses and *CP* phases. For large m_3 the three neutrinos are approximately degenerate. The only possibility for a mass hierarchy is to have m_3 smaller than the other two. A hierarchy with m_3 largest is not allowed. Small *CP* violation is possible only near two special values of m_3. Also, the neutrinoless double beta decay parameter is approximately bounded as $0.020\ \mathrm{eV} < |m_{ee}| < 0.185\ \mathrm{eV}$. As a by-product of looking at physical amplitudes we discuss an alternative parametrization of the lepton mixing matrix which results in simpler formulas. The physical meaning of this parametrization is explained.

DOI: 10.1103/PhysRevD.71.093005 PACS numbers: 14.60.Pq, 11.30.Er, 13.15.+g

I. INTRODUCTION

The remarkable experimental achievements (for some recent examples see Refs. [1–3]) relating to neutrino oscillations [4] have brought much closer to reality the goal of determining the "light" neutrino masses and the presumed 3×3 lepton mixing matrix. It is possible that more than three light neutrinos are required in order to understand the results of the LSND Collaboration experiment [5]. However, we consider it reasonable, before deciding on this, to wait for further supporting evidence as should be supplied soon by the MiniBooNE Collaboration [6]. The mixing matrix contains three mixing angles and, if the neutrinos are considered to be Dirac type fermions, a single *CP* violation phase. That would be completely analogous to the situation prevailing in the quark sector of the electroweak theory. But it seems very interesting to consider the possibility that the three light neutrinos are Majorana type fermions. This involves only half as many fermionic degrees of freedom and would be mandated if neutrinoless double beta decay were to be conclusively established. The Majorana neutrino scenario implies the existence of two additional *CP* violation phases [7–10]. Then nine quantities (beyond the charged lepton masses) would be required for the specification of the lepton sector: three neutrino masses, three mixing angles and three *CP* violation phases.

According to a recent analysis [11] it is possible to extract from the data to good accuracy, two squared neutrino mass differences: $m_2^2 - m_1^2$ and $|m_3^2 - m_2^2|$, and two intergenerational mixing angle squared sines: s_{12}^2 and s_{23}^2. Furthermore the intergenerational mixing parameter s_{13}^2 is found to be very small. Thus 5 out of 9 quantities needed to describe the leptonic sector in the Majorana neutrino scenario can be considered as "known." For many purposes it is desirable to get an idea of the remaining 4 parameters. As an aid in partially determining the other parameters, a so-called "complementary ansatz" was proposed [12–15]. The name arises from the fact that if *CP* violation is neglected, the ansatz determines (up to two different cases) all three neutrino masses, given the two known squared mass differences.

This complementary ansatz simply reads

$$\mathrm{Tr}\,(M_\nu) = 0. \qquad (1)$$

Here M_ν is the symmetric, but in general complex, prediagonal Majorana neutrino mass matrix. It is brought to diagonal form by the transformation

$$U^T M_\nu U = \mathrm{diag}(m_1, m_2, m_3) = \hat{M}_\nu, \qquad (2)$$

where U is a unitary matrix and the m_i may be chosen as real, positive. We impose the condition in a basis where the

*Electronic address: masood@oswego.edu
†Electronic address: snasri@physics.umd.edu
‡Electronic address: schechte@phy.syr.edu

charged leptons are diagonal so that U gets identified with the lepton mixing matrix. This choice is briefly motivated below and in somewhat more detail elsewhere [16]. Once we have made this choice the ansatz, of course, will not hold in any other basis. We nowhere need to impose basis invariance of the ansatz since, once it is imposed in the given basis, the physical Lagrangian is already specified.

To count parameters let us rewrite Eq. (2) as $M_\nu = U^* \hat{M}_\nu U^\dagger$. M_ν has in general 12 arbitrary real parameters which equal the sum of the three real parameters of $\hat{M}_\nu$ and the nine real parameters of a general unitary matrix U. Now, as will be reviewed in the next section, the conventional lepton mixing matrix K has only six parameters—three angles and three phases. Three additional phases are needed to get the most general U. These three phases can be included by multiplying K on the left by a diagonal matrix of phases. However, these additional phases may be eliminated by a rephasing of the charged lepton fields which sit to the left of K in the standard weak Lagrangian. This means that if a matrix U which diagonalizes M_ν is found, there will always exist a physically equivalent situation in which the three phases to the left of K are eliminated. In this rephased basis the ansatz may be restated as $\mathrm{Tr}(K^* \hat{M}_\nu K^\dagger) = 0$.

Since Eq. (1) comprises two real equations, it gives two conditions on the unknown 4 parameters of the lepton sector. In other words, the lepton sector is described by a two parameter family of solutions. This can be approximately simplified [12,14,15] by noting that the effects of one CP violation phase get suppressed when the small quantity s_{13} vanishes. Then there is a one parameter family of solutions describing the lepton sector and it is straightforward to compute physical quantities for parameter choices which span this family. The main purpose of the present paper is to find the general solutions of Eq. (1) without making this approximation. This gives a two parameter family which allows one to study the interplay of all three CP violation phases.

A plausibility argument supporting the complementary ansatz is concisely presented in Secs. 2 and 3 of [16]. It is based on using the SO(10) grand unification group in the approximation that the nonseesaw neutrino mass term dominates. The Higgs fields which can contribute to fermion masses at tree level are the **10**, **120**, and the **126**. If only a single **126** appears (but any number of the others) one has the relation

$$\mathrm{Tr}\,(M^D - rM^E) \propto Tr(M_\nu), \tag{3}$$

where M^D and M^E are, respectively, the prediagonal mass matrices of the charge $-1/3$ quarks and charge -1 leptons, while $r \approx 3$ takes account of running masses from the grand unified scale to about 1 GeV. Now one of the major surprises generated by the neutrino oscillation experiments is that, unlike the quark mixing matrix which has the form $\mathrm{diag}(1, 1, 1) + O(\epsilon)$, the lepton mixing matrix is not at all close to the unit matrix. This suggests a further approximation in which one takes M^D and M^E to be diagonal but allows the neutrino mass matrix to be far from the unit matrix. Then the left hand side of Eq. (3) is approximately $(m_b - 3m_\tau)$, which is in turn close to zero.

As we will see, the model makes a number of characteristic predictions for the neutrino mass spectrum which should enable it to be readily tested in the near future. A very recent review of many other models is given in Ref. [17].

For convenience, our notation (essentially the standard one) for the lepton mixing matrix and the corresponding parametrized ansatz is given in Sec. II.

In Sec. III the ansatz is solved in the sense of providing a geometrical algorithm which, given the two input quantities m_3 (third neutrino mass, taken positive) and δ (conventional CP violation phase in the lepton mixing matrix), enables one to predict the other two neutrino masses as well as the other two CP violation phases. Of course, the experimental knowledge on the neutrino squared mass differences and CP conserving intergenerational mixing angles are taken to be "known." We separate the solutions into two types I and II, depending, respectively, on whether m_3 is the largest or the smallest of the neutrino masses. In addition, there is a discrete ambiguity corresponding to reflecting a triangle involved in the algorithm. A "panoramic" view of the predictions as functions of m_3 and δ are presented in a convenient tabular form. The greatest allowed value of m_3 is determined by a cosmology bound. As m_3 decreases, a point is reached at which the type I solutions no longer exist. As m_3 decreases even further, the type II solutions also cease to exist. The corresponding values of m_3 at which these solutions become disallowed depend on the assumed value of the input δ. This correlation is studied analytically.

Some physical considerations are discussed in Sec. IV. First, the dependence on the experimentally bounded squared mixing angle s_{13}^2 is investigated. We present also a chart showing the dependence of the neutrinoless double beta decay parameter $|m_{ee}|$ on the input parameters m_3 and δ. Even though the inputs are varying over a fairly large range, the rather restrictive approximate bounded range $0.020\ \mathrm{eV} < |m_{ee}| < 0.185\ \mathrm{eV}$ emerges from the ansatz.

After calculating observable quantities in the model one observes that they depend more simply on certain linear combinations of the conventional "Dirac" and "Majorana" CP violation phases. In Sec. V we discuss an alternative parametrization of the lepton mixing matrix in which these combinations occur directly. In this parametrization the three phases just correspond to the three possible intergenerational mixings. An "invariant" combination of these three corresponds to the usual Dirac phase δ.

We conclude in Sec. VI which contains a brief summary and a discussion of results which emphasize some unique features of the present work.

II. PARAMETERIZED COMPLEMENTARY ANSATZ

We define the lepton mixing matrix K from the charged gauge boson interaction term in the leptonic sector of the electroweak Lagrangian:

$$\mathcal{L} = \frac{ig}{\sqrt{2}} W_\mu^- \bar{e}_L \gamma_\mu K \nu + \text{H.c.} \tag{4}$$

Note that the "mass diagonal" neutrino fields ν_i are related to the fields ρ_i in the prediagonal mass basis by the matrix equation

$$\rho = U\nu. \tag{5}$$

We adopt essentially what seems to be the most common parametrization:

$$K = K_{\exp}\omega_0^{-1}(\tau), \tag{6}$$

where a unimodular diagonal matrix of phases is defined as

$$\omega_0(\tau) = \text{diag}(e^{i\tau_1}, e^{i\tau_2}, e^{i\tau_3}), \qquad \tau_1 + \tau_2 + \tau_3 = 0. \tag{7}$$

The remaining factor $K_{\exp}$ which is the only part needed for describing ordinary neutrino oscillations is written as the product of three successive two dimensional unitary transformations,

$$K_{\exp} = \omega_{23}(\theta_{23}, 0)\omega_{13}(\theta_{13}, -\delta)\omega_{12}(\theta_{12}, 0), \tag{8}$$

with three mixing angles and the CP violation phase δ. For example in the (12) subspace one has

$$\omega_{12}(\theta_{12}, \phi_{12}) = \begin{bmatrix} \cos\theta_{12} & e^{i\phi_{12}}\sin\theta_{12} & 0 \\ -e^{-i\phi_{12}}\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

with clear generalization to the (13) and (23) transformations. Multiplying out yields

$$K_{\exp} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}, \tag{10}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

Identifying K with U in Eq. (2), the ansatz of Eq. (1) now reads

$$\text{Tr}\,(\hat{M}_\nu K_{\exp}^{-1} K_{\exp}^* \omega_0(2\tau)) = 0, \tag{11}$$

where Eqs. (2) and (6) were used. With the parametrized mixing matrix of Eq. (10) the ansatz finally becomes

$$m_1 e^{2i\tau_1}[1 - 2i(c_{12}s_{13})^2 \sin\delta e^{-i\delta}] + m_2 e^{2i\tau_2}[1 - 2i(s_{12}s_{13})^2 \sin\delta e^{-i\delta}] + m_3 e^{2i\tau_3}[1 + 2i(s_{13})^2 \sin\delta e^{i\delta}] = 0. \tag{12}$$

In this equation we can choose the diagonal masses m_1, m_2, m_3 to be real positive. Notice that setting the mixing parameter s_{13} to zero eliminates the dependence on the CP violation phase δ. Then Eq. (12) goes over to the simpler form studied previously [15].

III. SOLVING THE ANSATZ IN THE GENERAL CASE

For definiteness we will use the following best fit values for the differences of squared neutrino masses obtained in Ref. [11]:

$$A \equiv m_2^2 - m_1^2 = 6.9 \times 10^{-5} eV^2, \qquad B \equiv |m_3^2 - m_2^2| = 2.6 \times 10^{-3} eV^2. \tag{13}$$

The uncertainty in these determinations is roughly 25%. Similarly for definiteness we will adopt the best fit values for s_{12}^2 and s_{23}^2 obtained in the same analysis:

$$s_{12}^2 = 0.30, \quad s_{23}^2 = 0.50. \tag{14}$$

These mixing angles also have about a 25% uncertainty. The experimental status of s_{13}^2 is less accurately known. At present only the 3σ bound,

$$s_{13}^2 \leq 0.047, \tag{15}$$

is available. For our discussion we will consider s_{13}^2 to be known at a "typical" value satisfying this bound and examine the sensitivity to changing it. Of course, the experimental determination of s_{13} is a topic of great current interest.

Previously [15], the (positive) mass of the third neutrino m_3 was considered to be the free parameter. It was varied to obtain, via the simplified ansatz equation, a "panoramic view" of the two independent Majorana phases (say τ_1 and τ_2). In the present case, we shall not neglect the CP violation phase δ and consider it too as a free parameter to be varied. It is necessary to specify a suitable algorithm to treat the full ansatz. Previously it was noted that the simplified ansatz could be pictured as a vector triangle in the complex plane having sides equal to corresponding neutrino masses (see Fig. 1 of [15]). The three internal angles were found by trigonometry and related to the three angles made by the sides with respect to the positive real axis. Those in turn were twice the three (constrained) Majorana phases. The orientation of the triangle got determined (up to a reflection) by the constraint in Eq. (7). In the present case we will also rewrite the ansatz equation as a vector triangle in the complex plane. However, the sides will differ from the neutrino masses. In addition the angles will differ from twice the constrained Majorana phases.

To start, we choose a value for m_3 and a value for the phase δ. Then we can obtain from Eqs. (13) two different solutions for the other masses m_1 and m_2. We call the solution where m_3 is the largest neutrino mass, the type I case. The case where m_3 is the smallest neutrino mass is designated type II. m_1 will be determined from the assumed value of m_3 as

$$m_1^2 = m_3^2 - A \mp B, \tag{16}$$

where the upper and lower sign choices, respectively, refer to the type I and type II cases. In either case we find m_2 as

$$m_2^2 = A + m_1^2. \tag{17}$$

Next, we redefine variables so that each of the three terms in the ansatz equation, (12) is characterized by a single magnitude m_i' and a single phase $2\tau_i'$. Equation (12) then reads,

$$m_1' e^{2i\tau_1'} + m_2' e^{2i\tau_2'} + m_3' e^{2i\tau_3'} = 0. \tag{18}$$

This equation evidently represents a vector triangle in the complex plane, as illustrated in Fig. 1. However, the lengths are not the physical neutrino masses and the phases are not twice the physical Majorana *CP* violation phases. The auxiliary, primed, masses are seen to be related to the (now known) physical masses by

$$m_i' = G_i m_i, \tag{19}$$

where,

$$\begin{aligned} G_1 &= [1 - 4(c_{12}s_{13})^2\sin^2\delta + 4(c_{12}s_{13})^4\sin^2\delta]^{1/2}, \\ G_2 &= [1 - 4(s_{12}s_{13})^2\sin^2\delta + 4(s_{12}s_{13})^4\sin^2\delta]^{1/2}, \\ G_3 &= [1 - 4s_{13}^2\sin^2\delta + 4s_{13}^4\sin^2\delta]^{1/2}. \end{aligned} \tag{20}$$

Notice that, since δ has already been specified, the relations between the m_i' and the m_i are now known. Similarly, the physical phases are related to the primed ones by

$$\tau_i = \tau_i' + F_i, \tag{21}$$

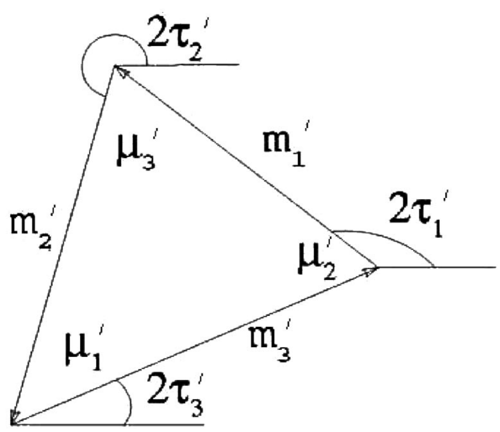

FIG. 1. Vector triangle representing Eq. (18).

where,

$$\begin{aligned} F_1 &= \frac{1}{2}\arctan\frac{(c_{12}s_{13})^2\sin(2\delta)}{1 - 2(c_{12}s_{13})^2\sin^2\delta}, \\ F_2 &= \frac{1}{2}\arctan\frac{(s_{12}s_{13})^2\sin(2\delta)}{1 - 2(s_{12}s_{13})^2\sin^2\delta}, \\ F_3 &= -\frac{1}{2}\arctan\frac{s_{13}^2\sin(2\delta)}{1 + 2s_{13}^2\sin^2\delta}. \end{aligned} \tag{22}$$

Again, note that, since δ has been specified, the relations between the τ_i and the τ_i' are now known. Referring to Fig. 1 we can determine the internal angles μ_i' by using the law of cosines. For example,

$$\cos\mu_1' = \frac{-(m_1')^2 + (m_2')^2 + (m_3')^2}{2m_2'm_3'}. \tag{23}$$

Next, the auxiliary phases τ_i' can be related to the internal angles just obtained as

$$\begin{aligned} \tau_1' &= \frac{1}{6}(\pi - \mu_1' - 2\mu_2') + \rho, \\ \tau_2' &= \frac{1}{6}(\pi + 2\mu_1' + \mu_2') + \rho, \\ \tau_3' &= \frac{1}{6}(-2\pi - \mu_1' + \mu_2') + \rho. \end{aligned} \tag{24}$$

The remaining still unknown parameter here is ρ which we added to the right hand side of each equation. It represents the effect of an arbitrary rotation of the whole triangle, which should not be determinable from the internal angles. It can be determined, however, by making use of the constraint on the physical phases $\sum_i \tau_i = 0$. Notice that there is no corresponding constraint for $\sum_i \tau_i'$. Using Eq. (21) we get,

$$\rho = -\frac{1}{3}\sum_i F_i, \tag{25}$$

where the F_i are to be read from Eqs. (22). Now the masses m_1, m_2 and the phases τ_1, τ_2 have been determined by a simple algorithm, upon specification of m_3 and δ.

As remarked above, the dependence on the *CP* violation phase δ is suppressed in the limit that the mixing parameter s_{13}^2 vanishes. Hence, to illustrate this new feature, we will consider a value $s_{13}^2 = 0.04$, close to the 3σ upper bound of 0.047 [11]. The predictions of the neutrino masses (m_1, m_2) and two independent phases (τ_1, τ_2), from the ansatz for various assumed values of m_3 and δ are given in Table I. Representative values of δ were chosen to lie between 0 and π since it may be observed from Eqs. (20) and (22) that the solutions will have a periodicity of π with respect to δ. Just from the ansatz there is no upper bound on the value of m_3. However, there is a recent cosmological bound [18] which requires

$$|m_1| + |m_2| + |m_3| < 0.7 \text{ eV}. \tag{26}$$

Thus values of m_3 greater than about 0.3 eV are physically

TABLE I. Panorama of solutions as m_3 is lowered from about the highest value which is experimentally reasonable to about the lowest value imposed by the model. For each value of m_3, predictions are given for (m_1, m_2) and for (τ_1, τ_2) in the cases where $\delta = 0, 0.5, 1.0, 1.5, 2.0, 2.5$. All phases are measured in radians. Here the value $s_{13}^2 = 0.04$ was adopted. In the type I solutions m_3 is the largest mass while in the type II solutions m_3 is the smallest mass.

Type	m_1, m_2, m_3 in eV	$\tau_1, \tau_2(\delta = 0)$	$\tau_1, \tau_2(\delta = 0.5)$	$\tau_1, \tau_2(\delta = 1.0)$	$\tau_1, \tau_2(\delta = 1.5)$	$\tau_1, \tau_2(\delta = 2.0)$	$\tau_1, \tau_2(\delta = 2.5)$
I	0.2955, 0.2956, 0.3	0.0043, 1.0428	0.0126, 1.0495	0.0058, 1.0536	−0.0108, 1.0510	−0.0210, 1.0440	−0.0153, 1.0394
II	0.3042, 0.3043, 0.3	−0.0041, 1.0512	0.0043, 1.0577	−0.0023, 1.0615	−0.0189, 1.0587	−0.0291, 1.0518	−0.0235, 1.0476
I	0.0856, 0.0860, 0.1	0.0486, 0.9975	0.0566, 1.0049	0.0489, 1.0106	0.0318, 1.0091	0.0219, 1.0015	0.0285, 0.9953
II	0.1119, 0.1123, 0.1	−0.0311, 1.0774	−0.0226, 1.0835	−0.0289, 1.0863	−0.0453, 1.0828	−0.0556, 1.0763	−0.0503, 1.0731
I	0.0305, 0.0316, 0.06	0.3913, 0.6543	0.3873, 0.6748	0.3578, 0.7048	0.3288, 0.7167	0.3258, 0.7013	0.3530, 0.6720
II	0.0783, 0.0787, 0.06	−0.0669, 1.1119	−0.0583, 1.1174	−0.0644, 1.1188	−0.0806, 1.1145	−0.0911, 1.1085	−0.0860, 1.1066
II	0.0643, 0.0648, 0.04	−0.1064, 1.1494	−0.0978, 1.1541	−0.1040, 1.1538	−0.1203, 1.1483	−0.1307, 1.1430	−0.1255, 1.1428
II	0.0541, 0.0548, 0.02	−0.1747, 1.2115	−0.1669, 1.2142	−0.1751, 1.2095	−0.1928, 1.2012	−0.2024, 1.1976	−0.1951, 1.2019
II	0.0506, 0.0512, 0.005	−0.2601, 1.2620	−0.2603, 1.2611	−0.2914, 1.2276	−0.3251, 1.2035	−0.3250, 1.2094	−0.2950, 1.2369
II	0.0503, 0.0510, 0.001	−0.3830, 1.1805					

disfavored. Table I shows that at this value both type I and type II solutions exist. This is true also for higher values of m_3. The picture remains very similar down to around $m_3 = 0.1$ eV but as one gets closer to roughly 0.06 eV, there is a marked change. If one further lowers m_3, it is found that the type I solution no longer exists. On the other hand the type II solution persists and does not change much until m_3 approaches the neighborhood of 0.001 eV. There are no solutions for m_3 below this region.

Note that the columns in Table I with $\delta = 0$ correspond to the previous case, discussed in some detail in Sec. IV of [15]. In this case, m_i' and τ_i', respectively, coincide with m_i and τ_i so we can identify the vectors of the triangle with the physical masses and phases. As one decreases the value of m_3 the type I triangle goes from being close to equilateral to the degenerate situation with three collinear vectors. In this limiting case the vectors representing neutrino one and neutrino two are approximately equal and add up to exactly cancel the vector representing neutrino 3. The precise orientation of the straight line is due to imposing the constraint in Eq. (7). This is actually a *CP* conserving case [19]. Then one can find the m_3 value (a little below 0.06 eV) for this situation by looking for a real solution of $m_1 + m_2 + m_3 = 0$ together with Eqs. (13) [See Eq. (4.4) of [12]]. Clearly there can be no type I solutions below this value of m_3. The type II solutions can exist below this value but similarly end (a little below $m_3 = 0.001$ eV) when the triangle becomes degenerate in a different way. For the type II degenerate triangle, the neutrino 1 and neutrino 2 vectors are collinear but oppositely directed and the small neutrino 3 vector adds to the neutrino 1 vector to cancel the neutrino 2 vector. This is also a *CP* conserving case.

When the effects of δ not equal to zero are included, it is not possible to make a triangle out of the physical neutrino masses and phases. The relevant auxiliary triangle is made, as illustrated, using the primed masses and phases. Thus the limiting values of m_3, where the type I and type II cases each end, correspond to this primed triangle becoming degenerate. We can get the limiting value by looking for real solutions of $\sum_i G_i m_i = 0$, together with Eqs. (13). The limiting value $(m_3)_{\rm min}$ is found to be

$$(m_3)^2_{\rm min} = \frac{1}{2\alpha}\Big[-\beta - (\beta^2 - 4\alpha\gamma)^{1/2}\Big], \qquad (27)$$

where,

$$\begin{aligned}
\alpha &= (G_1^2 + G_2^2 - G_3^2)^2 - 4G_1^2G_2^2,\\
\beta &= -2(G_1^2 + G_2^2 - G_3^2)(AG_1^2 \pm B(G_1^2 + G_2^2))\\
&\quad + 4G_1^2G_2^2(A \pm 2B),\\
\gamma &= (AG_1^2 \pm B(G_1^2 + G_2^2))^2 \mp 4G_1^2G_2^2B(A \pm B).
\end{aligned} \qquad (28)$$

Here the upper and lower sign choices, respectively, refer to the type I and type II cases.

The computed values of $(m_3)_{\rm min}$ as a function of δ are shown in Table II. Looking at this table, one can see why the entries in Table I for $m_3 = 0.001$ eV and nonzero values of δ are missing. Simply, for those cases, $(m_3)_{\rm min} > 0.001$ eV. Clearly, this correlation of the allowed input values of m_3 with the input values of δ must be respected in studying the present model. This correlation is imposed by the ansatz itself. For large values of m_3 there is no constraint from the ansatz but Eq. (26) gives an experimental constraint. It should be remarked that the collinear auxiliary triangles with nonzero δ correspond to *CP* violation. In these cases the *CP* violation arises from nontrivial phases τ_i in addition to the assumed nonzero δ. Actually, a better measure of *CP* violation involves the (two independent) phase differences $\tau_i - \tau_j$. These are the objects essentially related to the internal angles in Fig. 1.

As noted in Ref. [15], the possibility of reflecting the triangle about any line in the plane gives another set of solutions corresponding to reversing the signs of all the phase differences $\tau_i - \tau_j$. In the present case, where δ is not zero, reflecting the "unphysical" triangle about any line in the plane will give an alternate solution in which the $\tau_i' - \tau_j'$ are reversed in sign. More specifically, one should

TABLE II. Minimum allowed value of the input mass m_3 as a function of the input CP violation phase δ. These correspond to the cases where the triangle in Fig. 1 becomes degenerate. Here, the choice $s_{13}^2 = 0.04$ has been made.

Type	$(m_3)_{\min}$ ($\delta = 0$) in eV	$(m_3)_{\min}$ ($\delta = 0.5$)	$(m_3)_{\min}$ ($\delta = 1.0$)	$(m_3)_{\min}$ ($\delta = 1.5$)	$(m_3)_{\min}$ ($\delta = 2.0$)	$(m_3)_{\min}$ ($\delta = 2.5$)
I	0.059 271 6	0.059 096 7	0.058 717 8	0.058 479 9	0.058 620 3	0.058 997 1
II	0.000 681 1	0.001 0546 1	0.001 902 4	0.002 463 6	0.002 129 4	0.001 272 3

TABLE III. Panorama of solutions, using the reflected triangle, as m_3 is lowered from about the highest value which is experimentally reasonable to about the lowest value imposed by the model. Notice that the predicted masses m_1 and m_2 have not been given since they are the same as in Table I.

Type	m_3 in eV	$\tau_1, \tau_2(\delta = 0)$	$\tau_1, \tau_2(\delta = 0.5)$	$\tau_1, \tau_2(\delta = 1.0)$	$\tau_1, \tau_2(\delta = 1.5)$	$\tau_1, \tau_2(\delta = 2.0)$	$\tau_1, \tau_2(\delta = 2.5)$
I	0.3	−0.0043, −1.0428	0.0113, −1.0393	0.0210, −1.0422	0.0151, −1.0492	−0.0015, −1.0536	−0.0123, −1.0512
II	0.3	0.0041, −1.0512	0.0196, −1.0475	0.0291, −1.0500	0.0232, −1.0569	0.0066, −1.0614	−0.0040, −1.0593
I	0.1	−0.0486, −0.9975	−0.0326, −0.9947	−0.0221, −0.9992	−0.0275, −1.0073	−0.0444, −1.0111	−0.0560, −1.0070
II	0.1	0.0311, −1.0774	0.0465, −1.0732	0.0557, −1.0748	0.0495, −1.0810	0.0331, −1.0859	0.0228, −1.0848
I	0.06	−0.3913, −0.6543	−0.3634, −0.6645	−0.3310, −0.6934	−0.3246, −0.7149	−0.3483, −0.7109	−0.3806, −0.6837
II	0.06	0.0669, −1.1119	0.0822, −1.1071	0.0912, −1.1073	0.0849, −1.1127	0.0685, −1.1180	0.0584, −1.1183
II	0.04	0.1064, −1.1494	0.1217, −1.1438	0.1308, −1.1423	0.1246, −1.1465	0.1082, −1.1526	0.0979, −1.1506
II	0.02	0.1747, −1.2115	0.1909, −1.2040	0.2019, −1.1981	0.1971, −1.1994	0.1799, −1.2072	0.1676, −1.2136
II	0.005	0.2601, −1.2620	0.2853, −1.2447	0.3182, −1.2162	0.3294, −1.2017	0.3024, −1.2190	0.2694, −1.2486
II	0.001	0.3830, −1.1805					

reverse the signs of the first terms on the right hand sides in Eqs. (24). The physical phases τ_i for this alternate solution will then depend on δ as illustrated in Table III.

Unlike the $\delta = 0$ case, the phase differences for the reflected triangle solution are now only approximately the negatives of those for the original solution. For example, in the case of a type I triangle with $m_3 = 0.3$ and $\delta = 1.0$, Table I shows $\tau_1 - \tau_2 = -1.0478$ for the original solution while Table III shows $\tau_1 - \tau_2 = +1.0632$ for the reflected triangle solution.

It should be remarked that the number of decimal places to which we are calculating is chosen in order to be able to compare various solutions of the ansatz with each other for precisely fixed values of the input mass differences and mixing angles. The experimental accuracy of the inputs must, of course, be kept in mind.

IV. PHYSICAL APPLICATIONS

It is very interesting to note the dependence of our results on the value of the necessarily small quantity s_{13}^2, which can be seen from the ansatz Eq. (12) to modulate the δ dependence. For this purpose let us consider, instead of the value 0.04, the value 0.01. The resulting analog of Table I is presented in Table IV.

Notice that Table IV has fewer missing solutions for the case $m_3 = 0.001$ than does Table I. This is because decreasing s_{13}^2 brings the G_i in Eq. (19) closer to unity, which in turn brings the physical neutrino masses closer to the auxiliary m_i'. The modified lower limits for m_3 are illustrated in Table V.

The implications of this model are relevant to experiments which are designed to search for evidence of neu-

TABLE IV. Panorama of solutions as in Table I but with $s_{13}^2 = 0.01$. Notice that the predicted masses m_1 and m_2 have not been given since they are the same as in Table I.

Type	m_3 in eV	$\tau_1, \tau_2(\delta = 0)$	$\tau_1, \tau_2(\delta = 0.5)$	$\tau_1, \tau_2(\delta = 1.0)$	$\tau_1, \tau_2(\delta = 1.5)$	$\tau_1, \tau_2(\delta = 2.0)$	$\tau_1, \tau_2(\delta = 2.5)$
I	0.3	−0.0043, −1.0428	0.0063, 1.0445	0.0046, 1.0455	0.0007, 1.0448	−0.0018, 1.0432	−0.0006, 1.0420
II	0.3	0.0041, −1.0512	−0.0020, 1.0528	−0.0037, 1.0537	−0.0076, 1.0530	−0.0101, 1.0514	−0.0089, 1.0503
I	0.1	−0.0486, −0.9975	0.0505, 0.9994	0.0486, 1.0008	0.0446, 1.0004	0.0421, 0.9985	0.0436, 0.9970
II	0.1	0.0311, −1.0774	−0.0290, 1.0789	−0.0306, 1.0796	−0.0345, 1.0787	−0.0370, 1.0772	−0.0358, 1.0763
I	0.06	−0.3913, −0.6543	0.3900, 0.6597	0.3816, 0.6681	0.3738, 0.6718	0.3736, 0.6676	0.3813, 0.6592
II	0.06	0.0669, −1.1119	−0.0608, 1.1132	−0.0634, 1.1136	−0.0702, 1.1125	−0.0727, 1.1111	−0.0716, 1.1106
II	0.04	0.1064, −1.1494	−0.1043, 1.1505	−0.1058, 1.1504	−0.1097, 1.1492	−0.1122, 1.1479	−0.1111, 1.1478
II	0.02	0.1747, −1.2115	−0.1728, 1.2122	−0.1748, 1.2110	−0.1789, 1.2091	−0.1813, 1.2082	−0.1797, 1.2091
II	0.005	0.2601, −1.2620	−0.2605, 1.2602	−0.2673, 1.2538	−0.2744, 1.2486	−0.2750, 1.2496	−0.2687, 1.2558
II	0.001	0.3830, −1.1805	0.4036, 1.1593	−0.4828, 1.0840			−0.4252, 1.1420

TABLE V. Minimum allowed value of the input mass m_3 as a function of the input CP violation phase δ as in Table II but with $s_{13}^2 = 0.01$.

Type	$(m_3)_{\min}$ $(\delta = 0)$ in eV	$(m_3)_{\min}$ $(\delta = 0.5)$	$(m_3)_{\min}$ $(\delta = 1.0)$	$(m_3)_{\min}$ $(\delta = 1.5)$	$(m_3)_{\min}$ $(\delta = 2.0)$	$(m_3)_{\min}$ $(\delta = 2.5)$
I	0.059 271 6	0.059 226 3	0.059 131 2	0.059 073 5	0.059 107 4	0.059 200 9
II	0.000 681 1	0.000 775 5	0.000 976 1	0.001 099 3	0.001 026 8	0.000 828 7

TABLE VI. The neutrinoless double beta decay amplitude factor $|m_{ee}|$ in eV as a function of the input CP violation phase δ. Here, the choice $s_{13}^2 = 0.04$ has been made.

Type	m_1, m_2, m_3 in eV	$|m_{ee}|(\delta = 0)$	$|m_{ee}|(\delta = 0.5)$	$|m_{ee}|(\delta = 1.0)$	$|m_{ee}|(\delta = 1.5)$	$|m_{ee}|(\delta = 2.0)$	$|m_{ee}|(\delta = 2.5)$
I	0.2955, 0.2956, 0.3	0.164	0.174	0.183	0.181	0.170	0.162
II	0.3042, 0.3043, 0.3	0.167	0.177	0.185	0.183	0.172	0.164
I	0.0856, 0.0860, 0.1	0.051	0.055	0.057	0.057	0.054	0.051
II	0.1119, 0.1123, 0.1	0.058	0.062	0.065	0.064	0.060	0.057
I	0.0305, 0.0316, 0.06	0.026	0.028	0.029	0.030	0.029	0.027
II	0.0783, 0.0787, 0.06	0.038	0.040	0.042	0.041	0.039	0.037
II	0.0643, 0.0648, 0.04	0.029	0.031	0.032	0.031	0.029	0.028
II	0.0541, 0.0548, 0.02	0.022	0.023	0.023	0.023	0.022	0.021
II	0.0506, 0.0512, 0.005	0.019	0.020	0.020	0.019	0.019	0.019
II	0.0503, 0.0510, 0.001	0.019					

trinoless double beta decay. The amplitudes for these processes contain a factor m_{ee}, which is independent of the nuclear wave functions. Its magnitude is given by

$$|m_{ee}| = |m_1(K_{\exp 11})^2 e^{-2i\tau_1} + m_2(K_{\exp 12})^2 e^{-2i\tau_2} + m_3(K_{\exp 13})^2 e^{-2i\tau_3}|, \quad (29)$$

which appears to require, for its evaluation, a full knowledge of the neutrino masses, mixing angles, and CP violation phases. The present experimental bound [20] on this quantity is

$$|m_{ee}| < (0.35 \rightarrow 1.30) \text{ eV}. \quad (30)$$

A very recent review of neutrinoless double beta decay is given in Ref. [21]. Using the general parametrization of Eq. (6) one finds.

$$m_{ee} = \sqrt{C^2 + D^2}, \quad (31)$$

wherein

$$\begin{aligned} C &= m_1(c_{12}c_{13})^2 + m_2(s_{12}c_{13})^2 \cos[2(\tau_2 - \tau_1)] \\ &\quad + m_3(s_{13})^2 \cos[2(\tau_3 - \tau_1 + \delta)], \\ D &= m_2(s_{12}c_{13})^2 \sin[2(\tau_2 - \tau_1)] \\ &\quad + m_3(s_{13})^2 \sin[2(\tau_3 - \tau_1 + \delta)]. \end{aligned} \quad (32)$$

The dependence of $|m_{ee}|$ on the input values of m_3 and δ, obtained by using the ansatz of present interest, is displayed in Table VI for the same choices as in Table I. There is noticeable dependence on the input CP phase δ for the larger values of m_3.

For the reflected triangle solutions discussed above, the predictions of $|m_{ee}|$ are given below in Table VII. Again

TABLE VII. The neutrinoless double beta decay amplitude factor $|m_{ee}|$ in eV as a function of the input CP violation phase δ using the reflected triangle. Notice that the predicted masses m_1 and m_2 have not been given since they are the same as in Table VI.

Type	m_3 in eV	$|m_{ee}|(\delta = 0)$	$|m_{ee}|(\delta = 0.5)$	$|m_{ee}|(\delta = 1.0)$	$|m_{ee}|(\delta = 1.5)$	$|m_{ee}|(\delta = 2.0)$	$|m_{ee}|(\delta = 2.5)$
I	0.3	0.164	0.161	0.167	0.178	0.183	0.177
II	0.3	0.167	0.163	0.169	0.180	0.186	0.180
I	0.1	0.051	0.051	0.053	0.056	0.058	0.056
II	0.1	0.058	0.057	0.059	0.063	0.065	0.063
I	0.06	0.026	0.027	0.029	0.030	0.030	0.028
II	0.06	0.038	0.037	0.038	0.040	0.042	0.041
II	0.04	0.029	0.028	0.029	0.030	0.032	0.031
II	0.02	0.022	0.021	0.022	0.022	0.023	0.023
II	0.005	0.019	0.019	0.019	0.019	0.019	0.020
II	0.001	0.019					

TABLE VIII. The neutrinoless double beta decay amplitude factor $|m_{ee}|$ in eV as a function of the input CP violation phase δ as in Table VI but with $s_{13}^2 = 0.01$. Notice that the predicted masses m_1 and m_2 have not been given since they are the same as in Table VI.

Type	m_3 in eV	$\lvert m_{ee}\rvert(\delta = 0)$	$\lvert m_{ee}\rvert(\delta = 0.5)$	$\lvert m_{ee}\rvert(\delta = 1.0)$	$\lvert m_{ee}\rvert(\delta = 1.5)$	$\lvert m_{ee}\rvert(\delta = 2.0)$	$\lvert m_{ee}\rvert(\delta = 2.5)$
I	0.3	0.177	0.180	0.181	0.182	0.179	0.176
II	0.3	0.179	0.182	0.184	0.183	0.181	0.179
I	0.1	0.056	0.057	0.057	0.057	0.056	0.056
II	0.1	0.063	0.064	0.064	0.064	0.063	0.062
I	0.06	0.029	0.029	0.030	0.030	0.030	0.029
II	0.06	0.041	0.041	0.042	0.041	0.041	0.040
II	0.04	0.031	0.031	0.031	0.031	0.031	0.031
II	0.02	0.023	0.023	0.023	0.023	0.023	0.023
II	0.005	0.020	0.020	0.020	0.020	0.020	0.020
II	0.001	0.020	0.020	0.020			0.020

there is a noticeable dependence on δ for the larger values of m_3. However, the peak values occur at different values of δ compared to Table VI.

The effects of lowering s_{13}^2 to 0.01 are finally illustrated below, for the nonreflected triangle case, in Table VIII.

The main conclusion of this model for neutrinoless double beta decay, obtained by looking at all three tables above and noting the smooth dependence of $|m_{ee}|$ on the inputs m_3 and δ for each of the type I and type II solutions, is that $|m_{ee}|$ should satisfy the restrictive approximate bounds:

$$0.020\ \text{eV} < |m_{ee}| < 0.185\ \text{eV}. \tag{33}$$

Here, the lower bound is intrinsic to the model but the upper bound reflects the experimental bound on the sum of neutrino masses quoted in Eq. (26) and might be improved upon. We also note that, when both type I and type II solutions exist for a given value of m_3, the type II solution gives somewhat larger $|m_{ee}|$. The main dependence of $|m_{ee}|$ is, of course, on the input parameter m_3.

We plan to discuss elsewhere other physical applications including beta decay end point spectra [22] and leptogenesis [23] using the approach of Ref. [15].

V. SYMMETRICAL PARAMETRIZATION

In the parametrization of the leptonic mixing matrix given by Eq. (6), which is similar to the one usually adopted, there appears to be an important distinction between the phase δ and the phases τ_i in the sense that only δ survives when one considers the ordinary (overall lepton number conserving) neutrino oscillation experiments. However, this distinction may be preserved in a different way while using a more symmetrical parametrization. Thus consider,

$$K_S = \omega_{23}(\theta_{23}, \phi_{23})\omega_{13}(\theta_{13}, \phi_{13})\omega_{12}(\theta_{12}, \phi_{12}), \tag{34}$$

which contains the same mixing angles θ_{ij} as before but now has the three associated CP violation phases ϕ_{ij}. The $\omega_{ij}(\theta_{ij}, \phi_{ij})$ were defined as in Eq. (9). Writing out the whole matrix yields,

$$K_S = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13}e^{i\phi_{12}} & s_{13}e^{i\phi_{13}} \\ -s_{12}c_{23}e^{-i\phi_{12}} - c_{12}s_{13}s_{23}e^{i(\phi_{23}-\phi_{13})} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i(\phi_{12}+\phi_{23}-\phi_{13})} & c_{13}s_{23}e^{i\phi_{23}} \\ s_{12}s_{23}e^{-i(\phi_{12}+\phi_{23})} - c_{12}s_{13}c_{23}e^{-i\phi_{13}} & -c_{12}s_{23}e^{-i\phi_{23}} - s_{12}s_{13}c_{23}e^{i(\phi_{12}-\phi_{13})} & c_{13}c_{23} \end{bmatrix}. \tag{35}$$

To relate this form to the previous one, we can use the identity [7],

$$\begin{aligned} \omega_0^{-1}(\tau)K_S\omega_0(\tau) = {} & \omega_{23}(\theta_{23}, -\tau_2 + \phi_{23} + \tau_3) \\ & \times \omega_{13}(\theta_{13}, -\tau_1 + \phi_{13} + \tau_3) \\ & \times \omega_{12}(\theta_{12}, -\tau_1 + \phi_{12} + \tau_2), \end{aligned} \tag{36}$$

where the diagonal matrix of phases $\omega_0(\tau)$ was defined in Eq. (7). Now choose the τ_i's (two are independent) so that the transformed (12) and (23) phases vanish. Then, with the identifications,

$$\begin{gathered} \phi_{12} = \tau_1 - \tau_2, \qquad \phi_{23} = \tau_2 - \tau_3, \\ \phi_{13} = \tau_1 - \tau_3 - \delta, \end{gathered} \tag{37}$$

we notice that K_S is related to K of Eq. (6) as

$$K_S = \omega_0(\tau)K. \tag{38}$$

Since K sits in the weak interaction Lagrangian, Eq. (4) with the charged lepton field row vector on its left, all physical results will be unchanged if K is multiplied by a diagonal matrix of phases on its left. Thus K and K_S are equivalent and the relations between the CP violation phases of the two parametrizations are given by Eq. (37). By construction, δ is the only CP violation phase which can appear in the description of ordinary neutrino oscillations. Solving the Eqs. (37) for δ in terms of the three ϕ_{ij}'s gives,

$$\delta = \phi_{12} + \phi_{23} - \phi_{13}, \tag{39}$$

which shows that in the symmetrical parametrization, the "invariant phase" [24] combination $I_{123} = \phi_{12} + \phi_{23} - \phi_{13}$ is the object which measures *CP* violation for ordinary neutrino oscillations. It has the desired property of intrinsically spanning three generations, as needed for *CP* violation in ordinary neutrino oscillation or in the quark mixing analog. Furthermore, it can be seen [25] to have an interesting mathematical structure and to be useful for extension to the case where there are more than three generations of fermions

The convenience of the symmetrical parametrization can already be seen in Eqs. (32) needed for obtaining $|m_{ee}|$. These equations simplify when one observes that the combinations of phases occurring within them are simply ϕ_{12} and ϕ_{13}. These are evidently the two phases which describe the coupling of the first lepton generation. Similarly, the estimates of *CP* violation needed for the treatment of leptogenesis made in Sec. V of [15] also simplify when expressed in terms of the ϕ_{ij}.

VI. SUMMARY AND DISCUSSION

Assuming that the squared neutrino mass differences and the three (*CP* conserving) lepton mixing angles are known, the mass of one neutrino and three *CP* violation phases remain to be determined. The complementary ansatz, $\mathrm{Tr}(M_\nu) = 0$, provides two real conditions on the parameters of the lepton system. Here, we took m_3 and δ as input parameters and then determined the other two masses and the other two *CP* violation phases according to this ansatz. A geometric algorithm was presented based on a modification of an earlier treatment [15] in which the Dirac *CP* violation phase δ was neglected, but the other two Majorana type *CP* violation phases were retained. That is a reasonable first approximation because the effect of delta is always suppressed by s_{13} which is known experimentally to be small. However, there is great interest in the determination of δ so it should not be ignored. Additionally in Ref. [15] it was suggested that small *CP* violation scenarios might be close to the physical case. The present algorithm is exact and does not require the assumption that any parameters are small.

The ansatz yields a characteristic pattern for the neutrino masses and the *CP* violation phases. Because s_{13}^2 is small, the main cause of change is the assumed value of the input parameter m_3. First one notes that the small experimental value of A in Eq. (13) always forces the neutrino 1 and neutrino 2 masses to be almost degenerate (See Table I). For the largest allowed [from the cosmological bound Eq. (26)] value of m_3, around 0.3 eV, there is an approximate threefold degeneracy of all the neutrino masses. This is understandable since when the mass scale becomes large, both A and B can be considered negligible. Then the triangle of Fig. 1 becomes approximately equilateral. The internal angles of the triangle approximately measure the strength of the *CP* phases $\tau_i - \tau_j$ and are clearly large in this situation. As m_3 decreases, a point around 0.06 eV is reached at which the type I solutions (m_3 largest) no longer exist. At this point the *CP* violation vanishes for the $\delta = 0$ case and becomes small when $\delta \neq 0$. Also at this point the almost degenerate neutrino 1 and neutrino 2 masses decrease to about half the neutrino 3 mass in the type I case. In the present model neutrinos 1 and 2 never go below about half the mass of neutrino 3. The situation is a little different for the type II cases (m_3 smallest). The type II solutions exist from a maximum of m_3 about 0.3 eV to a minimum of about 0.001 eV. At the minimum the *CP* violation ceases for $\delta = 0$ and has small effects when $\delta \neq 0$. Furthermore, at the minimum neutrinos 1 and 2 are about 50 times heavier than neutrino 3. Thus a possible hierarchy of neutrino masses can only exist in one way for the present model, with m_3 considerably smaller than the other two.

Some further technical details of the model were displayed in Tables II, III, and IV. These describe the δ dependence of the limiting values of the input parameter m_3 just mentioned, the solution corrresponding to a reflected triangle and the effect of varying s_{13}^2.

The model might be handy for getting an idea about the range of predictions for various leptonic phenomena, since it gives a plausible two parameter complete set of neutrino masses, mixing angles and *CP* violation phases. For example, in Sec. V an application is made to the quantity $|m_{ee}|$ which characterizes neutrinoless double beta decay. The results for about the largest allowed s_{13}^2, the reflected solution and a smaller s_{13}^2 choice are shown in Tables VI, VII, and VIII. It may be noted that the solutions vary rather smoothly with m_3 and δ for a solution of given type. Thus, even though one might initially expect the result of allowing a two parameter family choice to be rather weak, it turns out that one gets fairly restrictive upper and lower bounds on $|m_{ee}|$ as expressed in Eq. (33). These approximate bounds may be considered also as a test of the present model. Of course, any direct determination of a neutrino mass, say from a beta decay end point experiment, will also provide a test of the model.

In Sec. V we took up a question which is independent of the present ansatz. How should one parametrize the lepton mixing matrix? Of course, this is fundamentally a question of choice. However, we notice in the present work that the physical quantities we calculate depend in the simplest way not on the conventional phases δ and τ_i (where $\sum \tau_i = 0$) but on the quantities ϕ_{ij} given in Eq. (37). Such a dependence arises naturally if one uses the alternative mixing matrix parametrization given by K_S in Eq. (34). This may be understood physically in the following way. The ϕ_{ij}'s by definition [see Eq. (9), for example] span two generations. It is known [10] that *CP* violation begins at the two generation level for Majorana neutrinos.

Thus it seems appropriate that the ϕ_{ij}'s should appear. However, if $\delta = 0$, the three ϕ_{ij}'s are not linearly independent according to Eq. (39). This takes care of the two Majorana phases. When $\delta \neq 0$ the three ϕ_{ij}'s are of course independent and the "invariant phase" combination I_{123} discussed some time ago [24,25] intrinsically spans three generations, as expected for Dirac type *CP* violation.

ACKNOWLEDGMENTS

S. S. M would like to thank the Physics Department at Syracuse University for their hospitality. The work of S. N is supported by National Science Foundation Grant No. PHY-0099. The work of J. S. is supported in part by the U.S. DOE under Contract No. DE-FG-02-85ER 40231.

[1] K. Eguchi *et al.* (KamLAND Collaboration), Phys. Rev. Lett. **90**, 021802 (2003).
[2] Q. R. Ahmad *et al.* (SNO Collaboration), Phys. Rev. Lett. **92**, 181301 (2004).
[3] M. H. Ahn *et al.* (K2K Collaboration), Phys. Rev. Lett. **90**, 041801 (2003).
[4] For recent reviews see, for examples, S. Pakvasa and J. W. F. Valle, Proc. Indian Natl. Sci. Acad. A **70**, 189 (2004); V. Barger, D. Marfatia and K. Whisnant, Int. J. Mod. Phys. E **12**, 569 (2003).
[5] C. Athanassopoulos *et al.* (LSND Collaboration), Phys. Rev. Lett. **81**, 1774 (1998).
[6] Recent summaries of the status of this experiment are given by H. Ray *et al.* (MiniBooNE Collaboration), hep-ex/0411022 [Int. J. Mod. Phys. A (to be published)]; A. Aguilar-Arevalo, hep-ex/0408074; See also R. Tayloe *et al.* (MiniBooNE Collaboration), Nucl. Phys. B Proc. Suppl. **118**, 157 (2003).
[7] J. Schechter and J. W. F. Valle, Phys. Rev. D **22**, 2227 (1980).
[8] S. M. Bilenky, J. Hosek, and S. T. Petcov, Phys. Lett. **94B**, 495 (1980).
[9] M. Doi, T. Kotani, H. Nishiura, K. Okuda, and E. Takasugi, Phys. Lett. **102B**, 323 (1981).
[10] J. Schechter and J. W. F. Valle, Phys. Rev. D **23**, 1666 (1981); A. de Gouvea, B. Kayser, and R. N. Mohapatra, Phys. Rev. D **67**, 053004 (2003).
[11] M. Maltoni, T. Schwetz, M. A. Tortola, and J. W. F. Valle, Phys. Rev. D **68**, 113010 (2003). For the choice $m_2 > m_1$ see A. de Gouvea, A. Friedland, and H. Murayama, Phys. Lett. B **490**, 125 (2000).
[12] D. Black, A. H. Fariborz, S. Nasri, and J. Schechter, Phys. Rev. D **62**, 073015 (2000).
[13] X.-G. He and A. Zee, Phys. Rev. D **68**, 037302 (2003).
[14] W. Rodejohann, Phys. Lett. B **579**, 127 (2004).
[15] S. Nasri, J. Schechter, and S. Moussa, Phys. Rev. D **70**, 053005 (2004).
[16] S. Nasri, J. Schechter, and S. Moussa, Int. J. Mod. Phys. A **19**, 5367 (2004).
[17] R. N. Mohapatra *et al.*, hep-ph/0412099.
[18] D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003); S. Hannestad, J. Cosmol. Astropart. Phys. 05 (2003) 004.
[19] See the discussion in Sec. IV of [12] above and L. Wolfenstein, Phys. Lett. B **107**, 77 (1981).
[20] H. V. Klapdor-Kleingrothaus *et al.*, Eur. Phys. J. A **12**, 147 (2001).
[21] C. Aalseth *et al.*, hep-ph/0412300.
[22] Y. Farzan, O. L. G. Peres and A. Yu. Smirnov, Nucl. Phys. **B612**, 59 (2001).
[23] M. Fukugita and T. Yanagida, Phys. Lett. B **174**, 45 (1986). The original baryogenesis mechanism is given in A. D. Sakharov, Pis'ma Zh. Eksp. Teor. Fiz. **5**, 32 (1967) [JETP Lett. **5**, 32 (1967)].
[24] M. Gronau and J. Schechter, Phys. Rev. Lett. **54**, 385 (1985); **54**, 1209(E) (1985).
[25] M. Gronau, R. Johnson, and J. Schechter, Phys. Rev. D **32**, 3062 (1985). The sense in which I_{123} is entitled to be called an invariant phase for the quark mixing matrix or equivalently for the ordinary neutrino oscillations is discussed in Eqs. (2′), (4′), (13a), and (13b) of this reference.