Generalized Boltzmann formalism for oscillating neutrinos

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In the standard approaches to neutrino transport in the simulation of core-collapse supernovas, one will often start from the classical Boltzmann equation for the neutrino's spatial, temporal, and spectral evolution. For each neutrino species, and its antiparticle, the classical density in phase space, or the associated specific intensity, will be calculated as a function of time. The neutrino radiation is coupled to matter by source and sink terms on the ''right-hand side'' of the transport equation and together with the equations of hydrodynamics this set of coupled partial differential equations for classical densities describes, in principle, the evolution of core collapse and explosion. However, with the possibility of neutrino oscillations between species, a purely quantum-physical effect, how to generalize this set of Boltzmann equations for classical quantities to reflect oscillation physics has not been clear. To date, the formalisms developed have retained the character of quantum operator physics involving complex quantities and have not been suitable for easy incorporation into standard supernova codes. In this paper, we derive generalized Boltzmann equations for quasiclassical, real-valued phase-space densities that retain all the standard oscillation phenomenology, including the matter-enhanced resonant flavor conversion (Mikheev-Smirnov-Wolfenstein effect), neutrino self-interactions, and the interplay between decohering matter coupling and flavor oscillations. With this formalism, any code(s) that can now handle the solution of the classical Boltzmann or transport equation can easily be generalized to include neutrino oscillations in a quantum-physically consistent fashion.

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I. INTRODUCTION

Particle oscillations are fundamental for a wide range of interesting physics: quark mixing by the Cabbibo-Kobayashi-Maskawa matrix, its leptonic analog for massive neutrinos [1,2], hypothetical photon-axion and photon-graviton oscillations in the presence of external magnetic fields [3,4], and $K^0 - \bar{K}^0$ oscillations [5]. Physically, these quantum systems are coupled to macroscopic systems and through external interaction their quantum evolution is altered. One prominent astrophysical context in which such oscillation for macroscopic systems is important involves the neutrinos in supernova cores that may execute flavor oscillations while simultaneously interacting with ambient supernova matter [6–11]. The primary motivation of the present paper is to provide a straightforward generalization of the Boltzmann formalism with which to analyze the kinetics of oscillating neutrinos with collisions. By taking ensemble-averaged matrix elements of quantum field operators for mixed particles, following the pioneering work of [12–16], we obtain quasiclassical phase-space densities that satisfy real-valued Boltzmann equations with coupling terms that account for the neutrino oscillations. The formalism is clear, numerically tractable, and does not contain operators, complex quantities, or wave functions.

In Sec. II, we introduce the Wigner phase-space density operator approach from which we derive our formalism involving classical phase-space neutrino flavor densities and their off-diagonal, overlap correlates. The latter couple the different flavor states to account for neutrino oscillations. In Sec. III, we demonstrate with several simple examples that the set of equations reproduces (i) standard flavor oscillations in a vacuum, (ii) flavor oscillations with absorptive matter coupling (''quantum decoherence'' [17]), and (iii) resonant matter-induced flavor conversion (the MSW effect) for a neutrino beam [18–20]. In Sec. IV, we summarize the salient features of our new approach. We show in Appendix A how one might include neutrino and antineutrino self-interactions into our formalism.

II. GENERALIZED BOLTZMANN EQUATIONS

To analyze a multiparticle system and reflect its inherent statistics, one usually sets up the density matrix of the system [15,16]. For classical systems this is the phase-space density. For quantum fields, the conceptually most similar analog is the Wigner phase-space density operator [21,22]

$$\rho(\mathbf{r}, \mathbf{p}, t) = \int \frac{d^3\mathbf{r}'}{(2\pi\hbar)^3} e^{-i\mathbf{p}\mathbf{r}'} \psi^\dagger\left(\mathbf{r} - \frac{1}{2}\mathbf{r}', t\right)\psi\left(\mathbf{r} + \frac{1}{2}\mathbf{r}', t\right), \tag{1}$$

where $\psi^\dagger$ and ψ denote the creation and annihilation

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operators, respectively, for the mixing particles of interest [12,16]. ρ works for fermions or bosons (resulting in different commutation relations of the creation and annihilation operators) and massive or massless particles. Here, we average over an ensemble of two neutrino species, for example ν_e's and ν_μ's and take the matrix elements in the number-density basis of Fock space:

$$\mathcal{F} = \langle n_i|\rho|n_j\rangle = \begin{pmatrix} f_{\nu_e} & f_{e\mu} \\ f^*_{e\mu} & f_{\nu_\mu} \end{pmatrix}, \tag{2}$$

where the indices i, j run over the neutrino flavor, and $*$ means complex conjugation. A generalization to more particle species is straightforward. The diagonal terms are real-valued and denote quasiclassical phase-space densities. The off-diagonal entries are complex-valued macroscopic overlap functions. For completely decohered ensembles and for nonmixing ensembles the off-diagonal entries vanish. A Heisenberg-Boltzmann-type equation [12,15,17,23,24], which can heuristically be derived by merging the Boltzmann equation with the Heisenberg equation, governs the time evolution of the matrix elements of the Wigner phase-space density operator:

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{1}{2}\left\{\mathbf{v}, \frac{\partial \mathcal{F}}{\partial \mathbf{r}}\right\} + \frac{1}{2}\left\{\dot{\mathbf{p}}, \frac{\partial \mathcal{F}}{\partial \mathbf{p}}\right\} = -i[\Omega, \mathcal{F}] + C, \tag{3}$$

where curly brackets denote matrix anticommutators, square brackets are matrix commutators, C is the collision matrix and Ω is the mixing Hamiltonian. We componentize this equation for two oscillating neutrino species interacting with a background medium. Then,

$$C = \begin{pmatrix} C_{\nu_e} & 0 \\ 0 & C_{\nu_\mu} \end{pmatrix}, \tag{4}$$

and the following general mixing Hamiltonian reads:

$$\Omega(\mathbf{r}, \mathbf{p}, t) = \Omega(\varepsilon, \mathbf{r}) + \Omega_{\nu\nu}(\mathbf{r}, \mathbf{p}, t) - \Omega_{\nu\bar{\nu}}(\mathbf{r}, \mathbf{p}, t), \tag{5}$$

where $\Omega(\varepsilon, \mathbf{r})$ encompasses the vacuum mixing and the neutrino-matter interaction amplitude for the ν_e neutrino connected with the matter-induced mass. $\Omega_{\nu\nu}$ is the off-diagonal mixing contribution from neutrino-neutrino self-interactions and $\Omega_{\nu\bar{\nu}}$ is the analog for neutrino-antineutrino interactions. We show how to include neutrino self-interactions and neutrino-antineutrino interaction into our formalism in Appendix A and work here with the vacuum and ordinary matter contribution only. Therefore, we have for $\Omega(\varepsilon, \mathbf{r})$ the expression

$$\Omega(\varepsilon, \mathbf{r}) = \frac{\pi c}{L}\begin{pmatrix} -\cos 2\theta + 2A & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{6}$$

where θ is the neutrino vacuum mixing angle between ν_e and ν_μ and A is given by [25,26]

$$A = \left(\frac{L}{\pi c}\right)\frac{2\sqrt{2}G_F}{\hbar} n_e(\mathbf{r}), \tag{7}$$

where L is the vacuum neutrino oscillation length:

$$L = \frac{4\pi\hbar c\varepsilon}{\Delta m^2 c^4}. \tag{8}$$

G_F denotes Fermi's constant, $n_e(\mathbf{r})$ the electron number density, ε the neutrino energy, m_1 and m_2 are the masses of the neutrino mass eigenstates, and $\Delta m^2 = m_2^2 - m_1^2$. The other variables have their standard meanings. For antineutrinos the sign of A is reversed. Note that A, through n_e, is in general a function of spatial position. In this paper, we do not take into account the effects of microscopic density fluctuations on the mixing in the ensemble [27]. In other words, we assume that the scale of the spatial variation of the phase-space density is larger than the neutrino de Broglie wavelength. Similarly, we assume small external forces. With these reasonable assumptions, we can ignore the off-diagonal terms on the left-hand side of Eq. (3). Defining the real part of the off-diagonal macroscopic overlap in Eq. (2) as

$$f_r = \frac{1}{2}(f_{e\mu} + f^*_{e\mu}), \tag{9}$$

and the corresponding imaginary part as

$$f_i = \frac{1}{2i}(f_{e\mu} - f^*_{e\mu}), \tag{10}$$

respectively, we find the generalized Boltzmann equations for real-valued quantities:

$$\begin{aligned}
&\frac{\partial f_{\nu_e}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\nu_e}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_{\nu_e}}{\partial \mathbf{p}} = -\frac{2\pi c}{L} f_i \sin 2\theta + C_{\nu_e} \\
&\frac{\partial f_{\nu_\mu}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\nu_\mu}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_{\nu_\mu}}{\partial \mathbf{p}} = \frac{2\pi c}{L} f_i \sin 2\theta + C_{\nu_\mu} \\
&\frac{\partial f_r}{\partial t} + \mathbf{v}\cdot\frac{\partial f_r}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_r}{\partial \mathbf{p}} = -\frac{2\pi c}{L} f_i(\cos 2\theta - A) \\
&\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_i}{\partial \mathbf{p}} = \frac{2\pi c}{L}\left(\frac{f_{\nu_e} - f_{\nu_\mu}}{2}\sin 2\theta \right. \\
&\qquad\qquad \left. + f_r(\cos 2\theta - A)\right).
\end{aligned} \tag{11}$$

The neutrino oscillations are incorporated with new sink and source terms indirectly coupling the standard Boltzmann equations for f_{ν_e} and f_{ν_μ} through the off-diagonal macroscopic overlap functions f_r and f_i. The number of equations is increased from two to four. In principle, one can insert blocking factors for both the collision terms and the oscillation terms. Note that in the absence of collisions, Liouville's theorem for the total lepton specific intensity, $d/dt(f_{\nu_e} + f_{\nu_\mu}) = 0$, is recovered.

It is instructive to rewrite Eq. (11) in terms of the specific intensities of the neutrino radiation field. We use the one-to-one relation [28,29] between the invariant phase-space densities, f_{ν_e}, f_{ν_μ} and the specific intensities, I_{ν_e}, I_{ν_μ}, and define accordingly:

$$I_{\nu_e} = \frac{\varepsilon^3 f_{\nu_e}}{(2\pi\hbar)^3 c^2}, \qquad I_{\nu_\mu} = \frac{\varepsilon^3 f_{\nu_\mu}}{(2\pi\hbar)^3 c^2}. \tag{12}$$

For the off-diagonal macroscopic overlap analogs one defines

$$\mathcal{R}_{e\mu} = \frac{\varepsilon^3 f_r}{(2\pi\hbar)^3 c^2}, \qquad \mathcal{I}_{e\mu} = \frac{\varepsilon^3 f_i}{(2\pi\hbar)^3 c^2}. \tag{13}$$

The generalized Boltzmann equations in the laboratory (Eulerian frame) for the radiation field of two oscillating neutrino species with collisions are then:

$$\begin{aligned}
\frac{1}{c}\frac{\partial I_{\nu_e}}{\partial t} + \frac{\mathbf{v}}{c}\cdot\frac{\partial I_{\nu_e}}{\partial \mathbf{r}} + \frac{\varepsilon^3 \dot{\mathbf{p}}}{c}\cdot\frac{\partial(I_{\nu_e}\varepsilon^{-3})}{\partial \mathbf{p}} &= -\frac{2\pi}{L}\mathcal{I}_{e\mu}\sin 2\theta + C'_{\nu_e}\\
\frac{1}{c}\frac{\partial I_{\nu_\mu}}{\partial t} + \frac{\mathbf{v}}{c}\cdot\frac{\partial I_{\nu_\mu}}{\partial \mathbf{r}} + \frac{\varepsilon^3 \dot{\mathbf{p}}}{c}\cdot\frac{\partial(I_{\nu_\mu}\varepsilon^{-3})}{\partial \mathbf{p}} &= \frac{2\pi}{L}\mathcal{I}_{e\mu}\sin 2\theta + C'_{\nu_\mu}\\
\frac{1}{c}\frac{\partial \mathcal{R}_{e\mu}}{\partial t} + \frac{\mathbf{v}}{c}\cdot\frac{\partial \mathcal{R}_{e\mu}}{\partial \mathbf{r}} + \frac{\varepsilon^3 \dot{\mathbf{p}}}{c}\cdot\frac{\partial(\mathcal{R}_{e\mu}\varepsilon^{-3})}{\partial \mathbf{p}} &= -\frac{2\pi}{L}(\cos 2\theta - A)\mathcal{I}_{e\mu}\\
\frac{1}{c}\frac{\partial \mathcal{I}_{e\mu}}{\partial t} + \frac{\mathbf{v}}{c}\cdot\frac{\partial \mathcal{I}_{e\mu}}{\partial \mathbf{r}} + \frac{\varepsilon^3 \dot{\mathbf{p}}}{c}\cdot\frac{\partial(\mathcal{I}_{e\mu}\varepsilon^{-3})}{\partial \mathbf{p}} &= \frac{2\pi}{L}\left(\frac{I_{\nu_e} - I_{\nu_\mu}}{2}\sin 2\theta + (\cos 2\theta - A)\mathcal{R}_{e\mu}\right).
\end{aligned} \tag{14}$$

These equations and Eqs. (11) are our major results. The collision terms can be conveniently written as [28]

$$C'_{\nu_e} = -\kappa^s_{\nu_e} I_{\nu_e} + \kappa^a_{\nu_e}\left(\frac{B_{\nu_e} - I_{\nu_e}}{1 - \mathcal{F}^{\rm eq}_{\nu_e}}\right) + \frac{\kappa^s_{\nu_e}}{4\pi}\int \Phi_{\nu_e}(\Omega, \Omega') I_{\nu_e}(\Omega') d\Omega', \tag{15}$$

where Φ_{ν_e} is a phase function for scattering into the beam integrated over the solid angle $d\Omega'$, $\kappa^a_{\nu_e}$ is the sum of all absorption processes $\sum_i n_i \sigma^a_i$, where n_i is the number density of matter species i and σ^a_i denotes the absorption cross sections (for scattering processes the superscript a is replaced with s), $\mathcal{F}^{\rm eq}_{\nu_e}$ is the equilibrium Fermi-Dirac occupation probability, and B_{ν_e} is the corresponding blackbody specific intensity. Changing the subscript from ν_e to ν_μ yields the corresponding parameters for the ν_μ's. For sterile neutrinos, one substitutes ν_s for ν_μ and sets the scattering and absorption terms to zero. Similarly, one can write a set of equations for antineutrinos with different collision terms and with the sign of A reversed. Neutrino and antineutrino evolution are implicitly coupled through pair processes. If self-interactions are included as we show in Appendix A, neutrino and antineutrino evolution are nonlinearly coupled.

III. SIMPLE TESTS OF THE NEW FORMALISM

A. Oscillations with absorptive matter coupling

It is straightforward to show that our set of generalized coupled Boltzmann equations behaves as we would expect from our experience with the standard wave function approach. As our first example, we solve Eq. (14) for ν_e–ν_μ oscillations in box of isotropic neutrinos that can also experience decohering absorption on matter. Note that in reality neutrino interactions only play a role at densities where observable neutrino oscillations are suppressed [11]. To demonstrate the expected limiting behavior of our formalism we artificially "turn off" matter suppression by setting the matter term A to zero. This is not the situation found in nature. We define the approximate oscillation time

$$t_{\rm osc} \simeq \frac{L}{2\pi c} = \frac{2\hbar\varepsilon}{\Delta m^2 c^4}, \tag{16}$$

and set the interaction rate of the ν_e's equal to the interaction rate of the ν_μ's to define the characteristic absorption time

$$t^{\nu_e}_{\rm col} = \frac{1}{\kappa^{a*}_{\nu_e} c} = \frac{(1 - \mathcal{F}^{\rm eq}_{\nu_e})}{c N_A \rho Y_n \sigma^a_{\nu_e n}}. \tag{17}$$

N_A denotes Avogadro's number, Y_n is the neutron fraction per nucleon, and $\sigma^a_{\nu_e n}$ is the cross section for absorption on neutrons (see Appendix B). We define the ratio of the oscillation to the absorption time scale:

$$\alpha = \frac{t_{\rm osc}}{t^{\nu_e}_{\rm col}}, \tag{18}$$

and the dimensionless time coordinate τ:

$$\tau = \frac{t}{t_{\rm osc}}. \tag{19}$$

Furthermore, we set $B_{\nu_e} = B_{\nu_\mu}$ and denote the dimensionless specific intensities and the off-diagonal macroscopic overlap functions that are normalized to the blackbody intensity with a hat. The resulting dimensionless version of Eq. (14) reads:

$$\begin{aligned}
\frac{\partial \hat{I}_{\nu_e}}{\partial \tau} &= -\hat{\mathcal{I}}_{e\mu}\sin 2\theta + \alpha(1-\hat{I}_{\nu_e}) \\
\frac{\partial \hat{I}_{\nu_\mu}}{\partial \tau} &= \hat{\mathcal{I}}_{e\mu}\sin 2\theta + \alpha(1-\hat{I}_{\nu_\mu}) \\
\frac{\partial \hat{\mathcal{R}}_{e\mu}}{\partial \tau} &= -\hat{\mathcal{I}}_{e\mu}\cos 2\theta \\
\frac{\partial \hat{\mathcal{I}}_{e\mu}}{\partial \tau} &= \frac{\hat{I}_{\nu_e}-\hat{I}_{\nu_\mu}}{2}\sin 2\theta + \hat{\mathcal{R}}_{e\mu}\cos 2\theta,
\end{aligned} \tag{20}$$

where the scattering in and out of the beam has been canceled due to the assumption of isotropy. In a vacuum ($\alpha = 0$) and with the initial conditions $\hat{I}_{\nu_e} = 1$, $\hat{I}_{\nu_\mu} = 0$, and consequently no off-diagonal overlap terms at $\tau = 0$, we obtain

$$\begin{aligned}
\hat{I}_{\nu_e}(\tau) &= 1 - \sin^2 2\theta \sin^2\left(\frac{\tau}{2}\right) \\
\hat{I}_{\nu_\mu}(\tau) &= \sin^2 2\theta \sin^2\left(\frac{\tau}{2}\right) \\
\hat{\mathcal{R}}_{e\mu}(\tau) &= \frac{1}{2}\sin 2\theta\cos 2\theta(\cos\tau - 1) \\
\hat{\mathcal{I}}_{e\mu}(\tau) &= \frac{1}{2}\sin 2\theta \sin\tau.
\end{aligned} \tag{21}$$

This behavior of the radiation field is unambiguously identical to the probability density obtained by squaring the amplitude of a single-neutrino wave function in a beam. The off-diagonal terms representing the macroscopic overlap peak when mixing of ν_e and ν_μ neutrinos is maximal and vanish when the ensemble is single-flavored. In matter and for the initial conditions $\hat{I}_{\nu_\mu} = \hat{\mathcal{R}}_{e\mu} = \hat{\mathcal{I}}_{e\mu} = 0$ and $\hat{I}_{\nu_e} \neq 0$, one can derive an harmonic oscillator equation for the early rate of evolution of $\hat{I}_{\nu_e}$:

$$\frac{\partial^2 \hat{I}_{\nu_e}}{\partial \tau^2} + \left(\frac{1}{2} - \alpha^2\right)\hat{I}_{\nu_e} = \text{const.} \tag{22}$$

As expected, for $\alpha \ll 1$, the early time dependence of the solution is predominantly sinusoidal: neutrino oscillations dominate. For $\alpha \gg 1$, an exponential decay/increase dominates. The time scale then is $1/\alpha$.

In Fig. 1, we depict the solutions to Eq. (20) for oscillating ν_e's in a box with nucleons. Initial conditions are $\hat{I}_{\nu_\mu} = \hat{\mathcal{R}}_{e\mu} = \hat{\mathcal{I}}_{e\mu} = 0$ and $\hat{I}_{\nu_e} = 0.8$. For this example, flavor oscillations and collisions happen on the same time scale. The ensemble is guided to flavor and radiative equilibrium. Coherent flavor oscillations are disrupted by absorptive collisions. Asymptotically, the diagonal specific intensities for the ν_e's and ν_μ's equilibrate. Absorption on neutrons, and by detailed balance, the resulting emissivity, drive the ν_e's to the blackbody intensity. The oscillation amplitude decreases with time; the quantum evolution of the system is decohered through absorptive coupling with matter ("quantum decoherence" [17]). The real part of the off-diagonal overlap, $\hat{\mathcal{R}}_{e\mu}$, takes predominantly negative values whereas the imaginary part, $\hat{\mathcal{I}}_{e\mu}$, oscillates symmetrically around zero. Both vanish asymptotically and no oscillations persist.

B. Matter-enhanced resonant flavor conversion

To demonstrate that our formalism contains the MSW effect [18–20], we solve Eq. (14) for a monoenergetic one-dimensional neutrino beam propagating down a density profile for which resonant matter-enhanced flavor conversion takes place. We define the dimensionless distance coordinate in terms of the oscillation length:

$$\hat{x} = x\frac{2\pi}{L}, \tag{23}$$

and the dimensionless matter-induced mass term in terms of its resonance value:

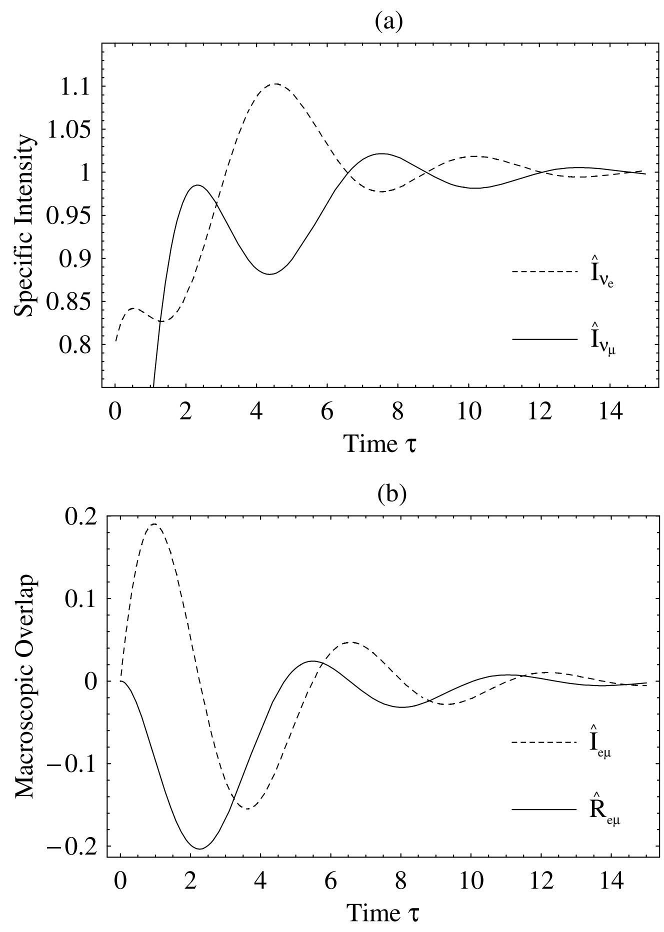

FIG. 1. ν_e–ν_μ oscillations of boxed isotropic neutrinos with absorptive matter coupling. (a) Specific intensities. (b) Off-diagonal macroscopic overlap functions. Parameters: $\varepsilon_{\nu_e} = \varepsilon_{\nu_\mu} = 10$ MeV, $\rho = 8 \times 10^{12}$ g cm^{-3}, $T = 5$ MeV, and from the large-mixing-angle solution (LMA) [30]: $\sin 2\theta = 0.9$, and $\Delta m^2 = 6.9 \times 10^{-5}$ eV^2. A is artificially set to zero.

$$\hat{A} = \frac{A}{A_{\text{res}}} = \frac{A}{\cos 2\theta}. \tag{24}$$

The beam passes the resonance density for $\hat{A} = 1$. We analyze the following dimensionless version of Eq. (14):

$$\begin{aligned} \frac{\partial I_{\nu_e}}{\partial \hat{x}} &= -\mathcal{I}_{e\mu} \sin 2\theta \\ \frac{\partial I_{\nu_\mu}}{\partial \hat{x}} &= \mathcal{I}_{e\mu} \sin 2\theta \\ \frac{\partial \mathcal{R}_{e\mu}}{\partial \hat{x}} &= -\cos 2\theta (1 - \hat{A}) \mathcal{I}_{e\mu} \\ \frac{\partial \mathcal{I}_{e\mu}}{\partial \hat{x}} &= \frac{I_{\nu_e} - I_{\nu_\mu}}{2} \sin 2\theta + \cos 2\theta (1 - \hat{A}) \mathcal{R}_{e\mu}, \end{aligned} \tag{25}$$

where C'_{ν_e} and C'_{ν_μ} have been set to zero.

In Fig. 2, we depict the solutions to Eq. (25) for a density profile of $A = \beta/\hat{x}^2$. We set $\beta = 900$ and thus ensure that the scale of spatial inhomogeneities is large compared to the microscopic length scales such as the neutrino de Broglie wavelength and the oscillation length. Initially, the beam contains only ν_e neutrinos. The mixing angle is arbitrarily taken to be $\sin^2 2\theta = 0.18$ (at present the LMA is favored [30]). From Fig. 2, it is clear that at the resonance density, $\hat{x} = \sqrt{\beta} = 30$, the flavor composition of the beam is radically altered. For higher values of $\hat{x}$, the beam executes vacuum oscillations. In this illustrative problem the density at production is much greater than the resonance density. Then, the spatially averaged survival probability of a ν_e neutrino going from matter to free space should be [31]

$$\langle P(\nu_e \to \nu_e) \rangle \approx (1 - P_x) \sin^2\theta + P_x \cos^2\theta, \tag{26}$$

where P_x is the Landau-Zener probability for nonadiabatic transitions. For the chosen density profile, propagation is adiabatic and $P_x = 0$. The averaged ν_e survival probability in Fig. 2 converges toward ~ 0.05, which is congruent with the value predicted in Eq. (26): $\sin^2\theta \sim 0.05$. For high densities ($\hat{x} \leq 20$), matter suppression is severe. No flavor oscillations happen and the off-diagonal overlap functions $\mathcal{R}_{e\mu}$ and $\mathcal{I}_{e\mu}$ are close to zero. In free space for $\hat{x} > 60$, the imaginary part $\mathcal{I}_{e\mu}$ oscillates symmetrically around zero whereas the real part $\mathcal{R}_{e\mu}$ is positive.

The solution given in Fig. 2 is numerically equivalent to the solution obtained using the standard wave function formalism [18–20,26]:

$$i \frac{\partial}{\partial \hat{x}} \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\cos 2\theta (1 - 2\hat{A}) & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \end{pmatrix}, \tag{27}$$

when we identify the specific intensities with the probability densities

$$I_{\nu_e} \leftrightarrow |\psi_{\nu_e}|^2 \qquad I_{\nu_\mu} \leftrightarrow |\psi_{\nu_\mu}|^2, \tag{28}$$

and the macroscopic overlap functions with

$$\begin{aligned} \mathcal{R}_{e\mu} &\leftrightarrow \frac{1}{2} (\psi_{\nu_e} \psi^*_{\nu_\mu} + \psi^*_{\nu_e} \psi_{\nu_\mu}) \\ \mathcal{I}_{e\mu} &\leftrightarrow \frac{1}{2i} (\psi_{\nu_e} \psi^*_{\nu_\mu} - \psi^*_{\nu_e} \psi_{\nu_\mu}). \end{aligned} \tag{29}$$

Thus, our Boltzmann formalism is completely consistent with the existing description.

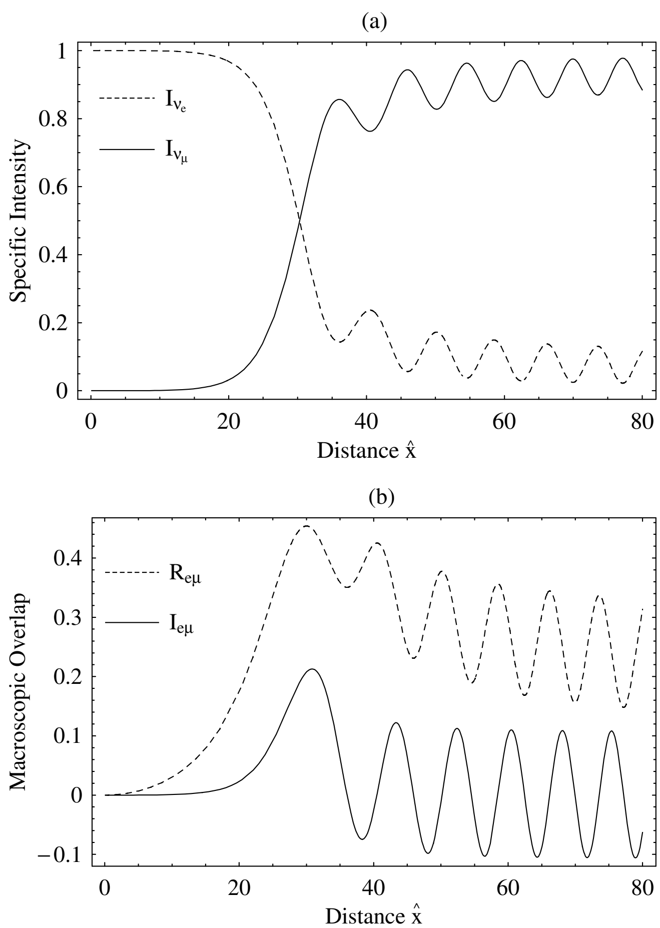

FIG. 2. MSW effect. (a) Specific intensities. (b) Off-diagonal macroscopic overlap functions. An initially ν_e beam propagates down the density profile $\beta/\hat{x}^2$, where $\beta = 900$.

IV. CONCLUSIONS

In this paper, we have derived a generalized set of Boltzmann equations for real-valued phase-space densities of oscillating neutrinos interacting with a background medium. The off-diagonal functions of the Wigner phase-space density matrix representing macroscopic overlap are explicitly included and serve to couple the flavor states to reflect neutrino oscillation physics. Conceptually, we have reduced the time evolution of creation and annihilation operators to that of real-valued phase-space densities without losing quantum-physical accuracy. Important quantum effects such as matter-enhanced resonant flavor conversion and "decoherence" [17] through matter coupling are correctly incorporated. The generalized Boltzmann equations are simple and very similar to the equations of classical transport theory. Neutrino oscillations are incorporated by new sink and source terms that indirectly couple the expanded set of equations. We have shown how to include neutrino self-interactions in our formalism. The self-interactions nonlinearly couple neutrino and antineutrino evolution and therefore for the most generic case one has to deal with eight nontrivially coupled equations for a two-flavor ensemble and their antiparticles.

Using this formalism, codes that now solve the standard Boltzmann equations for the classical neutrino phase-space density (f_{ν_i}), or which address its angular and/or energy moments, can straightforwardly be reconfigured by the simple addition of source terms and similar transport equations for overlap densities that have the same units as f_{ν_i}, to incorporate neutrino oscillations in a quantum-physically consistent fashion.

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APPENDIX A: NEUTRINO SELF-INTERACTIONS

Evolution in a neutrino background is nontrivial [32–35]. To address this, one can retain the one-particle character of description and neglect any effects arising from coherent many-body state formation or flavor entanglement [36–40]. The local low-energy four Fermi-interaction Hamiltonian of neutrino-neutrino scattering is given by

$$\Omega_{\nu\nu}^{\mathbf{pq}} = \frac{G_F}{\sqrt{2}}\left(\sum_i \bar{\psi}_{\mathbf{q}}^i \gamma^\mu \psi_{\mathbf{q}}^i\right)\left(\sum_j \bar{\psi}_{\mathbf{q}}^j \gamma_\mu \psi_{\mathbf{q}}^j\right), \tag{A1}$$

where the sum is over all neutrino flavors. This effective Hamiltonian must satisfy $U(N)$ flavor symmetry for a system of N flavors [33]. For a two-flavor system consisting of ν_e's and ν_μ's one can rewrite this Hamiltonian to accentuate its off-diagonal character [34,36]:

$$\Omega_{\nu\nu}^{\mathbf{pq}} = \beta(1-\cos\theta^{\mathbf{pq}}) \times \left[|\psi_{\nu_e}^{\mathbf{q}}|^2 + |\psi_{\nu_\mu}^{\mathbf{q}}|^2 + \begin{pmatrix} |\psi_{\nu_e}^{\mathbf{q}}|^2 & \psi_{\nu_e}^{\mathbf{q}}\psi_{\nu_\mu}^{\mathbf{q}*} \\ \psi_{\nu_e}^{\mathbf{q}*}\psi_{\nu_\mu}^{\mathbf{q}} & |\psi_{\nu_\mu}^{\mathbf{q}}|^2 \end{pmatrix} \right], \tag{A2}$$

where the coupling coefficient includes the angle between the "test neutrino" with momentum $\mathbf{p}$ and the background neutrino with momentum $\mathbf{q}$ and the coupling strength is given by $\beta = (\sqrt{2}G_F)/\hbar$. The neutrino fields of the background neutrino with momentum $\mathbf{q}$ are normalized such that

$$\int dV(|\psi_{\nu_e}^{\mathbf{q}}|^2 + |\psi_{\nu_\mu}^{\mathbf{q}}|^2) = 1. \tag{A3}$$

We now match the above expressions and convert them into our quasiclassical phase-space densities and the macroscopic overlap functions. The ν–ν mixing Hamiltonian for test neutrinos with momentum $\mathbf{p}$ in ensemble-averaged form is denoted by

$$\Omega_{\nu\nu}(\mathbf{p}, \mathbf{r}, t) = \begin{pmatrix} B_{\nu_e} & B_r + iB_i \\ B_r - iB_i & B_{\nu_\mu} \end{pmatrix}, \tag{A4}$$

where, while throwing away the overall phase term in Eq. (A2) proportional to the identity matrix, we have for the diagonal elements:

$$\begin{aligned} B_{\nu_e}(\mathbf{p}, \mathbf{r}, t) &= \beta \int d^3\mathbf{q}(1-\cos\theta^{\mathbf{pq}}) f_{\nu_e}(\mathbf{q}, \mathbf{r}, t) \\ B_{\nu_\mu}(\mathbf{p}, \mathbf{r}, t) &= \beta \int d^3\mathbf{q}(1-\cos\theta^{\mathbf{pq}}) f_{\nu_\mu}(\mathbf{q}, \mathbf{r}, t). \end{aligned} \tag{A5}$$

We used the "matching" from our formalism to the wave function formalism as prescribed in Eqs. (28) and (29). The momentum integration goes over all momenta $\mathbf{q}$ in the ensemble. For the off-diagonal elements we write in our notation,

$$\begin{aligned} B_r(\mathbf{p}, \mathbf{r}, t) &= \beta \int d^3\mathbf{q}(1-\cos\theta^{\mathbf{pq}}) f_r(\mathbf{q}, \mathbf{r}, t) \\ B_i(\mathbf{p}, \mathbf{r}, t) &= \beta \int d^3\mathbf{q}(1-\cos\theta^{\mathbf{pq}}) f_i(\mathbf{q}, \mathbf{r}, t), \end{aligned} \tag{A6}$$

where we have used the variables of Eqs. (9) and (10). For the neutrino-antineutrino interaction there is

$$\Omega_{\nu\bar{\nu}}(\mathbf{p}, \mathbf{r}, t) = \begin{pmatrix} \tilde{B}_{\nu_e} & \tilde{B}_r + i\tilde{B}_i \\ \tilde{B}_r - i\tilde{B}_i & \tilde{B}_{\nu_\mu} \end{pmatrix}, \tag{A7}$$

where the B's are now defined in terms of the antineutrino phase-space densities $\tilde{f}_{\nu_e}$, $\tilde{f}_{\nu_\mu}$, $\tilde{f}_r$, and $\tilde{f}_i$. Note that in accordance with Eq. (5) the coupling coefficient in front of the integrals has to be implemented in the equations for neutrinos with reversed sign. For two mixing neutrino species and their antiparticles interacting with a background medium and with neutrino-neutrino interactions included, the generalized Boltzmann equations in their most generic form are

$$\begin{aligned}
\frac{\partial f_{\nu_e}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\nu_e}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_{\nu_e}}{\partial \mathbf{p}} &= -f_i\left[\frac{2\pi c}{L}\sin 2\theta + 2(B_r + \tilde{B}_r)\right] + 2f_r(B_i + \tilde{B}_i) + C_{\nu_e} \\
\frac{\partial f_{\nu_\mu}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\nu_\mu}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_{\nu_\mu}}{\partial \mathbf{p}} &= f_i\left[\frac{2\pi c}{L}\sin 2\theta + 2(B_r + \tilde{B}_r)\right] - 2f_r(B_i + \tilde{B}_i) + C_{\nu_\mu} \\
\frac{\partial f_r}{\partial t} + \mathbf{v}\cdot\frac{\partial f_r}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_r}{\partial \mathbf{p}} &= f_i\left[\frac{2\pi c}{L}(A - \cos 2\theta) + B_{\nu_e} - \tilde{B}_{\nu_e} - B_{\nu_\mu} + \tilde{B}_{\nu_\mu}\right] + (f_{\nu_e} - f_{\nu_\mu})(\tilde{B}_i - B_i) \\
\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial f_i}{\partial \mathbf{p}} &= (f_{\nu_e} - f_{\nu_\mu})\left[\frac{\pi c}{L}\sin 2\theta + (B_r - \tilde{B}_r)\right] - f_r\left[\frac{2\pi c}{L}(A - \cos 2\theta) + B_{\nu_e} - \tilde{B}_{\nu_e} - B_{\nu_\mu} + \tilde{B}_{\nu_\mu}\right].
\end{aligned} \tag{A8}$$

In a very similar fashion one can write the corresponding antiparticle analog. We need to interchange tildes and reverse the sign for A to complete our set of equations:

$$\begin{aligned}
\frac{\partial \tilde{f}_{\nu_e}}{\partial t} + \mathbf{v}\cdot\frac{\partial \tilde{f}_{\nu_e}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \tilde{f}_{\nu_e}}{\partial \mathbf{p}} &= -\tilde{f}_i\left[\frac{2\pi c}{L}\sin 2\theta + 2(\tilde{B}_r + B_r)\right] + 2\tilde{f}_r(\tilde{B}_i + B_i) + \tilde{C}_{\nu_e} \\
\frac{\partial \tilde{f}_{\nu_\mu}}{\partial t} + \mathbf{v}\cdot\frac{\partial \tilde{f}_{\nu_\mu}}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \tilde{f}_{\nu_\mu}}{\partial \mathbf{p}} &= \tilde{f}_i\left[\frac{2\pi c}{L}\sin 2\theta + 2(\tilde{B}_r + B_r)\right] - 2\tilde{f}_r(\tilde{B}_i + B_i) + \tilde{C}_{\nu_\mu} \\
\frac{\partial \tilde{f}_r}{\partial t} + \mathbf{v}\cdot\frac{\partial \tilde{f}_r}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \tilde{f}_r}{\partial \mathbf{p}} &= \tilde{f}_i\left[\frac{2\pi c}{L}(-A - \cos 2\theta) + \tilde{B}_{\nu_e} - B_{\nu_e} - \tilde{B}_{\nu_\mu} + B_{\nu_\mu}\right] + (\tilde{f}_{\nu_e} - \tilde{f}_{\nu_\mu})(B_i - \tilde{B}_i) \\
\frac{\partial \tilde{f}_i}{\partial t} + \mathbf{v}\cdot\frac{\partial \tilde{f}_i}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \tilde{f}_i}{\partial \mathbf{p}} &= (\tilde{f}_{\nu_e} - \tilde{f}_{\nu_\mu})\left[\frac{\pi c}{L}\sin 2\theta + (\tilde{B}_r - B_r)\right] - \tilde{f}_r\left[\frac{2\pi c}{L}(-A - \cos 2\theta) + \tilde{B}_{\nu_e} - B_{\nu_e} - \tilde{B}_{\nu_\mu} + B_{\nu_\mu}\right].
\end{aligned} \tag{A9}$$

Neutrino and antineutrino evolution are nonlinearly coupled. The collision terms differ for neutrinos and antineutrinos. The sign reversal of A means that, dependent on the mass hierarchy, only neutrinos or antineutrinos execute the MSW resonance. The above nontrivially coupled set of eight equations is entirely real-valued; yet they contain all the quantum-mechanical oscillation phenomenology. This is the complete set of kinetic equations that include neutrino self-interactions for the real-valued neutrino phase-space densities f_{ν_e} and f_{ν_μ} and the corresponding antineutrino phase-space densities $\tilde{f}_{\nu_e}$ and $\tilde{f}_{\nu_\mu}$.

APPENDIX B: CROSS SECTION: $\nu_e + n \rightarrow e^- + p$

A convenient reference neutrino cross section is σ_o, given by

$$\sigma_o = \frac{4G_F^2(m_e c^2)^2}{\pi(\hbar c)^4} \simeq 1.705 \times 10^{-44}\ \mathrm{cm}^2, \tag{B1}$$

where G_F is the Fermi weak coupling constant ($\simeq 1.436 \times 10^{-49}$ ergs cm^3). The total $\nu_e - n$ absorption cross section for the reaction $\nu_e + n \rightarrow e^- + p$ is then given by

$$\sigma^a_{\nu_e n} \sim \sigma_0\left(\frac{1 + 3g_A^2}{4}\right)\left(\frac{\varepsilon_{\nu_e} + \Delta_{np}}{m_e c^2}\right)^2 \times \left[1 - \left(\frac{m_e c^2}{\varepsilon_{\nu_e} + \Delta_{np}}\right)^2\right]^{1/2}, \tag{B2}$$

where g_A is the axial-vector coupling constant (~ -1.23), and $\Delta_{np} = m_n c^2 - m_p c^2 = 1.29332$ MeV.

[1] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. **59**, 671 (1987).
[2] S. M. Bilenky and B. Pontecorvo, Phys. Lett. B **61**, 248 (1976).
[3] E. Moertsell, L. Bergstroem, and A. Goobar, Phys. Rev. D **66**, 047702 (2002).
[4] G. Raffelt and L. Stodolsky, Phys. Rev. D **37**, 1237 (1988).
[5] E. Akhmedov, A. Barroso, and P. Keraenen, Phys. Lett. B **517**, 355 (2001).
[6] K. Takahashi, K. Sato, A. Burrows, and T. A. Thompson, Phys. Rev. D **68**, 113009 (2003).
[7] T. A. Thompson, A. Burrows, and J. E. Horvath, Phys. Rev. C **62**, 035802 (2000).
[8] A. Burrows and R. Gandhi, Phys. Lett. B **243**, 149 (1990).
[9] G. Raffelt, *Stars as Laboratories for Fundamental Physics* (University of Chicago Press, Chicago, 1996).
[10] J. T. Pantaleone and T. K. Kuo, Rev. Mod. Phys. **61**, 937 (1989).
[11] L. Wolfenstein, Phys. Rev. D **20**, 2634 (1979).
[12] G. Raffelt and G. Sigl, Nucl. Phys. **B406**, 423 (1993).

[13] G. Raffelt, G. Sigl, and L. Stodolsky, Phys. Rev. D **45**, 1782 (1992).
[14] G. Raffelt and G. Sigl, Astropart. Phys. **1**, 165 (1993).
[15] A. I. Akhiezer and S. V. Peletminskii, *Methods of Statistical Physics*, International Series on Natural Philosophy Vol. 104 (Pergamon, New York, 1981).
[16] S. de Groot, W. van Leeuwen, and C. G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
[17] G. Raffelt, G. Sigl, and L. Stodolsky, Phys. Rev. Lett. **70**, 2363 (1993).
[18] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
[19] H. Bethe, Phys. Rev. Lett. **56**, 1305 (1986).
[20] S. P. Mikheev and A. Yu. Smirnov, Sov. Phys. JETP **64**, 4 (1986).
[21] E. Wigner, Phys. Rev. **40**, 749 (1932).
[22] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner Phys. Rep. **106**, 121 (1984).
[23] L. D. Landau, Sov. Phys. JETP **5**, 101 (1957).
[24] M. A. Rudzsky, Astrophys. Space Sci. **165**, 65 (1990).
[25] R. N. Mohapatra and P. B. Pal, *Massive Neutrinos in Physics and Astrophysics*, Lecture Notes in Physics Vol. 72 (World Scientific, Singapore, 2003).
[26] C. Giunti, Neutrino Mixing and Oscillations, http://www.nu.to.infn.it/, 2004.
[27] R. F. Sawyer, Phys. Rev. D **42**, 3908 (1990).
[28] A. Burrows, T. Young, P. Pinto, R. Eastman, and T. A. Thompson, Astrophys. J. **539**, 865 (2000).
[29] D. Mihalas and B. Weibel-Mihalas, *Foundations of Radiation Hydrodynamics* (Dover, New York, 1999).
[30] S. Fukuda *et al.* (Super-Kamiokande Collaboration), Phys. Lett. B **539**, 179 (2002).
[31] S. J. Parke, Phys. Rev. Lett. **57**, 1275 (1986).
[32] Y.-Z. Qian and G. Fuller, Phys. Rev. D **51**, 1479 (1995).
[33] J. Pantaleone, Phys. Lett. B **287**, 128 (1992).
[34] J. Pantaleone, Phys. Rev. D **46**, 510 (1992).
[35] J. Pantaleone, Phys. Lett. B **342**, 250 (1995).
[36] A. Friedland and C. Lunardini, Phys. Rev. D **68**, 013007 (2003).
[37] A. Friedland and C. Lunardini, J. High Energy Phys. 10 (2003) 043.
[38] N. F. Bell, A. A. Rawlinson, and R. F. Sawyer, Phys. Lett. B **573**, 86 (2003).
[39] R. F. Sawyer, hep-ph/0408265.
[40] R. F. Sawyer, hep-ph/0503013.