Flat currents of the Green-Schwarz superstrings in $AdS_5 \times S^1$ and $AdS_3 \times S^3$ backgrounds

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We construct a one-parameter family of flat currents in $AdS_5 \times S^1$ and $AdS_3 \times S^3$ Green-Schwarz superstrings, which would naturally lead to a hierarchy of classical conserved nonlocal charges. In the former case we rewrite the $AdS_5 \times S^1$ string using a new $\mathbb{Z}_4$-graded base of the superalgebra $su(2,2|2)$. In both cases the existence of the $\mathbb{Z}_4$ grading in the superalgebras plays a key role in the construction. As a result, we find that the flat currents, when formally written in terms of the $\mathcal{G}_0$-gauge invariant lowercase 1-forms, take the same form as the one in the $AdS_5 \times S^5$ case.

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I. INTRODUCTION

The AdS/CFT correspondence provides a lot of deep insights into both string theories and gauge theories [1–3]. This duality realizes a fairly old idea, originated from 't Hooft [4], that a large N gauge theory should be equivalent to a string theory. A main progress in this subject in recent years is the discovery of integrable structures on both sides. It was found [5–7] that the one-loop dilatation operator can be interpreted as the Hamiltonian of an integrable spin chain. On the string side, a hierarchy of infinite nonlocal charges are constructed [8] in the Green-Schwarz superstring on $AdS_5 \times S^5$ space-time [9], implying that the world-sheet sigma model is integrable. Subsequently it was shown [10] that such charges also exist in the pure spinor formalism. The analog nonlocal charges were also found [11] in the ten dimensional pp-wave background and they can be identified as the Penrose limit of the nonlocal charges in $AdS_5 \times S^5$ string. It was shown in [12] that the integrability unravelled in the string side could be intimately connected with the Yangian symmetry constructed in the $\mathcal{N} = 4$ superconformal Yang-Mills theory in the free-field limit (But see also [13]). Actually, AdS/CFT correspondence implies that the Yangian symmetry should persist in the $\mathcal{N} = 4$ superconformal Yang-Mills theory in the planar limit even when $g^2_{\rm YM}N$ is nonvanishing. By using the supertwistor correspondence [14] of the $\mathcal{N} = 4$ sypersymmetric Yang-Mills (SYM), the author of [15] studied the supertwistor construction of hidden symmetry algebras of the self-dual SYM equations. Recently, in the pure spinor formalism of superstrings on $AdS_5 \times S^5$, it was proved in [16] that the nonlocal charges in the string theory are Becchi-Rouet-Stora-Tyutin (BRST)-invariant and physical, and in [17] it was shown that there exists an infinite set of nonlocal BRST-invariant currents at the quantum level. Some recent developments on nonlocal charges could be found in [18,19].

It is important to study the AdS/CFT correspondence in the cases with less supersymmetry, in order to better understand the correspondence and be close to the real world. There are various ways to partially break the supersymmetry. By calculating the 1-loop β function Polyakov [20] proposed that the noncritical $AdS_p \times S^q$ string theories should be dual to the gauge theories with less or no supersymmetry. Last year he briefly constructed several sigma models in [21] and argued that they are conformal invariant and also completely integrable,[1] just as the critical case. Soon after, Klebanov and Maldacena [22] found the $AdS_5 \times S^1$ solution in the low energy supergravity effective action of six-dimensional noncritical string theory with N units of RR flux and in the presence of N_f space-time filling $D5$-branes. This solution has the right structure to be dual to $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theories with N_f flavors, in agreement with the proposal in [21]. Several other $AdS_p \times S^q$ solutions were also found [23] in the context of the six-dimensional noncritical superstring theory.

In this paper, as the first step to understanding the integrable structure in Green-Schwarz superstring on $AdS_5 \times S^1$ [21], we try to construct the flat currents of this model, which would naturally lead to infinite nonlocal charges. It has been shown that the coset structure and the $\mathbb{Z}_4$ grading of $psu(2,2|4)$, which is the symmetry algebra of the $AdS_5 \times S^5$ string, are essential to construct the nonlocal charges [8]. The Green-Schwarz superstring on $AdS_5 \times S^1$ is a sigma model with the target space $SU(2,2|2)/(SO(4,1) \times SO(3))$. To find the flat currents we need first determine whether the superalgebra $su(2,2|2)$ has the similar grading structure. Unlike $su(2,2|4) \simeq psu(2,2|4) \oplus u(1)$ can be decomposed into a

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[1]In [21] the author asserted that the sigma models described in that paper are completely integrable with explicit construction of the flat currents in a $OSp(2|4)/(SO(3,1) \times SO(2))$ model.

direct sum of two algebras, the algebra $su(2,2|2)$ is already simple by itself. The $u(1)$ part has the nonzero (anti)commutators with other generators of $su(2,2|2)$. In spite of this difference we find that the algebra $su(2,2|2)$ also has a $\mathbb{Z}_4$ grading structure by explicitly constructing the graded generators, and that the denominator of the supercoset $SU(2,2|2)/(SO(4,1)\times SO(3))$ corresponds to the part of grade zero. In terms of these new graded generators, we rewrite the κ-symmetric action of the $AdS_5\times S^1$ string and derive the equations of motions. The construction of the flat currents follows [8] in a straightforward way. In terms of the $\mathcal{G}_0$-gauge invariant lowercase forms, the Maurer-Cartan equations and the equations of motion take the same form as the ones in the $AdS_5\times S^5$ string. As a result, the flat currents we construct, when written in terms of the gauge invariant 1-forms, have the same form as those of the $AdS_5\times S^5$ case.

Another supercoset model with $\mathbb{Z}_4$ grading is the Green-Schwarz superstring in $AdS_3\times S^3$ background, which was constructed in [24]. We construct the flat currents and find it also has the same form as the ones in the $AdS_5\times S^5$ string.

This paper is organized as follows. We first study the $AdS_5\times S^1$ string. In Sec. II we construct the $\mathbb{Z}_4$-graded generators of the superalgebra $su(2,2|2)$ explicitly, then we define the Maurer-Cartan 1-forms with respect to these generators and write down the Maurer-Cartan equations that these forms must satisfy. In Sec. III we rewrite the action of $AdS_5\times S^1$ string using the graded generators and then get the equations of motion. Then we explicitly construct a one-parameter family of the flat currents in $AdS_5\times S^1$ superstring. In Sec. IV we construct the flat currents in $AdS_3\times S^3$ superstring. In Sec. V we review our results and discuss some future directions in this subject. In Appendix A we list our conventions in this paper, which are mainly used in the construction of the $\mathbb{Z}_4$-graded generators of $su(2,2|2)$. In Appendix B we check the κ-symmetry of the action of the $AdS_5\times S^1$ superstring. In Appendix C we write down the closed expressions of the Maurer-Cartan 1-forms of the coset space $SU(2,2|2)/(SO(4,1)\times SO(3))$.

II. $su(2,2|2)$ SUPERALGEBRA

Our starting point is the extended superconformal algebra $su(2,2|2)$, which is the symmetry algebra of the $AdS_5\times S^1$ superstring. This is a simple superalgebra. Its bosonic part is $su(2,2)\oplus su(2)\oplus u(1)$. We denote $M_{\bar a\bar b}, T_{a'}, R$ as the generators of $su(2,2)$, $su(2)$, $u(1)$ respectively. Notice the isomorphisms $su(2,2)\simeq so(4,2)$ and $su(2)\simeq so(3)$, then

$$\begin{aligned}[M_{\bar a\bar b}, M_{\bar c\bar d}] &= \eta_{\bar a\bar d}M_{\bar b\bar c}+\eta_{\bar b\bar c}M_{\bar a\bar d}-\eta_{\bar a\bar c}M_{\bar b\bar d}\\ &\quad -\eta_{\bar b\bar d}M_{\bar a\bar c},\\ [T_{a'},T_{b'}] &= \varepsilon_{a'b'c'}T_{c'}.\end{aligned}\tag{2.1}$$

Here $(\eta_{\bar a\bar b})=\mathrm{diag}(-++++-)$, $(\eta_{a'b'})=\mathrm{diag}(+++)$ with $\bar a,\bar b=0,1,2,3,4,5$ and $a',b'=1,2,3$.

The fermionic part of $su(2,2|2)$ lies in a $(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\mathbf{4}^*,\mathbf{2}^*,\mathbf{1}^*)$ representation of the bosonic part under the adjoint action. If we denote $F_{\alpha\alpha'}$ and $\bar F^{\alpha\alpha'}$ as the fermionic generators with $\alpha,\beta=1,2,3,4$ and $\alpha',\beta'=1,2$, then

$$\begin{aligned}[F^{\alpha\alpha'}, M_{\bar a\bar b}] &= \frac{1}{2}(\Gamma_{\bar a\bar b})^\alpha_\beta F^{\beta\alpha'},\\ [\bar F_{\alpha\alpha'}, M_{\bar a\bar b}] &= -\frac{1}{2}\bar F_{\beta\alpha'}(\Gamma_{\bar a\bar b})^\beta_\alpha,\\ [F^{\alpha\alpha'}, T_{a'}] &= \frac{1}{2}(\tau_{a'})^{\alpha'}_{\beta'}F^{\alpha\beta'},\\ [\bar F_{\alpha\alpha'}, T_{a'}] &= -\frac{1}{2}\bar F_{\alpha\beta'}(\tau_{a'})^{\beta'}_{\alpha'},\\ [F^{\alpha\alpha'}, R] = \frac{\mathrm{i}}{2}F^{\alpha\alpha'}, &\qquad [\bar F_{\alpha\alpha'}, R] = -\frac{\mathrm{i}}{2}\bar F_{\alpha\alpha'}.\end{aligned}\tag{2.2}$$

Here $\Gamma_{\bar a\bar b}$ are 4×4 matrices, their explicit form can be found in Appendix A, and $\tau_{a'}=-\mathrm{i}\sigma_{a'}$ with $\sigma_{a'}(a'=1,2,3)$ being the three Pauli matrices. In the above formulae we choose the convention that the upper indices of Γ's and τ's are the row indices, and that the lower ones are the column indices.

At last the anticommutator of two fermionic generators are

$$\begin{aligned}\{F^{\alpha\alpha'}, \bar F_{\alpha\alpha'}\} &= -\frac{\mathrm{i}}{2}(\Gamma^{\bar a\bar b})^\alpha_\beta\delta^{\alpha'}_{\beta'}M_{\bar a\bar b}-\delta^\alpha_\beta\delta^{\alpha'}_{\beta'}R\\ &\quad +2\mathrm{i}\delta^\alpha_\beta(\tau^{a'})^{\alpha'}_{\beta'}T_{a'}.\end{aligned}\tag{2.3}$$

The structure of the right hand side of the above anticommutator is the consequence of the covariance, and the coefficients in front of each terms are fixed by the Jacobi identities. The commutation relations (2.1), (2.2), and (2.3) entirely define the superalgebra $su(2,2|2)$ in the standard way. The author of [21] used just these generators to construct the action of $AdS_5\times S^1$ superstring. The grading structure is implicit.

A. Construction of the graded generators

Now we have the commutation relations of $su(2,2|2)$, but we cannot see clearly whether there is a $\mathbb{Z}_4$ grading in terms of the generators $(M_{\bar a\bar b}, T_{a'}, R, F, \bar F)$. The grading structure of the symmetry algebra is an important ingredient in the study of nonlocal charges of the $AdS_5\times S^5$ string. In this subsection we demonstrate the $\mathbb{Z}_4$-grading of the superalgebra $su(2,2|2)$ by constructing the graded generators explicitly.

The fermionic generators $F^{\alpha\alpha'}$ and $\bar F_{\alpha\alpha'}$ are 4-component spinors with respect to the index α. We split them into 2-component spinors as $F^{\alpha\alpha'}=(U^{A\alpha'}, V_{\dot A}{}^{\alpha'})$ and $\bar F_{\alpha\alpha'}=(V_{A\alpha'}, U^{\dot A}{}_{\alpha'})$, where $V_{\dot A}{}^{\alpha'}=C^{\alpha'\beta'}(V_{A\beta'})^\dagger$ and $U^{\dot A}{}_{\alpha'}=C_{\alpha'\beta'}(U^{A\beta'})^\dagger$. Next we define $P_a=M_{a5}$, $J_{ab}=M_{ab}(a,b=0,1,2,3,4)$ and $Q^I_{\alpha\alpha'}=(Q^I_{A\alpha'}, Q^{I\dot A}{}_{\alpha'})I=1,2$

by

$$
\begin{aligned}
Q^1_{A\alpha'} &= \varepsilon_{AB} C_{\alpha'\beta'} U^{B\beta'} - V_{A\alpha'} \\
Q^{1\dot{A}}{}_{\alpha'} &= -\varepsilon^{\dot{A}\dot{B}} C_{\alpha'\beta'} V_{\dot{B}}{}^{\beta'} - U^{\dot{A}}{}_{\alpha'} \\
Q^2_{A\alpha'} &= \mathrm{i}(\varepsilon_{AB} C_{\alpha'\beta'} U^{B\beta'} + V_{A\alpha'}) \\
Q^{2\dot{A}}{}_{\alpha'} &= \mathrm{i}(-\varepsilon^{\dot{A}\dot{B}} C_{\alpha'\beta'} V_{\dot{B}}{}^{\beta'} + U^{\dot{A}}{}_{\alpha'}).
\end{aligned} \tag{2.4}
$$

In terms of the new generators $(P_a, J_{ab}, R, T_{a'}, Q^I_{\alpha\alpha'})$, the commutation relations of the superalgebra change to :

$$
\begin{aligned}
&[P_a, P_b] = J_{ab}, \qquad [P_a, J_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, \qquad [T_{a'}, T_{b'}] = \varepsilon_{a'b'c'} T_{c'}, \\
&[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \qquad [Q^I_{\alpha\alpha'}, P_a] = -\frac{\mathrm{i}}{2}\varepsilon^{IJ} Q^J_{\beta\alpha'}(\gamma_a)^\beta_\alpha, \qquad [Q^I_{\alpha\alpha'}, R] = \frac{1}{2}\varepsilon^{IJ} Q^J_{\alpha\alpha'}, \\
&[Q^I_{\alpha\alpha'}, J_{ab}] = -\frac{1}{2} Q^I_{\beta\alpha'}(\gamma_{ab})^\beta_\alpha, \qquad [Q^I_{\alpha\alpha'}, T_{a'}] = -\frac{1}{2} Q^I_{\alpha\beta'}(\tau_{a'})^{\beta'}_{\alpha'}, \\
&\{Q^I_{\alpha\alpha'}, Q^J_{\beta\beta'}\} = \delta^{IJ}[-2\mathrm{i}(C\gamma^a)_{\alpha\beta} C_{\alpha'\beta'} P_a + 2C_{\alpha\beta} C_{\alpha'\beta'} R] + \varepsilon^{IJ}[(C\gamma^{ab})_{\alpha\beta} C_{\alpha'\beta'} J_{ab} - 4C_{\alpha\beta}(C'\tau^{a'})_{\alpha'\beta'} T_{a'}].
\end{aligned} \tag{2.5}
$$

Here $a, b = 0, 1, 2, 3, 4$ which can be lowered and raised by the 5-dimensional metric $(\eta_{ab}) = (\eta^{ab}) = \mathrm{diag}(- + + + +)$, and $a', b' = 1, 2, 3$ which can be lowered and raised by the 3-dimensional metric $(\eta_{a'b'}) = (\eta^{a'b'}) = \mathrm{diag}(+ + +)$. The indices $I, J = 1, 2$ and the Levi-Civita symbol is defined by $\varepsilon^{12} = -\varepsilon^{21} = 1$. The gamma-matrices are

$$
\begin{aligned}
\gamma^a &= \begin{cases} \gamma^\mu, & a = \mu, \\ \bar{\gamma}, & a = 4, \end{cases} \\
\gamma^{ab} &= \frac{1}{2}[\gamma^a, \gamma^b], \\
\tau^{a'} &= -\mathrm{i}\sigma^{a'}.
\end{aligned} \tag{2.6}
$$

Their explicit expressions can be found in Appendix A. The matrices γ^a and $\tau^{a'}$ are the generators of 5-dimensional and 3-dimensional Clifford algebra, C and C' are their charge conjugation matrices[2] respectively, which satisfy

$$
C\gamma^a = (\gamma^a)^T C, \quad C'\tau^{a'} = -(\tau^{a'})^T C'. \tag{2.7}
$$

Their explicit form can be seen in (A11) and (A13).

Now we can see the $\mathbb{Z}_4$-grading of $su(2, 2|2)$ explicitly from (2.5). Therefore $su(2, 2|2) = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$ with

$$
\begin{aligned}
\mathcal{G}_0 &= \{J_{ab}, T_{a'}\}, \qquad \mathcal{G}_1 = \{Q^1_{\alpha\alpha'}\}, \\
\mathcal{G}_2 &= \{P_a, R\}, \qquad \mathcal{G}_3 = \{Q^2_{\alpha\alpha'}\},
\end{aligned} \tag{2.8}
$$

where $\mathcal{G}_i$ denotes the component of grade i. The target space of $AdS_5 \times S^1$ superstring is the supercoset space $SU(2, 2|2)/(SO(4, 1) \times SO(3))$. As in the $AdS_5 \times S^5$ case, the algebra of the isotropy group, which would be gauged away, is just the grade zero part $\mathcal{G}_0$ of the whole algebra. This similarity determines that the $AdS_5 \times S^1$ string has many common properties with the intensely studied $AdS_5 \times S^5$ case. For instance it enables us to use the methods of [8] to construct the flat currents which would lead to the classical conserved nonlocal charges.

[2]To simplify the the notation, the prime on C' is explicitly written only when it does not carry the indices. So $C_{\alpha'\beta'}$ are the elements of the matrix C'.

B. Maurer-Cartan 1-forms

It is convenient to use the Maurer-Cartan 1-forms to construct the world sheet action. Here our target space is the coset space $SU(2, 2|2)/(SO(4, 1) \times SO(3))$. The Maurer-Cartan 1-form $G^{-1}dG$ of the supergroup $SU(2, 2|2)$ is pulled back to the coset space by the local section $G = G(x, \theta)$, where the coordinates (x, θ) parametrize the coset space and $G(x, \theta)$ is an element of the supergroup $SU(2, 2|2)$. In the following we shall call the pullback of the Maurer-Cartan 1-form of the supergroup the Maurer-Cartan 1-form of the coset space.

The left invariant Maurer-Cartan 1-form $\mathbf{L} \equiv G(x, \theta)^{-1} dG(x, \theta)$ of the coset space takes its value in the superalgebra $su(2, 2|2)$, so we can expand it using the generators of this algebra

$$
\begin{aligned}
\mathbf{L} &\equiv G(x, \theta)^{-1} dG(x, \theta) \\
&\equiv P_a L^a + R L^R + \frac{1}{2} J_{ab} L^{ab} + T_{a'} L^{a'} + Q^I_{\alpha\alpha'} L^{I\alpha\alpha'},
\end{aligned} \tag{2.9}
$$

where L^a and L^R are the 5-beins and 1-beins of AdS_5 and S^1 respectively. $L^{I\alpha\alpha'}$ are the two $(I = 1, 2)$ spinorial 8-beins which are two Majorana spinors of $SO(4, 1) \times SO(3)$. L^{ab} and $L^{a'}$ are the $SO(4, 1)$ and $SO(3)$ connections, respectively. The components of the Lie algebra valued 1-form $\mathbf{L}$ are super 1-forms. They are both differential 1-forms and the functions of Grassmann variables θ. Among these components L^a, L^R, L^{ab} and $L^{a'}$ are

Grassmann even quantities, while $L^{I\alpha\alpha'}$ are Grassmann odd ones. So we have $L^a \wedge L^{I\alpha\alpha'} = -L^{I\alpha\alpha'} \wedge L^a$, but $L^{I\alpha\alpha'} \wedge L^{J\beta\beta'} = +L^{J\beta\beta'} \wedge L^{I\alpha\alpha'}$.

The Maurer-Cartan 1-form satisfies the zero-curvature equation $d\mathbf{L} = -\frac{1}{2}[\mathbf{L}, \mathbf{L}]$. Then we get the following Maurer-Cartan equations

$$dL^a = -L^b \wedge L^{ba} - \mathrm{i}\bar{L}^I \gamma^a \wedge L^I, \tag{2.10}$$

$$dL^R = \bar{L}^I \wedge L^I, \tag{2.11}$$

$$\begin{aligned} dL^I = &\frac{\mathrm{i}}{2}\varepsilon^{IJ}(L^a\gamma^a + \mathrm{i}L^R) \wedge L^J \\ &- \frac{1}{4}(L^{ab}\gamma^{ab} + 2L^{a'}\tau^{a'}) \wedge L^I. \end{aligned} \tag{2.12}$$

Here we define the Majorana conjugation $\bar{L}^I_{\beta\beta'} = L^{I\alpha\alpha'}C_{\alpha\beta}C_{\alpha'\beta'}$. Then we have

$$\begin{aligned} d\bar{L}^I = &-\frac{\mathrm{i}}{2}\varepsilon^{IJ}\bar{L}^J \wedge (L^a\gamma^a + \mathrm{i}L^R) \\ &- \frac{1}{4}\bar{L}^I \wedge (L^{ab}\gamma^{ab} + 2L^{a'}\tau^{a'}). \end{aligned} \tag{2.13}$$

III. FLAT CURRENTS OF $AdS_5 \times S^1$ SUPERSTRING

The action of the $AdS_5 \times S^1$ superstring is

$$S = S_0 + S_1, \tag{3.1}$$

$$S_0 = -\frac{1}{2}\int d^2\sigma\sqrt{-g}g^{ij}(L^a_i L^a_j + L^R_i L^R_j), \tag{3.2}$$

$$S_1 = k\int \bar{L}^1 \wedge L^2. \tag{3.3}$$

Here we choose the Wess-Zumino term in a quadratic form. This kind of action has been discussed in [21,25]. The coefficient k will be determined by requiring κ-symmetry of the whole action S (see Appendix B). It is fixed to be -2.

To check the κ-symmetry and also to derive the equations of motion, we need the variations of the Maurer-Cartan forms. Suppose that we make the variation $G \to G' \simeq G(1+\omega)$ by right multiplication, where $\omega \equiv P_a\omega^a + R\omega^R + \frac{1}{2}J_{ab}\omega^{ab} + T_{a'}\omega^{a'} + Q^I_{\alpha\alpha'}\omega^{I\alpha\alpha'}$ is an infinitesimal element of the algebra (not a differential form). By definition $\mathbf{L} = G^{-1}dG$ we know $\delta\mathbf{L} = d\omega + [\mathbf{L}, \omega]$. So the variations of the Maurer-Cartan 1-forms are

$$\delta L^a = d\omega^a + L^{ab}\omega^b + L^b\omega^{ba} + 2\mathrm{i}\bar{L}^I\gamma^a\omega^I, \tag{3.4}$$

$$\delta L^R = d\omega^R - 2\bar{L}^I\omega^I, \tag{3.5}$$

$$\begin{aligned} \delta L^I = d\omega^I &+ \frac{\mathrm{i}}{2}\varepsilon^{IJ}(\omega^a\gamma^a + \mathrm{i}\omega^R)L^J \\ &- \frac{\mathrm{i}}{2}\varepsilon^{IJ}(L^a\gamma^a + \mathrm{i}L^R)\omega^J \\ &- \frac{1}{4}(\omega^{ab}\gamma^{ab} + 2\omega^{a'}\tau^{a'})L^I \\ &+ \frac{1}{4}(L^{ab}\gamma^{ab} + 2L^{a'}\tau^{a'})\omega^I. \end{aligned} \tag{3.6}$$

From above equations we get the variation of the Wess-Zumino term ($s^{IJ} = \mathrm{diag}(1, -1)$)

$$\begin{aligned} \delta(\bar{L}^1 \wedge L^2) = &d(\bar{\omega}^1 L^2 - \bar{L}^1\omega^2) - \frac{\mathrm{i}}{2}s^{IJ}\bar{L}^I(\omega^a\gamma^a + \mathrm{i}\omega^R) \\ &\wedge L^J + \mathrm{i}s^{IJ}\bar{\omega}^I(L^a\gamma^a + \mathrm{i}L^R) \wedge L^J. \end{aligned} \tag{3.7}$$

Here we have used the variations of $\bar{L}^I$

$$\begin{aligned} \delta\bar{L}^I = d\bar{\omega}^I &+ \frac{\mathrm{i}}{2}\varepsilon^{IJ}\bar{L}^J(\omega^a\gamma^a + \mathrm{i}\omega^R) \\ &- \frac{\mathrm{i}}{2}\varepsilon^{IJ}\bar{\omega}^J(L^a\gamma^a + \mathrm{i}L^R) \\ &+ \frac{1}{4}\bar{L}^I(\omega^{ab}\gamma^{ab} + 2\omega^{a'}\tau^{a'}) \\ &- \frac{1}{4}\bar{\omega}^I(L^{ab}\gamma^{ab} + 2L^{a'}\tau^{a'}). \end{aligned} \tag{3.8}$$

By use of Eqs. (3.4), (3.5), (3.6), and (3.7) and fixing $k = -2$ we get the equations of motion in terms of the Maurer-Cartan 1-forms as following

$$\sqrt{-g}g^{ij}(\nabla_i L^a_j + L^{ab}_i L^b_j) + \mathrm{i}\varepsilon^{ij}s^{IJ}\bar{L}^I_i\gamma^a L^J_j = 0, \tag{3.9}$$

$$\sqrt{-g}g^{ij}\nabla_i L^R_j - \varepsilon^{ij}s^{IJ}\bar{L}^I_i L^J_j = 0, \tag{3.10}$$

$$(\sqrt{-g}g^{ij}\delta^{IJ} - \varepsilon^{ij}s^{IJ})(L^a_i\gamma^a + \mathrm{i}L^R_i)L^J_j = 0. \tag{3.11}$$

Here ∇_i is the covariant derivative with respect to the world sheet metric g_{ij}. As usual there is another constraint that should be supplemented to the above three equations

$$L^a_i L^a_j + L^R_i L^R_j = \frac{1}{2}g_{ij}g^{kl}(L^a_k L^a_l + L^R_k L^R_l). \tag{3.12}$$

This equation is the consequence of the vanishing of $\delta S/\delta g^{ij}$.

A. Flat currents

In this subsection we follow the method of [8] to find the flat currents which would lead to the classical nonlocal charges of the $AdS_5 \times S^1$ superstring. We decompose the left invariant Maurer-Cartan form as

$$-\mathbf{L} \equiv -G^{-1}dG = \mathbf{H} + \mathbf{P} + \mathbf{Q}^1 + \mathbf{Q}^2 \tag{3.13}$$

with

$$\mathbf{H} = -\frac{1}{2}J_{ab}L^{ab} - T_{a'}L^{a'}, \qquad \mathbf{P} = -P_a L^a - RL^R,$$
$$\mathbf{Q}^I = -Q^I_{\alpha\alpha'}L^{I\alpha\alpha'} \quad \text{(no summation over } I\text{)}, \tag{3.14}$$
$$\mathbf{Q} \equiv \mathbf{Q}^1 + \mathbf{Q}^2, \qquad \mathbf{Q}' \equiv \mathbf{Q}^1 - \mathbf{Q}^2.$$

To find the flat currents it is convenient to transform the above Lie superalgebra valued 1-forms denoted by capital letters to the ones denoted by lowercase letters by conjugation $\mathbf{x} = G\mathbf{X}G^{-1}$. Notice that $\mathbf{H}$ transforms as a connection under the $\mathcal{G}_0$-gauge transformations, while $\mathbf{P}, \mathbf{Q}, \mathbf{Q}'$ transform in the adjoint representation of $\mathcal{G}_0$. So the lowercase forms $\mathbf{p}, \mathbf{q}, \mathbf{q}'$ are $\mathcal{G}_0$-gauge invariant. Although the capital forms correspond to the decomposition of the Maurer-Cartan forms under the grading of the superalgebra, the lowercase forms, however, do not reflect the grading.

The Maurer-Cartan equations in terms of this lowercase quantities are

$$\begin{aligned} d\mathbf{h} &= -\mathbf{h}\wedge\mathbf{h} + \mathbf{p}\wedge\mathbf{p} - (\mathbf{h}\wedge\mathbf{p} + \mathbf{p}\wedge\mathbf{h}) \\ &\quad - (\mathbf{h}\wedge\mathbf{q} + \mathbf{q}\wedge\mathbf{h}) + \frac{1}{2}(\mathbf{q}\wedge\mathbf{q} - \mathbf{q}'\wedge\mathbf{q}'), \\ d\mathbf{p} &= -2\mathbf{p}\wedge\mathbf{p} - (\mathbf{p}\wedge\mathbf{q} + \mathbf{q}\wedge\mathbf{p}) + \frac{1}{2}(\mathbf{q}\wedge\mathbf{q} + \mathbf{q}'\wedge\mathbf{q}'), \\ d\mathbf{q} &= -2\mathbf{q}\wedge\mathbf{q}, \\ d\mathbf{q}' &= -2(\mathbf{p}\wedge\mathbf{q}' + \mathbf{q}'\wedge\mathbf{p}) - (\mathbf{q}\wedge\mathbf{q}' + \mathbf{q}'\wedge\mathbf{q}). \end{aligned} \tag{3.15}$$

And the equations of motions could be rewritten as

$$\begin{aligned} d*\mathbf{p} &= (\mathbf{p}\wedge *\mathbf{q} + *\mathbf{q}\wedge\mathbf{p}) + \frac{1}{2}(\mathbf{q}\wedge\mathbf{q}' + \mathbf{q}'\wedge\mathbf{q}), \\ 0 &= \mathbf{p}\wedge(*\mathbf{q} - \mathbf{q}') + (*\mathbf{q} - \mathbf{q}')\wedge\mathbf{p}, \\ 0 &= \mathbf{p}\wedge(\mathbf{q} - *\mathbf{q}') + (\mathbf{q} - *\mathbf{q}')\wedge\mathbf{p}. \end{aligned} \tag{3.16}$$

The Maurer-Cartan equations (3.15) and the equations of motions (3.16) have the same form as the corresponding ones in the $AdS_5 \times S^5$ case [8]. So there is also a family of flat currents $\mathbf{a} = \alpha\mathbf{p} + \beta * \mathbf{p} + \gamma\mathbf{q} + \delta\mathbf{q}'$ in $AdS_5 \times S^1$ superstring parametrized by

$$\begin{aligned} \alpha &= -2\sinh^2\lambda, \\ \beta &= \mp 2\sinh\lambda\cosh\lambda, \\ \gamma &= 1 \pm \cosh\lambda, \\ \delta &= \sinh\lambda, \end{aligned} \tag{3.17}$$

by requiring the zero-curvature equation $d\mathbf{a} + \mathbf{a}\wedge\mathbf{a} = 0$. This one-parameter family of flat currents would naturally lead to a hierarchy of classical conserved nonlocal charges by the standard method [8,26,27]. This fact is a characteristic property that the $2d$ sigma model of $AdS_5 \times S^1$ is completely integrable at least at classical level.

In [27], it has been argued that there exists another class of flat currents in the $AdS_5 \times S^5$ case, which respect the $\mathbb{Z}_4$ symmetry explicitly. It was later realized [28] that the two currents are equivalent.

IV. FLAT CURRENTS OF $AdS_3 \times S^3$ SUPERSTRING

The Green-Schwarz superstring in $AdS_3 \times S^3$ space is constructed in [24]. In this section we shall use the notation of that paper. The action is defined as a two-dimensional sigma model with the target space $SU(1,1|2)^2/(SO(1,2)\times SO(3))$. We define the Maurer-Cartan 1-forms of $SU(1,1|2)^2$

$$\mathbf{L} \equiv G^{-1}dG \equiv L^{\hat{a}}P_{\hat{a}} + \frac{1}{2}L^{\hat{a}\hat{b}}J_{\hat{a}\hat{b}} + \frac{1}{2}\bar{L}^I Q^I + \frac{1}{2}\bar{Q}^I L^I. \tag{4.1}$$

A main difference between the $AdS_3 \times S^3$ string and the former $AdS_5 \times S^1$ string is $\bar{Q}$ and Q are independent generators in the former case, while in the latter case $\bar{Q} = Q(C\otimes C')$. The $SU(1,1|2)^2$ algebra has a $\mathbb{Z}_4$-grading structure with

$$\begin{aligned} \mathcal{G}_0 &= \{J_{ab}, J_{a'b'}\}, \\ \mathcal{G}_1 &= \{Q^1, \bar{Q}^1\}, \\ \mathcal{G}_2 &= \{P_a, P_{a'}\}, \\ \mathcal{G}_3 &= \{Q^2, \bar{Q}^2\}. \end{aligned} \tag{4.2}$$

The grade zero part corresponds to the denominator $SO(1,2)\times SO(3)$ of the supercoset. We decompose the Maurer-Cartan 1-forms with respect to the grading

$$\mathbf{H} = -\frac{1}{2}L^{\hat{a}\hat{b}}J_{\hat{a}\hat{b}}, \qquad \mathbf{P} = -L^{\hat{a}}P_{\hat{a}},$$
$$\mathbf{Q}^I = -\frac{1}{2}\bar{L}^I Q^I - \frac{1}{2}\bar{Q}^I L^I, \qquad \mathbf{Q} = \mathbf{Q}^1 + \mathbf{Q}^2, \tag{4.3}$$
$$\mathbf{Q}' = \mathbf{Q}^1 - \mathbf{Q}^2.$$

and define the corresponding lowercase forms $\mathbf{x} = G\mathbf{X}G^{-1}$ which will be used to construct the flat currents. They satisfy the following Maurer-Cartan equations

$$\begin{aligned} d\mathbf{h} &= -\mathbf{h}\wedge\mathbf{h} + \mathbf{p}\wedge\mathbf{p} - (\mathbf{h}\wedge\mathbf{p} + \mathbf{p}\wedge\mathbf{h}) \\ &\quad - (\mathbf{h}\wedge\mathbf{q} + \mathbf{q}\wedge\mathbf{h}) + \frac{1}{2}(\mathbf{q}\wedge\mathbf{q} - \mathbf{q}'\wedge\mathbf{q}'), \\ d\mathbf{p} &= -2\mathbf{p}\wedge\mathbf{p} - (\mathbf{p}\wedge\mathbf{q} + \mathbf{q}\wedge\mathbf{p}) + \frac{1}{2}(\mathbf{q}\wedge\mathbf{q} + \mathbf{q}'\wedge\mathbf{q}'), \\ d\mathbf{q} &= -2\mathbf{q}\wedge\mathbf{q}, \\ d\mathbf{q}' &= -2(\mathbf{p}\wedge\mathbf{q}' + \mathbf{q}'\wedge\mathbf{p}) - (\mathbf{q}\wedge\mathbf{q}' + \mathbf{q}'\wedge\mathbf{q}). \end{aligned} \tag{4.4}$$

which are the same as (3.15) and also (3.10) of Ref. [8].

The action of the Green-Schwarz superstring in $AdS_3 \times S^3$ space is

$$S = \int(\mathfrak{L}_0 + \mathfrak{L}_1), \tag{4.5}$$

$$\mathfrak{L}_0 = -\frac{1}{2}(L^a \wedge *L^a + L^{a'} \wedge *L^{a'}), \tag{4.6}$$

$$\mathfrak{L}_1 = \frac{1}{2}(\bar{L}^1 \wedge L^2 + \bar{L}^2 \wedge L^1). \tag{4.7}$$

Here we use a quadratic form of the Wess-Zumino term. The differential of $\mathfrak{L}_1$ coincides with the Wess-Zumino term in a cubic form used in [24]. It should be noticed that $\bar{L}^1 \wedge L^2 = \bar{L}^2 \wedge L^1$ in the $AdS_5 \times S^1$ case, while here $\bar{L}^2 \wedge L^1$ is the Hermitian conjugation of $\bar{L}^1 \wedge L^2$. It is also not difficult to check the κ-symmetry of this action. The variations of the Maurer-Cartan 1-forms are

$$\delta L^a = d\omega^a + L^{ab}\omega^b + L^b\omega^{ba} + \frac{\mathrm{i}}{2}(\bar{L}^I\gamma^a\omega^I - \bar{\omega}^I\gamma^a L^I), \tag{4.8}$$

$$\delta L^{a'} = d\omega^{a'} + L^{a'b'}\omega^{b'} + L^{b'}\omega^{b'a'} - \frac{1}{2}(\bar{L}^I\gamma^{a'}\omega^I - \bar{\omega}^I\gamma^{a'}L^I), \tag{4.9}$$

$$\delta L^I = d\omega^I + \frac{1}{2}\varepsilon^{IJ}\omega^{\hat{a}}\tilde{\gamma}^{\hat{a}}L^J - \frac{1}{2}\varepsilon^{IJ}L^{\hat{a}}\tilde{\gamma}^{\hat{a}}\omega^J - \frac{1}{4}L^{\hat{a}\hat{b}}\tilde{\gamma}^{\hat{a}\hat{b}}\omega^I + \frac{1}{4}\omega^{\hat{a}\hat{b}}\tilde{\gamma}^{\hat{a}\hat{b}}L^I, \tag{4.10}$$

$$\delta\bar{L}^I = d\bar{\omega}^I + \frac{1}{2}\varepsilon^{IJ}\omega^{\hat{a}}\bar{L}^J\tilde{\gamma}^{\hat{a}} - \frac{1}{2}\varepsilon^{IJ}L^{\hat{a}}\bar{\omega}^J\tilde{\gamma}^{\hat{a}} + \frac{1}{4}L^{\hat{a}\hat{b}}\bar{\omega}^I\tilde{\gamma}^{\hat{a}\hat{b}} - \frac{1}{4}\omega^{\hat{a}\hat{b}}\bar{L}^I\tilde{\gamma}^{\hat{a}\hat{b}}, \tag{4.11}$$

where

$$\tilde{\gamma}^{\hat{a}} = \begin{cases} -\mathrm{i}\gamma^a & \hat{a} = a \\ \gamma^{a'} & \hat{a} = a' \end{cases} \quad \text{and}$$

$$\tilde{\gamma}^{\hat{a}\hat{b}} = \begin{cases} -\gamma^{ab} & \hat{a}\hat{b} = ab \\ -\gamma^{a'b'} & \hat{a}\hat{b} = a'b' \end{cases}$$

The variation of the Wess-Zumino term is

$$\delta\mathfrak{L}_1 = \frac{1}{2}[-s^{IJ}\bar{L}^I\omega^{\hat{a}}\tilde{\gamma}^{\hat{a}} \wedge L^J + s^{IJ}\bar{\omega}^I L^{\hat{a}}\tilde{\gamma}^{\hat{a}} \wedge L^J + s^{IJ}\bar{L}^I \wedge L^{\hat{a}}\tilde{\gamma}^{\hat{a}}\omega^J + \sigma^{IJ}d(\bar{\omega}^I L^J - \bar{L}^I\omega^J)], \tag{4.12}$$

where

$$(s^{IJ}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$(\sigma^{IJ}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The equations of motions of $AdS_3 \times S^3$ are

$$\begin{aligned} &\sqrt{-g}g^{ij}(\nabla_i L_j^a + L_i^{ab}L_j^b) + \frac{\mathrm{i}}{2}\varepsilon^{ij}s^{IJ}\bar{L}_i^I\gamma^a L_j^J = 0, \\ &\sqrt{-g}g^{ij}(\nabla_i L_j^{a'} + L_i^{a'b'}L_j^{b'}) - \frac{1}{2}\varepsilon^{ij}s^{IJ}\bar{L}_i^I\gamma^{a'}L_j^J = 0, \\ &(L_i^a\gamma^a + \mathrm{i}L_i^{a'}\gamma^{a'})(\sqrt{-g}g^{ij}\delta^{IJ} - \varepsilon^{ij}s^{IJ})L_j^J = 0, \\ &\bar{L}_i^I(L_i^a\gamma^a + \mathrm{i}L_i^{a'}\gamma^{a'})(\sqrt{-g}g^{ij}\delta^{IJ} + \varepsilon^{ij}s^{IJ}) = 0. \end{aligned} \tag{4.13}$$

In terms of the lowercase 1-forms the equations of motions are translated into

$$\begin{aligned} d * \mathbf{p} &= (\mathbf{p} \wedge *\mathbf{q} + *\mathbf{q} \wedge \mathbf{p}) + \frac{1}{2}(\mathbf{q} \wedge \mathbf{q}' + \mathbf{q}' \wedge \mathbf{q}), \\ 0 &= \mathbf{p} \wedge (*\mathbf{q} - \mathbf{q}') + (*\mathbf{q} - \mathbf{q}') \wedge \mathbf{p}, \\ 0 &= \mathbf{p} \wedge (\mathbf{q} - *\mathbf{q}') + (\mathbf{q} - *\mathbf{q}') \wedge \mathbf{p}. \end{aligned} \tag{4.14}$$

which also have the same form as (3.16) and the ones of the $AdS_5 \times S^5$ string. Because of the $\mathbb{Z}_4$-grading structure and the similar forms of the equations of motions the existence of a family of one-parameter flat currents is evident, and their expressions in terms of the lowercase 1-forms are the same as the ones of the $AdS_5 \times S^5$ string.[3]

V. DISCUSSIONS

In this paper we construct a one-parameter family of nonlocal currents from the κ-symmetric actions of Green-Schwarz superstring in both $AdS_5 \times S^1$ and $AdS_3 \times S^3$ backgrounds. In the former case, although it is briefly described in [21], here we use a new $\mathbb{Z}_4$-graded base of the superalgera $su(2, 2|2)$ to rewrite the action. We take the quadratic Wess-Zumino term and determine the coefficient by requiring the κ-symmetry of the whole action. Then we construct explicitly a one-parameter family of flat currents which would lead to classical nonlocal charges. This fact suggests that the sigma model of $AdS_5 \times S^1$ string is really completely integrable, just as is asserted in [21]. Similarly, we discuss the construction of flat currents in $AdS_3 \times S^3$ superstring, whose symmetry algebra also has a $\mathbb{Z}_4$ grading. From the construction, we notice that in terms of the $\mathcal{G}_0$-gauge invariant lowercase forms, the Maurer-Cartan equations and the equations of motion take the same form as the ones in the $AdS_5 \times S^5$ case. As a result, the flat currents in all three supercoset models with $\mathbb{Z}_4$ grading take the similar form.

It is a notorious fact that the superstring in $AdS_5 \times S^5$ with RR-backgrounds is difficult to be quantized. One promising approach is the covariant quantization in the pure spinor formalism [30]. It would be very interesting to investigate whether one can have a pure spinor formulation for $AdS_5 \times S^1$ superstring and check whether the nonlocal charges survive the quantization.

[3]The nonlocal conserved currents in the $AdS_3 \times S^3$ *NSR*-superstring has been studied in Ref. [29].

On the other hand, it would be illuminating to set up the dictionary in the correspondence between the noncritical superstring in $AdS_5 \times S^1$ space and $\mathcal{N} = 1$ super-Yang-Mills with flavors. Actually, the pure $\mathcal{N} = 1$ super-Yang-Mills could not be conformal invariant and so cannot be the dual of closed superstring in $AdS_5 \times S^1$ space-time. However, the recent discovery in [22] implies that a superstring theory with closed and open string degrees of freedom in $AdS_5 \times S^1$ space-time, which is different from the one studied in this paper, may dual to a conformal invariant $\mathcal{N} = 1$ super-Yang-Mills with flavors. The correspondence and the possible role played by the integrable structure deserves further investigation.

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Note added.—While we are finishing the paper, there appears a paper [28] in arXiv, which has some overlap with our paper. Our result is consistent with its argument. One remarkable fact is that in [28], the coefficient of the Wess-Zumino term is determined by requiring the existence of a one-parameter family of flat currents, without taking into account of κ-symmetry. In this paper, we treat the κ-symmetry seriously and fix the Wess-Zumino term from it. It seems that the κ-symmetry implies the existence of the flat currents. The relation between κ-symmetry and flat currents deserves further study.

APPENDIX A: NOTATION

The convention of this paper is listed in this appendix. The barred Latin letters $\bar{a}, \bar{b} = 0, 1, 2, 3, 4, 5$ are the $so(4, 2)$ vector indices. The ordinary Latin letters $a, b = 0, 1, 2, 3, 4$ are the $so(4, 1)$ vector indices (AdS_5 tangent space). The primed Latin letters $a', b' = 1, 2, 3$ are the $so(3)$ vector indices. The ordinary Greek letters $\alpha, \beta = 1, 2, 3, 4$ are the $so(4, 2)$ spinor indices (also the $so(4, 1)$ spinor indices). The primed Greek letters $\alpha', \beta' = 1, 2$ are the $so(3)$ spinor indices. The six-dimensional metric is $(\eta_{\bar{a}\bar{b}}) = \mathrm{diag}(- + + + + -)$, the five-dimensional metric is $(\eta_{ab}) = \mathrm{diag}(- + + + +)$, and the four-dimensional metric is $(\eta_{\mu\nu}) = \mathrm{diag}(- + + +)$.

We choose the representation of 4-dimensional Dirac matrices γ^μ as the following

$$\gamma^\mu = -\mathrm{i}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{with} \quad \sigma^\mu = (\mathbf{1}, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i). \tag{A1}$$

They satisfy the standard anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and σ^i are three Pauli matrices. Then we define the $\bar{\gamma}$ as

$$\bar{\gamma} = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A2}$$

Now we use the above γ_μ and $\bar{\gamma}$ to construct the generators $\Gamma_{\bar{a}\bar{b}}$ of $so(4, 2)$:

$$\begin{aligned} \Gamma_{\mu\nu} &= \frac{1}{2}[\gamma_\mu, \gamma_\nu] = -\frac{1}{2}\begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix}, \\ \Gamma_{\mu 4} &= \gamma_\mu \bar{\gamma} = -\mathrm{i}\begin{pmatrix} 0 & \sigma_\mu \\ -\bar{\sigma}_\mu & 0 \end{pmatrix}, \\ \Gamma_{\mu 5} &= \gamma_\mu = -\mathrm{i}\begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \\ \Gamma_{45} &= \bar{\gamma} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{A3}$$

It is not difficult to check that $\Gamma_{\bar{a}\bar{b}}$ satisfies the standard commutation relations:

$$\left[\frac{1}{2}\Gamma_{\bar{a}\bar{b}}, \frac{1}{2}\Gamma_{\bar{c}\bar{d}}\right] = \frac{1}{2}\eta_{\bar{a}\bar{d}}\Gamma_{\bar{b}\bar{c}} + \frac{1}{2}\eta_{\bar{b}\bar{c}}\Gamma_{\bar{a}\bar{d}} - \frac{1}{2}\eta_{\bar{a}\bar{c}}\Gamma_{\bar{b}\bar{d}} - \frac{1}{2}\eta_{\bar{b}\bar{d}}\Gamma_{\bar{a}\bar{c}}. \tag{A4}$$

When we construct the graded generators of $su(2, 2|2)$ we split the $so(4, 1)$ 4-spinors into its 2-component forms:

$$\Psi_\alpha = (\phi^A, \chi_{\dot{A}}). \tag{A5}$$

Here $A, B = 1, 2$ and $\dot{A}, \dot{B} = 3, 4$. Define

$$\begin{aligned} (\varepsilon_{AB}) &= (\varepsilon_{\dot{A}\dot{B}}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ (\varepsilon^{AB}) &= (\varepsilon^{\dot{A}\dot{B}}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{A6}$$

So we have

$$\varepsilon^{AB}\varepsilon_{BC} = \delta^A_C, \qquad \varepsilon^{\dot{A}\dot{B}}\varepsilon_{\dot{B}\dot{C}} = \delta^{\dot{A}}_{\dot{C}}. \tag{A7}$$

The 2-spinor indices can be raised and lowered by the above ε-tensors

$$\begin{aligned} \psi^A &= \varepsilon^{AB}\psi_B, \qquad \psi_A = \varepsilon_{AB}\psi^B; \\ \psi^{\dot{A}} &= \varepsilon^{\dot{A}\dot{B}}\psi_{\dot{B}}, \qquad \psi_{\dot{A}} = \varepsilon_{\dot{A}\dot{B}}\psi^{\dot{B}}. \end{aligned} \tag{A8}$$

The index structures of σ^μ and $\bar{\sigma}^\mu$ are

$$\sigma^\mu = (\sigma^\mu)^{A\dot{B}}, \qquad \bar{\sigma}^\mu = (\bar{\sigma}^\mu)_{\dot{A}B}. \tag{A9}$$

It is true that

$$(\bar{\sigma}^\mu)_{\dot{A}B} = (\sigma^\mu)_{B\dot{A}} \equiv \varepsilon_{BC}\varepsilon_{\dot{A}\dot{D}}(\sigma^\mu)^{C\dot{D}}. \tag{A10}$$

We can use the ε-tensors to construct the 4×4 charge conjugation matrix of $so(4, 1)$ Clifford algebra

$$C = (C_{\alpha\beta}) = \begin{pmatrix} \varepsilon_{AB} & 0 \\ 0 & -\varepsilon^{\dot{A}\dot{B}} \end{pmatrix}, \tag{A11}$$

which has the characteristic property that

$$C\gamma^a C^{-1} = (\gamma^a)^T. \tag{A12}$$

We also define the 2×2 charge conjugation matrix of $so(3)$ Clifford algebra

$$C' = (C_{\alpha'\beta'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A13}$$

which has the similar property

$$C'\tau^{a'}(C')^{-1} = -(\tau^{a'})^T. \tag{A14}$$

APPENDIX B: κ-SYMMETRY OF THE $AdS_5 \times S^1$ SUPERSTRING

Now we check that when $k = -2$ the whole action $S = S_0 + S_1$ is invariant under the local κ-transformations

$$\omega^a_\kappa = \omega^R_\kappa = 0, \tag{B1}$$

$$\begin{aligned} \omega^1_\kappa &= P^{ij}_-(L^a_i\gamma^a - \mathrm{i}L^R_i)\kappa^1_j, \\ \omega^2_\kappa &= P^{ij}_+(L^a_i\gamma^a - \mathrm{i}L^R_i)\kappa^2_j, \end{aligned} \tag{B2}$$

$$\delta_\kappa(\sqrt{-g}g^{ij}) = -8\mathrm{i}\sqrt{-g}(P^{il}_- P^{jk}_- \bar{L}^1_k\kappa^1_l + P^{il}_+ P^{jk}_+ \bar{L}^2_k\kappa^2_l). \tag{B3}$$

Here we have defined two projectors $P^{ij}_\pm = \frac{1}{2} \times (g^{ij} \pm \frac{1}{\sqrt{-g}}\varepsilon^{ij})$. They have the following useful properties

$$P^{ij}_\pm = P^{ji}_\mp, \tag{B4}$$

$$P^{ij}_\pm P^{kl}_\pm = P^{kj}_\pm P^{il}_\pm. \tag{B5}$$

Let $k = -2$ in the action. By using the variations of the Maurer-Cartan 1-forms we have

$$\delta_\kappa S = \int d^2\sigma(\Delta_1 + \Delta_2), \tag{B6}$$

$$\Delta_1 = -\frac{1}{2}\delta_\kappa(\sqrt{-g}g^{ij})(L^a_i L^a_j + L^R_i L^R_j), \tag{B7}$$

$$\begin{aligned} \Delta_2 &= -2\mathrm{i}(\sqrt{-g}g^{ij}\delta^{IJ} + \varepsilon^{ij}s^{IJ})\bar{L}^I_i(L^a_j\gamma^a + \mathrm{i}L^R_j)\omega^J_\kappa, \\ &= -4\mathrm{i}\sqrt{-g}[P^{ij}_+ P^{kl}_- \bar{L}^1_i(L^a_j\gamma^a + \mathrm{i}L^R_j)(L^b_k\gamma^b - \mathrm{i}L^R_k)\kappa^1_l \\ &\quad + P^{ij}_- P^{kl}_+ \bar{L}^2_i(L^a_j\gamma^a + \mathrm{i}L^R_j)(L^b_k\gamma^b - \mathrm{i}L^R_k)\kappa^2_l]. \end{aligned} \tag{B8}$$

By virtue of (B4) and (B5) it is not very difficult to check that Δ_2 is just the opposite of Δ_1. So the whole action has the κ- symmetry when the coefficient k in front of the Wess-Zumino term is set to be -2.

APPENDIX C: MAURER-CARTAN 1-FORMS OF $SU(2,2|2)/(SO(4,1)\times SO(3))$

We can use the methods of [31] to write down the explicit forms of L^a, L^R, L^I as follows

$$L^a = e^a_\mu dx^\mu - \mathrm{i}\bar{\theta}^I\gamma^a\bigg[\bigg(\frac{\sinh\mathcal{M}/2}{\mathcal{M}/2}\bigg)^2 D\theta\bigg]^I, \tag{C1}$$

$$L^R = e^R_\mu dx^\mu - \mathrm{i}\bar{\theta}^I\bigg[\bigg(\frac{\sinh\mathcal{M}/2}{\mathcal{M}/2}\bigg)^2 D\theta\bigg]^I, \tag{C2}$$

$$L^I = \bigg[\bigg(\frac{\sinh\mathcal{M}}{\mathcal{M}}\bigg)D\theta\bigg]^I, \tag{C3}$$

where

$$\begin{aligned} (\mathcal{M}^2)^{IJ} &= \varepsilon^{IK}(-\gamma^a\theta^K\bar{\theta}^J\gamma^a + \theta^K\bar{\theta}^J) \\ &\quad + \frac{1}{2}\varepsilon^{KJ}(\gamma^{ab}\theta^I\bar{\theta}^K\gamma^{ab} - 4\tau^{a'}\theta^I\bar{\theta}^K\tau^{a'}), \end{aligned} \tag{C4}$$

and

$$\begin{aligned} (D\theta)^I &= d\theta^I + \frac{1}{4}(A^{ab}\gamma^{ab} + 2A^{a'}\tau^{a'})\theta^I \\ &\quad - \frac{\mathrm{i}}{2}\varepsilon^{IJ}(e^a\gamma^a + \mathrm{i}e^R)\theta^J. \end{aligned} \tag{C5}$$

Here e^a and e^R are the bosonic vielbeins corresponding to AdS_5 directions and S^1 directions, A^{ab} and $A^{a'}$ are the bosonic connections. Notice that the last term in (C4) is different from the $AdS_5 \times S^5$ case, in which the coefficient is -1, not -4. Using these formulae we can find the θ-expansion of the action of the $AdS_5 \times S^1$ superstring.

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