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Space-time symmetries of noncommutative spaces

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We define a noncommutative Lorentz symmetry for canonical noncommutative spaces. The noncommutative vector fields and the derivatives transform under a deformed Lorentz transformation. We show that the star product is invariant under noncommutative Lorentz transformations. We then apply our idea to the case of actions obtained by expanding the star product and the fields taken in the enveloping algebra via the Seiberg-Witten maps and verify that these actions are invariant under these new noncommutative Lorentz transformations. We finally consider general coordinate transformations and show that the metric is undeformed.

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Lorentz symmetry plays a central role in any realistic quantum field theory. Recently, due to progress in string/M theory [1], the idea that space-time could involve at short distances some nontrivial noncommutative coordinates was revived. But these quantum field theories typically violate Lorentz invariance. In the first paper on space-time noncommutativity [2], Snyder argued that Lorentz invariance is not incompatible with a discrete space-time and he gave a concrete noncommutative algebra that allows one to recover Lorentz invariance, but not Poincaré invariance.

Noncommutative gauge theories are very interesting since they represent simple examples of models with a minimal length and it has recently been established that quantum mechanics considered together with classical general relativity imply the existence of a minimal length in nature [3]. Nevertheless gauge theories formulated on a canonical noncommutative space-time violate Lorentz invariance. Although it is known how to formulate the standard model on a noncommutative space-time [4] (see also [5] for another approach), there is no obvious way to preserve Lorentz invariance and the bounds on the noncommutative nature of space-time are actually derived from bounds on Lorentz invariance violation [6]. One way to consider Lorentz invariant noncommutative models is to consider space-time dependent noncommutativity [7], but this approach has not yet been studied in great details and it remains a speculation. In this work we show that noncommutative theories formulated on a canonical space-time have an underlying exact symmetry that corresponds to Lorentz invariance in the limit $\theta^{\mu\nu} \rightarrow 0$. We call this symmetry noncommutative Lorentz invariance. Let us consider the noncommutative algebra:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{1}$$

where i, j run from 1 to 3 and where we set $\theta^{0i} = 0$, i.e. we assume that the space coordinates commute with the time coordinate. It will soon become obvious why we restrict our considerations to that case. Furthermore one has the Heisenberg algebra:

$$[\hat{x}^i, p^j] = i\hbar\delta^{ij} \tag{2}$$

and

$$[p^i, p^j] = 0. \tag{3}$$

One could try to introduce a noncommutative Lorentz symmetry by imposing a transformation $\hat{x}^i = \Lambda^i_j \hat{x}^j$, but that is not consistent with the algebra (1) since it would require that θ^{ij} transforms[1] as $\Lambda^i_k \Lambda^i_l \theta^{kl}$ which makes little sense since it is by definition a constant and thus should remain invariant.

It is easy to see that one can introduce a new operator x_c^i defined by

$$x_c^i = \hat{x}^i + \frac{1}{2\hbar}\theta^{ij}p_j, \tag{4}$$

which leads to the following algebras:

$$[x_c^i, x_c^j] = 0, [x_c^i, p^j] = i\hbar\delta^{ij} \quad \text{and} \quad [p^i, p^j] = 0, \tag{5}$$

i.e. x_c^i are commuting coordinates. Since t is not an operator in quantum mechanics, one cannot eliminate the constraint (1) for space-time noncommutativity; this explains our previous assumption $\theta^{0i} = 0$. This condition has to be imposed in the string/M theory approach [9] to noncommutative gauge theories to avoid problems with unitarity [10]. We can now treat the problem in a covariant way and introduce Greek variables which are running from 0 to 3. Given the algebras (5), we can define a transformation

$$x_c^\mu = \Lambda^\mu{}_\nu x_c^\nu \tag{6}$$

that leaves the interval $s^2 = \eta_{\mu\nu} x_c^\mu x_c^\nu$ invariant if $\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$. Notice that p^μ transforms as an usual Lorentz vector, i.e.

$$p^\mu = \Lambda^\mu{}_\nu p^\nu. \tag{7}$$

[1]This approach has been considered in [8], but we wish to treat θ^{ij} as a universal constant tensor, just like the speed of light in special relativity on commutative spaces.

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One thus finds that the transformation (4) implies that $\hat{x}^\mu$ transforms as

$$\hat{x}^{\mu\prime} = x_c^{\mu\prime} - \frac{1}{2\hbar}\theta^{\mu\nu}p'_\nu = \Lambda^\mu{}_\nu x_c^\nu - \frac{1}{2\hbar}\theta^{\nu\rho}\Lambda_\rho{}^\sigma p_\sigma \quad (8)$$

or

$$\hat{x}^{\mu\prime} = \Lambda^\mu{}_\nu \hat{x}^\nu + \frac{1}{2\hbar}\Lambda^\mu{}_\nu\theta^{\nu\rho}p_\rho - \frac{1}{2\hbar}\theta^{\mu\nu}\Lambda_\nu{}^\rho p_\rho \quad (9)$$

which defines the noncommutative Lorentz transformation. Note that the second term $\Lambda^\mu{}_\nu\theta^{\nu\rho}p_\rho$ is not a transformation of the noncommutative parameters $\theta^{\mu\nu}$, but of $\theta^{\mu\nu}p_\nu$. It is easy to verify that the algebra (1) is left invariant by this transformation and that there is a smooth limit to the Lorentz transformation on classical space-time by taking the limit $\theta^{\mu\nu} \to 0$. We now define the noncommutative invariant length. The square of the invariant length for the commutative coordinate x_c^μ is

$$s^2 = \eta_{\mu\nu}x_c^\mu x_c^\nu. \quad (10)$$

Using the variable transformation (4), one finds that the square of the noncommutative invariant length is given by

$$s_{nc}^2 = \hat{x}^\mu \hat{x}_\mu + \frac{1}{\hbar}\theta_{\mu\nu}\hat{x}^\mu p^\nu + \frac{1}{4\hbar^2}\theta^{\mu\alpha}\theta_{\mu\beta}p_\alpha p^\beta. \quad (11)$$

It is easy to verify that s_{nc}^2 is left invariant by the noncommutative Lorentz transformation (9). This is the way we define the noncommutative Lorentz transformations; those are the transformations that leave s_{nc}^2 invariant.

It is straightforward to extend our results to a Poincaré transformation since a shift by a constant of the noncommutative coordinates is compatible with the algebra (1). Let us now consider an infinitesimal noncommutative Poincaré transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $a^\mu = \epsilon^\mu$. It is implemented by the operator

$$U(1+\omega, \epsilon) = 1 + \frac{1}{2}i\omega_{\rho\sigma}J^{\rho\sigma} - i\epsilon_\rho p^\rho + \ldots \quad (12)$$

with $J^{\mu\nu} = x_c^\mu p^\nu - x_c^\nu p^\mu$. The operator is undeformed. The Lie algebra of the Lorentz group is also undeformed:

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}, \quad (13)$$

$$i[p^\mu, J^{\rho\sigma}] = \eta^{\mu\rho}p^\sigma - \eta^{\mu\sigma}p^\rho, \quad (14)$$

$$[p^\mu, p^\rho] = 0. \quad (15)$$

We note that our approach is different from the twisted Poincaré symmetry considered in [11]. It is also different from the κ-Poincaré quasi group where the Poincaré symmetry is deformed [12].

We shall now consider field theories. We need to introduce a derivative. Derivatives have to be defined in such a way that they do not lead to new relations for the coordinates. In the canonical case, it is easy to show that $\hat{x}^\alpha - i\theta^{\alpha\rho}\hat{\partial}_\rho$ with $\hat{\partial}_\rho\hat{x}^\mu = \delta_\rho^\mu + \hat{x}^\mu\hat{\partial}_\rho$ commutes with all coordinates [13]. One thus finds $\hat{\partial}_\mu f = -i\theta^{-1}_{\mu\nu}[\hat{x}^\nu, f]$. In our case we need a derivative which is compatible with the noncommutative Lorentz symmetry. We define the derivative in the following way:

$$i\theta_{\mu\nu}\hat{\partial}^\nu f = 2[\hat{x}_\mu + \frac{1}{2\hbar}\theta_{\mu\alpha}p^\alpha, f] \quad (16)$$

with $[p^\mu, f] = -i\hbar\partial^\mu f$. Note that the left-hand side of the equation is covariant. One finds that the derivative $\hat{\partial}_\nu$ transforms as

$$\hat{\partial}'_\nu = \theta^{-1}_{\nu\alpha}\Lambda^\alpha{}_\beta\theta^{\beta\rho}\hat{\partial}_\rho \quad (17)$$

under a noncommutative Lorentz transformation. We can thus write a noncommutative Lorentz invariant free field action for a noncommutative scalar field:

$$S = \int d^4x(\hat{\partial}_\mu\Phi\hat{\partial}^\mu\Phi - m^2\Phi\Phi - \lambda\Phi\Phi\Phi\Phi). \quad (18)$$

Note that the one-particle states are classified according to the eigenvectors of the four-momentum which transforms as usual under Lorentz transformations. The scalar, vector, and spinor fields thus transform in the usual way under Lorentz transformations. The Weyl quantization procedure can be applied to map the noncommutative fields $\Phi(\hat{x})$ to the commutative ones $\Phi(x)$. As usual, this corresponds to a replacement of the multiplication operation by a star product given by $f(x) \star g(x) = f(x)\exp(-i\partial_\mu\theta^{\mu\nu}\partial_\nu)g(x)$. It is easy to verify that the star product is invariant under noncommutative Lorentz transformations. The noncommutative gauge theories inspired by string theory are thus invariant under these transformations.

The noncommutative Lorentz transformation is compatible with gauge transformations. Remember that one has to introduce a covariant coordinate $\hat{X}^\mu$ [14] such that $\hat{\delta}_{\hat{\Lambda}}(\hat{X}^\mu\hat{\Psi}(\hat{x})) = \hat{\Lambda}\hat{X}^\mu\hat{\Psi}(\hat{x})$ where $\hat{\Lambda}$ is a noncommutative gauge transformation. One finds that $\hat{X}^\mu = \hat{x}^\mu + \hat{B}^\mu$ with $\hat{\delta}_{\hat{\Lambda}}\hat{B}^\mu = i[\hat{\Lambda}, \hat{B}^\mu] - i[\hat{x}^\mu, \hat{\Lambda}]$. The Yang-Mills gauge potential $\hat{A}^\mu$ is related to the gauge potential for the coordinate $\hat{B}^\mu$ by the relation $\hat{B}^\mu = \theta^{\mu\nu}\hat{A}_\nu$ and the covariant derivative $\hat{D}^\mu$ is given by $\hat{D}_\mu = -i\theta^{-1}_{\mu\nu}\hat{X}^\nu$. The coordinate gauge potential $\hat{B}_\mu$ transforms as $\hat{B}'_\mu = \Lambda^\nu_\mu\hat{B}_\nu$; one thus finds that the noncommutative Yang-Mills potential transforms as

$$\hat{A}'_\mu = \theta^{-1}_{\mu\nu}\Lambda^\nu{}_\rho\theta^{\rho\sigma}\hat{A}_\sigma. \quad (19)$$

The noncommutative covariant derivative transforms as

$$\hat{D}'_\mu = \theta^{-1}_{\mu\rho}\Lambda^\rho{}_\sigma\theta^{\sigma\alpha}\hat{D}_\alpha \quad (20)$$

under a noncommutative Lorentz transformation. The field strength $\hat{F}_{\mu\nu}$ is given by $\hat{F}_{\mu\nu} = i[\hat{D}_\mu, \hat{D}_\nu]$; it transforms as

$$\hat{F}'_{\mu\nu} = \theta^{-1}_{\mu\rho}\Lambda^{\rho}{}_{\sigma}\theta^{\sigma\alpha}\theta^{-1}_{\nu\kappa}\Lambda^{\kappa}{}_{\xi}\theta^{\xi\beta}\hat{F}_{\alpha\beta} \tag{21}$$

under a noncommutative Lorentz transformation. The noncommutative spinor field $\hat{\Psi}$ transforms as

$$\hat{\Psi}' = \exp\left(-\frac{i}{2}w^{\alpha\beta}S_{\alpha\beta}\right)\hat{\Psi}, \tag{22}$$

with $S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$. Note that if the fields are taken in the enveloping algebra, the leading order field of the Seiberg-Witten expansion [15], i.e. the classical field, also transforms according to (20)–(22).

Up to this point our considerations were completely general and did not assume a specific approach to space-time noncommutativity. We now apply our results to a specific framework, namely, we consider fields to be in the enveloping algebra. Given (20)–(22) it is easy to verify that the effective action obtained in the leading order in θ after the expansion of the noncommutative fields via the Seiberg-Witten map and of the star product,

$$\begin{aligned} S = \int d^4x\bigg[& \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}\theta^{\mu\nu}\bar{\psi}F_{\mu\nu}(i\not{D} - m)\psi \\ & - \frac{1}{2}\theta^{\mu\nu}\bar{\psi}\gamma^{\rho}F_{\rho\mu}iD_{\nu}\psi - \frac{1}{2}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \\ & + \frac{1}{4}\theta^{\mu\nu}\,\mathrm{Tr}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \theta^{\mu\nu}\,\mathrm{Tr}F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma}\bigg] \\ & + \mathcal{O}(\theta^2), \end{aligned} \tag{23}$$

is invariant under noncommutative Lorentz transformations.

There are implications for the bounds on space-time noncommutativity [6,16]. The 10 TeV bound on space-time noncommutativity when fields are taken in the enveloping algebra comes from atomic clock comparison studies. These studies search for a difference between two atomic transition frequencies, searching for variations as the Earth rotates [17]. The 10 TeV bound was obtained in [16] assuming that the fermionic sector of (23) transforms according to the classical Lorentz transformations. If we posit that the noncommutative Lorentz invariance is a symmetry of nature, one should use the noncommutative Lorentz transformations described in this work to compare the laboratory frame to the laboratory frame rotating with the Earth. The bounds on space-time noncommutativity coming from atomic clock comparison have to be reconsidered. A noncommutative Lorentz transformation corresponding to a 2π rotation would not take a system back to the same point; one could imagine testing this symmetry by measuring the spectrum of some transition in e.g. a nuclei and by studying how the spectrum is affected if the complete experiment is rotated by 2π. We emphasize that in our case θ is a constant in all reference frames, i.e. our symmetry is not spontaneously broken. Tests of Lorentz invariance, in the framework of noncommutative gauge theories, usually assume that θ changes from one reference frame to another and thus breaks Lorentz invariance spontaneously. It is thus not obvious how to use bounds on spontaneous violations of Lorentz to constrain our symmetry. A detailed analysis of the phenomenological consequences of this symmetry will appear elsewhere and is beyond the scope of this article.

The noncommutative Lorentz symmetry also has implications for the bounds relevant to the string/M theory approach to space-time noncommutativity. In that case the bounds come from the operators $O_1 = m_e\theta^{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi$, $O_2 = \theta^{\mu\nu}\bar{\psi}D_{\mu}\gamma_{\nu}\psi$, $O_3 = \lambda_3\theta^{\mu\alpha}\theta_{\alpha\nu}F_{\mu\rho}F^{\rho\nu}$, $O_4 = \lambda_4/8(\theta^{\mu\nu}F_{\mu\nu})^2$, and $O_5 = \theta_{\mu\rho}F^{\rho\sigma}\theta_{\sigma\gamma}F^{\gamma\mu}$ which are typically generated at two loops [18]. It is however easy to verify that these operators are not invariant under the transformations (21) and (22) and are thus an artifact of the cutoff used to regularize the divergent integrals. In that case again, the bounds on space-time noncommutativity are affected if we postulate that the noncommutative Lorentz symmetry is a symmetry of nature. On the other hand one finds that the effective cutoff responsible for the UV/IR phenomenon, $\Lambda^2_{\text{eff}} = (1/\Lambda^2 - p_{\mu}\theta^2_{\mu\nu}q_{\nu})^{-1}$, [19] is invariant under the deformed Lorentz symmetry. The UV/IR mixing phenomenon is thus not related to a symmetry of the noncommutative space-time.

It is straightforward to extend our results to the case of general coordinate transformations. As for the case of Lorentz transformations, we can consider general coordinate transformations of the commutative variable x^{μ}_c. One finds that the infinitesimal length interval

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}_c dx^{\nu}_c \tag{24}$$

is invariant under a general coordinate transformation $x_{\mu} \to \xi_{\mu}$, if the metric transforms as $g_{\mu\nu} = g_{\alpha\beta}(\partial x^{\alpha}/\partial\xi^{\mu})(\partial x^{\beta}/\partial\xi^{\nu})$. Applying the variable transformation (4) to the infinitesimal length interval, we find

$$ds^2 = g_{\mu\nu}(x)\frac{\partial x^{\mu}_c}{\partial\hat{x}^{\alpha}}d\hat{x}^{\alpha}\frac{\partial x^{\nu}_c}{\partial\hat{x}^{\beta}}d\hat{x}^{\beta}. \tag{25}$$

Using $(\partial x^{\mu}_c/\partial\hat{x}^{\alpha}) = -i2\theta^{-1}_{\alpha\nu}[\hat{x}^{\nu} + \frac{1}{2\hbar}\theta^{\nu\sigma}p_{\sigma}, x^{\mu}_c] = \delta^{\mu}{}_{\alpha}$, we find

$$ds^2 = g_{\mu\nu}(\hat{x})d\hat{x}^{\mu}d\hat{x}^{\nu}. \tag{26}$$

The noncommutative metric is therefore undeformed. This does not imply that the noncommutative Einstein action will itself be undeformed [20].

In summary we have defined space-time transformations for noncommutative spaces. The basic idea is to define these transformations for a commutative variable and to feed back these transformations to the noncommutative sector via a variable transformation. We have shown that the θ-expanded action is invariant under noncommutative Lorentz transformations and we have applied the same idea to general coordinate transformations and shown that the metric remains undeformed; this might not be a surprise since in string/M theory, gravity is determined by closed strings that do not feel the noncommutativity.

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