

**Matrix model for a quantum Hall droplet with manifest particle-hole symmetry**

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We find that a gauged matrix model of rectangular fermionic matrices (a matrix version of the fermion harmonic oscillator) realizes a quantum hall droplet with manifest particle-hole symmetry. The droplet consists of free fermions on the topology of a sphere. It is also possible to deform the Hamiltonian by double trace operators, and we argue that this device can produce two body potentials which might lead the system to realize a fractional quantum hall state on the sphere. We also argue that a single gauged fermionic quantum mechanics of Hermitian matrices realizes a droplet with an edge that has  $c = 1/2$  CFT on it.

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**I. INTRODUCTION**

Calculating the quantum levels of a nonrelativistic electron in a uniform magnetic field (in two dimensions) is a well known problem in quantum mechanics. The classical motion in the magnetic field is given by circular orbits, which are centered at two coordinates  $\tilde{q}_1$  and  $\tilde{q}_2$ , built out of the phase space variables.

When we turn the problem to quantum mechanics,  $\tilde{q}_1$  and  $\tilde{q}_2$  are noncommuting operators, which parametrize the degeneracy of each Landau level. They can also be described as the phase space of a single classical variable (for the sake of argument it can be  $\tilde{q}_1$ ). Since these two coordinates do not commute, there is a minimum area that the wave function occupies in the phase space determined by  $\tilde{q}_1, \tilde{q}_2$ , which can be identified with  $\hbar$  after rescalings. When we consider a system of free fermions (lets say  $k$ ), the lowest lying state of the system will have all the fermions in the lowest Landau level, but at different values of  $\tilde{q}_1$  and  $\tilde{q}_2$ , for which there is a lot of degeneracy.

In order to count states, we can break the degeneracy of the system by introducing a small perturbing potential which is a function of  $\tilde{q}_1$  and  $\tilde{q}_2$  only, which for simplicity we take to be quadratic in  $\tilde{q}_1$  and  $\tilde{q}_2$ , but with very small coefficients, so that the energy differences associated to different levels of the  $\tilde{q}_1$  and  $\tilde{q}_2$  oscillator are much smaller than the ones associated to going to a higher Landau level. We also require this of the Fermi level. We have chosen the function so that the level sets of the Hamiltonian are compact. We could have also chosen it so that the level sets are given by hyperbolas. This is the setup that corresponds to the  $c = 1$  matrix model (see [1] for an introductory review).

Fermi statistics will force the particles to be located at different values of  $\tilde{q}_1$  and  $\tilde{q}_2$ . With our choice for potential for the  $\tilde{q}_i$ , the Fermi surface will be a circle in the  $(\tilde{q}_1, \tilde{q}_2)$  plane, and the fermions form a droplet of constant density in this plane. This is the quantum hall droplet (see for example [2]). The fact that this droplet is described in a

phase space of a single classical variable has been used to provide some hydrodynamical description in terms of a noncommutative  $U(1)$  gauge theory [3].

There are various ways to describe the excitations of the system. One can consider the collective motion of particles that deforms the Fermi surface, and one can also consider taking individual particles from the top of the Fermi surface and giving them a lot of energy. The deformations of the Fermi surface are given by a free (chiral) boson in  $1 + 1$  dimensions, and this is an example of bosonization (see [4–6] for a description of the edge physics). One can also create collective excitations of the particles that describe holes in the Fermi surface. When we are near the top of the Fermi sea, particle and hole states behave very similar, so one has a symmetry that can exchange particle states and hole states. This symmetry only appears after quantization. Classically, one has no holes, and the particles occupy zero area.

This symmetry between holes and particles is a property of the quantized system, but it is not a property of the classical dynamics of the theory. In every event that one has a symmetry in a quantum system, one would like to describe the system in such a way that the symmetry is manifest, so that one does not have to solve the dynamics before seeing if it is there or not. This is also important if one tries to use arguments based on particle-hole symmetry in a system where this is not manifest. Indeed, such arguments are used when describing the fractional quantum hall effect with filling fractions  $\nu$  and  $1 - \nu$ , which are argued to be equivalent systems, and also as part of the idea of the fractional quantum hall hierarchy by [7]. Although we have great confidence in this type of symmetry arguments, it is always useful to see the symmetries explicitly realized in the “classical” system. In this case the arguments by symmetry are exact, so long as the Hamiltonian preserves the symmetry. This is one of the main motivations to write this paper. However, as described above, in usual setups this symmetry will only appear after quantization. Thus to have a system which has this symmetry manifestly, it should be a quantum system to begin with. To describe our proposal, it is important to first describe a

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second approach to arrive at this system based on different ideas.

A second route that eventually leads to the same system is to consider a gauged  $U(N)$  matrix model for a Hermitian matrix  $X$ , with the  $U(N)$  group action by conjugation on  $X$ , and whose action is

$$S = \frac{1}{2} \int dt \operatorname{tr}((D_t X)^2 - X^2) \quad (1)$$

The system classically allows a separation of variables into the eigenvalues of  $X$  (as one can always gauge transform a Hermitian matrix to a diagonal one by a unitary transformation), which we label by  $\lambda$ . Quantum mechanically this is still true, and due to measure factors the eigenvalues are actually noninteracting Fermions [8]. Each such eigenvalue has a Hamiltonian which is given by

$$H = \frac{1}{2} \int dt \operatorname{tr}(p_\lambda^2 + \lambda^2) \quad (2)$$

so we end up in the same system that we described above: free fermions in a harmonic oscillator potential, which we choose to describe in the phase space of the system.

The wave functions of the  $\lambda$  can be described in terms of a complete basis of symmetric polynomials in the  $\lambda$  times the VanderMonde determinant of the  $\lambda$  times an universal factor  $\exp(-\lambda^2/2)$ . There are various choices of such symmetric polynomials. One which is very easy to write down is the set of all polynomials in the sums of the powers of  $\lambda$ . Each such sum over  $\lambda$  is of trace form  $\sum_\lambda \lambda^n \sim \operatorname{tr}(X^n)$ . A less obvious basis is given in terms of Schur polynomials for the  $\lambda$  [9]. These can be related to characters of  $X$  in various representations of  $U(N)$ , which is a very natural idea in the matrix model. This is analogous to Wilson lines in various representations of the gauge group in 2D QCD (see [10–12]). This last description actually offers a very simple description of particles (as characters of completely symmetric representations of  $U(N)$ ) and holes (as completely antisymmetric representations of  $U(N)$ ) [9,13]). At this level, the particle-hole symmetry corresponds to changing symmetry types of representations.

From the point of view of the Young tableaux that characterize the representations of  $U(N)$ , this corresponds to performing a mirror image of the tableaux along the diagonal (we will refer to this as flipping the tableaux).

The symmetry under permutations of the eigenvalues (the statistics of the particles) is embedded in the  $U(N)$  group, and it is the residual gauge symmetry after we have chosen  $X$  to be diagonal. In this system, we have a finite droplet of quantum hall liquid in an infinite sea of holes. If we want a system that describes finitely many particles and holes, so that they can appear symmetrically, we would like to see the statistics under permutation of particle wave functions particles and the statistics of hole wave functions in the same way as above: embedded in the gauge group.

This suggests having a theory with a  $U(N) \times U(M)$  symmetry, which after taking eigenvalues appropriately reduces to a  $S_N \times S_M$  symmetry of permutations of eigenvalues. The easiest way to connect the  $U(N)$  and  $U(M)$  symmetries by a matrix, is to consider a pair of rectangular matrices  $X, Y$  that transform in the  $(N, \bar{M})$  and  $(\bar{N}, M)$  representation of the group. We can make the system 'Hermitian' if we consider  $X = Y^\dagger$ , and then we can consider the gauged matrix quantum mechanics of  $X, Y$  with a simple quadratic action. Such constructions have appeared in studies of the  $c = 1$  matrix model, when one considers orbifolds of the matrix model in the dual non-critical string theory [14].

However, the idea of the particle hole-like symmetry that would exchange  $M \leftrightarrow N$  does not work for that system, the first one that one would consider. We show that fact in this paper. However, if we let  $X, Y$  be fermionic oscillators instead of bosons, then the symmetry under the exchange  $M \leftrightarrow N$  exchanges particles and holes, once these states have been identified in the theory. This second option can be also suggested by realizing that if we have only finitely many holes and particles, then the total phase space should have area  $N + M$  and be compact. Thus the quantum system should only have a finite dimensional Hilbert space describing the system. This is one feature that the fermionic matrix model realizes automatically. As an aside, droplets of quantum hall liquids on various Riemann surfaces and higher dimensional spaces have been considered recently in [15–17].

This paper describes how this symmetry can be understood as particle-hole symmetry in detail for the above system. The paper is organized as follows. In Sec. II we describe the relationship between matrix models and the quantum hall effect in detail. This is review material. Next, in Sec. III we describe the system of Hermitian rectangular bosonic matrices, which we label 0A harmonic oscillator to follow the conventions from string theory. Some of these results are probably not new, but I am not aware of work where this is described in the way I present it. Here we pay special attention to a  $SL(2, \mathbb{R})$  symmetry of the system, of which the Hamiltonian is one of the generators. We show that single particle states are uniquely characterized by one irreducible representation of the algebra. This idea becomes central later on in Sec. IV when we describe the fermionic matrix model of rectangular matrices, so that we can map the system to free fermions on a sphere. In Sec. V we suggest a possible route to make the particles interact so that one can in principle describe a fractional quantum Hall effect (FQHE) system on a sphere. In Sec. VI we describe for completeness the gauged fermionic matrix model for square matrices. We show that in this system one has gauged the particle-hole symmetry. However, the system still has an edge, which is described by a free chiral fermion on a circle with antiperiodic boundary conditions. We then conclude.

## II. THE GAUGED $U(N)$ HARMONIC OSCILLATOR AND THE QHE

Let us consider the  $U(N)$  matrix quantum mechanics where  $X$  is a Hermitian  $N \times N$  matrix, and where  $X$  transforms in the adjoint representation of  $U(N)$  by matrix conjugation. We wish to consider the gauged quantum mechanics of  $X$ , where we choose the Lagrangian to be given by

$$L = \int dt \operatorname{tr} \left[ \frac{1}{2} (DX)^2 + V(X) \right]. \quad (3)$$

We will be interested in the potential  $V(X) = \frac{1}{2} X^2$  later on, but for the time being we will comment on general  $V(X)$ . There are two ways to solve the system. Choose the gauge  $A = 0$ , solve the system and impose the gauge constraint. Another way to solve the system is to eliminate the gauge redundancy as much as is possible and solve the system in terms of gauge invariant functions of the variables.

As is well known, this second route can be performed if we choose the gauge where  $X$  is a diagonal matrix with real entries. All Hermitian matrices are conjugate to these matrices by  $U(N)$  transformation. Under these conditions the off-diagonal components of  $X$  and  $\dot{X}$  can be set to zero identically. Therefore the dynamics of the system reduces to the dynamics of the eigenvalues of  $X$ . Classically all we have to do is replace a diagonal ansatz for  $X$  in the Lagrangian to get the dynamics of the eigenvalues, and a straight forward calculation shows that they are classically independent of each other.

Quantum mechanically, we have to consider the change of variables from  $X$  generic to  $X$  diagonal in the wave functions of  $X$ . This produces a change of measure for the eigenvalues of  $X$  which is the square of the Van der Monde determinant [8].

$$d\mu = \Delta(X)^2 \prod d\lambda_i \quad (4)$$

We also have to remember that there is an unbroken symmetry of permutations of the eigenvalues of  $X$  which preserves the form of the ansatz, and which can be embedded into the gauged  $U(N)$  symmetry. This symmetry is gauged, so all of the eigenvalues are treated as identical bosons and with measure given by  $d\mu$ . This measure dependence can be absorbed in the wave functions of the eigenvalues  $\lambda$ ,  $\psi' = \psi \Delta(X)$ , with a new measure  $d\mu' = \prod d\lambda$ . In terms of these wave functions the system describes totally antisymmetric wave functions of the eigenvalues, and we have a system of  $N$  identical noninteracting fermions in the potential  $V(\lambda)$ . From here, the solvability of the model depends on the particular form of the potential  $V(\lambda)$ . For our purposes  $V(\lambda)$  will be the harmonic oscillator. The system in the ground state will fill the first  $N$  energy levels of the harmonic oscillator.

We can now equally well consider this system as a set of noninteracting particles in a strong magnetic field. The

main idea is that when we reduce a system of particles to the lowest Landau level, the degeneracy of states is captured by a noncommutative plane of magnetic translations. As described in the introduction, this is equivalent to the algebra associated to the phase space of a single quantum variable  $X$ . The idea is to identify this noncommutative plane with the phase space of an eigenvalue of  $X$ . The quantum of area is determined by the magnetic field, which will be identified with  $\hbar$  after a suitable rescaling of units. Here we think of the system as having a complex matrix  $X + iP$ , and its complex conjugate  $X - iP$  as a set of conjugate variables.

The degeneracy of states can be broken by a small potential, which is identified with the Hamiltonian for the eigenvalues. This serves to localize the wave functions of the particles in the phase space.

Any Hamiltonian function will generically break the degeneracy, but it will not be solvable. Choosing the harmonic oscillator has the benefit of producing a rotationally invariant potential with a solvable spectrum. The lowest energy state will give a circular droplet whose radius is determined by the number of particles in the droplet. This is due to the Fermi statistics of the eigenvalues. The ground state energy for  $N$  particles is exactly  $\frac{1}{2} N^2$ , and it coincides with the ground state energy of the  $N^2$  harmonic oscillators in the matrix model. In this system the energy measures the total angular momentum of the system on the plane.

### A. Description of the excitations of the system

So far we have described the system in terms of  $N$  free fermions in the harmonic oscillator potential, and we have calculated the energy of the ground state.

We now want to describe all excited states of the system. The mathematics of this setup have been recently reviewed in a work of the author in [18], see also [19]. The complete set of wave functions can be given by a Slater determinant of wave functions of the Harmonic oscillator. These wave functions are labeled by their occupation numbers  $n_1, \dots, n_N$ . They are all different and we can use the permutation symmetry so that

$$n_1 > n_2 > \dots > n_N \geq 0. \quad (5)$$

The lowest energy configuration has  $n_i = N - i$ . We will call these values  $n_i^0$ . The first eigenvalue is chosen at the top of the Fermi sea, and then we go down.

If we introduce the quantities  $\tilde{n}_i = n_i - n_i^0$ , to this state we can associate a Young tableaux with up to  $N$  rows, where on row  $i$  we put  $\tilde{n}_i$  boxes. Young tableaux can also be related to the irreducible representations of  $U(N)$  built by tensoring multiple copies of the defining representation. The energy above the ground state is then the number of boxes of the tableaux.

A second way to describe the spectrum is given by choosing the gauge  $A = 0$  first. Then we reduce the system to  $N^2$  free harmonic oscillators, which can be described by

a set of creation and annihilation operators  $(a^\dagger)_j^i$  and  $a_j^i$  with commutation relations given by

$$[(a^\dagger)_j^i, a_m^l] = -\delta_m^i \delta_j^l \quad (6)$$

and the Hamiltonian of the system is  $H = \text{tr}(a^\dagger a) + \frac{1}{2}N^2$ . The vacuum is  $U(N)$  invariant and given by the state  $|0\rangle$ , such that  $a_j^i|0\rangle = 0$  for all pairs  $i, j$ . To build excited states with energy  $k$  we act with  $k$  raising operators in the vacuum and form linear combinations of the states so obtained

$$A_{i_1 \dots i_k}^{j_1 \dots j_k} (a^\dagger)_{j_1}^{i_1} (a^\dagger)_{j_2}^{i_2} \dots (a^\dagger)_{j_k}^{i_k} |0\rangle. \quad (7)$$

Now we need to impose the gauge constraint on these states. This boils down to all upper indices being contracted with all lower indices in some order, so that the state is a singlet under  $U(N)$  transformations. We can use matrix multiplication to write these states as follows

$$|(s_1, n_1), (s_2, n_2) \dots (s_m, n_m)\rangle \\ = \text{tr}((a^\dagger)^{s_1})^{n_1} \dots \text{tr}((a^\dagger)^{s_m})^{n_m} |0\rangle \quad (8)$$

and we can commute these past each other so that  $s_1 > s_2 \dots > s_m$ .

This description gives the same counting of states of the eigenvalue description, provided that we consider  $s_i \leq N$ . This constraint can be seen from the fact that the matrix  $a^\dagger$  is a  $N \times N$  matrix, and therefore  $\text{tr}((a^\dagger)^{N+1})$  can be written algebraically in terms of lower traces. This second basis looks like a Fock space of states with one oscillator per integer  $0 < i \leq N$ , namely  $\text{tr}((a^\dagger)^i)$ , with energy  $i$ . This basis is not orthogonal however, so the Fock space structure is only an approximation. It also follows that this basis cannot coincide with the basis determined before with Slater determinants, because that basis is orthogonal.

In the thermodynamic limit (large  $N$ ), this approximation of a Fock space is very good (the failure of orthogonality of states is small, of order  $1/N^2$ , this is done by following the 't Hooft idea of counting nonplanar diagrams [20].) and these can be described as a free field theory of collective excitations of the quantum hall droplet. The states described above change the shape of the droplet. These are the edge states of the droplet [9]. The oscillator  $i$  can be interpreted as a wave on the edge of the quantum Hall droplet with  $i$  units of angular momentum. This coincides exactly the spectrum of a relativistic chiral boson on a circle with periodic boundary conditions.

We need a way to relate these two descriptions of the states of the system. This is provided by the identification of states with energy  $k$  and irreducible representations of  $U(N)$  with  $k$  boxes. The idea is that we can make a new basis of states by thinking of the matrix  $a^\dagger$  as a matrix in  $GL(N, \mathbb{C})$ . The character of  $g$  in a representation  $R$  is given by the trace of  $g$  in the given representation,  $\chi_R(g) = \text{tr}(g)_R$  and it is gauge invariant. Therefore we can make a list of states based on the irreducible representations of  $U(N)$ , by taking the combination  $\chi_R(a^\dagger)$ . These are the

Schur polynomials. Many details of these computations can be found in [21]. This basis of states coincides with the eigenvalue basis of Slater determinants that we described first.

With the dictionary between Young tableaux and Slater determinants, it is easy to see that single particle and single hole states are described by completely symmetric and completely antisymmetric representations. When we are not too far from the Fermi sea, there is a particle-hole symmetry. This symmetry can be understood by noticing that a Young tableaux can be flipped about the diagonal to produce a new Young tableaux. However, if the Young tableaux is sufficiently wide, then when we flip it, it will no longer be an allowed tableaux for a  $U(N)$  representation.

Also if we look at the droplet system as made of two nonmixing liquid systems, we have a finite droplet of particles, and an infinite droplet of holes. The symmetry between particles and holes is clearly broken.

The problem we are concerned with in this paper is to find a realization of the quantum hall droplet which has this symmetry manifest in the description of the system, and where it does not appear only after we have solved the system in a low energy approximation.

Making the symmetry between holes and particles more manifest requires us to have either both infinite numbers of holes and particles, or finitely many of each. We want to deal with a finite system, so we take the second option. Finitely many particles and holes filling space requires a change in topology of the system: the area must be finite. This gives us a phase space of finite volume, so that the degeneracy of the Landau levels is finite, and we will only be able to access a Hilbert space of states of finite dimension. We will see that the matrix model we propose will be realizing the second option of possibilities.

### III. THE TYPE 0A HARMONIC OSCILLATOR MATRIX MODEL.

Let us now consider an orbifold of the matrix model we discussed in the previous section, the so called type 0A matrix model [14], but we will consider it in the harmonic oscillator case, as opposed to in the  $c = 1$  matrix model. The construction of the orbifold is performed by following the ideas in [22], and basically leads to a matrix model for a pair of rectangular matrices. This will provide us with some techniques to deal with the fermionic matrix model we will introduce later on, and the physical interpretation does not change with respect to the previous discussion.

The idea is to orbifold the harmonic oscillator matrix model by the  $\mathbb{Z}_2$  action  $x \rightarrow -x$ . This produces a quiver diagram theory with gauge group  $U(N) \times U(N + M)$ , and  $X$  is split into two matrices that transform as the  $(N, (N + M))$  and the  $(N + M, \bar{M})$  representations of the group [22]. These are complex conjugate to each other, so we can obtain a Hermitian matrix model by thinking of a matrix

$X$  as

$$X \sim \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix} \quad (9)$$

and imposing  $X = X^\dagger$ .

We can get a complex matrix model with two rectangular matrices of size  $N \times (M + N)$  and  $(N + M) \times M$  respectively, together with their adjoints as their conjugate variables in phase space by adding the momenta conjugate to  $X$ .

We can again describe the system in terms of eigenvalues of the composite matrix  $X_1 X_2 = \text{diag}(\lambda_i^2)$  which is positive definite as  $X_2 = X_1^\dagger$ , so that we can pick a gauge where the gauge group is broken down to a diagonal  $U(1)^N \times U(M)$ ,

$$\begin{aligned} X_1 &\sim \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ \vdots & \ddots & 0 & \dots \\ 0 & \dots & \lambda_N & \dots \end{pmatrix}, \\ X_2 &\sim \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & \lambda_N \\ 0 & \ddots & \vdots \end{pmatrix}. \end{aligned} \quad (10)$$

Again, permutations of the eigenvalues can be embedded in the gauge group, so we are left with symmetric functions of the  $\lambda_i^2$ . There is also a Van der Monde-like determinant (the volume of the gauge orbit) which leads to a measure of the form

$$d\mu \sim \prod \lambda_i^{2M} d\lambda_i \prod_{i < j} (\lambda_i^2 - \lambda_j^2)^2 \quad (11)$$

this has been calculated explicitly in [14,23,24] and see also [25]. The integration region for the measure is given by  $\lambda_i \geq 0$ , as there is no distinction between  $\lambda_i$  and  $-\lambda_i$  (these are identified by the  $\mathbb{Z}_2$  orbifold action).

After absorbing the square root of the measure in the wave functions, we get antisymmetric wave functions of the  $\lambda_i^2$ , and the effective quantum mechanical system gives rise to the following Hamiltonian

$$H = \frac{1}{2} \sum p_i^2 + \lambda_i^2 + (M^2 - 1/4)/\lambda_i^2 \quad (12)$$

with the restriction that  $\lambda_i \geq 0$ . This has been conjectured to describe a string theory in  $AdS_2$  [26], and we borrow freely some facts from that paper.

The important point we want to focus on is that system has an  $SL(2)$  algebra for each eigenvalue. Define

$$D = \frac{1}{2} (\lambda_i p_i + p_i \lambda_i) = \lambda_i \partial_{\lambda_i}, \quad (13)$$

$$P = \frac{1}{2} (p_i^2 + (M^2 - 1/4)\lambda_i^{-2}), \quad (14)$$

$$K = \frac{1}{2} \lambda_i^2, \quad (15)$$

and it is easy to see that these three operators satisfy an  $SL(2, \mathbb{R})$  algebra. A different basis for the algebra is provided by

$$\begin{aligned} L_0 = H &= \frac{P + K}{2}, & L_+ &= \frac{1}{2} (P - K - iD), \\ L_- &= \frac{P - K + iD}{2} \end{aligned} \quad (16)$$

This algebra is similar to the  $SL(2, \mathbb{R})$  algebra of a single harmonic oscillator, with generators  $a^\dagger a + 1/2$ ,  $(a^\dagger)^2$  and  $a^2$ .

We can again think of the system as describing a quantum hall droplet. Because of the  $SL(2, \mathbb{R})$  symmetry we can consider it as giving the holomorphic quantization of wave functions on the Poincare disc (or under a conformal transformation by the upper half plane), which is  $SL(2, \mathbb{R})/U(1)$ .

Now we can repeat the procedure for the system as we did in the last section, and go again to the gauge  $A = 0$ . For each component of  $X_{1,2}$  we get a harmonic oscillator. The creation operator and annihilation operators for  $X_1$  and  $X - 2 = X_1^\dagger$  are given by  $a^\dagger, a$  and  $b^\dagger, b$ .

The  $SL(2, \mathbb{R})$  algebra is given by

$$L_0 = H = \text{tr}(a^\dagger a + b^\dagger b) + M(M + N), \quad (17)$$

$$L_+ = \text{tr}(a^\dagger b^\dagger), \quad (18)$$

$$L_- = \text{tr}(ab). \quad (19)$$

Again, we get a description in terms of waves on the edge of the droplet by considering traces  $\text{tr}((a^\dagger b^\dagger)^n)$ , each of which represents one quantum with angular momentum  $n$  on the edge, and energy  $2n$ . This description becomes accurate in the thermodynamic limit of large  $N$ . Notice that the normalization of  $L_0$  and  $H$  differ by a factor of 2 from the standard  $SL(2)$  normalization.

We can also describe the system in terms of representations of  $U(N)$  with  $k$  boxes. This works very similar to the previous discussion in the ordinary matrix model. The only difference is that we need to take the composite matrix  $a^\dagger b^\dagger$  as the  $N \times N$  matrix. Here, we get representations of  $U(N)$  and  $U(N + M)$  to work with. We take properly symmetrized products of  $(a^\dagger)_{j_k}^{i_k}$  in the upper  $U(N)$  indices characterized by a given Young tableaux. The fact that these operators all commute with each other means that whatever symmetry the upper indices have, it is mirrored in the lower indices. To make singlets we need to contract the lower  $U(N + M)$  indices with those of  $b^\dagger$ , which will have the same symmetry properties as those for the upper indices for the  $a^\dagger$ . Therefore the Young tableaux for  $U(N)$  and  $U(N + M)$  are identical. The orthogonality of the basis

thus constructed is fairly easy to prove. This basis should coincide with the eigenvalue basis.

Now we want to look at the exchange of  $N \leftrightarrow M + N$ , or  $M \rightarrow -M$ . As described above, the Young tableaux for both groups ended up being the same, so this exchanges particles by particles and holes by holes and fails to give us what we want. This is why we said in the introduction that this model does not work to describe particles and holes in a symmetric way.

The lesson we should learn after the two examples we have studied is that traces always represent edge states of the system, and that Schur functions (characters of composite operator on irreducible representations) represent Slater determinant wave functions.

#### IV. A FERMIONIC MATRIX QUANTUM MECHANICS WITH $U(N) \times U(M)$ SYMMETRY

Now we are ready to describe another matrix model, which is the main result of this paper. The basic idea is very simple: consider the type 0A harmonic oscillator matrix model, except that we use fermionic oscillators instead of bosonic ones. Thus we have a system with two rectangular matrices of fermionic creation operators of size  $N \times M$  and  $M \times N$ , and we impose a  $U(N) \times U(M)$  invariance on the allowed wave functions. Ordinary matrix models for fermionic variables have been studied in [27,28] and they have similar properties to ordinary matrix models. Here we are studying the fermionic quantum mechanics for the matrix harmonic oscillator with rectangular matrices.

The first thing we should notice is that the fact that we have fermions makes various changes to the system. First, there are only finitely many degrees of freedom (without using the gauge invariance there are  $2^{2NM}$  states). The second point is that in some sense fermionic variables can only be interpreted in terms of operator algebras, but they cannot be thought of as numbers. Because of this, it is not possible to diagonalize an  $m \times m$  matrix of fermions to discover the eigenvalues. The reason for this is that as operators, the entries of the matrices do not commute, they anticommute. Therefore they cannot be diagonalized simultaneously as operators to obtain a matrix of c-numbers on states which can be diagonalized. This makes some aspects of the description of the system a little bit awkward, because we have lost part of the semiclassical description.

We still have the second alternative of using the gauge  $A = 0$  and writing gauge invariant wave functions by taking traces. This gives us the same behavior as the edge of a quantum hall droplet: we obtain one oscillator of the edge per integer  $n$  in the thermodynamic limit  $N \sim M$  large. The creation operator for such an edge state of momentum  $n$  is given by  $\text{tr}((a^\dagger b^\dagger)^n)$ . Notice that these are bosons, because they are made out of an even number of fermionic operators, thus these operators commute. Again one can show that in the thermodynamic limit there is a similarity to a

Fock space of bosons made out of these states, and that they are approximately orthogonal to leading order in  $1/NM$ , so long as we keep the energy finite and not scaling in the limit. We do see that the system is describing an edge of some type of quantum hall droplet. Our final purpose is to investigate this in more detail. In this paper we are interested in the case  $N, M$  finite, and not in the thermodynamic limit of the system itself.

Here, the ground state of the system has energy  $-NM$ . The negative number is the standard fact that free fermions contribute oppositely to the zero point energy than free bosons.

Since these states built out of traces are not orthogonal to each other, they are not a good basis for wave functions of the system. As we have seen, there is a second basis of orthogonal states which correspond to a basis of eigenvalues. These are obtained using characters of the groups  $U(N)$  and  $U(M)$  associated to different irreducibles of  $U(N)$  and  $U(M)$ . We will proceed with this description now.

#### A. Description of the spectrum in terms of Young tableaux.

Let us consider  $\chi_R(a^\dagger b^\dagger)$  for  $R$  a young tableaux of  $U(N)$ . The Fermi statistics of the  $(a^\dagger)_j^i$  show that if we have two upper indices which are symmetrized, then the two lower indices are antisymmetrized. Similarly if the upper indices are antisymmetrized, then the lower indices are symmetrized. This simple observation is at the heart of our claims about the properties of this system.

This means that the representations  $R$  of  $U(N)$  and  $\tilde{R}$  of  $U(M)$  are correlated

$$\chi_R(a^\dagger b^\dagger) = \chi_{\tilde{R}}(b^\dagger a^\dagger) \tag{20}$$

where we have to consider two Young tableaux which are mirror images under the reflection on the diagonal. Notice that above  $a^\dagger b^\dagger$  and  $b^\dagger a^\dagger$  are matrices of size  $N \times N$  and  $M \times M$  respectively. This is presented in Fig. 1.

In particular, the restrictions on the tableaux from being allowed both in  $U(N)$  and  $U(M)$  tell us that the tableaux for

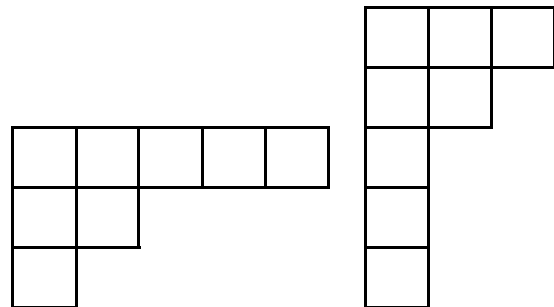


FIG. 1. Flipping the tableaux: upper  $U(N)$  indices and lower  $U(M)$  indices of tensors made of products of  $a^\dagger$  are related by Fermi statistics by flipping the tableaux through the diagonal.

$U(N)$  have columns of length less than  $N$  and rows of length less than  $M$ , while the ones for  $U(M)$  have columns of length less than  $M$  and rows of length less than  $N$ . Each box adds two units of energy to the system. The description of the spectrum is essentially symmetric in the exchange  $M \leftrightarrow N$ , except that we also have to flip the Young tableaux along the diagonal.

In particular, the state with the maximum number of boxes allowed has energy  $(-NM) + 2NM = NM$ , which is the highest energy state of the ungauged matrix model.

The system has an underlying  $SU(2)$  algebra, similar to the  $SL(2, R)$  algebra for the type 0A matrix model described previously. This is generated by the Hamiltonian,  $H \sim L_0$ ,  $L_+ = \text{tr}(a^\dagger b^\dagger)$ , and  $L_- = \text{tr}(ab)$ . The difference between  $SU(2)$  and  $SL(2, \mathbb{R})$  is the sign with which  $L_0$  appears on the commutator of  $L_+$  and  $L_-$ . Notice that  $(a^\dagger, b)$  and  $(-b^\dagger, a)$  form two doublets of operators under this  $SU(2)$  symmetry.

The Hamiltonian is  $L_0$ , so it breaks the  $SU(2)$  symmetry of the system. However, the Casimir operator  $L^2$  commutes with  $H$ , so it is a good quantum number. Thus we can describe the spectrum in terms of the representation theory of  $SU(2)$ . If we want to associate this symmetry with a two-dimensional topology, it is naturally the isometry group of a 2-sphere, so it suggests that the associated quantum hall droplet should be a state on a sphere.

We can easily see that the system is being given by  $N$  bosons in the spin  $M/2$  representation of  $SU(2)$  by counting the degeneracies of  $L_0$ . This is the same as having  $M$  bosons in the spin  $N/2$  representation. Roughly speaking, each of the bosons is given by the rows of the Young tableaux with respect to  $U(N)$ .

We write this Hilbert space as follows

$$S^N(V_{M/2}) \sim S^M(V_{N/2}). \quad (21)$$

In this description we can think of the system as being given  $N$  free charged bosons on the lowest Landau level of a sphere with magnetic monopole of strength  $M$ , in the presence of a small amount of gravity (or an electric field pointed on the same direction: this is the source of the potential proportional to  $L_0$ ) that makes them settle to the bottom of the sphere. This description is dual to  $M$  free bosons on the lowest Landau level on a sphere with a magnetic monopole field of strength  $N$  in the presence of gravity. In these two descriptions, we see that we have a duality where we exchange number of particles with flux on a two sphere.

We should point out that in the description of the original matrix model we get that  $N$  and  $M$  appear symmetrically. This symmetry is broken depending only on how we choose to interpret the system: either  $N$  or  $M$  bosons on geometries with different monopole backgrounds.

The total number  $n$  of states is

$$n = \binom{N+M}{M} = \binom{N+M}{N} \quad (22)$$

which also suggests a description in terms of  $N$  or  $M$  fermions with  $N + M$  states available to each. This is the description we will look for now.

### B. Fermionization of the “eigenvalues” and $SU(2)$ .

So far we have obtained a description of the Hilbert space in terms of identical bosons. We can fermionize this description. The idea is to think of the system as  $N$  fermions on a sphere with monopole background determined by the number of states available per fermion. To get  $N + M$  states from which we occupy  $N$ , we need to start with the spin  $(N + M - 1)/2$  representation of  $SU(2)$ . This corresponds to a monopole background on a two sphere of charge  $N + M$ . Remember that we can interpret the 2-sphere as the homogeneous space  $SU(2)/U(1)$ . We also have a small electric field breaking the degeneracy of the states.

It is easy to show that the representation theory of  $SU(2)$  found before is given by antisymmetric tensors of the  $V_{(M+N-1)/2}$  representation.

$$S^N(V_{M/2}) \sim \Lambda^N(V_{(M+N-1)/2}) \sim \Lambda^M(V_{(M+N-1)/2}) \quad (23)$$

Because of Fermi statistics, the  $N$  fermions will fill the bottom of the  $S^2$  sphere, leaving  $M$  holes at the top as we have a total of  $N + M$  states available to each fermion. So now we see that  $N$  and  $M$  can be interpreted as the number of particles and holes on the sphere, respectively, and we have a droplet of quantum hall liquid for particles at the bottom and for holes at the top. The interface between the two liquids will correspond to the edge of the droplet. We already have the candidate states that describe the changes in shape for the droplet, given by traces of the powers of  $a^\dagger b^\dagger$ . These do not distinguish between  $U(N)$  and  $U(M)$  symmetry because of the cyclic property of the trace (there is a  $(-1)$  sign associated to this operation which will become relevant later).

The energy of the ground state will be (if we count the particle states)

$$\sum_{i=1}^N (-M - N + 2i + 1) = -(M + N)N + N^2 = -MN \quad (24)$$

which coincides with the matrix model. This is just the  $SU(2)$  angular momentum of the lowest weight state in the antisymmetric tensor product.

Now, the next step is to decide how to interpret the particle and hole excitations in the system. Following our previous discussion of the one matrix model in terms of Young tableaux, we create particles by considering symmetric representations of  $U(N)$ , and holes by considering antisymmetric representations of  $U(N)$ , these are equivalent to antisymmetric and symmetric representations of  $U(M)$  respectively, from the pairing of Young tableaux.

Again, we see that the exchange  $M \leftrightarrow N$  exchanges the notions of 'holes' and 'particles'.

The first ones should build one large eigenvalue for the  $N \times N$  matrix  $a^\dagger a$ , while the second ones build a large eigenvalue for the matrix  $b^\dagger b$ .

It should be clear by now that we have a matrix model which describes a quantum hall droplet on the topology of the sphere, and which is also in the presence of gravity or an electric field. In the thermodynamic limit it has an exact  $c = 1$  CFT on the edge. Moreover in the matrix model the particles and holes ( $N$  and  $M$ ) appear symmetrically in terms of the matrix degrees of freedom.

The coordinates of the particles and holes themselves (as considered by the eigenvalues of  $a^\dagger a$  and  $b^\dagger b$ ) appear as different composite fields of gauge variant fields, which are not directly observable. This is what makes possible the symmetry between particles and holes to be present in this formulation of the model. Neither one of them is fundamental: both are composite objects.

This description is in terms of free fermions:  $N$  particles or  $M$  holes, and we can go back and forth between these by the identification of the following naturally dual vector spaces

$$\Lambda^N V_{\frac{M+N-1}{2}} = \Lambda^M V_{\frac{M+N-1}{2}}^* \quad (25)$$

because their vector space  $\Lambda^{(N+M)} V_{\frac{M+N-1}{2}}$  has dimension one. We then use the Hilbert space norm to identify  $V$  and  $V^*$ . Now that we have identified a matrix model which has all the states to describe a finite number of fermions on a finite geometry we can perturb the Hamiltonian to obtain other interesting models, which can include interactions between the fermions.

## V. TOWARDS THE FQHE: ADDING INTERACTIONS.

The fractional quantum hall effect can be obtained by considering fermions which fill a fraction of a Landau level and which have repulsive interactions [29]. We also have to go to the thermodynamic limit so that the number of particles and holes both scale the same way as we take the number of particles to infinity, keeping  $N/M$  finite and rational. In this paper we so far have ignored the details of the large  $N, M$  limit, so we will try to provide a method for studying the system at finite values of  $N, M$ . For the topology of a sphere, the fractional quantum hall state has also been considered in [7].

In our case we already have described a system of free fermions, so now we need to add interactions between them by perturbing the Hamiltonian of the model. Since we have the correct Hilbert space to describe all wave functions in the corresponding Landau level, there exists a perturbation that will produce the desired effect on all of the states. What is not clear is how simply this perturbation is described in terms of the natural variables we are work-

ing with (the  $a, b$  fermionic oscillators), and how much of the details of the FQHE depend on the exact form of the Hamiltonian.

We would want to perturb the system by adding as few terms as possible, and we definitely want to do it in such a way that the terms that we add are  $SU(2)$  invariant and that they involve only two particles at a time. This last part is where this formulation might become cumbersome. The only term which would break the  $SU(2)$  symmetry is the unperturbed Hamiltonian, and this will favor the state with largest negative eigenvalue of  $L_0$ .

The requirement of invariance under  $SU(2)$  transformations places various constraints on the perturbation. In particular, it has to commute with  $L_0$ , so the additional term in the Hamiltonian and  $L_0$  are always mutually diagonalizable. The Hamiltonian will then mix states with the same number of boxes (and which also belong to the same eigenvalues of  $L^2$ ), and so long as the perturbation is small, the lowest energy state will be a state with a small number of boxes (compared to  $N_1$  and  $N_2$ ), so it can be analyzed as a small perturbation of the edge of the quantum hall droplet.<sup>1</sup>

This also means that in the matrix basis, the perturbation will always have the same number of raising and lowering operators.

For the description to be simple in terms of the matrix variables, we would want to have a polynomial with few terms in the fermionic fields. The simplest terms that we can add involve two raising and two lowering operators.

In the description we had above, all states with same values of  $L_0$  were degenerate in energy, irrespective of their total  $SU(2)$  quantum number. The simplest operator we can add that breaks this degeneracy is a term proportional to the total angular momentum

$$\delta H = \alpha L^2 = \alpha(L_+ L_- + L_0^2 + 2L_0). \quad (26)$$

We also have the freedom to add a constant term to  $\delta H$  so that the energy of the lowest state we had before is not altered. It is easy to see that this perturbation of the Hamiltonian has exactly two raising and two lowering operators.

This perturbation is of double trace type, as each of  $L_+, L_-$  and  $L_0$  is of single trace type. Single trace operators for diagonal matrices cannot produce interactions between the eigenvalues (this is a statement one makes with classical diagonal matrices). Double trace operators produce interactions between pairs of eigenvalues, triple trace operators produce interactions between three eigenvalues at a time and so on. In general, for two body interactions we would expect that the Hamiltonian is of

<sup>1</sup>This procedure would not realize the fractional quantum hall state as the lowest lying state, which has  $L^2 = 0$  and energy  $MN/2$  over the vacuum in the free particle limit.



double trace type. Double trace deformations of large  $N$  systems are also solvable .

The first term we add clearly lifts some of the degeneracies. It removes all degeneracies for a system with just two particles or two holes, but in general for higher numbers of particles this will not be enough, as we will have various representations with the same value of  $L^2$ . This will be true for any function  $f(L^2)$ , but the restriction of being double trace basically forces us to consider only the term above.

The next thing we can do is classify all the single trace operators according to their  $SU(2)$  quantum numbers. Let us consider a trace with only creation operators. For example

$$\text{tr}((a^\dagger b^\dagger)^n). \quad (27)$$

This state is the highest weight state of a spin  $n$  representation of  $SU(2)$ . Acting with lowering operators changes some of the  $a^\dagger$  and  $b^\dagger$  for  $b, a$  respectively.

Now, since we have matrix-valued variables, the order of operators inside the trace matters. In principle this means that the two operators

$$\text{tr}(a^\dagger a a^\dagger b^\dagger), \quad \text{tr}(a^\dagger b^\dagger b b^\dagger) \quad (28)$$

would be linearly independent.

However, we are imposing the gauge constraint on the system, which reads

$$:a^\dagger \cdot a + b \cdot b^\dagger: = 0 \quad (29)$$

$$:a \cdot a^\dagger + b^\dagger \cdot b: = 0 \quad (30)$$

The multiplication above is matrix multiplication. The operators written above are the generators of the  $U(N) \times U(M)$  symmetry. They are normal ordered, but we keep the matrix order as written above.

On physical states, the above operators vanish. This means that when we find these operators inside a trace, we can use these relations to replace one matrix operator by another one.

We see this way that different orderings of the letters  $a, b$  do not really matter too much, and the invariant of the collection of such operators is the length of the word and the spin  $L_0$ . The length tells us that the word is obtained from the highest weight state (27) by commutators with  $L_-$ .

In essence, the total set of single trace operators that we can consider fall into the representations

$$0 \oplus 1 \oplus 2 \oplus \dots \oplus \tilde{N} \quad (31)$$

where  $\tilde{N}$  is the smallest of  $M, N$ . After we get to  $\text{tr}((a^\dagger b^\dagger)^{\tilde{N}})$  there are no more algebraically independent traces that we can consider.

From here, the double trace operators will come from considering the singlets in the  $k \otimes k$  representation of

$SU(2)$ , one coefficient for each of the representations above, in Eq. (31).

Our hamiltonian will contain one coefficient per representation of  $SU(2)$ . These should be related to the potentials that Haldane [7] used written in terms of  $L_i \cdot L_j$  for different particles.

Let us now explore the symmetry between particles and holes further. It is easy to see that changing  $M \leftrightarrow N$  changes the sign of the highest weight single trace operator:

$$\text{tr}((a^\dagger b^\dagger)^n) \leftrightarrow \text{tr}((b^\dagger a^\dagger)^n) = -\text{tr}((a^\dagger b^\dagger)^n) \quad (32)$$

where we have used the cyclic property of the trace. The same is true for the lowest weight. However, we see that this operation also changes  $L_+ \rightarrow -L_+$  and  $L_- \rightarrow -L_-$ , so not all single trace operators change their sign. If we take for example

$$\begin{aligned} \text{tr}(a^\dagger a a^\dagger b^\dagger) &\rightarrow \text{tr}(b^\dagger b b^\dagger a^\dagger) = -\text{tr}(b b^\dagger a^\dagger b^\dagger) \\ &\simeq \text{tr}(a^\dagger a a^\dagger b^\dagger), \end{aligned} \quad (33)$$

where in the last operation we have used the gauge constraint  $:a^\dagger a + b b^\dagger: = 0$ . This is also compatible with the change in sign of  $L_+$  and  $L_-$ . All together we find that the perturbed Hamiltonian we have written is actually invariant under exchange of particles and holes as well, because the  $(-1)$  signs we get on the operators square always to one in the Hamiltonian.

Theoretically this is very convenient. As discussed in the introduction, this gives us a manifest equality between the system at filling fraction  $\nu$  and  $1 - \nu$ . We take this result as strong evidence that the perturbations we are considering are exactly of the right type to describe the fractional quantum hall effect.

It would be interesting to determine how many of these coefficients should be needed to realize a fractional quantum hall state, this is currently under investigation. The observation that one expects a fractional quantum hall system when the interactions between the particles are short range and repulsive should indicate that we need to go to high spin for the operators to mimic this effect. The terms with low spin seem to be related to the long range potentials between the particles. One can make this a little bit more precise: if we have a total of  $M + N$  levels on a sphere of radius  $R$ , then each one occupies an area of order  $A \sim R^2/(M + N)$ . The typical short distance scale between two particles (or a particle and a hole) is therefore of order

$$l \sim \sqrt{A} \sim R/\sqrt{M + N} \quad (34)$$

which tells us that we probably do not need the most general form of the Hamiltonian that we wrote above, but only consider coefficients with  $L$  in the trace up to order  $\sqrt{N + M}$ . These should be enough to characterize any short distance shape of the potential. Curiously, purely from the matrix model point of view, it is also exactly at

this order where planar diagrams stop being useful, because nonplanar diagrams start competing with planar diagrams due to combinatorics, see the discussion in [18]. If this is a correct argument, then it suggests that planar diagrams are still a useful notion to describe the new ground state of the system and its perturbations. This is also suggestive of a description in terms of some type of string, possibly topological. However, we already know of such type of connections between the fractional quantum hall effect and two-dimensional conformal field theories, see for example [31].

We have not yet been able to understand in detail what features of the perturbation potential will help to form the fractional quantum hall effect as opposed to some other type of condensate. It should be clear that to favor systems with  $L \sim 0$  we would want the coefficient of  $L^2$  to be greater than zero. This is still under investigation.

### VI. A SYSTEM WITH A $c = 1/2$ EDGE

We have so far described a matrix model which represents a quantum hall droplet on a sphere. This model is an ‘‘orbifold’’ by a  $\mathbb{Z}_2$  action, just like the OA matrix model was an orbifold by a  $\mathbb{Z}_2$  action. For completeness, we should describe the associated fermionic system without orbifolding. This is just the gauged fermionic quantum mechanics, based on one fermionic matrix oscillator pair  $f, f^\dagger$ .

The discussion is not too distinct from the our original fermionic matrix model. The spectrum does not allow for fermionic eigenvalues, so we also need to be careful to describe the spectrum correctly.

From the point of view of traces, we get the states

$$c_n = \text{tr}((f^\dagger)^n). \tag{35}$$

Notice that by the cyclic property of the trace,  $c_n = (-)^{n+1}c_n$ , so that the only values of  $n$  allowed are the odd integers. One also uses this property to show that  $\{c_n, c_m\} = 0$ , so that these states are fermions.

There is one fermion mode per odd integer. This is similar with the mode structure of a chiral fermion on a circle with antiperiodic boundary conditions. That system has a  $c = 1/2$  central charge. We conclude that this fermionic matrix model is describing an ‘edge’ with a  $c = 1/2$  central charge. It is not usual that one would find such a system in a FQHE, as the central charge is usually greater than or equal to one to accommodate for the possibility of adding and removing charge from the system. This leads to a  $U(1)$  current algebra that measures this charge and has central charge  $c = 1$ , see [32–34] for example, and more recently [35]. There can be additional degrees of freedom which might raise the central charge even further (for example, the Pfaffian state of Moore and Read has this property [31]).

From the point of view of Young tableaux, we need to remember that for fermions, as described previously, the

upper and lower indices in a Young tableaux transform in representations that are related by flipping along the diagonal. To make a gauge invariant state, the upper and lower indices should correspond to the same tableaux, so only those tableaux that are symmetric along the diagonal are allowed, this is shown in Fig. 2.

Notice that the counting of states coincides with the counting via traces. This is because one can count the energy of a fermion by a hook centered on the diagonal of the tableaux. In the example in Fig. 2, one has two hooks of length 9 and 3, respectively. It is easy to see also that no pair of these hooks has the same length in any tableaux. From the point of view of particles and holes (rows and columns of the tableaux), the Young tableaux indicate that whenever we give a large amount of energy to a particle we also are giving the same amount of energy to a hole. The system is therefore describing correlated particle-hole states. Notice also that the system does not have the  $SU(2)$  symmetry anymore either. This is because the generators  $L^+$  and  $L_-$  from the previous section are not allowed operators.

One can consider a third description of the system, based on a different approach to understanding the fundamental degrees of freedom as fermionic matrices. The idea is to remember that one can think of differentials as fermions, so here we have a set of differential forms which are Lie algebra valued for the Lie algebra of  $U(N)$  matrices. If we consider the set of  $U(N)$  equivariant forms for the Lie algebra, we will be describing the gauge invariant states. The relations between the Lie algebra and the Lie group itself can be used to rethink the problems in terms of the (differential) cohomology groups of the  $U(N)$  manifold itself. It is known that  $SU(N)$  can be considered as a sphere bundle  $S^{2N-1}$  over the group  $SU(N - 1)$ , and that the cohomology of  $SU(N)$  is the cohomology of the above sphere tensored with the cohomology for  $SU(N - 1)$ . We can proceed by induction to get algebraic generators given by a  $S^3$ , an  $S^5$ , etc. Here our Hamiltonian is the degree of the differential form over  $SU(N)$ . To consider

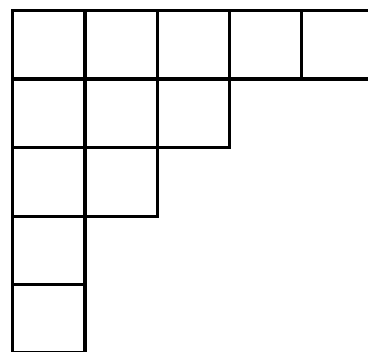


FIG. 2. An allowed Young tableaux for the fermionic matrix model.

$U(N) \sim U(1) \times SU(N)/Z_N$ , we remember that it is essentially a product space, so we also get a generator for degree one from the  $U(1)$  circle. Thus the system can also be considered as a topological model on the  $U(N)$  group manifold. The counting of states we obtain this way is the same as the one we described previously. There is one fermion oscillator for each odd integer.

The orbifold of this system is obtained by the  $\mathbb{Z}_2$  identification  $f \rightarrow -f$ , and this recovers the quantum hall droplet on the sphere. An orbifold system by an Abelian symmetry also has a quantum symmetry that one can orbifold by and recover the original system. This acts by exchanging  $a \leftrightarrow b$  in the Sec. IV, so that one ends up identifying the two. One can recover the original formulation by thinking of this as an orbifold of an orbifold, along the lines of [36]. Notice also that this particular  $\mathbb{Z}_2$  symmetry is exactly the one that identifies particles with holes. The  $\mathbb{Z}_2$  quantum symmetry does not commute with the  $SU(2)$  action however, as it does not identify the doublets properly. This is in contrast to the case of the bosonic matrix model, where the  $SL(2, \mathbb{R})$  symmetry is present in both the type 0A and type 0B theory.

It would also be interesting to investigate if it is possible to deform the theory by double trace operators and get exotic states which are somewhat analogous to a fractional quantum hall state: a quantum liquid with a gap, but with no charged excitations.

## VII. CONCLUSION

We have argued in this paper that a particular fermionic gauged matrix quantum mechanics, which describes of a pair of rectangular matrices, provides a matrix model description of a quantum hall droplet of noninteracting particles. This quantum hall droplet lives on a geometry with the topology of a sphere, and the model has a manifest particle-hole symmetry.

The way our system is able to do this is that neither the holes nor the particles are manifest in the model. Rather, it is by looking at the spectrum of the model that one is able to identify particle and hole wave functions in the description. Indeed, these can be considered as eigenvalues of two different matrix-valued composite operators.

Since we have this particle-hole symmetry manifestly in the system, it is possible to gauge it by performing an ‘‘orbifold of an orbifold construction,’’ as in [36]. Doing this we recover a single gauged matrix model for a single fermion. This gave rise to a quantum droplet with an edge which has a  $c = 1/2$  CFT, namely, a theory with a free fermion on a circle. It is always interesting to ask if it is possible to find some realization of the periodic boundary condition as well, as in the CFT this would correspond to the description of twist operators.

We have also argued that it is possible to include interactions between the particles, which does not look too complicated and maybe it is enough to describe fractional quantum hall phases in the thermodynamic limit. We have shown that our prescription for introducing interactions manifestly preserves the particle-hole symmetry of the original system.

Recently, there has also been a string theory dual proposal for the gauged harmonic oscillator system [37], and see also [18]. It would not be surprising if it is also possible to find such a dual string theory for the fermionic oscillator we have described here.

## ACKNOWLEDGMENTS

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