Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. III. Radiation reaction for binary systems with spinning bodies

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(Received 10 February 2005; published 26 April 2005)

Using post-Newtonian equations of motion for fluid bodies that include radiation-reaction terms at 2.5 and 3.5 post-Newtonian (PN) order ($O[(v/c)^5]$ and $O[(v/c)^7]$ beyond Newtonian order), we derive the equations of motion for binary systems with spinning bodies. In particular we determine the effects of radiation reaction coupled to spin-orbit effects on the two-body equations of motion, and on the evolution of the spins. For a suitable definition of spin, we reproduce the standard equations of motion and spin-precession at the first post-Newtonian order. At 3.5 PN order, we determine the spin-orbit induced reaction effects on the orbital motion, but we find that radiation damping has *no* effect on either the magnitude or the direction of the spins. Using the equations of motion, we find that the loss of total energy and total angular momentum induced by spin-orbit effects precisely balances the radiative flux of those quantities calculated by Kidder *et al.* The equations of motion may be useful for evolving inspiraling orbits of compact spinning binaries.

DOI: 10.1103/PhysRevD.71.084027 PACS numbers: 04.30.–w, 04.25.Nx

I. INTRODUCTION AND SUMMARY

This is the third in a series of papers on motion and gravitational radiation in the post-Newtonian approximation to general relativity (GR). In paper I [1], we developed a method of direct integration of the relaxed Einstein equations, whereby we wrote the Einstein equations as an inhomogeneous flat-spacetime wave equation together with a harmonic gauge condition and solved them formally in terms of retarded integrals over the past null cone of a given field point. We obtained formal solutions both for the far-zone gravitational waves and for the near-zone fields needed to obtain equations of motion. We then developed the near-zone fields in a post-Newtonian expansion, in powers of $\epsilon \sim (v/c)^2 \sim Gm/rc^2$, where v, m and r represent typical velocities, masses and separations in the system, and G and c are the gravitational constant and speed of light (both set equal to unity henceforth). Each power of ϵ represents one "post-Newtonian" (PN) order in the series ($\epsilon^{1/2}$ represents one-half, or 0.5 PN orders). The near-zone metric was evaluated through 3.5 PN order in terms of instantaneous, Poisson-like integrals over distributions of perfect fluid.

In paper II [2] we specialized the equations of motion to binary systems of nonrotating, suitably spherical balls of pressureless fluid, whose size is small compared to their separation, and derived the two-body equations of motion through 2.5 PN order and including the 3.5 PN terms (calculation of the 3 PN terms is ongoing). Through 2 PN order, the equations of motion are conservative, and our results were in complete agreement with calculations of others [3–8]. The 2.5 and 3.5 PN terms represent the effects of gravitational-radiation damping and their post-Newtonian corrections. We showed that they lead to losses of orbital energy and angular momentum that correctly match the losses calculated from the gravitational-wave flux to infinity [9,10]. They are also consistent with a recent calculation of 3.5 PN terms in the equations of motion using the post-Minkowskian approach [11].

In addition to the formal question of deriving suitably well-defined equations of motion in general relativity, a problem that dates back to the earliest days of the subject, this work is motivated by practical considerations. The operation of a network of ground-based, kilometer-scale gravitational-wave interferometers, and the research and development toward a space-based interferometer have made it critical to obtain highly accurate theoretical models for the evolution of and gravitational-wave emission from two-body systems. One of the leading candidate sources for detection by interferometers is the inspiral of a binary system containing black holes or neutron stars, and the preferred method of detection, optimal matched filtering, requires theoretical "template" waveforms that are accurate to fractions of a wave cycle over the potentially thousands of cycles in a "chirp" signal whose frequency and amplitude increase with time as the binary inspirals. This is why calculations to high PN order are needed.

However, the two-body equations of motion of paper II (and indeed of all high-PN-order calculations done to date) have a shortcoming—they assume nonspinning bodies. But spin may play a critical role in binary inspiral, particularly involving black holes. Spin-orbit and spin-spin coupling leads to precessions of the spins of the bodies and of the orbital plane, the latter effect resulting in modulations of the amplitude of the gravitational waveform received at

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a detector. Furthermore, spin effects contribute directly to the gravitational waveform, and to the overall emission of energy and angular momentum from the system [12,13].

From the point of view of gravitational-wave data analysis, spin complicates matters by increasing the number of parameters that must be estimated in a binary-inspiral matched filter; this can significantly decrease the accuracy with which any parameters can be measured, and can increase the computational burden [14–18].

Thus, it is desirable to have as complete a theoretical picture as is reasonable of the effects of spin in relativistic binaries. Formally, spin effects first enter at the 1 PN level. By analogy with quantum mechanics, the spin-orbit contribution to the energy is of order $\mathbf{L}\cdot\mathbf{S}/r^3$, where $\mathbf{L}$ is the orbital angular momentum. This can be reexpressed roughly in the form $\delta E \sim (mrv)(mRV)/r^3 \sim (m^2/r)\times(R/r)vV \sim \epsilon E_N$, where R and V are the radius and rotational velocity of the spinning body, and $E_N \sim m^2/r$ is the Newtonian orbital energy [19]. Indeed, the 1 PN effects of spin have been derived by numerous authors from a variety of points of view, ranging from formal developments of the GR equations of motion in multipole expansions [20,21], to post-Newtonian calculations [22], to treatments of linearized GR as a spin-two quantum theory, with the concomitant spinning-body interaction potentials [23,24]. For a review of these various approaches, see [25].

The resulting two-body equations of motion can be written in the form

$$\mathbf{a} = -\frac{m}{r^2}\mathbf{n} + \mathbf{a}_{\rm PN} + \mathbf{a}_{\rm SO} + \ldots, \tag{1.1}$$

where $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ is the relative acceleration, and where the 1 PN point-mass and spin-orbit contributions are given by

$$\mathbf{a}_{\rm PN} = -\frac{m}{r^2}\left\{\mathbf{n}\left[(1+3\eta)v^2 - 2(2+\eta)\frac{m}{r} - \frac{3}{2}\eta\dot r^2\right] - 2(2-\eta)\dot r\mathbf{v}\right\}, \tag{1.2a}$$

$$\mathbf{a}_{\rm SO} = \frac{1}{r^3}\left\{\frac{3}{2}\frac{\mathbf{n}}{r}\tilde{\mathbf{L}}_{\rm N}\cdot(4S+3\boldsymbol{\xi}) - \mathbf{v}\times(4S+3\boldsymbol{\xi}) + \frac{3}{2}\dot r\mathbf{n}\times(4S+3\boldsymbol{\xi})\right\}, \tag{1.2b}$$

where $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$, $r \equiv |\mathbf{x}|$, $\mathbf{n} \equiv \mathbf{x}/r$, $m \equiv m_1 + m_2$, $\eta \equiv m_1 m_2/m^2$, $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$, $\dot r = dr/dt = \mathbf{n}\cdot\mathbf{v}$, $S = S_1 + S_2$, $\boldsymbol{\xi} = (m_2/m_1)S_1 + (m_1/m_2)S_2$, and $\tilde{\mathbf{L}}_{\rm N} = \mathbf{x}\times\mathbf{v}$. The 1 PN equations of spin precession are given by

$$\dot S_1 = \frac{\eta m}{r^3}\tilde{\mathbf{L}}_{\rm N}\times S_1\left(2 + \frac{3}{2}\frac{m_2}{m_1}\right), \tag{1.3a}$$

$$\dot S_2 = (1 \rightleftharpoons 2). \tag{1.3b}$$

These equations assume a specific definition of the center of mass and spin of each spinning body, given by $\mathbf{x}_1 \equiv m_1^{-1}\int_1 \rho^* \mathbf{x} d^3x$, and $S_1 \equiv \mathbf{S}_1(1 + \frac{1}{2}v_1^2 + 3m_2/r) - \frac{1}{2}\mathbf{v}_1\times(\mathbf{v}_1\times\mathbf{S}_1)$, with analogous expressions for body 2, where ρ^* is the so-called "conserved" or baryonic density [Eq. (2.1)], m_1 is the baryonic mass, and $\mathbf{S}_1 \equiv \int_1 \rho^* \bar{\mathbf{x}}\times\bar{\mathbf{v}} d^3x$, is the baryonic spin, with $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}_1$. The spin-orbit terms in Eq. (1.2b) and the spin precession equations (1.3) derived by our method are in complete agreement with earlier derivations [25].

Turning now to radiation-reaction effects, it is clear that no spin contributions can occur at the 2.5 PN order corresponding to the leading-order damping effects. The simplest way to see this is to recall that, in a certain gauge, the 2.5 PN radiation-reaction acceleration may be expressed in the form

$$a^i_{2.5\,\rm PN} = -\frac{2}{5}x^j\frac{d^5}{dt^5}\mathcal{I}^{ij}, \tag{1.4}$$

where $\mathcal{I}^{ij}$ is the trace-free moment of inertia of the system, and where we sum over repeated spatial indices. But since, to lowest order, $\mathcal{I}^{ij}$ contains no velocities, it cannot contain spin terms at lowest order, even if the bodies are rotating; in other words, spin terms will be of higher, 3.5 PN order. Similarly, in calculating $\dot{\mathbf{S}}_1$ from $\mathbf{x}\times\mathbf{a}_{2.5\,\rm PN}$ it is also clear that no spin terms can arise at lowest order. Hence, the leading contributions of spin in radiation reaction must be at 3.5 PN order.

We therefore make use of the 2.5 *and* 3.5 PN equations of motion for fluid systems derived in paper I and calculate the equations of motion and spin-precession for two spinning bodies. These are our central results: the 2.5 PN contributions to radiation reaction are given by the standard "point-mass" result

$$\mathbf{a}_{2.5\,\rm PN} = \frac{8}{5}\eta\frac{m^2}{r^3}\left[\dot r\mathbf{n}\left(3v^2 + \frac{17}{3}\frac{m}{r}\right) - \mathbf{v}\left(v^2 + 3\frac{m}{r}\right)\right], \tag{1.5}$$

and the spin-orbit contributions to radiation reaction are given by

$$\begin{aligned}\mathbf{a}_{3.5\,\rm PN-SO} = -\frac{\eta m}{5r^4}\bigg\{&\frac{\dot r\mathbf{n}}{r}\left[\left(120v^2 + 280\dot r^2 + 453\frac{m}{r}\right)\tilde{\mathbf{L}}_{\rm N}\cdot S + \left(285v^2 - 245\dot r^2 + 211\frac{m}{r}\right)\tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\right]\\ &+\frac{\mathbf{v}}{r}\left[\left(87v^2 - 675\dot r^2 - \frac{901}{3}\frac{m}{r}\right)\tilde{\mathbf{L}}_{\rm N}\cdot S + 4\left(6v^2 - 75\dot r^2 - 41\frac{m}{r}\right)\tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\right] - \frac{2}{3}\dot r\mathbf{v}\times S\left(48v^2 + 15\dot r^2 + 364\frac{m}{r}\right)\\ &-\frac{1}{3}\dot r\mathbf{v}\times\boldsymbol{\xi}\left(375v^2 - 195\dot r^2 + 640\frac{m}{r}\right) + \frac{1}{2}\mathbf{n}\times S\left(31v^4 - 260v^2\dot r^2 + 245\dot r^4 - \frac{689}{3}v^2\frac{m}{r} + 537\dot r^2\frac{m}{r} + \frac{4}{3}\frac{m^2}{r^2}\right)\\ &-\frac{1}{2}\mathbf{n}\times\boldsymbol{\xi}\left(29v^4 - 40v^2\dot r^2 - 245\dot r^4 + 211v^2\frac{m}{r} - 1019\dot r^2\frac{m}{r} - 80\frac{m^2}{r^2}\right)\bigg\}.\end{aligned} \tag{1.6}$$

The point-mass, 3.5 PN contributions are as given in paper II, Eqs. (1.2) and (1.3d). We also find the radiation-reaction contribution to spin precession

$$\dot{S}_1^{3.5\ \mathrm{PN-SO}} = 0, \qquad \dot{S}_2^{3.5\ \mathrm{PN-SO}} = 0. \tag{1.7}$$

Although the latter result arises effectively from a cancellation of many terms, it should really come as no surprise. Our bodies are assumed to be axially symmetric, and as such, should not couple to gravitational radiation on the basis of their rotation alone. Their *precessions* could lead to gravitational radiation and to back reaction if, for example, they were rotationally flattened, but those precessions and the flattening are themselves 1 PN-order effects, and so should result in radiation-reaction effects of at least 4.5 PN order, beyond the level at which we are working.

We have verified that these equations lead to a loss of total energy and total angular momentum that matches precisely the energy and angular momentum flux at infinity from binary systems of spinning bodies, as calculated by Kidder *et al.* [12,13].

In the conventional terminology of spinning bodies in general relativity, the center of mass of each body is specified by a so-called "spin supplementary condition" (Appendix A), given by a parameter k_{SSC} whose value is typically zero, one or 1/2. Our center-of-mass definition corresponds to the value $k_{\mathrm{SSC}} = 1/2$. Appendix B displays the final equations of motion corresponding to the "covariant" spin supplementary condition (SSC) given by $k_{\mathrm{SSC}} = 1$.

The remainder of this paper provides details. In Sec. II we derive the equations of motion and spin precession to 1 and 2.5 PN order, and demonstrate explicitly the absence of observable radiation-reaction spin effects at 2.5 PN order. In Sec. III we complete the derivation to 3.5 PN order, and verify the agreement with energy and angular momentum flux. A variety of technical matters and detailed equations are relegated to appendixes.

II. POST-NEWTONIAN AND 2.5 PN EQUATIONS OF MOTION

A. Foundations

We wish to analyze a binary system consisting of balls of perfect fluid that are sufficiently small compared to their separation that tidal interactions (and their relativistic generalizations) can be ignored, but that are sufficiently extended that they can support a finite rotational angular momentum, or spin. At Newtonian order, the result is essentially trivial: the equation of motion for body 1 is $d^2\mathbf{x}_1/dt^2 = -m_2\mathbf{x}/r^3 + O(mR^2/r^4)$, where R is the characteristic size of the bodies. Spin plays no role whatsoever, because the Newtonian interaction does not depend on velocity. But at post-Newtonian order, there are velocity-dependent accelerations of the schematic form mv^2/r^2, and thus, taking into account the finite size of the body and expanding about its center of mass, one could anticipate acceleration terms of the form $(mVR)v/r^3 \sim Sv/r^3$. However, the combination of finite size and spin introduces an ambiguity in the definition of the center of mass of each body. At Newtonian order, the center of mass is defined naturally by $\mathbf{x}_A = m_A^{-1}\int_A \rho\mathbf{x}d^3x$, where ρ is the density and where the integral is over the body in question. But at post-Newtonian order, depending on the choice of "density" used, there could be correction terms in the center of mass of the form $m^{-1}\int \rho v^2\mathbf{x}d^3x \sim vS/m$. Because "center of mass" is an arbitrarily chosen point, these differences have no physical content in the end, as long as results are expressed in terms of measurable quantities (such as the total energy of the system, or the radar range to the surface of the body), but they do result in equations of motion with different explicit forms. This has given rise to the concept of SSC, a statement about which center-of-mass definition is being used; this concept is discussed in Appendix A.

Because we intend to work to post-Newtonian orders that include 1, 2.5 and 3.5 PN terms, we need to define centers of mass and spins in a way that is physically reasonable, calculationally simple, and easily transformable to definitions based on other criteria. In post-Newtonian theory, the simplest density available is the so-called conserved density, given by

$$\rho^* \equiv \rho\sqrt{-g}u^0, \tag{2.1}$$

where ρ is the mass-energy density as measured by an observer in a local inertial frame momentarily at rest with respect to the fluid, g is the determinant of the metric, and u^0 is the time component of the fluid four velocity. Making the reasonable assumption that ρ is directly proportional to the baryon number density, then, by virtue of conservation of baryons, we have that $(\rho u^\alpha)_{;\alpha} = 0$, where the semicolon denotes covariant derivative, and the index α is summed over all spacetime values. This is equivalent to the *exact* continuity equation

$$\partial\rho^*/\partial t + \nabla\cdot(\rho^*\mathbf{v}) = 0, \tag{2.2}$$

where $v^i = u^i/u^0$ is the ordinary (coordinate) velocity of the fluid. The use of ρ^* in defining such quantities as center of mass is convenient because of the fact, based on the equation of continuity, that

$$\frac{\partial}{\partial t}\int \rho^*(t,\mathbf{x}')f(\mathbf{x},\mathbf{x}')d^3x' = \int \rho^*(t,\mathbf{x}')\mathbf{v}'\cdot\nabla' f(\mathbf{x},\mathbf{x}')d^3x', \tag{2.3}$$

for any suitably regular function $f(\mathbf{x},\mathbf{x}')$ and for integration over a complete body.

Accordingly, we will define the baryonic mass, center of mass and baryonic spin of each body in our system to be

$$m_A \equiv \int_A \rho^* d^3x, \tag{2.4a}$$

$$\mathbf{x}_A \equiv m_A^{-1} \int_A \rho^* \mathbf{x} d^3x, \tag{2.4b}$$

$$\mathbf{S}_A \equiv \epsilon^{ijk} \int_A \rho^* \bar{x}^j \bar{v}^k d^3x, \tag{2.4c}$$

where $\bar{x}^j = x^j - x_A^j$ and $\bar{v}^j = v^j - v_A^j$. For future use we will also define a tensorial spin quantity

$$S_A^{ij} \equiv 2 \int_A \rho^* \bar{x}^{[i} \bar{v}^{j]} d^3x, \qquad S_A^{ij} = \epsilon^{ijk} S_A^k, \qquad S_A^i = \frac{1}{2} \epsilon^{ijk} S_A^{jk}, \tag{2.5}$$

where [ ] around indices denotes antisymmetrization. We will demonstrate in Appendix A that this definition of center of mass corresponds to the SSC value $k_{\rm SSC} = 1/2$. With these definitions, the baryonic mass m_A is constant, and the velocity, acceleration and rate of change of spin of body A are given by

$$\mathbf{v}_A = m_A^{-1} \int_A \rho^* \mathbf{v} d^3x, \tag{2.6a}$$

$$\mathbf{a}_A = m_A^{-1} \int_A \rho^* \mathbf{a} d^3x, \tag{2.6b}$$

$$dS_A^i/dt = \epsilon^{ijk} \int_A \rho^* \bar{x}^j a^k d^3x. \tag{2.6c}$$

Note that, because $\int_A \rho^* \bar{\mathbf{x}} d^3x = 0$, we do not need the "barred" version of the acceleration in Eq. (2.6c).

B. Baryonic equations of motion and spin precession

We begin by working to 1 and 2.5 PN order, reproducing a number of well-known 1 PN formulas for spinning bodies, and establishing some results that will be useful when we go on to 3.5 PN order. Since we are only interested in radiation-reaction aspects of spin, we can ignore the 2 PN terms in the equations of motion; these produce only PN corrections to the spin equations of motion. Incorporating the 2 PN terms would require a more accurate definition of spin, including PN corrections. Such a framework has not been developed to date and is beyond the scope of this paper. Initially, we derive everything in terms of our baryonic definitions of center of mass and spin, then later, we transform to more relevant definitions.

We use the continuum equations of motion derived in paper II, Eqs. (2.23), (2.24a), and (2.24b), with all quantities expressed in terms of the conserved density ρ^*. They are given by

$$d^2x^i/dt^2 = U^{,i} + a^i_{\rm PN} + a^i_{2.5\ \rm PN}, \tag{2.7}$$

where

$$a^i_{\rm PN} = v^2 U^{,i} - 4v^i v^j U^{,j} - 4UU^{,i} - 3v^i \dot{U} + 4\dot{V}^i + 8v^j V^{[i,j]} + \frac{3}{2}\Phi_1^{,i} - \Phi_2^{,i} + \frac{1}{2}\ddot{X}^{,i}, \tag{2.8a}$$

$$a^i_{2.5\ \rm PN} = \frac{3}{5} x^j \left(\overset{(5)}{I}{}^{ij} - \frac{1}{3}\delta^{ij} \overset{(5)}{I}{}^{kk} \right) + 2v^j \overset{(4)}{I}{}^{ij} + 2U^{,j} \overset{(3)}{I}{}^{ij} + \frac{4}{3} U^{,i} \overset{(3)}{I}{}^{kk} - X^{,ijk} \overset{(3)}{I}{}^{jk}, \tag{2.8b}$$

where commas denote partial derivatives, overdots denote partial time derivatives, and (n) above quantities denotes the number of total time derivatives. The potentials used here and elsewhere in the paper are given by the general definitions

$$\Sigma(f) \equiv \int \frac{\rho^*(t, \mathbf{x}') f(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad X(f) \equiv \int \rho^*(t, \mathbf{x}') f(t, \mathbf{x}') |\mathbf{x} - \mathbf{x}'| d^3x', \quad Y(f) \equiv \int \rho^*(t, \mathbf{x}') f(t, \mathbf{x}') |\mathbf{x} - \mathbf{x}'|^3 d^3x', \tag{2.9}$$

with specific potentials given by

$$U = \Sigma(1), \quad V^i = \Sigma(v^i), \quad \Phi_1 = \Sigma(v^2), \quad \Phi_1^{jk} = \Sigma(v^j v^k), \quad \Phi_2 = \Sigma(U), \quad X = X(1), \quad X^i = X(v^i), \quad X_1 = X(v^2), \quad X_2 = X(U), \quad Y = Y(1). \tag{2.10}$$

The multipole moments of the system I^{ij}, I^{ijk}, and J^{qj}, and additional moments that will be relevant at 3.5 PN order are defined in Appendix C. In Eq. (2.8b), we have eliminated terms that are purely functions of time whose main effect is to generate an overall center-of-mass motion for the system that is irrelevant for our purposes. This is discussed in Appendix D.

We now multiply the equation of motion (2.7) by ρ^* and integrate over body 1, expressing the variables $\mathbf{x}$ and $\mathbf{v}$ as $\mathbf{x} = \mathbf{x}_A + \bar{\mathbf{x}}$ and $\mathbf{v} = \mathbf{v}_A + \bar{\mathbf{v}}$, where $A = 1, 2$, depending on the body in which the point lies. To get the acceleration of body 1, we divide the result by m_1. We use Eqs. (2.4) and (2.6) to simplify where possible. We expand the various potentials in powers of $\bar{x}/r$, and keep terms proportional to $\bar{v} \times \bar{x}$, as well as "internal" terms proportional to $\bar{v}^2$, and $\rho^*/\bar{r}$. The random part of the "$\bar{v}^2$" terms can be thought of as pressure contributions to the internal structure, so we no longer are treating the bodies as purely pressureless. We make extensive use of virial relations derived in Appendix E to simplify expressions dependent on the internal structure of each body. We also assume that the bodies are approximately spherically symmetric in their own rest frames (we ignore centrifugal and Lorentz-boost flattening, which will be higher-PN-order effects), so that integrals of internal expressions with an odd number of spatial indices (corresponding to odd-l spherical harmonics) can be set to zero. We also ignore internal body terms that vary as $\bar{x}^2$, which represent quadrupole and higher "tidal" effects, and which vanish as the bodies' sizes tend to zero.

The Newtonian term gives $a^i_{\rm N} = -m_2 x^i_{12}/r^3$, where, in this paragraph, we denote $x^i_{12} \equiv x^i_1 - x^i_2$, $r \equiv |\mathbf{x}_{12}|$, and $\mathbf{n} = \mathbf{x}_{12}/r$. To illustrate the approach, we show two examples of the terms that arise at 1 PN order, namely

$$\begin{aligned}\frac{1}{m_1}\int_1 \rho^* v^2 U^{,i} d^3x &= -\frac{1}{m_1}\int_1 \rho^*(v_1^2 + 2\mathbf{v}_1\cdot\bar{\mathbf{v}} + \bar{v}^2)d^3x\left[\int_1 \frac{\rho^{*\prime}(x-x')^i}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' + \frac{m_2 x^i_{12}}{r^3} + \frac{m_2\bar{x}^j(\delta^{ij}-3n^in^j)}{r^3} + \dots\right]\\ &= \frac{2v_1^j}{m_1}\mathcal{H}_1^{ji} - m_2 v_1^2\frac{x^i_{12}}{r^3} - \frac{2\mathcal{T}_1}{m_1}\frac{m_2 x^i_{12}}{r^3} - \frac{S_1^{jk}v_1^k}{m_1}\frac{m_2(\delta^{ij}-3n^in^j)}{r^3},\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\frac{1}{m_1}\int_1 \rho^* U U^{,i} d^3x &= -\frac{1}{m_1}\int_1 \rho^* d^3x\left[\int_1 \frac{\rho^{*\prime\prime}}{|\mathbf{x}-\mathbf{x}''|}d^3x'' + \frac{m_2}{r} - \frac{m_2}{r^3}\bar{\mathbf{x}}\cdot\mathbf{x}_{12} + \dots\right]\\ &\quad\times\left[\int_1 \frac{\rho^{*\prime}(x-x')^i}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' + \frac{m_2 x^i_{12}}{r^3} - \frac{m_2\bar{x}^j(\delta^{ij}-3n^in^j)}{r^3} + \dots\right]\\ &= -\frac{\Omega_1^{ij}}{m_1}\frac{m_2 x^j_{12}}{r^3} + 2\frac{\Omega_1}{m_1}\frac{m_2 x^i_{12}}{r^3} - \frac{m_2^2 x^i_{12}}{r^4},\end{aligned} \tag{2.12}$$

where $\mathcal{T}_1$, Ω_1^{ij}, Ω_1 and $\mathcal{H}_1^{ij}$ are defined in Appendix E. These entirely internal, structure-dependent terms can be eliminated via virial relations; however they can and do generate spin terms at 1 PN and 3.5 PN order. Detailed expressions for these virial relations may be found in Appendix E.

In the combination of 1 PN terms $4\dot{V}^i + \frac{1}{2}\ddot{X}^{,i}$ in Eq. (2.8a), the time derivatives generate accelerations inside the potentials. To the order needed for our purposes, we must therefore substitute the Newtonian and 2.5 PN continuum terms for those accelerations and carry out the same procedures for the integrals as described above. Those Newtonian and 2.5 PN acceleration terms inserted into 1 PN expressions will then generate respective 1 PN and 3.5 PN point-mass and spin-orbit contributions.

Turning to the 2.5 PN terms, Eq. (2.8b), the integrations lead to no explicit internal or spin terms; however for our 3.5 PN accurate work, the multipole moments and their time derivatives must be evaluated to 1 PN order, and in those 1 PN corrections, spin terms will appear (see Appendix C).

The resulting equation of motion for body 1 is

$$a_1^i = -\frac{m_2 n^i}{r^2} + (a_1^i)_{\rm PN} + (a_1^i)_{\rm SO} + (a_1^i)_{2.5\ \rm PN}, \tag{2.13}$$

where

$$(a_1^i)_{\rm PN} = \frac{m_2 n^i}{r^2}\left[5\frac{m_1}{r} + 4\frac{m_2}{r} - v_1^2 + 4\mathbf{v}_1\cdot\mathbf{v}_2 - 2v_2^2 + \frac{3}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right] - \frac{m_2}{r^2}(v_1 - v_2)^i(3\mathbf{v}_2\cdot\mathbf{n} - 4\mathbf{v}_1\cdot\mathbf{n}), \tag{2.14a}$$

$$(a_1^i)_{\rm SO} = \frac{m_2}{r^3}(\delta^{ij} - 3n^in^j)\left[2(v_2^k\tilde{S}_1^{jk} - v_1^k\tilde{S}_2^{jk}) - \frac{3}{2}(v_1^k\tilde{S}_1^{jk} - v_2^k\tilde{S}_2^{jk})\right] - \frac{m_2}{r^3}(\delta^{jk} - 3n^jn^k)(v_1 - v_2)^j\left(\frac{3}{2}\tilde{S}_1^{ik} + 2\tilde{S}_2^{ik}\right), \tag{2.14b}$$

$$(a_1^i)_{2.5\ \rm PN} = \frac{3}{5}x_1^j\left(\overset{(5)}{\mathcal{I}}{}^{ij} - \frac{1}{3}\delta^{ij}\overset{(5)}{\mathcal{I}}{}^{kk}\right) + 2v_1^j\overset{(4)}{\mathcal{I}}{}^{ij} - \frac{1}{3}\frac{m_2}{r^2}n^i\overset{(3)}{\mathcal{I}}{}^{kk} - 3\frac{m_2}{r^2}n^in^jn^k\overset{(3)}{\mathcal{I}}{}^{jk}, \tag{2.14c}$$

where $\tilde{S}_A^{ij} \equiv S_A^{ij}/m_A$, and where the "mass" m_2 in the Newtonian term is now given, to PN order, by the baryonic mass plus $\frac{1}{2}\Omega_2$, which corresponds to the total mass-energy of the body.

We calculate the precession of the spin in a similar manner. Starting with $dS_1^i/dt = \epsilon^{ijk}\int_1 \rho^*\bar{x}^j a^k d^3x$, we expand about the baryonic centers of mass, keeping terms that depend on $\bar{x}\times\bar{v}$, dropping internal terms with odd numbers of spatial indices, and throwing away terms that vary as R^2 or higher. Notice that, even in Newtonian theory, such $(m_2/r^3)(m_1R_1^2)$ terms occur, and represent standard quadrupole coupling; in the Earth-Moon system, these lead to the precession of the equinoxes. However, as we wish to deal with compact bodies, we shall ignore such effects. At 1 PN order, the only terms in Eq. (2.8a) that contribute are those that have explicit velocity dependence. The result, at 1 PN order is

$$(\dot{S}_1^i)_{\rm PN} = \frac{m_2}{r^2}\Big[5S_1^i(\mathbf{v}_1\cdot\mathbf{n}) - 3S_1^i(\mathbf{v}_2\cdot\mathbf{n}) - 2n^i(\mathbf{S}_1\cdot\mathbf{v}) + (v_1 - 2v_2)^i(\mathbf{S}_1\cdot\mathbf{n})\Big]. \tag{2.15}$$

Strangely, however, there *is* a 2.5 PN contribution to the spin equation of motion, arising from the v-dependent term in Eq. (2.8b), and given by

$$(\dot{S}_1^i)_{2.5\ \rm PN} = S_1^i\overset{(4)}{\mathcal{I}}{}^{jj} - S_1^j\overset{(4)}{\mathcal{I}}{}^{ij}. \tag{2.16}$$

However, we notice that, since the spin is constant to lowest order, then, to 2.5 PN order, the right-hand side of Eq. (2.16) is a total time derivative, and thus can be moved to the left-hand side and absorbed into a redefinition of the spin. In any case, it is clear from the argument made in Sec. I that this is a spurious effect, because there is a gauge in which the 2.5 PN terms in the equation of motion have no explicit velocity dependence. This 2.5 PN gauge, sometimes called Burke-Thorne gauge [26], may be obtained from our gauge by the coordinate transformation

$$x^i \rightarrow x^i - x^j \overset{(3)}{I}{}^{ij} + \frac{1}{3} x^i \overset{(3)}{I}{}^{jj}, \qquad t \rightarrow t + \frac{2}{3} \overset{(2)}{I}{}^{jj}. \tag{2.17}$$

Applying this transformation to our definition of baryonic spin Eq. (2.4c), and recalling that $\rho^* d^3x$ is an invariant quantity, we find that

$$(S_1^i)_{\text{Burke-Thorne}} = S_1^i - S_1^i \overset{(3)}{I}{}^{jj} + S_1^j \overset{(3)}{I}{}^{ij}. \tag{2.18}$$

Hence, absorbing the 2.5 PN terms into the definition of the spin is equivalent to defining our spin in the Burke-Thorne gauge. Although this redefinition eliminates 2.5 PN terms from the spin precession, it *will* have consequences at 3.5 PN order.

C. Post-Newtonian conserved quantities

It is useful to verify that, at 1 PN order, the equations of motion and spin precession we have found admit suitable conserved quantities for energy, angular momentum and momentum or center-of-mass motion. Taking the equations of motion (2.13), (2.14a) and (2.14b), contracting with $m_1\mathbf{v}_1$, summing over both bodies, and using the equations of motion to extract time derivatives (see Appendix F), one obtains a conserved total energy. Doing the same procedure with a cross product with $m_1 x_1$, and combining with the spin-precession equations, one obtains a conserved total angular momentum. Finally, multiplying by m_1, summing over both bodies and extracting time derivatives, one obtains an expression for the system center of mass. These 1 PN-conserved quantities are given by

$$E = \frac{1}{2} m_1 v_1^2 - \frac{1}{2}\frac{m_1 m_2}{r} + \frac{3}{8} m_1 v_1^4 + \frac{3}{2} m_1 v_1^2 \frac{m_2}{r} + \frac{1}{2}\frac{m_1 m_2^2}{r^2} - \frac{1}{4}\frac{m_1 m_2}{r}(7\mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n}) + (1 \rightleftharpoons 2), \tag{2.19a}$$

$$\begin{aligned}\mathbf{J} = {}& m_1(\mathbf{x}_1 \times \mathbf{v}_1)\left(1 + \frac{1}{2}v_1^2 - \frac{1}{2}\frac{m_2}{r}\right) - \frac{1}{2}\frac{m_1 m_2}{r}(7\mathbf{x}\times\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{n}\mathbf{x}_1\times\mathbf{n}) + S_1\left(1 + \frac{1}{2}v_1^2 + 3\frac{m_2}{r}\right) \\ & - \frac{1}{2}\mathbf{v}_1 \times (\mathbf{v}_1 \times S_1) + \frac{m_2}{r}\left(2\mathbf{n} - \frac{1}{2}\frac{\mathbf{x}_1}{r}\right)\times(\mathbf{n}\times S_1) + (1 \rightleftharpoons 2),\end{aligned} \tag{2.19b}$$

$$\mathbf{I} = m_1\mathbf{x}_1\left(1 + \frac{1}{2}v_1^2 - \frac{1}{2}\frac{m_2}{r}\right) + \frac{1}{2}\mathbf{v}_1 \times \mathbf{S}_1 + (1 \rightleftharpoons 2), \tag{2.19c}$$

where, to the 1 PN order needed, the masses shown are the total masses of each body, given by $m_1 + \frac{1}{2}\Omega_1$ and $m_2 + \frac{1}{2}\Omega_2$. The first thing to notice about these conserved quantities is the absence of a spin-orbit contribution to the total energy. This well-known result is merely a consequence of our choice of SSC. Converting from our SSC to the covariant SSC, for example, gives the standard spin-orbit term (see Appendix B). These conserved quantities can also be derived from global definitions of energy, momentum and angular momentum, as discussed in Appendix G. It is straightforward to verify, by taking a time derivative of Eqs. (2.19) and substituting the 1 PN equations of motion (2.14a) and (2.14b) and spin precession (2.15), that E, $\mathbf{J}$ and $\dot{\mathbf{I}}$ are constant to 1 PN order.

D. The proper spin

The total angular momentum $\mathbf{J}$, Eq. (2.19b), has been written in a form that appears to have an orbital piece, plus 1 PN corrections, a spin piece, plus 1 PN corrections, and a final, spin-orbit piece. Although the split is somewhat arbitrary, it is useful in that, if we identify the "proper" spin of each body by the collection of "spin" terms in Eq. (2.19b), then Eq. (2.15) is equivalent to the standard spin-precession Eq. (1.3). In addition, we found that, in our gauge, there was a 2.5 PN contribution to the spin-precession [Eq. (2.16)], but since that contribution was a total time derivative to 2.5 PN order, it could be absorbed into a redefinition of the spin. We therefore define the proper spin of each body to be

$$\begin{aligned} S_1^i \equiv {}& S_1^i\left(1 + \frac{1}{2}v_1^2 + 3\frac{m_2}{r}\right) - \frac{1}{2}[\mathbf{v}_1 \times (\mathbf{v}_1 \times S_1)]^i \\ & - S_1^i \overset{(3)}{I}{}^{jj} + S_1^j \overset{(3)}{I}{}^{ij}, \\ S_2 \equiv {}& (1 \rightleftharpoons 2), \end{aligned} \tag{2.20}$$

With this definition, the spins S_A satisfy Eq. (1.3), with *no* 2.5 PN contributions.

E. System center of mass and transformation to relative coordinates

Choosing our coordinates so that the "center of mass" quantity $\mathbf{I}$ vanishes, and defining the transformation from individual to relative coordinates to 1 PN order by

$$x_1^i = \frac{m_2}{m}x^i + \delta x^i, \qquad x_2^i = -\frac{m_1}{m}x^i + \delta x^i, \tag{2.21}$$

we obtain

$$\delta x^i = \frac{1}{2}\eta\frac{\delta m}{m}\left(v^2 - \frac{m}{r}\right)x^i - \frac{1}{2}\frac{\eta}{\delta m}\epsilon^{ijk}v^j(S^k - \xi^k), \tag{2.22}$$

where $\delta m \equiv m_1 - m_2$. These transformations do not affect the Newtonian term in the acceleration, of course. In the 1 PN and spin-orbit terms they will only produce 2 PN effects, which we ignore. The multipole moments that appear in the 2.5 PN terms in the equation of motion (2.14c) must also be converted to relative coordinates, keeping any PN and spin-orbit corrections generated by Eqs. (2.21); this is treated in Appendix C. In addition, in the 2.5 PN terms, multiple time derivatives of the multipole moments will generate accelerations, for which the 1 PN relative equations of motion including spin-orbit terms must be substituted; in explicitly 3.5 PN terms, the Newtonian equation of motion suffices.

Calculating the relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ using Eqs. (2.13) and (2.14), and converting to relative coordinates, we obtain the equations of motion given by Eqs. (1.1), (1.2), and (1.5).

In terms of these relative coordinates, the energy and angular momentum of the system to 1 PN order including spin terms then take the form,

$$E = \mu\left\{\frac{1}{2}v^2 - \frac{m}{r} + \frac{3}{8}(1-3\eta)v^4 + \frac{1}{2}(3+\eta)v^2\frac{m}{r} + \frac{1}{2}\eta\frac{m}{r}\dot{r}^2 + \frac{1}{2}\left(\frac{m}{r}\right)^2\right\}, \tag{2.23a}$$

$$\mathbf{J} = \mu\tilde{\mathbf{L}}_{\rm N}\left\{1 + \frac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{m}{r}\right\} + \mathcal{S} + \frac{1}{2}\frac{\mu}{r}\mathbf{n}\times[\mathbf{n}\times(4\mathcal{S} + 3\boldsymbol{\xi})], \tag{2.23b}$$

where $\mu = \eta m$ is the reduced mass. Notice that we do not keep the 2.5 PN contribution to $\mathbf{J}$ arising from the conversion from baryonic spin to proper spin using Eq. (2.20), since E and $\mathbf{J}$ are only well defined up to 2 PN order. In Sec. III C we will use these expressions together with the 3.5 PN equations of motion to compare $\dot{E}$ and $\dot{\mathbf{J}}$ with the corresponding fluxes of radiation to infinity.

III. 3.5 PN EQUATIONS OF MOTION

A. Equation of motion

To obtain the 3.5 PN contributions to the equations of motion including spin terms, we take the 3.5 PN fluid expressions shown in Appendix D, multiply by ρ^*, and integrate over body 1. We follow the same procedure as in Sec. II B, expanding potentials about the baryonic centers of mass of the bodies, keeping point-mass terms, internal self-energy terms, and spinlike terms (terms linear in $\bar{x}\,\bar{v}$) and discarding tidal-like terms. Many terms in Eq. (D4) make only point-mass contributions, such as terms of the form $r^2x^k(d^7I^{ik}/dt^7)$, $x^jU(d^5I^{ij}/dt^5)$, $v^2v^i(d^4I^{kk}/dt^4)$, $UU^{,j}(d^3I^{ij}/dt^3)$, and so on, because they do not have the proper mix of x and v to generate a spin. However terms such as $r^2v^k(d^6I^{ik}/dt^6)$, $x^j\dot{U}(d^4I^{ij}/dt^4)$ or $\Phi^{,i}(d^3I^{kk}/dt^3)$ will generate spins.

Some of these terms generate contributions of the form of an integral that is purely internal to body 1 multiplied by a multipole moment; examples include $\mathcal{T}_1^{ij}x_1^kd^5I^{jk}/dt^5$, $\Omega_1^{jk}v_1^id^4I^{jk}/dt^4$, $\mathcal{H}_1^{ji}v_1^kd^3I^{jk}/dt^3$. Virial relations must then be applied to these internal integrals, to see if any spin terms arise. But in the virial relation (E3) involving $\mathcal{T}_1^{ij}$ and Ω_1^{ij}, spin terms occur one PN order higher, hence there will be no contributions from these terms to the acceleration at 3.5 PN order. On the other hand, the virial relation (E4) involving $\mathcal{H}^{ij}$ *does* have a spin contribution at the same order, so those virial-induced terms must be kept.

To lowest order, the mass multipole moments $I^{ij\cdots}$ and their derivatives do not contain spin terms, however the current moments $J^{ij\cdots}$ do contain both point-mass and spin terms at lowest order.

In addition, the combination of 1 PN terms $4\dot{V}^i + \frac{1}{2}\ddot{X}^{,i}$ in Eq. (2.8a), will generate accelerations whose 2.5 PN terms will produce 3.5 PN point-mass and spin terms that must be included. We must also reexpress the 1 PN spin-orbit terms of Eq. (2.14b) in terms of the proper spin of Eq. (2.20); the 2.5 PN contributions there will generate 3.5 PN terms in the equation of motion. Finally, in the 2.5 PN accelerations of Eq. (2.14c), we must include the 1 PN corrections to the multipole moments as well as the 1 PN terms in the equations of motion that are generated by the many time derivatives; these corrections will also contain spin contributions (Appendix C).

The result for the 3.5 PN acceleration of body 1 is an expression too lengthy to reproduce here. The point-mass terms reproduce Eq. (4.2) of paper II, apart from small differences resulting from our use of the gauge transformation (D2) to remove purely time dependent terms from the 2.5 PN and 3.5 PN equations of motion. After transforming to relative coordinates and obtaining the relative acceleration, our point-mass terms match the corresponding expressions of paper II precisely. The expression involving spins is equally lengthy.

Calculating $a_1^i - a_2^i$ using Eqs. (2.14a)–(2.14c) and our 3.5 PN terms, substituting Eqs. (2.21) and the time-derivatives of the multipole moments (C2), and expressing the spin results in terms of the total spin $\mathcal{S}$ and the spin quantity $\boldsymbol{\xi}$, we obtain the final relative equation of motion terms as given in Eqs. (1.2), (1.5), and (1.6),.

B. Spin precession

We now want to calculate the precession of the proper spin $\mathcal{S}_1$ to 3.5 PN order. A time derivative of Eq. (2.20) gives

$$\dot{S}_1^i = \dot{\mathcal{S}}_1^i\left(1+v_1^2+3\frac{m_2}{r}\right)-\frac{1}{2}v_1^i(\mathbf{v}_1\cdot\dot{\mathbf{S}}_1) + \mathcal{S}_1^i\left(2\mathbf{v}_1\cdot\mathbf{a}_1-3\frac{m_2\dot{r}}{r^2}\right)-\frac{1}{2}a_1^i(\mathbf{v}_1\cdot\mathbf{S}_1)-\frac{1}{2}v_1^i(\mathbf{a}_1\cdot\mathbf{S}_1) - \mathcal{S}_1^i\overset{(4)}{\mathcal{I}}{}^{jj}+\mathcal{S}_1^j\overset{(4)}{\mathcal{I}}{}^{ij}-\dot{\mathcal{S}}_1^i\overset{(3)}{\mathcal{I}}{}^{jj}+\dot{\mathcal{S}}_1^j\overset{(3)}{\mathcal{I}}{}^{ij}. \tag{3.1}$$

We repeat the method of Sec. II B to determine the contributions of 3.5 PN fluid terms to the time derivative of the baryonic spin $\mathbf{S}_1$, by calculating $\epsilon^{ijk}\int_1\rho^*\bar{x}^j a^k_{3.5\ \mathrm{PN}}d^3x$. Only terms in $a^k_{3.5\ \mathrm{PN}}$ [Eq. (D4)] that have explicit v dependence will contribute a spin term. Notice that, as we have discussed, the 2.5 PN contribution to $\dot{\mathcal{S}}_1^i$ cancels the relevant terms in the last line of Eq. (3.1). For $\mathbf{a}_1$, which appears in the 1 PN terms in Eq. (3.1), we must substitute the 2.5 PN equations of motion; for $\dot{\mathcal{S}}_1^i$ in the final 2.5 PN terms in Eq. (3.1) we must substitute the 1 PN precession equations; finally we must use Eq. (2.20) to convert from $\mathbf{S}_1$ back to the proper spin $\mathcal{S}_1$ to the appropriate order.

The result is the 1 PN spin precession of Eq. (1.3), plus a lengthy 3.5 PN expression. However, using the fact that, to lowest order $\dot{\mathcal{S}}_1 = 0$, together with the identities listed in Appendix F, it is straightforward to show that our lengthy 3.5 PN expression is in fact a total time derivative, given by

$$(\dot{\mathcal{S}}_1)_{3.5\ \mathrm{PN}} = \frac{\eta^2m^2}{5}\frac{d}{dt}\left(\frac{\dot{r}}{r^2}\left\{\mathbf{S}_1\left[(14-13\alpha)v^2-(10+15\alpha)\dot{r}^2-\frac{2}{3}(27+11\alpha)\frac{m}{r}\right]+\mathbf{n}\mathbf{S}_1\cdot\mathbf{n}\left[15(4-\alpha)v^2-25(4+3\alpha)\dot{r}^2 -\frac{10}{3}(17+37\alpha)\frac{m}{r}\right]-\mathbf{v}\mathbf{S}_1\cdot\mathbf{v}(40+25\alpha)\right\}-\frac{1}{r^2}\mathbf{n}\mathbf{S}_1\cdot\mathbf{v}\left[(18-14\alpha)v^2-6(9+10\alpha)\dot{r}^2-\frac{1}{3}(34+161\alpha)\frac{m}{r}\right] -\frac{1}{r^2}\mathbf{v}\mathbf{S}_1\cdot\mathbf{n}\left[(26-20\alpha)v^2-(66+45\alpha)\dot{r}^2-\frac{5}{3}(50+59\alpha)\frac{m}{r}\right]\right), \tag{3.2}$$

where $\alpha = m_2/m_1$. As such, it can be moved to the left-hand side and absorbed into a meaningless, 3.5 PN term in the redefinition of the spin. As a result, we find, to little surprise, that radiation reaction makes *no* contribution to the precession of the spins [for a physical justification, see remarks following Eq. (1.7)].

C. Comparison with fluxes of energy and angular momentum

The fluxes of energy and angular momentum in gravitational waves from a binary with spin-orbit interactions were derived by Kidder *et al.* [12,13], and are given by

$$\frac{dE}{dt} = \dot{E}_{\mathrm{N}} + \dot{E}_{\mathrm{SO}}, \qquad \frac{d\mathbf{J}}{dt} = \dot{\mathbf{J}}_{\mathrm{N}} + \dot{\mathbf{J}}_{\mathrm{SO}}, \tag{3.3}$$

where we include only the lowest-order "Newtonian" and 1 PN spin-orbit contributions. After transforming from Kidder's $k_{\mathrm{SSC}} = 1$ formulas, Eqs. (3.25a), (3.25c), (3.28a) and (3.28c) of Ref. [13], to our $k_{\mathrm{SSC}} = 1/2$ using the transformation $\mathbf{x} \rightarrow \mathbf{x} - (\mathbf{v}\times\boldsymbol{\xi})/2m$, we obtain for the fluxes,

$$\dot{E}_N = -\frac{8}{15}\frac{\eta^2m^4}{r^4}(12v^2-11\dot{r}^2), \tag{3.4a}$$

$$\dot{E}_{SO} = -\frac{8}{15}\frac{\eta^2m^3}{r^6}\left[\tilde{\mathbf{L}}_{\mathrm{N}}\cdot\mathcal{S}\left(27\dot{r}^2-37v^2-12\frac{m}{r}\right)+\tilde{\mathbf{L}}_{\mathrm{N}}\cdot\boldsymbol{\xi}\left(18\dot{r}^2-19v^2-8\frac{m}{r}\right)\right], \tag{3.4b}$$

$$\dot{\mathbf{J}}_N = -\frac{8}{5}\frac{\eta^2m^3}{r^3}\tilde{\mathbf{L}}_{\mathrm{N}}\left(2v^2-3\dot{r}^2-2\frac{m}{r}\right), \tag{3.4c}$$

$$\dot{\mathbf{J}}_{SO} = -\frac{4}{5}\frac{\eta^2m^2}{r^3}\left\{\frac{2}{3}\frac{m}{r}(v^2-\dot{r}^2)(\mathcal{S}-\boldsymbol{\xi})-\frac{1}{3}\dot{r}\frac{m}{r}\mathbf{n}\times(7\mathbf{v}\times\mathcal{S}+5\mathbf{v}\times\boldsymbol{\xi}) + \frac{m}{r}\mathbf{n}\times\left[(\mathbf{n}\times\mathcal{S})\left(6\dot{r}^2-\frac{17}{3}v^2+2\frac{m}{r}\right)+(\mathbf{n}\times\boldsymbol{\xi})\left(6\dot{r}^2-6v^2+\frac{4}{3}\frac{m}{r}\right)\right] + \dot{r}\mathbf{v}\times\left[(\mathbf{n}\times\mathcal{S})\left(\frac{29}{3}\frac{m}{r}+24v^2-30\dot{r}^2\right)+5(\mathbf{n}\times\boldsymbol{\xi})\left(\frac{5}{3}\frac{m}{r}+4v^2-5\dot{r}^2\right)\right]+\mathbf{v}\times\left[\mathbf{v}\times\mathcal{S}\left(18\dot{r}^2-12v^2-\frac{23}{3}\frac{m}{r}\right) + \mathbf{v}\times\boldsymbol{\xi}\left(15\dot{r}^2-\frac{29}{3}v^2-7\frac{m}{r}\right)\right]+\frac{1}{r^2}\tilde{\mathbf{L}}_{\mathrm{N}}\tilde{\mathbf{L}}_{\mathrm{N}}\cdot\left[\mathcal{S}\left(30\dot{r}^2-18v^2-\frac{92}{3}\frac{m}{r}\right)+\boldsymbol{\xi}\left(20\dot{r}^2-13v^2-\frac{59}{3}\frac{m}{r}\right)\right]\right\}. \tag{3.4d}$$

We now calculate the time derivative of the energy and angular momentum expressions (2.23), and substitute the equations of motion, including 1 PN, spin-orbit, 2.5 PN and 3.5 PN spin-orbit terms, along with the 1 PN spin-precession equations (recall there are no 3.5 PN contributions to the spin precession). After recovering the fact that all 1 PN point-mass and spin-orbit contributions cancel, leaving E and $\mathbf{J}$ conserved to that order, we find that the changes in E and $\mathbf{J}$ due to 2.5 PN and 3.5 PN spin-orbit radiation reaction are obtained from the following expressions,

$$\dot{E} = \mu \mathbf{v} \cdot (\mathbf{a}_{2.5\text{PN}} + \mathbf{a}_{3.5\ \text{PN}-\text{SO}}),$$
$$\dot{\mathbf{J}} = \mu \mathbf{x} \times (\mathbf{a}_{2.5\ \text{PN}} + \mathbf{a}_{3.5\ \text{PN}-\text{SO}}). \tag{3.5}$$

Initially, the results do not match the flux expressions above. However, by making use of the identities listed in Appendix F, we can show that the difference between the expressions in all cases is a total time derivative. These can thus be absorbed into meaningless 2.5 and 3.5 PN corrections to the definition of total energy and angular momentum. Thus we have established a proper energy and angular momentum balance between the radiation flux and the evolution of the orbit, including spin-orbit effects.

IV. CONCLUSIONS

We have derived the equations of motion for binary systems of spinning bodies from first principles, including the effects of gravitational-radiation reaction, and incorporating the contributions of spin-orbit coupling at 3.5 PN order. We found that the spins themselves are unaffected by radiation reaction. The resulting equations of motion are instantaneous, dynamical equations, and do not rely on assumptions of energy balance, or orbital averaging. They may be used to study the effects of spin on the inspiral of compact binaries numerically. We have focussed attention on effects linear in the spins, corresponding to spin-orbit coupling; the effects of spin-spin coupling can in principle also be calculated with our approach. These issues will be the subject of future work.

ACKNOWLEDGMENTS

This work is supported in part by the National Science Foundation under Grants No. PHY 00-96522 and No. PHY 03-53180, and by the National Aeronautics and Space Administration under Grant No. NAG5-10186. Achamveedu Gopakumar made useful contributions to this work in its early phases. We are grateful to the Groupe Gravitation Relativiste et Cosmologie (GR*e*CO) of the Institut d'Astrophysique de Paris for its hospitality during the academic year 2003–2004, where this work was completed.

APPENDIX A: SPIN SUPPLEMENTARY CONDITIONS

When we deal with systems containing spinning bodies, the fact that the bodies have finite size introduces an ambiguity in the definition of each body's center of mass. This has given rise to the concept of SSC [20,21], which is a condition whose role is to fix the center of mass. One defines the antisymmetric tensor $S_A^{\mu\nu}$, given by

$$S_A^{\mu\nu} \equiv 2 \int_A (x^{[\mu} - x_A^{[\mu})\tau^{\nu]0} d^3x, \tag{A1}$$

where $\tau^{\alpha\beta}$ is the source stress-energy pseudotensor that appears in the "relaxed Einstein equations" given by $\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta}$, and is a combination of the stress-energy tensor of matter and terms quadratic in the gravitational fields $h^{\alpha\beta}$ (see Sec. IIA. of paper I), and where x_A^μ is to be identified with the world line of the center of mass of the body. The center of mass is then fixed by imposing the following condition:

$$S_A^{i0} - k_{\text{SSC}} S_A^{ij} v_A^j = 0, \tag{A2}$$

where k_{SSC} typically has the values 1, 1/2 or 0. The value $k_{\text{SSC}} = 1$ corresponds to the so-called covariant SSC, $S_A^{\mu\nu} u_{A\nu} = 0$.

In this paper, our baryonic center-of-mass definition $x_A^i = \int_A \rho^* x^i d^3x$ corresponds to $k_{\text{SSC}} = 1/2$, which can be seen as follows. From Eq. (G3a), we note that, to the appropriate order, $\tau^{00} = \rho^*(1 + \frac{1}{2}v^2 - \frac{1}{2}U)$. Calculating S^{i0} following the methods outlined in Sec. II, and including the spin term generated by $\int_A \rho^* x^i v^2 d^3x$, we find directly that $S_A^{i0} = \frac{1}{2} S_A^{ij} v_A^j$, and thus that $k_{\text{SSC}} = 1/2$.

One can also show that the relationship between the centers of mass for each value of k_{SSC} is given by

$$(x_A^i)^{(k')} = (x_A^i)^{(k)} + \frac{k - k'}{m_A} S_A^{ij} (v_A^j)^{(k)}. \tag{A3}$$

For further discussion of the role of the SSC in PN calculations, see [13,25,27].

APPENDIX B: EQUATIONS OF MOTION WITH SPIN IN THE COVARIANT SSC

We now transform our key equations to a form in which the centers of mass of the bodies are defined by the covariant spin supplementary condition, $k_{\text{SSC}} = 1$. That transformation is given by

$$\mathbf{x}_1^{(1/2)} = \mathbf{x}_1 + \frac{1}{2m_1} \mathbf{v}_1 \times \mathbf{S}_1, \tag{B1}$$

where the variables on the right-hand side are in terms of $k_{\text{SSC}} = 1$, and where $\mathbf{S}_1$ is the baryonic spin. For the relative coordinate, the transformation, to 1 PN order, is

$$\mathbf{x}^{(1/2)} = \mathbf{x} + \frac{1}{2m} \mathbf{v} \times \boldsymbol{\xi}, \tag{B2}$$

where $\boldsymbol{\xi}$ is again the baryonic spin variable. Converting from the baryonic spin to proper spin and keeping only the 2.5 PN correction from Eq. (2.20) (the PN corrections will not be relevant for this purpose), we have

$$\mathbf{x}^{(1/2)} = \mathbf{x} + \frac{1}{2m} \mathbf{v} \times \boldsymbol{\xi} - \frac{1}{2m} \mathbf{v} \times \boldsymbol{\xi} \overset{(3)}{I}{}^{kk} + \frac{1}{2m} \mathbf{v} \times (\xi^j \overset{(3)}{I}{}^{jk}) \mathbf{e}_k, \tag{B3}$$

where $\boldsymbol{\xi}$ now represents the proper spin variable. We substitute this transformation into the Newtonian, PN, spin-orbit and 2.5 PN terms in the equations of motion, Eqs. (1.1), (1.2), and (1.6), keeping only PN spin-orbit terms, 2.5 PN terms, and 3.5 PN spin-orbit terms. Where accelerations arise, *eg.* in transforming the velocities in the spin-orbit terms or in the 2.5 PN terms, we must employ suitably accurate expressions, in order to generate all appropriate 3.5 PN spin-orbit terms. The result is to change the form only of the 1 PN spin-orbit and 3.5 PN spin-orbit terms, which now become

$$\mathbf{a}_{\rm SO} = \frac{1}{r^3}\left\{6\frac{\mathbf{n}}{r}\tilde{\mathbf{L}}_{\rm N}\cdot(S+\boldsymbol{\xi}) - \mathbf{v}\times(4S+3\boldsymbol{\xi}) + 3\dot{r}\mathbf{n}\times(2S+\boldsymbol{\xi})\right\}, \tag{B4a}$$

$$\begin{aligned}\mathbf{a}_{3.5\,\rm PN-SO} = -\frac{\eta m}{5r^4}\bigg\{&\frac{\dot{r}\mathbf{n}}{r}\bigg[\bigg(120v^2+280\dot{r}^2+453\frac{m}{r}\bigg)\tilde{\mathbf{L}}_{\rm N}\cdot S + \bigg(120v^2+280\dot{r}^2+458\frac{m}{r}\bigg)\tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\bigg]\\ &+\frac{\mathbf{v}}{r}\bigg[\bigg(87v^2-675\dot{r}^2-\frac{901}{3}\frac{m}{r}\bigg)\tilde{\mathbf{L}}_{\rm N}\cdot S + 4\bigg(18v^2-150\dot{r}^2-66\frac{m}{r}\bigg)\tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\bigg]\\ &-\frac{2}{3}\dot{r}\mathbf{v}\times S\bigg(48v^2+15\dot{r}^2+364\frac{m}{r}\bigg) + \frac{1}{3}\dot{r}\mathbf{v}\times\boldsymbol{\xi}\bigg(291v^2-705\dot{r}^2-772\frac{m}{r}\bigg)\\ &+\frac{1}{2}\mathbf{n}\times S\bigg(31v^4-260v^2\dot{r}^2+245\dot{r}^4-\frac{689}{3}v^2\frac{m}{r}+537\dot{r}^2\frac{m}{r}+\frac{4}{3}\frac{m^2}{r^2}\bigg)\\ &+\frac{1}{2}\mathbf{n}\times\boldsymbol{\xi}\bigg(115v^4-1130v^2\dot{r}^2+1295\dot{r}^4-\frac{869}{3}v^2\frac{m}{r}+849\dot{r}^2\frac{m}{r}+\frac{44}{3}\frac{m^2}{r^2}\bigg)\bigg\}.\end{aligned} \tag{B4b}$$

The spin-precession equations are not affected by this SSC transformation, since we are working to linear order in spins. The energy and angular momentum in this SSC take the form

$$\begin{aligned}E = \mu\bigg\{&\frac{1}{2}v^2 - \frac{m}{r} + \frac{3}{8}(1-3\eta)v^4 + \frac{1}{2}(3+\eta)v^2\frac{m}{r}\\ &+\frac{1}{2}\eta\frac{m}{r}\dot{r}^2 + \frac{1}{2}\bigg(\frac{m}{r}\bigg)^2 + \frac{1}{r^3}\tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\bigg\},\end{aligned} \tag{B5a}$$

$$\begin{aligned}\mathbf{J} = \mu\tilde{\mathbf{L}}_{\rm N}&\bigg\{1+\frac{1}{2}(1-3\eta)v^2+(3+\eta)\frac{m}{r}\bigg\} + S\\ &+\frac{1}{2}\frac{\mu}{m}\bigg\{\frac{m}{r}\mathbf{n}\times[\mathbf{n}\times(4S+2\boldsymbol{\xi})]. - \mathbf{v}\times[\mathbf{v}\times\boldsymbol{\xi}]\bigg\}.\end{aligned} \tag{B5b}$$

The spin-orbit contributions to the energy and angular momentum fluxes, from Eqs. (3.25c) and (3.28c) of [13] are given by

$$\dot{E}_{SO} = -\frac{8}{15}\frac{\eta^2 m^3}{r^6}\bigg[\tilde{\mathbf{L}}_{\rm N}\cdot S\bigg(27\dot{r}^2-37v^2-12\frac{m}{r}\bigg) + \tilde{\mathbf{L}}_{\rm N}\cdot\boldsymbol{\xi}\bigg(51\dot{r}^2-43v^2+4\frac{m}{r}\bigg)\bigg], \tag{B6a}$$

$$\begin{aligned}\dot{\mathbf{J}}_{SO} = -\frac{4}{5}\frac{\eta^2 m^2}{r^3}\bigg\{&\frac{2}{3}\frac{m}{r}(v^2-\dot{r}^2)(S-\boldsymbol{\xi}) - \frac{1}{3}\dot{r}\frac{m}{r}\mathbf{n}\times(7\mathbf{v}\times S+5\mathbf{v}\times\boldsymbol{\xi}) + \frac{m}{r}\mathbf{n}\times\bigg[(\mathbf{n}\times S)\bigg(6\dot{r}^2-\frac{17}{3}v^2+2\frac{m}{r}\bigg)\\ &+(\mathbf{n}\times\boldsymbol{\xi})\bigg(9\dot{r}^2-8v^2-\frac{2}{3}\frac{m}{r}\bigg)\bigg] + \dot{r}\mathbf{v}\times\bigg[(\mathbf{n}\times S)\bigg(\frac{29}{3}\frac{m}{r}+24v^2-30\dot{r}^2\bigg)+5(\mathbf{n}\times\boldsymbol{\xi})\bigg(\frac{5}{3}\frac{m}{r}+4v^2-5\dot{r}^2\bigg)\bigg]\\ &+\mathbf{v}\times\bigg[\mathbf{v}\times S\bigg(18\dot{r}^2-12v^2-\frac{23}{3}\frac{m}{r}\bigg)+\mathbf{v}\times\boldsymbol{\xi}\bigg(18\dot{r}^2-\frac{35}{3}v^2-9\frac{m}{r}\bigg)\bigg]\\ &+\frac{1}{r^2}\tilde{\mathbf{L}}_{\rm N}\tilde{\mathbf{L}}_{\rm N}\cdot\bigg[S\bigg(30\dot{r}^2-18v^2-\frac{92}{3}\frac{m}{r}\bigg)+\boldsymbol{\xi}\bigg(35\dot{r}^2-19v^2-\frac{71}{3}\frac{m}{r}\bigg)\bigg]\bigg\}.\end{aligned} \tag{B6b}$$

By following the same steps as in Sec. III C, calculating the time derivative of E and $\mathbf{J}$, substituting the relevant contributions from the equations of motion and spin precession, and extracting total time derivatives from the result (using $k_{SSC}=1$ versions of the equations in Appendix F), we verify that the losses of orbital energy and angular momentum are completely equivalent to the fluxes above.

APPENDIX C: MULTIPOLE MOMENTS

The multipole moments that appear in the radiation-reaction terms in Eqs. (2.8b) and (D4) are defined by the general expressions (F1). Because the quadrupole moment $\mathcal{I}^{ij}$ appears in the 2.5 PN terms, it will be needed to 1 PN order, while the remaining moments will be needed to only the lowest, Newtonian order. Substituting the relevant expressions for $\tau^{\alpha\beta}$ from Eqs. (G3), and carrying out the standard procedure as in Appendix G, we obtain

$$\begin{aligned}
\mathcal{I}^{ij} &= m_1 x_1^{ij}\left(1+\frac{1}{2}v_1^2\right) - \frac{m_1 m_2}{r}\left(\frac{1}{2}x_1^{ij} - \frac{7}{4}r^2\delta^{ij}\right) + x_1^{(i}S_1^{j)k}v_1^k + (1 \rightleftharpoons 2),\\
\mathcal{I}^{ijk\ldots} &= m_1 x_1^{ijk\ldots} + (1 \rightleftharpoons 2),\\
\mathcal{J}^{ij} &= \epsilon^{iab}(m_1 v_1^b x_1^{aj} + x_1^{(a}S_1^{j)b}) + (1 \rightleftharpoons 2),\\
\mathcal{J}^{ijk} &= \epsilon^{iab}\left(m_1 v_1^b x_1^{ajk} + \frac{3}{2}x_1^{(jk}S_1^{a)b}\right) + (1 \rightleftharpoons 2),\\
\mathcal{M}^{ijkl} &= m_1 v_1^{ij}x_1^{kl} - \frac{1}{2}\frac{m_1 m_2}{r}n^{ij}x_1^{kl} + \frac{1}{12}m_1 m_2 r(n^{ijkl} - n^{ij}\delta^{kl} - n^{kl}\delta^{ij} + n^{i(k}\delta^{l)j} + n^{j(k}\delta^{l)i} - 2\delta^{i(k}\delta^{l)j} + 2\delta^{ij}\delta^{kl})\\
&\quad + 2x_1^{(k}S_1^{l)(i}v_1^{j)} + (1 \rightleftharpoons 2),
\end{aligned} \tag{C1}$$

where parentheses around indices denote symmetrization. Note that higher-order moments, such as $\mathcal{J}^{qjkl}$ or $\mathcal{M}^{kkjji}$ appear only in 3.5 PN terms that were transformed away in Appendix D, so their explicit forms are not needed.

Converting to relative coordinates, using the 1 PN correct transformation in the leading term of $\mathcal{I}^{ij}$, we obtain

$$\begin{aligned}
\mathcal{I}^{ij} &= \eta m x^{ij}\left(1 + \frac{1}{2}(1-3\eta)v^2 - \frac{1}{2}(1-2\eta)\frac{m}{r}\right) + \frac{7}{2}\eta m^2 r\delta^{ij} + \eta x^{(i}\xi^{j)k}v^k, \qquad \mathcal{I}^{ijk} = -\eta\delta m x^{ijk},\\
\mathcal{I}^{ijkl} &= \eta m(1-3\eta)x^{ijkl}, \qquad \mathcal{J}^{ij} = -\eta\delta m\tilde{L}_{\rm N}^i x^j - \eta\epsilon^{iab}x^{(j}\Delta^{a)b}, \qquad \mathcal{J}^{ijk} = \eta m(1-3\eta)\tilde{L}_{\rm N}^i x^{jk} + \frac{3}{2}\eta\epsilon^{iab}x^{(jk}\xi^{a)b},\\
\mathcal{M}^{ijkl} &= \eta m(1-3\eta)\left(v^{ij} - \frac{1}{3}\frac{m}{r}n^{ij}\right)x^{kl} - \frac{1}{6}\eta m^2 r(n^{ij}\delta^{kl} + n^{kl}\delta^{ij} - n^{i(k}\delta^{l)j} - n^{j(k}\delta^{l)i} + 2\delta^{i(k}\delta^{l)j} - 2\delta^{ij}\delta^{kl})\\
&\quad + 2\eta x^{(k}\xi^{l)(i}v^{j)},
\end{aligned} \tag{C2}$$

where $\tilde{\mathbf{L}}_{\rm N} \equiv \mathbf{x}\times\mathbf{v}$ is the orbital angular momentum per unit mass, $\xi^{ij} = (m_2/m_1)S_1^{ij} + (m_1/m_2)S_2^{ij}$ and $\Delta^{ij} \equiv m(S_2^{ij}/m_2 - S_1^{ij}/m_1) = (m/\delta m)(\xi^{ij} - S^{ij})$.

Time derivatives of the moments may be calculated using the relative equations of motion in place of $\ddot{x}^i$; 1 PN equations including spin contributions must be used in the leading term in $\mathcal{I}^{ij}$, while Newtonian equations are sufficient for the remaining terms. Since spin effects in the moments and their time derivatives are already at 1 PN order, the spins themselves may be treated as constants.

APPENDIX D: 3.5 PN TERMS IN THE FLUID EQUATIONS OF MOTION

We start with the 2.5 and 3.5 PN accelerations terms in the fluid equations of motion, written in terms of the conserved density ρ^*, Eqs. (2.24c) and (2.24d) of paper II. We then make a coordinate transformation at 2.5 PN and 3.5 PN order to eliminate the purely time dependent terms. These terms cancel when we compute the relative acceleration, but contribute to the gravitational-radiation induced recoil of the system. In other words, we choose coordinates that are fixed with respect to the recoiling center of mass of the system. This will simplify the transformation between the coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ of the individual bodies and the relative coordinate $\mathbf{x}$, Eq. (2.21), and will simplify our analysis of angular momentum (which can depend on the choice of center of mass) The required transformation is

$$x^i \to x^i + \delta x^i_{2.5\ PN} + \delta x^i_{3.5\ PN}, \tag{D1}$$

where

$$\delta x^i_{2.5\ PN} = \frac{2}{15}\overset{(3)}{\mathcal{I}}{}^{ijj} - \frac{2}{3}\epsilon^{qij}\overset{(2)}{\mathcal{J}}{}^{qj}, \tag{D2a}$$

$$\delta x^i_{3.5\ PN} = \frac{23}{4200}\overset{(5)}{\mathcal{I}}{}^{ijjkk} - \frac{2}{75}\epsilon^{qij}\overset{(4)}{\mathcal{J}}{}^{qjkk} - \frac{1}{30}\overset{(3)}{\mathcal{M}}{}^{kkjji}. \tag{D2b}$$

However, because of the velocity dependence in the PN terms in the equations of motion, the change in velocity at 2.5 PN order, $\delta v^i_{2.5\ \rm PN} = \delta\dot{x}^i_{2.5\ \rm PN}$ will induce additional 3.5 PN contributions in the equations of motion given by

$$\begin{aligned}
\delta a^i_{3.5\ \rm PN} &= \delta v^j_{2.5\ \rm PN}(2v^jU^{,i} + v^iU^{,j} - \dot{U}\delta^{ij} + V^{j,i} + \dot{X}^{,ij})\\
&\quad + \frac{1}{2}\delta\dot{v}^j_{2.5\ \rm PN}(X^{,ij} - 8U\delta^{ij}).
\end{aligned} \tag{D3}$$

When we add these changes to the 3.5 PN terms in Eq. (2.4d) of paper II, we obtain the final 3.5 PN contributions to the fluid equations of motion, given by

$$\begin{aligned}
a^i_{3.5\,PN} =& \frac{1}{210}(13r^2x^k\delta^{ij} - 4r^2x^i\delta^{jk} - x^ix^jx^k)\overset{(7)}{I}{}^{jk} + \frac{1}{30}(10r^2v^k\delta^{ij} + 4(v\cdot x)x^k\delta^{ij} - r^2v^i\delta^{jk} - 4x^ix^jv^k + 3x^jx^kv^i)\overset{(6)}{I}{}^{jk} \\
&+ \frac{1}{15}(10(v\cdot x)v^k\delta^{ij} - x^kv^2\delta^{ij} + 2(v\cdot x)v^i\delta^{jk} + 2x^iv^2\delta^{jk} - 5x^iv^jv^k + 4x^jv^iv^k)\overset{(5)}{I}{}^{jk} \\
&+ \frac{1}{15}(5r^2U^{,j} - 35x^jU + 59X^{,j})\overset{(5)}{I}{}^{ij} + \frac{1}{30}(5r^2U^{,i} + 30x^iU - 34X^{,i} - 6x^jX^{,ij})\overset{(5)}{I}{}^{kk} \\
&+ \frac{1}{90}(15x^jx^kU^{,i} - 6x^jX^{,ik} - 30x^iX^{,jk} - 15r^2X^{,ijk} - 4Y^{,ijk} + 5x^lY^{,ijkl})\overset{(5)}{I}{}^{jk} + \frac{1}{3}v^i(v^2\delta^{jk} - v^jv^k)\overset{(4)}{I}{}^{jk} \\
&- \frac{1}{3}(2x^j\dot{U} + 22V^j - 14\dot{X}^{,j} - 12v^kX^{,jk})\overset{(4)}{I}{}^{ij} + \frac{1}{18}(24x^jv^kU^{,i} + 12v^ix^jU^{,k} + 12x^jV^{k,i} + 54v^iX^{,jk} - 72v^jX^{,ik} \\
&+ 12x^j\dot{X}^{,ik} + 60X^{j,ki} - 72X^{i,jk} - 5\dot{Y}^{,ijk})\overset{(4)}{I}{}^{jk} - \frac{1}{3}(6v^iU - 8V^i - 2\dot{X}^{,i})\overset{(4)}{I}{}^{kk} + (2v^2U^{,j} - 8UU^{,j} - 8v^kV^{k,j} \\
&+ 3\Phi_1^{,j} - 2\Phi_2^{,j} + \ddot{X}^{,j})\overset{(3)}{I}{}^{ij} + \frac{1}{3}(2v^2U^{,i} - 8v^iv^jU^{,j} - 24UU^{,i} - 6v^i\dot{U} + 8\dot{V}^i + 16v^jV^{[i,j]} \\
&+ 3\Phi_1^{,i} - 6\Phi_2^{,i} + \ddot{X}^{,i})\overset{(3)}{I}{}^{kk} + \frac{1}{6}(48v^jV^{k,i} - 12v^jv^kU^{,i} - 6v^2X^{,ijk} + 24v^iv^lX^{,jkl} - 24\dot{X}^{i,jk} - 9X_1^{,ijk} + 6X_2^{,ijk} \\
&+ 18v^i\dot{X}^{,jk} - 48v^lX^{[i,l]jk} + 24UX^{,ijk} + 24U^{,i}X^{,jk} - 18\Phi_1^{jk,i} + 6\Sigma^{,i}(X^{,jk}) - \ddot{Y}^{,ijk})\overset{(3)}{I}{}^{jk} \\
&+ \frac{1}{630}(10x^ix^j\delta^{kl} - 25x^kx^l\delta^{ij} - 9r^2\delta^{ij}\delta^{kl})\overset{(7)}{I}{}^{jkl} + \frac{1}{45}(4x^{[i}v^{j]}\delta^{kl} - 2(v\cdot x)\delta^{ij}\delta^{kl} - 10x^kv^l\delta^{ij})\overset{(6)}{I}{}^{jkl} \\
&- \frac{1}{45}(v^2\delta^{ij}\delta^{kl} + 6v^iv^j\delta^{kl} + 5v^kv^l\delta^{ij})\overset{(5)}{I}{}^{jkl} - \frac{1}{45}(4U\delta^{jk} + 10x^jU^{,k} - 5X^{,jk})\overset{(5)}{I}{}^{ijk} + \frac{1}{45}(7X^{,ij} - 10x^jU^{,i})\overset{(5)}{I}{}^{jkk} \\
&+ \frac{1}{54}(6x^jX^{,ikl} - Y^{,ijkl})\overset{(5)}{I}{}^{jkl} + \frac{4}{45}(\dot{U}\delta^{ij} - v^iU^{,j} - 2v^jU^{,i} - V^{j,i} - \dot{X}^{,ij})\overset{(4)}{I}{}^{jkk} + \frac{1}{45}(3r^2\epsilon^{qik} + 2x^ix^j\epsilon^{qjk} \\
&+ 4x^jx^k\epsilon^{qij})\overset{(6)}{J}{}^{qk} - \frac{16}{45}x^jv^k\epsilon^{qjk}\overset{(5)}{J}{}^{qi} + \frac{2}{45}(2(v\cdot x)\epsilon^{qik} - 2x^iv^j\epsilon^{qjk} + 5x^jv^i\epsilon^{qjk} + 12x^jv^k\epsilon^{qij} \\
&+ 4x^kv^j\epsilon^{qij})\overset{(5)}{J}{}^{qk} - \frac{1}{9}(2U\epsilon^{qik} + 2x^jU^{,i}\epsilon^{qjk} - 4x^jU^{,k}\epsilon^{qij} + X^{,ij}\epsilon^{qjk} - 4X^{,jk}\epsilon^{qij} - 4x^lX^{,ijk}\epsilon^{qlj})\overset{(4)}{J}{}^{qk} \\
&+ \frac{2}{9}(4v^jv^k\epsilon^{qij} - v^2\epsilon^{qik})\overset{(4)}{J}{}^{qk} - \frac{4}{9}x^jU^{,k}\epsilon^{qjk}\overset{(4)}{J}{}^{qi} + \frac{2}{9}\dot{U}\epsilon^{qik}\overset{(3)}{J}{}^{qk} - \frac{2}{9}(v^iU^{,j} + 2v^jU^{,i} + V^{j,i} + \dot{X}^{,ij})\epsilon^{qjk}\overset{(3)}{J}{}^{qk} \\
&- \frac{1}{840}x^i\overset{(7)}{I}{}^{jjkk} + \frac{1}{35}x^j\overset{(7)}{I}{}^{ijkk} + \frac{1}{40}v^i\overset{(6)}{I}{}^{jjkk} + \frac{1}{24}U^{,i}\overset{(5)}{I}{}^{jjkk} - \frac{1}{30}x^j\epsilon^{qij}\overset{(6)}{J}{}^{qkk} - \frac{1}{15}x^j\epsilon^{qik}\overset{(6)}{J}{}^{qjk} \\
&+ \frac{1}{15}v^j\left(\epsilon^{qjk}\overset{(5)}{J}{}^{qik} - \epsilon^{qik}\overset{(5)}{J}{}^{qjk} - \epsilon^{qij}\overset{(5)}{J}{}^{qkk}\right) - \frac{1}{30}x^i\overset{(5)}{\mathcal{M}}{}^{kkjj} - \frac{1}{15}x^j\overset{(5)}{\mathcal{M}}{}^{kkij} - \frac{1}{6}v^i\overset{(4)}{\mathcal{M}}{}^{kkjj} + \frac{2}{3}v^j\overset{(4)}{\mathcal{M}}{}^{ijkk} \\
&+ \frac{1}{6}U^{,i}\overset{(3)}{\mathcal{M}}{}^{ijkk} + \frac{2}{3}U^{,j}\overset{(3)}{\mathcal{M}}{}^{ijkk} - \frac{1}{3}X^{,ijk}\overset{(3)}{\mathcal{M}}{}^{jkll},
\end{aligned} \tag{D4}$$

where the relevant potentials are given in Eqs. (2.10) and the multipole moments are given in Appendix C.

APPENDIX E: VIRIAL RELATIONS

Virial relations are statements about the internal structure of bound bodies that may be used to simplify the equations of motion. They assume that the body is either stationary or periodic over a suitable time scale. The simplest such virial relation, used in classical mechanics, states that $\ddot{I}/2 = 2T + \Omega$, where $I = \int \rho^* |\bar{x}|^2 d^3x$ is the scalar moment of inertia of the body, T is the internal kinetic energy, and Ω is the internal gravitational potential energy. For a stationary body, or when averaged over several internal time scales, we can set $\ddot{I} = 0$, and hence $2T + \Omega = 0$. Here we generalize this to a variety of tensorial virial relations, including spin effects and effects at 2.5 PN order.

We define the tensorial moment of inertia of body A by

$$I_A^{ij} \equiv \int_A \rho^* \bar{x}^i \bar{x}^j d^3x. \tag{E1}$$

Taking a time derivative and setting it to zero yields

$\dot{I}_A^{ij} = 2\int_A \rho^* \bar{x}^{(i}\bar{v}^{j)} d^3x = 0$. But from this and Eq. (2.5), we can conclude that

$$\int_A \bar{x}^i \bar{v}^j d^3x = \frac{1}{2} S_A^{ij}. \tag{E2}$$

A second time derivative gives $\ddot{I}_A^{ij} = 2\int_A \rho^* \bar{v}^i \bar{v}^j d^3x + 2\int_A \rho^* \bar{x}^{(i} a^{j)} d^3x$. For a^j, we substitute only the Newtonian, post-Newtonian and 2.5 PN terms from Eqs. (2.7) and (2.8). In the equations of motion, these virial relations will be needed only to simplify PN terms, so for this application, only the Newtonian and 2.5 PN terms will be needed. On the other hand, in the expression $E = \int_{\mathcal{M}} \tau^{00} d^3x$ for the conserved total energy to PN order, the virial relation will be needed to simplify the Newtonian contribution, and thus the 1 PN spin contributions to the virial relation will be needed. Expanding potentials about the center of mass of each body as described in Sec. II B, keeping PN and 2.5 PN spin terms and 2.5 PN internal terms, discarding terms that vanish as R^2 and higher as the body's size tends to zero, and setting time derivatives of the moment of inertia tensor to zero, we obtain the virial relation for body 1:

$$2T_1^{ij} + \Omega_1^{ij} = \frac{m_2}{r^2}(2n^k S_1^{k(i}(v_1^{j)} - v_2^{j)}) + n^{(i} S_1^{j)k}(v_1^k - 2v_2^k)) + S_1^{k(i} \overset{(4)}{I}{}^{j)k} - \frac{1}{3}\Omega_1^{ij} \overset{(3)}{I}{}^{kk} - 3\Omega_1^{ijkl} \overset{(3)}{I}{}^{kl}, \tag{E3}$$

The relevant internal quantities used here are defined below. A third derivative of I_A^{ij} gives $\dddot{I}_A^{ik} = 2\int_A \rho^*(3\bar{v}^{(i} a^{j)} + \bar{x}^{(i}\dot{a}^{j)}) d^3x$. The same procedure yields, for body 1,

$$2\mathcal{H}_1^{(ij)} - \frac{3}{2}\mathcal{K}_1^{ij} = -\frac{3}{2}\frac{m_2}{r^3} S_1^{k(i} n^{j)k} + \frac{1}{5} S_1^{k(i} \overset{(5)}{I}{}^{j)k} - 3\frac{m_2}{r^3} n^k n^{(i}(S_1^{j)l} - \frac{5}{2} S_1^{j)m} n^m n^l) \overset{(3)}{I}{}^{kl} + 2\Omega_1^{k(i} \overset{(4)}{I}{}^{j)k} - \frac{1}{6}\Omega_1^{ij} \overset{(4)}{I}{}^{kk} - \frac{3}{2}\Omega_1^{ijkl} \overset{(4)}{I}{}^{kl} - \frac{1}{2}(12\mathcal{K}_1^{(ij)kl} + 6\mathcal{K}_1^{klij} - 15\mathcal{K}_1^{*ijkl}) \overset{(3)}{I}{}^{kl}, \tag{E4}$$

where we have used the lowest-order virial relation from Eqs. (E3) and (E4), $2T_1^{ij} + \Omega_1^{ij} = 0$, and $2\mathcal{H}_1^{(ij)} - \frac{3}{2}\mathcal{K}_1^{ij} = -\frac{3}{2}\frac{m_2}{r^3} S_1^{k(i} n^{j)k}$ to simplify some of the 2.5 PN-order terms in Eq. (E4). The relevant quantities used in these virial relations are defined by

$$\begin{gathered}
\mathcal{T}_A^{ij} \equiv \frac{1}{2}\int_A \rho^* \bar{v}^i \bar{v}^j d^3x, \quad \Omega_A^{ij} \equiv -\frac{1}{2}\int_A\int_A \rho^* \rho^{*\prime} \frac{(x-x')^i (x-x')^j}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x', \\
\Omega_A^{ijkl} \equiv -\frac{1}{2}\int_A\int_A \rho^*\rho^{*\prime} \frac{(x-x')^i(x-x')^j(x-x')^k(x-x')^l}{|\mathbf{x}-\mathbf{x}'|^5} d^3x d^3x', \quad \mathcal{H}_A^{ij} \equiv \int_A\int_A \rho^*\rho^{*\prime} \frac{v'^i (x-x')^j}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x', \\
\mathcal{K}_A^{ij} \equiv \int_A\int_A \rho^*\rho^{*\prime} \frac{\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x-x')^i(x-x')^j}{|\mathbf{x}-\mathbf{x}'|^5} d^3x d^3x', \quad \mathcal{K}_A^{ijkl} \equiv \int_A\int_A \rho^*\rho^{*\prime} \frac{v'^i(x-x')^j(x-x')^k(x-x')^l}{|\mathbf{x}-\mathbf{x}'|^5} d^3x d^3x', \\
\mathcal{K}_A^{*ijkl} \equiv \int_A\int_A \rho^*\rho^{*\prime} \frac{\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x-x')^i(x-x')^j(x-x')^k(x-x')^l}{|\mathbf{x}-\mathbf{x}'|^7} d^3x d^3x'.
\end{gathered} \tag{E5}$$

APPENDIX F: EXTRACTING TOTAL TIME DERIVATIVES

Using the Newtonian equations of motion plus the 1 PN spin-orbit terms, it is straightforward to establish a number of identities, which may be used to extract time derivatives from 2.5 PN and 3.5 PN terms. For any non-negative integers s, p and q, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left\{pv^4 - (p+q)v^2\dot{r}^2 - 2s\dot{r}^2\frac{m}{r} - pv^2\frac{m}{r} + \frac{p}{2}\frac{v^2}{r^3}\tilde{\mathbf{L}}_{\rm N}\cdot(4\mathcal{S}+3\boldsymbol{\xi})\right\}, \\
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\tilde{\mathbf{L}}_{\rm N}\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left\{\left[pv^4 - (p+q)v^2\dot{r}^2 - 2s\dot{r}^2\frac{m}{r} - pv^2\frac{m}{r} + \frac{p}{2}\frac{v^2}{r^3}\tilde{\mathbf{L}}_{\rm N}\cdot(4\mathcal{S}+3\boldsymbol{\xi})\right]\tilde{\mathbf{L}}_{\rm N} \right. \\
&\quad \left. - \frac{v^2\dot{r}}{r}\mathbf{n}\times\left(\left[\mathbf{v} - \frac{3}{2}\dot{r}\mathbf{n}\right]\times(4\mathcal{S}+3\boldsymbol{\xi})\right)\right\}.
\end{aligned} \tag{F1}$$

Another set of identities, to be used only in 3.5 PN terms, require only the Newtonian equations of motion:

$$\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^{p}}{r^{q}}x^{i}x^{j}\right)=\frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left\{\left[pv^{4}-(p+q)v^{2}\dot{r}^{2}-2s\dot{r}^{2}\frac{m}{r}-pv^{2}\frac{m}{r}\right]x^{i}x^{j}+2v^{2}\dot{r}rx^{(i}v^{j)}\right\},$$
$$\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^{p}}{r^{q}}v^{i}v^{j}\right)=\frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left\{\left[pv^{4}-(p+q)v^{2}\dot{r}^{2}-2s\dot{r}^{2}\frac{m}{r}-pv^{2}\frac{m}{r}\right]v^{i}v^{j}-2m\frac{v^{2}\dot{r}}{r^{2}}x^{(i}v^{j)}\right\},\tag{F2}$$
$$\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^{p}}{r^{q}}x^{i}v^{j}\right)=\frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left\{\left[pv^{4}-(p+q)v^{2}\dot{r}^{2}-2s\dot{r}^{2}\frac{m}{r}-pv^{2}\frac{m}{r}\right]x^{i}v^{j}+v^{2}\dot{r}r\left(v^{i}v^{j}-\frac{m}{r}n^{i}n^{j}\right)\right\}.$$

APPENDIX G: TOTAL ENERGY AND ANGULAR MOMENTUM

In this appendix, we express the global definitions of mass, momentum, angular momentum and center of mass in a PN expansion, including spin contributions. In paper I, we defined the ‘‘source moments’’

$$P^{\mu}\equiv\int_{\mathcal{M}}\tau^{\mu 0}d^{3}x,\tag{G1a}$$
$$\mathcal{I}^{Q}\equiv\int_{\mathcal{M}}\tau^{00}x^{Q}d^{3}x,\tag{G1b}$$
$$\mathcal{J}^{iQ}\equiv\epsilon^{iab}\int_{\mathcal{M}}x^{aQ}\tau^{0b}d^{3}x,\tag{G1c}$$
$$\mathcal{M}^{ijQ}\equiv\int_{\mathcal{M}}\tau^{ij}x^{Q}d^{3}x,\tag{G1d}$$

where the capitalized superscript Q denotes a multi-index ($x^{Q}=x^{i_1}x^{i_2}\ldots x^{i_q}$). The integrals are to be taken over a constant-time hypersurface $\mathcal{M}$, which lies within the near zone, i.e. within one gravitational wavelength of the sources. The quantities P^0 and P^i are naturally interpreted as the total mass-energy and momentum of the system, respectively, $I^i\equiv\mathcal{I}^i(q=1)$ is the system dipole moment, which relates the center of mass to the origin of coordinates, and $J^i\equiv\mathcal{J}^i(q=0)$ is the total angular momentum. These quantities are conserved up to changes induced by a flux of gravitational radiation to infinity, that is,

$$\dot{P}^{\mu}=-\oint_{\partial\mathcal{M}}\tau^{\mu j}d^{2}S_{j},\qquad \dot{J}^{i}=-\epsilon^{iab}\oint_{\partial\mathcal{M}}\tau^{jb}x^{a}d^{2}S_{j},$$
$$\dot{I}^{i}=P^{i}-\oint_{\partial\mathcal{M}}\tau^{0j}x^{i}d^{2}S_{j}.\tag{G2}$$

where $\partial\mathcal{M}$ denotes the boundary of $\mathcal{M}$. Thus, these quantities are conserved and well defined only up to and including 2 PN order.

We now wish to evaluate these conserved quantities explicitly to 1 PN order for a many-body system. Explicit expressions for $\tau^{\alpha\beta}$ through the relevant order may be obtained from Eqs. (5.7) and (5.9a) of paper I, together with Eqs. (2.21) of paper II which serve to convert expressions to dependence on ρ^*. They are given by

$$\tau^{00}=\rho^{*}\left(1+\frac{1}{2}v^{2}-\frac{1}{2}U+\frac{3}{8}v^{4}+v^{2}U+\frac{3}{2}U^{2}+\frac{1}{2}\Phi_{1}-\Phi_{2}-2v^{j}V^{j}+\frac{1}{4}v^{j}\dot{X}^{,j}\right),\tag{G3a}$$
$$\tau^{0j}=\rho^{*}v^{j}\left(1+\frac{1}{2}v^{2}-U\right)-\frac{1}{2}\dot{X}^{,j}+\frac{1}{4\pi}\nabla^{k}\left(8UV^{[k,j]}+4U\dot{U}\delta^{jk}-\dot{X}^{,(j}U^{,k)}+\frac{1}{2}\delta^{jk}\dot{X}^{,m}U^{,m}\right),\tag{G3b}$$
$$\tau^{ij}=\rho^{*}v^{i}v^{j}+\frac{1}{4\pi}\left(U^{,i}U^{,j}-\frac{1}{2}\delta^{ij}\nabla U^{2}\right),\tag{G3c}$$

where we have extracted total divergences of functions, such as $\nabla\cdot(U\nabla U)$ from the expression for τ^{00}, since they will not affect the integral quantities P^0 and I^i, to the orders considered.

Expanding potentials about the baryonic center of mass of each body as described in Sec. II B, keeping spin terms and other internal terms, but discarding terms that vanish as R^2 and higher as the body's size tends to zero, and applying the virial relations of Appendix E, we obtain the same expressions (2.19) for E, $\mathbf{J}$, and $\mathbf{I}$, as we found from the equations of motion.

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