

**Naked shell singularities on the brane**

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By utilizing nonstandard slicings of 5-dimensional Schwarzschild and Schwarzschild-AdS manifolds based on isotropic coordinates, we generate static and spherically-symmetric braneworld spacetimes containing shell-like naked null singularities. For planar slicings, we find that the brane-matter sourcing the solution is a perfect fluid with an exotic equation of state and a pressure singularity where the brane crosses the bulk horizon. From a relativistic point of view, such a singularity is required to maintain matter infinitesimally above the surface of a black hole. From the point of view of the AdS/CFT conjecture, the singular horizon can be seen as one possible quantum correction to a classical black hole geometry. Various generalizations of planar slicings are also considered for a Ricci-flat bulk, and we find that singular horizons and exotic matter distributions are common features.

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**I. INTRODUCTION**

Recent advances in nonperturbative string theory have raised the prospect that our universe is a 4-dimensional hypersurface (brane) embedded within some higher-dimensional manifold with large extra dimensions. A phenomenological 5-dimensional realization of this idea was proposed by Randall & Sundrum (RS) in 1999 [1], which involved one or two 4-dimensional Minkowski branes embedded in an anti-de Sitter “bulk” 5-manifold (AdS<sub>5</sub>). One of the most attractive features of this “braneworld” model is the fact that the 5-dimensional graviton zero mode is sharply confined near the “visible brane” representing our universe, implying that the force of gravity has the appropriate Newtonian behavior at large distances. This automatically makes the one-brane model in excellent agreement with most astrophysical tests of general relativity in the weak gravity regime.

But this virtue is also somewhat of a detriment, because we must turn to strong gravity phenomena in order to test the model, and thereby the stringy ideas that motivated it. The appropriate formalism to deal with nontrivial curvature in the braneworld was developed by Shiromizu *et al.* [2] shortly after the RS model first appeared. They obtained an effective 4-dimensional Einstein equation that was in part sourced by the (traceless) projection  $\mathcal{E}_{\mu\nu}$  of the bulk Weyl tensor onto the brane. But this tensor did not come with a brane-based equation of motion, which means that the 4-dimensional effective theory is not closed—one needs to know about the geometry of the bulk to fully specify the dynamics of the brane. If one insists on using a purely brane-based formalism, the precise form of  $\mathcal{E}_{\mu\nu}$  is somewhat arbitrary.

It turns out this ambiguity is not a big problem for braneworld cosmology. If one has a cosmological brane which retains a Friedmann-Robertson-Walker (FRW) form

for all time, it follows that the bulk spacetime shares the same symmetries; i.e., the bulk is the product of  $\mathbb{R}^2$  with a maximally symmetric 3-space, and is sourced by a negative cosmological constant. Under such circumstances, the 5-dimensional version of Birkhoff’s theorem states that the bulk is necessarily isometric to the 5-dimensional Schwarzschild-anti-deSitter (AdS<sub>5</sub>) solution. This forces  $\mathcal{E}_{\mu\nu}$  to take the form of the stress-energy tensor of a cosmological radiation field whose amplitude is controlled by the mass of the bulk black hole.

But there is another strong gravity phenomenon that is at least as important as cosmology, namely, black holes. Spherically-symmetric black hole 4-metrics have fewer symmetries than their FRW counterparts, which implies that the bulk geometry is not nearly as constrained as it is for braneworld cosmology. In turn, this means that  $\mathcal{E}_{\mu\nu}$  is undetermined by simply specifying that the brane is spherically symmetric and devoid of matter. Stated in another way, there is no 4-dimensional Birkhoff uniqueness theorem for braneworld black holes; the bulk Weyl contribution acts as an arbitrary effective source.

Hence, there are many possible candidates for the “right” model of a braneworld black hole. One way to get at them is to set the matter content of the brane to zero and fine tune its tension, which makes the effective brane field equation  ${}^{(4)}R_{\mu\nu} = -\mathcal{E}_{\mu\nu}$ . This has been solved under spherically-symmetric conditions by a number of authors, but they all had to assume something about the form of  $\mathcal{E}_{\mu\nu}$ . For example, there is the tidal Reissner-Nordström solution of Dadhich *et al.* [3], or the line elements of Gregory *et al.* [4] that assume an equation of state for the “Weyl fluid.” A different line of attack comes from trying to solve the (scalar) field equation  ${}^{(4)}R = \text{constant}$  [5,6], which comes from the contracted Gauss-Codazzi equations. Recently, the so-called “gradient-expansion” method has been applied to problem in an effort to systematically include effects of the extra dimension on the brane metric [7]. Several workers have also looked at

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dynamical case of gravitational collapse on the brane, and have come to the conclusion that the exterior brane metric to a collapsing star cannot be static [8] and in some cases is not even a vacuum [9]. The cumulative effect of these efforts has been to create a veritable zoo of black hole candidates, some of which have reasonable physical properties.

But all of these models are somewhat unsatisfactory because of ignorance of the bulk geometry. One does not know if any singularities present extend off the brane, or the nature and shape of any 5-dimensional horizons. Hence, one cannot study the thermodynamics of such objects. Perturbations of these geometries are also ill-defined because of the under-determined nature of the effective theory. This means that we cannot address the stability of these models nor their gravity wave signatures, which may be an important observational test of extra dimensions [10]. It is in theory possible to obtain the bulk geometry by evolving the 4-metric off the brane, but the problem is analytically complicated [11,12] and robust numerical progress can only be made for “small” black holes [13–15].

However, there is at least one credible alternative to these brane-based approaches. Instead of trying to deal with effective field equations, one can take known 5-dimensional solutions and identify branes as slices embedded therein. The most successful example of this procedure is actually 4-dimensional. Emparan *et al.* [16] considered a simple slicing of the 4-dimensional C-metric. The slice had an extrinsic curvature proportional to its induced metric, implying a pure tension brane, and a (2 + 1)-dimensional black hole intrinsic geometry. Furthermore, the bulk was entirely regular. But unfortunately, there currently is no 5-dimensional generalization of the C-metric that allows the same construction for a (3 + 1)-dimensional brane black hole, despite concerted efforts to find one [17]. A different possibility for the bulk manifold is the 5-dimensional black string solution, which is a simple warped-product model where the brane metric is precisely Schwarzschild [18,19]. Unfortunately in a one-brane model, this metric is subject to the well-known long-wavelength Gregory-Laflamme (GL) instability [20]; however, one can engineer a two-brane scenario where the GL instability is cut off [10].

But why has it been so hard to find a brane localized black hole solution in 5 dimensions? Separately, Tanaka [21] and Emparan *et al.* [22] have conjectured that the reason has to do with the AdS/CFT correspondence ([23] and references therein), which states that the dynamics of an  $AdS_n$  manifold are formally dual to behavior of an  $(n - 1)$ -dimensional conformal field theory (CFT) living on its boundary. The authors noted that in the 4-dimensional model of Ref. [16], the  $\mathcal{E}_{\mu\nu}$  part of the brane’s effective stress-energy tensor took the form a quantum-corrected (2 + 1)-black hole. That is, the solution on the boundary

of  $AdS_4$  was derived from the backreaction of a quantum field on the classical lower-dimensional black hole geometry. Extending the logic to one dimension higher, we are led to believe that the (3 + 1)-braneworld black hole ought to take the form of Schwarzschild subject to quantum corrections [24]. The precise form of the correction depends on the choice made for the quantum vacuum. One possibility has the black hole radiating its mass away via the Hawking effect (which is what is conventionally regarded as the end-state of gravitational collapse on the brane) another involves the black hole in thermal equilibrium with a heat bath at infinity. Yet another choice yields a static configuration with a singularity where the horizon used to be. We now see the difficulty in finding the 4-dimensional brane black hole; all of these possibilities represent significant departures from the canonical Schwarzschild geometry.

The purpose of this paper is to develop spherically-symmetric and static braneworld models using methods inspired from the successful construction of 2-brane localized black holes. In particular, we will be considering various slicings of 5-dimensional black hole metrics, both with and without a negative cosmological constant  $\Lambda_5 = -6/\ell^2$ . We work in isotropic coordinates, which are developed in Sec. II. In Sec. III, we study the simplest possible braneworlds based on a planar slicing of the 5-manifold through the event horizon. The basic methodology is similar to the 4-dimensional “displace-cut-reflect” procedure for constructing thin-disk solutions to the Einstein equations [25]. Intriguingly, we find that brane 4-geometry involves a naked shell-singularity for all cases we consider; i.e., with  $\Lambda_5 \leq 0$ . This is as expected from the AdS/CFT considerations mentioned above. Because the planar slicing is selected on purely geometric grounds, the extrinsic curvature and matter content of the braneworlds is not freely specifiable, it is rather forced upon us. We find that the models are supported by a nontrivial perfect fluid with a pressure singularity where the brane intersects the 5-dimensional horizon. Such a singularity could have been predicted on physical grounds: One requires an infinite amount of force to keep matter suspended infinitesimally above the surface of a black hole. Hence, an infinite pressure gradient is needed to keep the brane-matter static. Finally, in Sec. IV we consider quite general nonplanar slices. These include slicings with vanishing Ricci scalar, radial pressure, tangential pressure, and extrinsic curvature, as well as slicings with isotropic pressure and pure tension branes. In the last case, the only solution we find corresponds to an Einstein-static universe coincident with the photon sphere of the bulk black hole. Sec. V is reserved for conclusions and final comments.

*Conventions*—We employ the “mostly positive” metric signature. Lowercase Latin indices run from 0 to 4 and lowercase Greek indices run from 0 to 3. Metric compatible covariant derivatives on 5-manifolds are denoted by  $\nabla_a$ ; while on 4-dimensional submanifolds (3-branes) they are denoted by  $\nabla_\alpha$ .

## II. TRANSFORMATIONS FROM SPHERICAL TO ISOTROPIC COORDINATES

The purpose of this section is to describe how isotropic coordinates can be constructed for a certain class of spherically-symmetric manifolds in an arbitrary number of dimensions, and to derive explicit coordinate transformations for two special 5-dimensional cases. These special cases will be used in the next section to construct brane-world shell solutions associated with vacuum and Schwarzschild-AdS bulk manifolds, respectively.

### A. General transformations for a class of $(d+2)$ -dimensional spherically-symmetric manifolds

We begin by considering a fairly wide  $(d+2)$ -dimensional class of spherically-symmetric manifolds  $(M, g)$  whose line element can be expressed as

$$ds_{(M)}^2 = -f(R)dt^2 + f^{-1}(R)dR^2 + R^2d\Omega_d^2, \quad (1)$$

where  $d\Omega_d^2$  is the interval on a unit  $d$ -sphere. Our goal is to find a coordinate transformation that puts this in the isotropic form

$$ds_{(M)}^2 = -H(\rho)dt^2 + G(\rho)[d\rho^2 + \rho^2d\Omega_d^2]. \quad (2)$$

This line element is called isotropic because a further simple coordinate transformation yields

$$ds_{(M)}^2 = -H(\rho)dt^2 + G(\rho) \sum_{i=1}^{d+1} dx_i^2, \quad (3)$$

with  $\rho = \sqrt{x_1^2 + x_2^2 + \dots + x_{d+1}^2}$ . In these coordinates each of the spatial directions is on the same footing, hence the moniker ‘‘isotropic.’’

It is easy to see that the coordinate transformation from (1) to (2) must satisfy

$$\left(\frac{dR}{d\rho}\right)^2 = f(R)G(\rho), \quad R^2 = G(\rho)\rho^2. \quad (4)$$

This set of equations is solved by

$$\rho(R) = \exp \int_{R_0}^R \frac{du}{\sqrt{u^2 f(u)}}, \quad (5)$$

which must be inverted to obtain  $R = R(\rho)$ . Here,  $R_0$  is some fiducial lower limit of integration that enforces  $\rho(R_0) = 1$ . Assuming that such an inversion is possible, we have the following implicit representations of the isotropic metric functions:

$$G(\rho) = \frac{R^2(\rho)}{\rho^2}, \quad H(\rho) = f(R(\rho)). \quad (6)$$

Hence the required coordinate transformation is found.

We make two comments before proceeding: First, it is straightforward to confirm that if we adopt the familiar 4-dimensional Schwarzschild solution with  $d = 2$  and

$f(R) = 1 - 2M/R$ , we obtain the usual isotropic coordinate patch found in standard textbooks. Second, we note that the integral in (5) is complex if  $f(R) < 0$  anywhere in the interval  $[R_0, R]$ . Therefore, if the line element (1) represents a black hole manifold, then the isotropic coordinate patch can only be used to cover the portion outside the horizon; i.e., the part of the manifold with  $f > 0$ . We will return to this point below.

### B. The 5-dimensional Schwarzschild black hole in isotropic coordinates

We now turn our attention to the 5-dimensional Schwarzschild black hole. Usually, this is expressed as

$$ds_{(M)}^2 = -\left(1 - \frac{R_0^2}{R^2}\right)d\mathcal{T}^2 + \left(1 - \frac{R_0^2}{R^2}\right)^{-1}dR^2 + R^2d\Omega_3^2, \quad (7)$$

Here,  $R_0$  represents the position of the black hole horizon and is also related to the ADM mass of the central object. For our purposes, it is useful to adopt dimensionless radial and time coordinates by making the changes  $\mathcal{R} \rightarrow \mathcal{R}_0 \times R$  and  $\mathcal{T} \rightarrow \mathcal{R}_0 \times t$ . If this is accompanied by a simultaneous scaling of the interval  $ds_{(M)}^2 \rightarrow \mathcal{R}_0^2 \times ds_{(M)}^2$ , we have the line element in the standard form (1) with

$$f(R) = 1 - 1/R^2, \quad d = 3. \quad (8)$$

Notice that when we are working in dimensionless coordinates, there are no freely specifiable parameters in the solution, and the horizon is always at  $R = 1$ .

By application of the formula (5) with  $R_0$  set to unity—as dictated by the horizon position in these coordinates—we obtain the transformations

$$\rho = R + \sqrt{R^2 - 1}, \quad R = \frac{\rho^2 + 1}{2\rho}. \quad (9)$$

From these, it is clear that the  $\rho$  coordinate is only well defined for  $R > 1$ ; i.e., outside the black hole horizon. The explicit form of the isotropic metric functions is

$$H(\rho) = \left(\frac{\rho^2 - 1}{\rho^2 + 1}\right)^2, \quad (10a)$$

$$G(\rho) = \left(\frac{\rho^2 + 1}{2\rho^2}\right)^2. \quad (10b)$$

In order to check our work, we have confirmed by direct calculation of the Einstein tensor that these metric functions represent a 5-dimensional vacuum solution. Intriguingly, they provide a solution for all  $\rho$ , not just  $\rho > 1$ —this will be important later on. It is also interesting to note that the Killing vector  $\partial_t$  in these coordinates becomes null at  $\rho = 1$ , but is nowhere spacelike. This is a direct affirmation of our previous conclusion that the isotropic coordinates do not cover the region inside the horizon, which is characterized by  $\partial_t \cdot \partial_t > 0$ .

### C. The 5-dimensional Schwarzschild-AdS black hole in isotropic coordinates

Moving on, we come to the case of a 5-dimensional black hole sourced by a negative cosmological constant; i.e., the Schwarzschild-AdS 5-manifold (AdS<sub>5</sub>). The conventional form of such a solution is

$$ds_{(M)}^2 = -\mathcal{F}d\mathcal{T}^2 + \mathcal{F}^{-1}d\mathcal{R}^2 + \mathcal{R}^2d\Omega_3^2, \quad (11a)$$

$$\mathcal{F} = \mathcal{F}(\mathcal{R}) = 1 - \frac{\mathcal{R}_0^2}{\mathcal{R}^2} + \frac{\mathcal{R}^2}{\ell^2}. \quad (11b)$$

Here,  $\mathcal{R}_0$  is again related to the ADM mass of the black hole while  $\ell$  is related to the (negative) cosmological constant. It is convenient to rewrite  $\mathcal{F}$  as

$$\mathcal{F} = \frac{(\mathcal{R}^2 + \mathcal{R}_+^2)(\mathcal{R}^2 - \mathcal{R}_-^2)}{\mathcal{R}^2\ell^2}, \quad (12)$$

where

$$\mathcal{R}_\pm^2 = \frac{\ell^2}{2} \left( \sqrt{\frac{4\mathcal{R}_0^2}{\ell^2} + 1} \pm 1 \right). \quad (13)$$

When the solution is written in this way, it is apparent that there is an event horizon at  $\mathcal{R} = \mathcal{R}_-$ . We again wish to make use of dimensionless coordinates, this time defined by the substitutions:

$$\begin{aligned} \mathcal{R} &\rightarrow R \times \mathcal{R}_-, & \mathcal{T} &\rightarrow t \times \mathcal{R}_-, \\ ds_{(M)}^2 &\rightarrow ds_{(M)}^2 \times \mathcal{R}_-^2. \end{aligned} \quad (14)$$

We must also define new parameters as follows:

$$\gamma = \frac{\ell}{\mathcal{R}_-}, \quad a_\gamma \equiv \frac{\mathcal{R}_+}{\mathcal{R}_-} = \sqrt{\gamma^2 + 1}. \quad (15)$$

With these manipulations, the S – AdS<sub>5</sub> line element can be expressed in the form of Eq. (1) with

$$f(R) = \frac{(R^2 + a_\gamma^2)(R^2 - 1)}{\gamma^2 R^2}, \quad d = 3, \quad (16)$$

and  $t$  and  $R$  as dimensionless coordinates. As in the vacuum example discussed in the previous subsection, the horizon is located at  $R = 1$ ; but unlike the Schwarzschild case, there is an adjustable parameter in the dimensionless solution, namely  $\gamma$ .

We now obtain the S – AdS<sub>5</sub> line element in isotropic coordinates. The first step is to put the S – AdS<sub>5</sub> expression for  $f$  into our general expression for  $\rho(R)$ ; i.e., Eq. (5). We again set the lower limit of integration  $R_0$  as the position of the horizon at  $R = 1$ . This results in

$$\rho(R) = \exp \left[ \frac{1}{\xi_\gamma} F \left( \sqrt{1 - \frac{1}{R^2}}, s_\gamma \right) \right]. \quad (17)$$

Here,

$$s_\gamma \equiv \sqrt{\frac{\gamma^2 + 1}{\gamma^2 + 2}}, \quad \xi_\gamma \equiv \sqrt{\frac{\gamma^2 + 2}{\gamma^2}}, \quad (18)$$

and  $F$  is an incomplete elliptic integral of the first kind, defined by<sup>1</sup>

$$F(z, k) \equiv \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \quad (19)$$

One comment about this coordinate transformation is in order: the limit of  $\rho(R)$  as  $R \rightarrow \infty$  is a constant value, namely

$$\rho_{\max} = \exp \left[ \frac{K(s_\gamma)}{\xi_\gamma} \right], \quad (20)$$

where  $K(k) = F(1, k)$  is a complete elliptic integral of the first kind. So, the  $R \rightarrow \rho$  transformation maps the semi-infinite interval  $R \in (1, \infty)$  onto some finite region  $\rho \in (1, \rho_{\max})$ . This is unlike the vacuum case above, since Eq. (9) implies that  $\rho \rightarrow 2R$  as  $R \rightarrow \infty$ .

The above expression (17) for  $\rho$  as a function of  $R$  is indeed invertible with the aid of the Jacobi sn and cn functions, which are implicitly defined by

$$\text{sn}(F(z, k), k) = z, \quad (21)$$

and

$$\text{cn}(z, k) \equiv \cos\{\arcsin[\text{sn}(z, k)]\}. \quad (22)$$

In many respects, these behave like the familiar trigonometric sine and cosine functions —, in particular, they are periodic in their first argument. The old radius  $R$  as a function of the isotropic radius  $\rho$  is then given by

$$R(\rho) = \text{nc}(\varphi(\rho), s_\gamma), \quad (23)$$

where  $\text{nc}(z, k) = 1/\text{cn}(z, k)$  and we have defined

$$\varphi(\rho) \equiv \xi_\gamma \ln \rho. \quad (24)$$

The periodic nature of the nc function in  $R(\rho)$  means that we should restrict  $\rho$  to lie within some finite interval in order to have a sensible coordinate transformation—however, this is no surprise because we have already determined from Eq. (17) that  $\rho(R) \in (1, \rho_{\max})$  for  $R \in (1, \infty)$ .

Finally, the isotropic metric functions  $G$  and  $H$  are easily found from Eqs. (6), (16), and (23):

<sup>1</sup>The reader should be wary of a common alternative definition of  $F$ , namely

$$F(z, m) \equiv \int_0^{\text{sn}z} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}.$$

Our definition (19) matches the one found in the MAPLE symbolic computation software.

$$G(\rho) = \frac{nc^2(\varphi(\rho), s_\gamma)}{\rho^2}, \quad (25a)$$

$$H(\rho) = \frac{sc^2(\varphi(\rho), s_\gamma) + a_\gamma^2 sn^2(\varphi(\rho), s_\gamma)}{\gamma^2}, \quad (25b)$$

where the Jacobi  $sc$  function is defined like a tangent; i.e.,  $sc(z, k) \equiv sn(z, k)/cn(z, k)$ . We note  $sn(0, k) = sc(0, k) = 0$  for all  $k$  and  $\varphi(1) = 0$ , therefore  $H(1) = 0$ . That is, at  $\rho = 1$  the Killing vector  $\partial_t$  becomes null. Elsewhere,  $H(\rho)$  is explicitly non-negative, which again confirms that the isotropic coordinates only cover the region outside the black hole horizon with  $\partial_t \cdot \partial_t < 0$ . Again, we have confirmed by direct computation that the above metric functions solve the 5-dimensional field equations:

$$G_{ab} = \frac{6}{\gamma^2} g_{ab}, \quad (26)$$

for all  $\rho$ .

### III. BRANEWORLDS FROM PLANAR SLICINGS OF ISOTROPIC CHARTS

In the previous section, we developed isotropic coordinate patches for a fairly wide class of spherically-symmetric manifolds and for two special 5-dimensional cases. We now attempt to generate braneworld models from these special cases by considering their planar slicings, first for the purely Schwarzschild bulk spacetime and then for the  $S - \text{AdS}_5$  manifold. While the latter is more technically complicated than the former, we will see that the basic physics associated with both cases is remarkably similar.

Before moving on to the particular cases, we comment on the general algorithm that we will employ. The basic strategy for the construction of braneworlds from bulk manifolds covered by isotropic coordinates is the same as the 4-dimensional “displace-cut-reflect” procedure for constructing thin-disk solutions to the Einstein equations [25]. The key is expressing the isotropic line element (2) as

$$ds_M^{(2)} = -H(\rho)dt^2 + G(\rho)[dr^2 + r^2 d\Omega_{d-1}^2 + dw^2], \quad (27)$$

where  $\rho = \sqrt{r^2 + w^2}$ . This is nothing more than a generalization of cylindrical coordinates on the isotropic spatial section of (2). To generate a braneworld model, we pick one of the  $w = \text{constant}$  hypersurfaces  $\Sigma_0$  to be the brane. Naturally, the  $\Sigma_0$  hypersurface will divide the bulk into two regions, one of which we discard and replace with the mirror image of the other half. In this way, we generate a  $\mathbb{Z}_2$  symmetric braneworld model. The metric on the brane is

$$ds_{(\Sigma_0)}^2 = -H(\sqrt{r^2 + w_0^2})dt^2 + G(\sqrt{r^2 + w_0^2})[dr^2 + r^2 d\Omega_{d-1}^2], \quad (28)$$

where  $w = w_0$  is the defining equation of  $\Sigma_0$ . We see that the brane’s geometry will necessarily be static and spherically symmetric. This procedure is diagrammed in Fig. 1, where we show the case of a planar braneworld intersecting a bulk black hole horizon.

One of the features of this procedure is that we have no control over the extrinsic curvature of the  $\Sigma_0$  hypersurface; it is essentially fixed by the bulk geometry and our choice of a planar braneworld geometry. Now, recall that in the general relativistic thin-shell formalism, the matter carried by a geometric defect such as  $\Sigma_0$  is related in a direct way to its extrinsic curvature. Therefore, the matter content of our braneworld is given to us from the model, rather than being something that we have input directly into the formalism. What exactly is the nature of the matter confined to  $\Sigma_0$ ? To answer this, we need the normal to the family of  $w = \text{constant}$  hypersurfaces  $\Sigma_w$ :

$$n_a = G^{1/2}(\sqrt{r^2 + w^2})\partial_a w. \quad (29)$$

We need the projection tensor and extrinsic curvature associated with  $\Sigma_w$

$$h_{ab} = g_{ab} - n_a n_b, \quad K_{ab} = h^c{}_a \nabla_c n_b, \quad (30)$$

which leads to the following expression for the stress-energy tensor of matter on the brane:

$$\kappa_5^2 S_{ab} = -2(K_{ab} - h_{ab} \text{Tr}K). \quad (31)$$

Here, evaluation at  $w = w_0$  is understood and  $\kappa_5^2$  is the 5-dimensional gravity-matter coupling. In these coordinates,

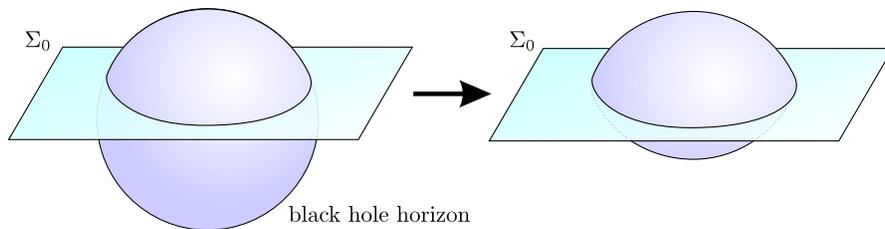


FIG. 1 (color online). An example of the displace–cut–reflect procedure when the  $\Sigma_0$  hypersurface is planar and intersects the horizon of the bulk black hole. In the picture, all but the  $r$ ,  $w$  and one of the angular coordinates in the  $S^{d-1}$  part of the metric (27) have been suppressed. Note that in this case, we have elected to retain the singularity-free half of the 5-manifold in the braneworld model.

we expect that  $S_{ww} = 0$  since  $S_{ab}n^a = 0$ . We will use this expression below to read off the properties of the brane matter in specific models.

### A. Braneworlds from a Schwarzschild bulk

We now apply our braneworld construction to the isotropic representation of the 5-dimensional Schwarzschild metric derived in Sec. II B. Using Eqs. (10) and (28), we obtain the following 4-geometry on the brane:

$$ds_{(\Sigma_0)}^2 = -\left(\frac{r^2 + w_0^2 - 1}{r^2 + w_0^2 + 1}\right)^2 dt^2 + \left[\frac{r^2 + w_0^2 + 1}{2(r^2 + w_0^2)}\right]^2 d\sigma_3^2, \quad (32)$$

where  $d\sigma_3^2 = dr^2 + r^2 d\Omega_2^2$  is the metric on flat Euclidean 3-space. We will denote the metric on  $\Sigma_0$  as  $h_{\alpha\beta}$ . This metric is static, spherically symmetric, and asymptotically flat in the  $r \rightarrow \infty$  limit. One of the first things that one notices about this metric is that if  $w_0 \in [-1, 1]$  there is a Killing horizon at  $r = \sqrt{1 - w_0^2} \equiv r_0$ —which we denote by  $\mathcal{H}$ —where the norm of  $\partial_t$  vanishes. It should be clear that  $\mathcal{H}$  is the intersection of the braneworld  $\Sigma_0$  with the 5-dimensional black hole horizon  $\rho = 1$ . Now, it is clear that the induced metric on  $\mathcal{H}$  is degenerate with signature  $(0 + +)$ , hence it is a null surface as must be true for all Killing horizons.

Another important feature of  $\mathcal{H}$  is that it is the location of a curvature singularity. To see this, consider the Kretschmann curvature scalar:

$$\mathcal{K} \equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{1024P(r, w_0)}{(r^2 + w_0^2 + 1)^8(r^2 + w_0^2 - 1)^2}, \quad (33)$$

where  $P(r, w_0)$  is a complicated 12th order polynomial in  $r$  and  $w_0$  satisfying  $P(\sqrt{1 - w_0^2}, w_0) = 12w_0^4$ . Therefore,  $\mathcal{K}$  diverges on  $\mathcal{H}$  signifying that the latter is a singular hypersurface.<sup>2</sup> Hence, in this spacetime a Killing horizon and a curvature singularity are coincident. This is an unusual, but not entirely unprecedented feature of this model. For example, one sees similar behavior in the extremal Dp-brane solutions of supergravity theory [26]. However, in the majority of those geometries the Killing horizon is also an event horizon. Is it the same true for this braneworld spacetime?

The answer is no, as we now demonstrate. The key is to show that there is a null geodesic of finite affine length that connects the singularity with arbitrary points in the exterior region. Radially outgoing geodesics in this spacetime have the following tangent vector field in the affine parametrization:

<sup>2</sup>For the moment, we exclude the  $w_0 = 0$  case.

$$k^\alpha \partial_\alpha = E\left(\frac{1}{H} \frac{\partial}{\partial t} + \frac{1}{\sqrt{GH}} \frac{\partial}{\partial r}\right), \quad k^\alpha \nabla_\alpha k^\beta = 0, \quad (34)$$

where  $E$  is the energy parameter. From this, we see that the affine length  $\Delta\lambda$  of a light ray travelling from  $\mathcal{H}$  to some  $r_1 > r_0$  is

$$\Delta\lambda = \frac{1}{E} \int_{r_0}^{r_1} \sqrt{GH} dr. \quad (35)$$

The integrand here is manifestly finite, hence  $\Delta\lambda$  is similarly finite and  $\mathcal{H}$  cannot be an event horizon.

Furthermore,  $\mathcal{H}$  is not even a trapping horizon.<sup>3</sup> To see this, we introduce the time and radial unit vectors:

$$\hat{t}^\alpha \partial_\alpha = H^{-1/2} \partial_t, \quad \hat{r}^\alpha \partial_\alpha = G^{-1/2} \partial_r. \quad (36)$$

Then, for every 2-sphere  $(t, r) = \text{constant}$  we can define vectors tangent to ingoing and outgoing radial null congruences as

$$\ell^\alpha = \frac{1}{\sqrt{2}}(\hat{t}^\alpha - \hat{r}^\alpha), \quad \tilde{k}^\alpha = \frac{1}{\sqrt{2}}(\hat{t}^\alpha + \hat{r}^\alpha), \quad (37)$$

respectively. Now, the induced metric on the 2-spheres is  $q_{\alpha\beta} = h_{\alpha\beta} + \hat{t}_\alpha \hat{t}_\beta - \hat{r}_\alpha \hat{r}_\beta$  and the expansion of the ingoing and outgoing congruences are

$$\theta_{(\ell)} = q^{\alpha\beta} \nabla_\alpha \ell_\beta, \quad \theta_{(\tilde{k})} = q^{\alpha\beta} \nabla_\alpha \tilde{k}_\beta, \quad (38)$$

respectively. Now, we want to know whether or not the 2-spheres that are the constant time slices of  $\mathcal{H}$  are apparent horizons. They will be if the outgoing expansion scalar vanishes for  $r = r_0$ . A quick calculation shows:

$$\theta_{(\tilde{k})} = \frac{rG_{,r} + 2G}{2rG^{3/2}}, \quad (39)$$

where  $G_{,r} = dG/dr$ . It is straightforward to verify that this reduces to

$$\theta_{(\tilde{k})} = \frac{w_0^2}{\sqrt{1 - w_0^2}}, \quad (40)$$

on  $\mathcal{H}$ . Since the expansion is clearly nonzero for  $w_0 \neq 0$ , we can conclude that  $\mathcal{H}$  is not a trapping horizon for such cases.

All this goes to show that when  $w_0^2 \in (0, 1]$ , we are dealing with a naked null singularity in this spacetime. Interestingly, we can find a coordinate system that is regular there. More precisely, the transformation

$$u = -e^{-(1/2)r_0(t-r_*)}, \quad v = e^{(1/2)r_0(t+r_*)}, \quad (41)$$

$$r_* = \frac{r}{2} - \frac{1}{2w_0} \arctan \frac{r}{w_0} + \frac{1}{r_0} \ln \frac{r - r_0}{r + r_0},$$

puts our metric in the form

<sup>3</sup>Defined as the world tube of a series of apparent horizons as in Ref. [27], for example.

$$\begin{aligned}
 ds_{(\Sigma_0)}^2 = & -\frac{4(r+r_0)^4 \exp(\frac{r_0}{2w_0} \arctan \frac{r}{w_0} - \frac{r_0 r}{2})}{r_0^2(r^2 - r_0^2 + 2)^2} du dv \\
 & + \left[ \frac{r(r^2 + w_0^2 + 1)}{2(r^2 + w_0^2)} \right]^2 d\Omega_2^2, \\
 r = & r(u, v).
 \end{aligned} \tag{42}$$

All the metric coefficients are well behaved at  $\mathcal{H}$ , despite the fact that there is a curvature singularity there.<sup>4</sup>

The final issue we want to address is the type of brane matter that sources this model. We can calculate the stress-energy tensor for the  $\Sigma_0$  hypersurface from the definitions leading up to Eq. (31); the result is

$$S^a_b = \text{diag}(-\epsilon, p, p, p, 0), \tag{43}$$

where

$$\kappa_5^2 \epsilon = \frac{24w_0}{(r^2 + w_0^2 + 1)^2}, \tag{44a}$$

$$\kappa_5^2 p = \frac{16w_0}{(r^2 + w_0^2 + 1)^2(r^2 + w_0^2 - 1)}. \tag{44b}$$

Therefore the brane matter admits a perfect fluid type description with energy density  $\epsilon$  and isotropic pressure  $p$ . For  $w_0 > 0$ , the density is finite and positive for all  $r$ . On the other hand, the pressure changes sign from positive to negative and diverges as  $r$  decreases across  $r = r_0$ . So, in addition to a curvature singularity at  $r = r_0$ , we also have a singularity in some of the matter properties.<sup>5</sup>

There are two comments to be made about the brane matter: The first centers around the observation that for  $w_0 < 0$ , the exterior density and pressure are both negative. The reason for this comes from an implicit assumption in our derivation; namely, we always discard the part of the bulk manifold with  $w < w_0$  when constructing our brane-world. If  $w_0 < 0$ , then there will be a 5-dimensional black hole on either side of the brane in static equilibrium. The only way to keep the black holes from crashing into each other is to separate them with a concentration of repulsive matter; i.e., matter with  $\epsilon + 3p < 0$ . Hence the negative energy when  $w_0 < 0$ .

Our second comment has to do with the pressure singularity on  $\mathcal{H}$ . In 5 dimensions, the brane can be thought of as a static thin disk of matter, and the disk's pressure provides support against gravitational collapse. But recall that an infinite amount of force is required to maintain a static matter distribution infinitesimally close to the surface

<sup>4</sup>A singularity associated with a regular metric in null coordinates is termed ‘‘weak’’ in the Tipler sense [28,29].

<sup>5</sup>As an interesting aside, we note that one can also get pressure singularities when a ‘‘bouncing’’ cosmological brane is embedded in a Schwarzschild bulk [30–33]. In that case, the origin of the singularity is a cusp in the embedding functions at the position of the bounce, which can occur within the bulk black hole event horizon [34].

of a black hole. Since  $\mathcal{H}$  is the intersection of the brane with the 5-dimensional horizon, we see that the pressure singularity is needed to prevent the disk matter from falling into the black hole. As viewed from the brane, we have a spherical distribution of matter on the verge of gravitational collapse supported by a shell-like pressure singularity.

To summarize, we have employed a planar slicing of the isotropic coordinate patch of the Schwarzschild 5-manifold derived in Sec. II B to derive a class of brane-world models (32). The models are static and spherically symmetric, and characterized by a singular null Killing horizon. We also derived the properties of the brane matter supporting the 4-geometry, for which there is an effective perfect fluid description. The energy density is well-behaved, but we found that the pressure had singular behavior on the Killing horizon  $\mathcal{H}$ . The pressure singularity is needed to prevent the collapse of the brane into the 5-dimensional event horizon.

## B. Braneworlds from a Schwarzschild-AdS bulk

We now move on to the case of Schwarzschild-AdS bulk manifolds. While the individual calculations are somewhat more involved than those of the previous section, the procedures and results are fairly similar. In this case the brane metric is

$$ds_{(\Sigma_0)}^2 = -H dt^2 + G(dr^2 + r^2 d\Omega_2^2), \tag{45a}$$

$$G = \frac{nc^2(\varphi, s_\gamma)}{r^2 + w_0^2}, \tag{45b}$$

$$H = \frac{sc^2(\varphi, s_\gamma) + a_\gamma^2 \text{sn}^2(\varphi, s_\gamma)}{\gamma^2}, \tag{45c}$$

$$\varphi = \frac{1}{2} \xi_\gamma \ln(r^2 + w_0^2). \tag{45d}$$

As in the last section, we will suppress the  $\sqrt{r^2 + w_0^2}$  argument of the various metric functions. It is useful to have series expansions of  $G$  and  $H$  about  $r = r_0 \equiv \sqrt{1 - w_0^2}$ , which are

$$\begin{aligned}
 G = & 1 - 2r_0(r - r_0) + [r_0^2(\xi_\gamma^2 + 4) - 1](r - r_0)^2 \\
 & + \mathcal{O}[(r - r_0)^3],
 \end{aligned} \tag{46a}$$

$$\begin{aligned}
 H = & \frac{\xi_\gamma^2 r_0^2}{\gamma^2} (a_\gamma^2 + 1)(r - r_0)^2 + \mathcal{O}[(r - r_0)^3].
 \end{aligned} \tag{46b}$$

It is immediately obvious from these series that the  $r = r_0$  hypersurface is again a Killing horizon  $\mathcal{H}$ . Also, since we can directly apply Eq. (39) to this situation, we can use the above series expansion for  $G$  to obtain the expansion of an outgoing null congruence on  $\mathcal{H}$ :

$$\theta_{(\bar{k})} = \frac{w_0^2}{\sqrt{1 - w_0^2}}. \tag{47}$$

This is precisely the same result as in the vacuum-bulk case and leads us to the same conclusion:  $\mathcal{H}$  is neither an event nor trapping horizon.

To determine if  $\mathcal{H}$  is the location of a curvature singularity, we can use the exact expressions for  $G$  and  $H$  to calculate the Kretschmann scalar, and then perform another expansion about  $r = r_0$ . The result is:

$$K = \frac{12(1 - r_0)^2(1 + r_0)^2}{r_0^2} (r - r_0)^{-2} + \mathcal{O}[(r - r_0)^{-1}]. \quad (48)$$

This clearly diverges as  $r \rightarrow r_0$ , so we have that  $\mathcal{H}$  is the site of a curvature singularity. Furthermore, when this fact is coupled with our knowledge of the fact that  $\mathcal{H}$  is not an event horizon, we conclude that it is a naked singularity, just as before.

One distinctive feature of this case is the asymptotic structure. Recall that when we derived the isotropic patch for  $S - \text{AdS}_5$ , the entirety of the region outside the black hole was covered by a finite interval of isotropic radius  $\rho \in (1, \rho_{\text{max}})$ . This would lead us to expect that there might be some special behavior of the 4-dimensional model at  $r_m = \sqrt{\rho_{\text{max}}^2 - w_0^2}$ . Now, our previous formula for  $\rho_{\text{max}}$  (20) gives us that  $\varphi = K(s_\gamma)$  at  $r = r_m$ , which allows us to expand our metric functions about  $r = r_m$ . Keeping leading order terms only, we have

$$ds_{(\Sigma_0)}^2 \sim \frac{r_m^2 + w_0^2}{r_m^2(r - r_m)^2} [-dt^2 + \gamma^2(dr^2 + r_m^2 d\Omega_2^2)]. \quad (49)$$

From this, it is clear that the proper distance between any point with  $r \in (r_0, r_m)$  and the  $r = r_m$  hypersurface is infinite. Hence, we should regard  $r = r_m$  as the spatial infinity of our 4-geometry. With this understanding, we can now interpret the plots of the brane's Kretschmann scalar versus  $r$  shown in Fig. 2. These show the expected divergence of  $K$  at  $r = r_0$ , but there are additional infinite features at greater values of  $r$ . As explained in the caption, these spikes always occur at  $r > r_m$  and hence are ‘‘beyond infinity’’; hence, they need not overly concern us.

We now determine the asymptotic behavior of the geometry as  $r \rightarrow r_m$  by calculating the limiting value of the Riemann tensor. To lowest order in  $(r - r_m)$ , we find

$$R^{\alpha\beta}_{\gamma\delta} = -\frac{r_m^2}{\gamma^2(r_m^2 + w_0^2)} (\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\beta_\gamma \delta^\alpha_\delta). \quad (50)$$

Hence, we have an asymptotically AdS-structure for the 4-geometry with total cosmological constant:<sup>6</sup>

<sup>6</sup>This was foreshadowed by the form of the asymptotic metric (49), which suggests that null geodesics could travel an infinite proper distance in a finite amount of coordinate time—a hallmark of AdS-space.

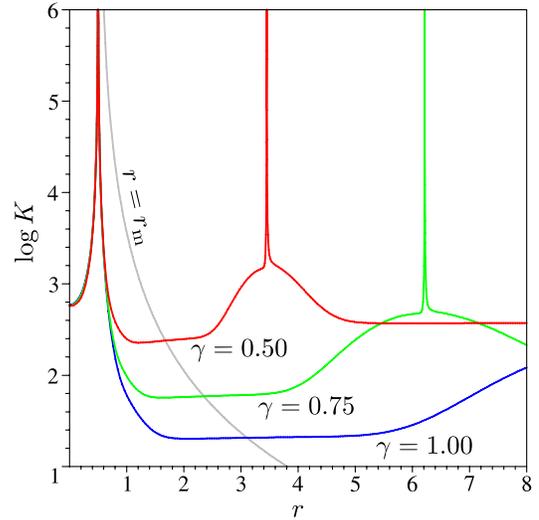


FIG. 2 (color online). The Kretschmann scalar as a function of  $r$  for various braneworlds in the case of an  $S - \text{AdS}_5$  bulk. Each of the sharp vertical peaks represent infinite spikes. We have selected  $w_0 = \sqrt{3}/2$  and as expected, we see the divergence of  $K$  at  $r = r_0 = 1/2$  in all instances. The intersection of the  $r = r_m$  line with each of the other curves gives the position of spatial infinity for each value of  $\gamma$ . Hence, we see that the rightmost spikes are ‘‘beyond infinity’’; i.e., there is only one divergence of  $\log K$  for  $r \in [r_0, r_m)$ .

$$\Lambda_4 = -\frac{3r_m^2}{\gamma^2(r_m^2 + w_0^2)}. \quad (51)$$

This can be compared with the 5-dimensional cosmological constant sourcing the bulk:

$$\Lambda_5 = -\frac{6}{\gamma^2}. \quad (52)$$

In situations such as these, there are standard formulae that relate  $\Lambda_5$  and  $\Lambda_4$  with the brane's tension  $\lambda$  (see Ref. [35], for example). In particular:

$$\lambda = \pm \sqrt{6(2\Lambda_4 - \Lambda_5)} = \pm \frac{6}{\gamma} \sqrt{\frac{w_0^2}{r_m^2 + w_0^2}}. \quad (53)$$

To remove the sign ambiguity in the sign of  $\lambda$ , we need to look at the properties of the brane matter sourcing the model, which are obtained using the general algorithm outlined above. The resulting expression for the brane's stress-energy tensor is extremely complicated and writing it down here will not convey much insight. But there are a few points worth mentioning:

- The brane stress-energy tensor is of the perfect fluid type with  $S_b^a = \text{diag}(-\epsilon, p, p, p, 0)$ , just as for the vacuum-bulk case.
- At  $r = r_0$ , the density and pressure behave like:

$$\lim_{r \rightarrow r_0} \kappa_5^2 \epsilon = 6w_0, \quad \lim_{r \rightarrow r_0} \kappa_5^2 p = \frac{2w_0}{r_0(r - r_0)}; \quad (54)$$

i.e., the density is finite at the position of the curvature singularity, while the pressure has an asymmetric pole such that  $p > 0$  for  $r > r_0$ . This exactly mirrors the vacuum-bulk case [cf. Eqs. (44)].

- (c) We have the following limiting behavior near spatial infinity:

$$\lim_{r \rightarrow r_m} \kappa_5^2 \epsilon = - \frac{6w_0}{\gamma \sqrt{r_m^2 + w_0^2}}, \quad (55a)$$

$$\lim_{r \rightarrow r_m} \kappa_5^2 p = + \frac{6w_0}{\gamma \sqrt{r_m^2 + w_0^2}}. \quad (55b)$$

Therefore, the asymptotic behavior of the brane matter is that of vacuum energy with cosmological constant

$$\lambda = - \frac{6w_0}{\gamma \sqrt{r_m^2 + w_0^2}}. \quad (56)$$

We identify this as the tension of our brane, which is *negative* for  $w_0 > 0$ . Of course it is in complete agreement with Eq. (53), which was obtained from a direct analysis of the asymptotic geometry.

All of these features are manifest in Fig. 3, which shows  $\epsilon$  and  $p$  for a particular choice of  $\gamma$  and  $w_0$ .

To summarize this section, we have used planar slicings of the isotropic coordinate map of  $S - \text{AdS}_5$  to generate braneworld models. The 4-manifolds we obtained share many properties with the ones derived from purely vacuum 5-manifolds in III A.; in particular, they involve null, shell-like, naked singularities where the brane crosses the bulk black hole horizon. The models approach AdS4 in the

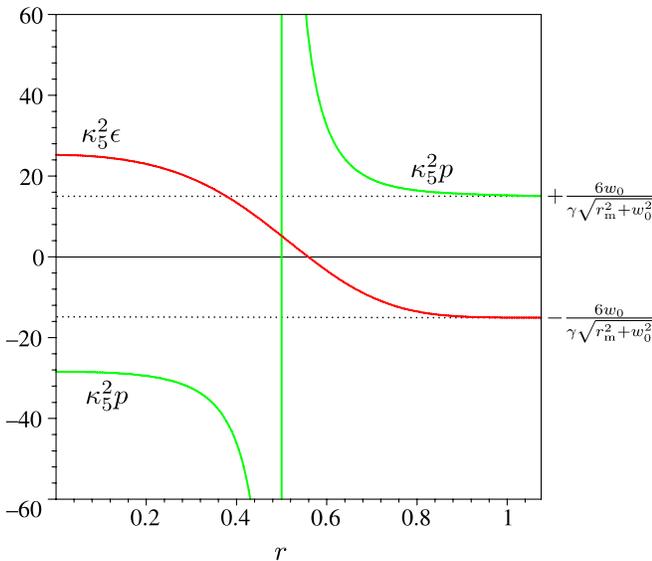


FIG. 3 (color online). Density and pressure of brane matter for  $\gamma = 1/4$  and  $w_0 = \sqrt{3}/2$ . Note for these parameters  $r_m \approx 1.08$ , which has been selected as the rightmost point on the  $r$  axis.

(appropriately identified) asymptotic region, and they are supported by perfect fluid brane matter. The pressure diverges at the position of the singularity while the density remains finite. By calculating the asymptotic behavior of  $\epsilon$  and  $p$ , we found an explicit expression for the brane tension, which is negative if  $w_0 > 0$ .

#### IV. BRANEWORLDS FROM NON-PLANAR SLICINGS

In the previous section, we saw that our consideration of purely planar slicings of black hole 5-manifolds led to brane matter whose properties were largely given to us by the geometry. In order to regain some control of the sources in our model, we now consider braneworlds formed from surfaces of revolution. For simplicity, we will limit our work to bulk vacuum-bulk manifolds, though much of what we do can be straightforwardly generalized to the  $S - \text{AdS}_5$  case.

We can define a surface of revolution in isotropic coordinates by  $w = w(r)$ , which induces the following metric on  $\Sigma_0$ :

$$ds_{(\Sigma_0)}^2 = -Hdt^2 + G[(1 + w_r^2)dr^2 + r^2d\Omega_2^2], \quad (57)$$

where  $w_r = dw/dr$  and  $H$  and  $G$  are the isotropic metric functions (10) evaluated at  $\rho = \sqrt{r^2 + w^2(r)}$ . Like the planar case, the 4-geometry is static and spherically symmetric. Now, let us calculate the Ricci scalar for this geometry:

$${}^{(4)}R = - \frac{8q_1}{r^2(r^2 + w^2 + 1)^3(r^2 + w^2 - 1)(1 + w_r^2)^2}, \quad (58)$$

where  $q_1 = q_1(r, w, w_r, w_{rr})$  is given in the Appendix and we have written  $w_{rr} = d^2w/dr^2$ . For generic choices of  $w(r)$ , the Ricci scalar will diverge at  $r = r_0$ , where  $r_0$  is the solution of  $r^2 + w^2(r) = 1$ .<sup>7</sup> We therefore identify a curvature singularity at the  $r = r_0$  hypersurface, just as in the planar case. We again expect this singularity to be naked, because in this case the affine length of a radial null geodesic with energy  $E$  travelling from  $r = r_1$  to  $r_0$  is

$$\Delta\lambda = \frac{1}{E} \int_{r_0}^{r_1} \sqrt{GH(1 + w_r^2)} dr, \quad (59)$$

which is generally finite, except perhaps for very special choices of  $w(r)$ .

<sup>7</sup>However, there is one important special case we should highlight: namely  $w(r) = \text{constant} \times r$ , which is equivalent to  $\chi = \chi_0$  in the original Schwarzschild coordinates. In this case, we find  $q_1$  vanishes identically; i.e.,  ${}^{(4)}R = 0$ . Indeed, the complete set of 4-dimensional curvature invariants is regular at  $r = r_0$ , so this braneworld likely does not have a shell singularity. I would like to thank Ken-ichi Nakao and Daisuke Ida for drawing this case to my attention.

Hence, the nonplanar case is similar to the planar case in that we can expect to find naked shell singularities. However, we now have an additional degree of freedom at our disposal to place certain conditions on the geometry, the brane matter, or both. Let us now consider a few examples of how this freedom can be used.

### A. Slicings with vanishing Ricci scalar

In the, admittedly short, history of the search for braneworld black holes, many workers have viewed solutions of  ${}^{(4)}R = 0$  as likely candidates. This is because of one of the contracted Gauss-Codazzi equations, which for a vacuum bulk reads:

$${}^{(4)}R = (\text{Tr}K)^2 - K^{ab}K_{ab}. \quad (60)$$

Since the brane's stress-energy tensor is essentially determined by the extrinsic curvature,  ${}^{(4)}R$  must vanish for models with no brane matter present. However, the reverse is not true; if  ${}^{(4)}R = 0$  we do not necessarily have that  $K_{\alpha\beta} = 0$ . In our case, the imposition of  ${}^{(4)}R = 0$  will not guarantee that the extrinsic curvature of the brane vanishes — it is actually impossible to get  $K_{\alpha\beta} = 0$  for a nontrivial slice (cf. Sec. IV E). However, it is still an interesting case to look at because it does place a constraint on the total effective matter on the brane, which includes contributions from both the brane and “Weyl” matter. In order to obtain  $w(r)$  we set  $q_1 = 0$  and solve the resulting second-order ODE numerically. Several representative solutions for  $w(r)$  are plotted in Fig. 4. In this plot, we note the planar solution  $w = 0$  along with more exotically shaped braneworlds.

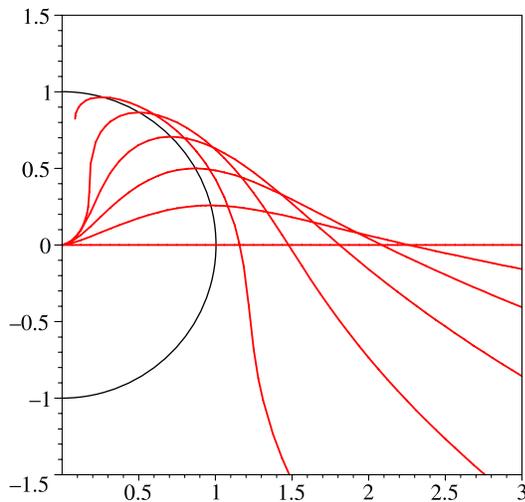


FIG. 4 (color online). Numeric solutions for  $w(r)$  associated with  ${}^{(4)}R = 0$  slicings. The semicircle indicates the position of the horizon. Note that the topmost curve is incomplete because it has a vertical tangent when inside the horizon, which caused the numeric integration to fail.

### B. Slicings with vanishing radial pressure

Another class of interesting braneworld are those with vanishing principle pressures. Using Eq. (31) with an extrinsic curvature calculated from

$$n_a dx^a = \frac{-w_{,r} dr + dw}{\sqrt{(w_{,r}^2 + 1)G}}, \quad (61)$$

we find that the brane's stress-energy tensor is of the form

$$S^a_b = \text{diag}(-\epsilon, p_r, p_\perp, p_\perp, 0), \quad (62)$$

where  $p_r \neq p_\perp$  in general. Explicitly, the radial pressure is

$$\kappa_5^2 p_r = \frac{q_2}{r(r^2 + w^2 + 1)^2 (r^2 + w^2 - 1) \sqrt{1 + w_{,r}^2}}, \quad (63)$$

where  $q_2 = q_2(r, w, w_{,r})$  is given in the Appendix. In this expression, we see the now familiar pressure singularity at  $r^2 + w^2(r) = 1$ . To find a braneworld with zero radial pressure, we need to solve the first-order ODE  $q_2 = 0$ . This is actually possible to do in a closed form, and the exact solution is given in the Appendix. We give a 3-dimensional representation of one possible braneworld obtained from this solution in Fig. 5. The plot gives the impression that the brane approaches a planar geometry from large  $r$ , which is actually not true since the limit of  $|w(r)|$  as  $r \rightarrow \infty$  is itself infinite.

### C. Slicings with vanishing tangential pressure

We now turn our attention to braneworlds where the brane matter satisfies  $p_\perp = 0$ . In general, we have

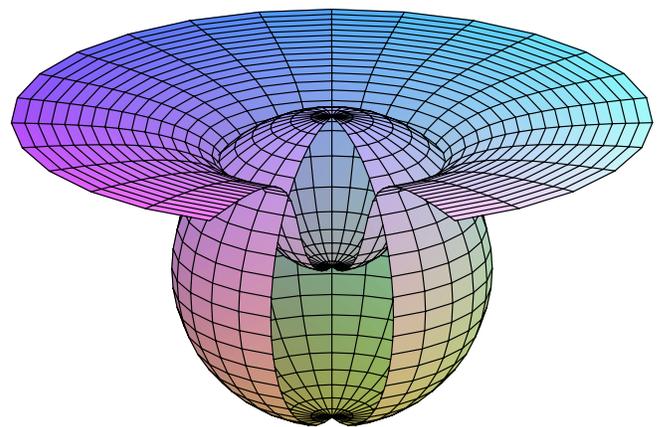


FIG. 5 (color online). The surface of revolution formed from our analytic solution for  $w(r)$  in the case of a  $p_r = 0$  slicing (a wedge has been removed to aid visualization). The spherical object indicates the position of the black hole horizon. The time and  $\theta$  coordinates have been suppressed, which means that each horizontal ruling on the surfaces actually represents a 2-sphere.

$$\kappa_5^2 p_\perp = \frac{-4q_3}{r(r^2 + w^2 + 1)^2(r^2 + w^2 - 1)(1 + w_{,r}^2)^{3/2}}, \quad (64)$$

where  $q_3 = q_3(r, w, w_{,r}, w_{,rr})$  is given in the Appendix. Obviously, the ODE to solve for  $p_\perp = 0$  is  $q_3 = 0$ , which is a rather complicated expression. We will content ourselves with numerical solutions, one of which is depicted in Fig. 6.

#### D. Slicings with isotropic pressure

We have already seen above that the planar slicings of 5-dimensional Schwarzschild space give rise to braneworlds with  $p_r = p_\perp$ ; i.e., with isotropic pressure. But are planar slicings the only ones that can be modeled as a perfect fluid? To answer this, consider:

$$\kappa_5^2(p_r - p_\perp) = \frac{4(r^2 + w^2)(rw_{,rr} - w_{,r} - w_{,r}^3)}{(w_{,r}^2 + 1)^{3/2}(w^2 + r^2 + 1)r}. \quad (65)$$

Setting this equal to zero, we find

$$c_1^2 = r^2 + (w - c_2)^2, \quad (66)$$

where  $c_1$  and  $c_2$  are arbitrary constants. Hence braneworlds with isotropic pressure have circular cross sections in the  $(r, z)$ -plane and look like off-center spheres as surfaces of revolution. In the limit of large radius ( $c_1 \rightarrow \infty$ ), we recover the planar result of Sec. III A.

#### E. Slicings with vanishing extrinsic curvature

We now turn our attention to braneworlds with  $K_{ab} = 0$ , which represent models with no matter confined to  $\Sigma_0$ . If

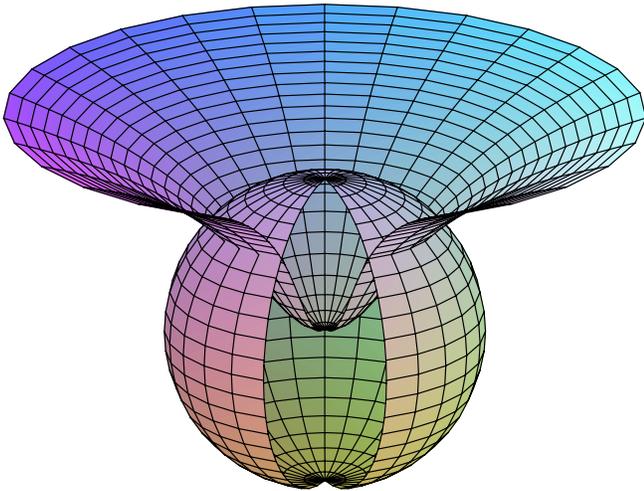


FIG. 6 (color online). A numeric solution for  $w(r)$  in the case of a  $p_\perp = 0$  slicing (see the caption of Fig. 5 for an explanation of the visualization scheme).

we calculate the extrinsic curvature explicitly, we find

$$K^t_t = \frac{8(r^2 + w^2)(rw_{,r} - w)}{\sqrt{w_{,r}^2 + 1}(1 - w^2 + r^2)(w^2 + r^2 + 1)^2}. \quad (67)$$

Setting this equal to zero yields  $w = cr$ , where  $c$  is a constant. Plugging this into  $K^a_b$ , we find the only non-vanishing components:

$$K^\theta_\theta = K^\phi_\phi = -\frac{2cr\sqrt{c^2 + 1}}{(c^2 + 1)r^2 + 1}. \quad (68)$$

Setting these identically equal to zero implies  $c = 0$ . Hence the only surface of revolution we can find with  $K_{ab} = 0$  is  $w = 0$ ; i.e., the equatorial plane of the black hole. This result makes intuitive sense, because we know that surfaces with vanishing extrinsic curvature must be symmetry surfaces of our spacetime, and the only way to symmetrically slice our 5-manifold is down the middle.

#### F. Slicings resulting in a pure tension brane: Einstein-static universe

The last type of slicing that we consider has the extrinsic curvature proportional to the induced metric, which represents a brane sourced by a cosmological constant (i.e. tension) only. Such a slicing has been previously sought by Chamblin *et al.* [18], but due to a particular choice of embedding scheme was not found.<sup>8</sup> Assuming  $K_{ab} = -\frac{1}{6}\sigma h_{ab}$  yields the solution:

$$r^2 + w^2 = 3 + 2\sqrt{2}, \quad \sigma = \pm 3. \quad (69)$$

Hence, the only pure tension brane solution takes the form of a static spherical shell of isotropic radius  $\rho = \sqrt{3 + 2\sqrt{2}}$ , which corresponds to a Schwarzschild radius of  $R = \sqrt{2}$ . The 4-metric in this case can be cast as

$$ds^2_{(\Sigma_0)} = -d\tau^2 + 2d\Omega_3^2, \quad (70)$$

where  $\tau = t/\sqrt{2}$ . This is the metric of the Einstein-static universe. In other words, pure tension static branes around 5-dimensional Schwarzschild black holes take the form of an Einstein-static universe. Note that the ‘‘dark radiation’’ plays the role that matter would in a 4-dimensional Einstein-static solution, as can be seen from the effective Friedman and Raychaudhuri equations of this ‘‘brane cosmology,’’ which in dimensionless coordinates read:

$$\frac{1}{R^2} \left( \frac{dR}{dt} \right)^2 = -\frac{1}{R^2} + \frac{1}{R^4} + \frac{\sigma^2}{36} \equiv 0, \quad \frac{d^2R}{dt^2} \equiv 0. \quad (71)$$

<sup>8</sup>See also the work of Kodama [36,37], which searched for pure tension branes in quite general bulk manifolds satisfying minimal symmetry assumptions.

An interesting observation is that in this case, the brane is coincident with the photon sphere of the bulk black hole. This fact could have been anticipated from the following fact: Any 5-dimensional null geodesic initially tangent to a pure tension brane  $\Sigma_0$  will remain confined to that brane. This can be seen by noting the following result from Ref. [38]: Given that a null geodesic is momentarily tangent to a hypersurface  $\Sigma_0$ , its acceleration orthogonal to that surface is proportional to  $K_{ab}k^ak^b$ , where  $k^a$  is the tangent vector. If  $\Sigma_0$  is a pure tension brane we have  $K_{ab} = -\frac{1}{6}\sigma h_{ab}$ , from which it follows  $K_{ab}k^ak^b = 0$  since  $k^an_a = k^ak_a = 0$ . Hence, there is no acceleration perpendicular to  $\Sigma_0$  and null geodesics are confined to pure tension branes. Surfaces such as this are known as “totally geodesic” with respect to null paths, which are a special type of umbilical surface [36].

## V. CONCLUSIONS

In this paper we have considered braneworld models obtained by nontrivial slicings of  $S - \text{AdS}_5$  manifolds defined in isotropic coordinates (Sec. II). We have succeeded in finding a number of static and spherically-symmetric configurations, but almost all of them are characterized by a naked pressure singularity where the brane crosses the horizon of the bulk black hole. From a relativity point of view, such a singularity is required to provide the infinite force supporting matter infinitesimally above an event horizon. From the AdS/CFT perspective, such a singularity can be interpreted as Boulware-type quantum correction to the horizon of the brane black hole. Generic models have nonzero matter content; for planar slicings we recover perfect fluid matter with an exotic equation of state (Sec. III). Different possible constraints on the 4-geometry were considered in Sec. IV in the simpler case of zero bulk cosmological constant. Branes with zero Ricci scalar, extrinsic curvature, vanishing principle pressures, and others were derived; but the only solution with vacuum (i.e., only brane tension) turned out to be the braneworld generalization of the Einstein-static universe residing on the 5-dimensional photon-sphere.

It must be said that none of the braneworlds derived can be considered as a black hole candidate. The ubiquitous matter content precludes that, but some interesting points have been raised nevertheless. We have explicitly seen how a regular bulk can easily give rise to a singular brane, and how singular 4-dimensional horizons are a persistent feature of our construction. Because this is from the divergence of tidal forces on static matter near the surface of a black hole, we expect it to generalize to any static brane with matter that intersects a bulk Killing horizon. Whether or not this extends to the “real” *static vacuum* braneworld black hole solution is an open question: it is unclear if one needs a pressure singularity to support the Weyl fluid certain to be present in such a model. If so, this provides strong support for, and physical insight into, the conjecture that braneworld black holes naturally incorporate quantum corrections.

One important issue that we have not addressed is the stability of these models. While it is true that the bulk geometries are stable, there is no guarantee that the inclusion of a brane boundary will not have a destabilization effect. Actually finding out if these models are stable is not an easy task, since all the branes considered tend to break the  $S^3$  symmetry of the bulk, which complicates the analysis of perturbation wave equations. The exception is the Einstein-static brane universe seen in Sec. IV F, which is prone to a relatively straightforward stability analysis. We will report on this case in a forthcoming paper.

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## APPENDIX

- (a) Definitions of various quantities associated with nonplanar slicings:

$$q_1 = [(8r^7w^2 + 2w^8r + 2r^9 + 12r^5w^4 + 8r^3w^6 + 2r^5 - 2w^4r)w_{,r} - 4r^2w^3 - 4r^4w]w_{,rr} + (w^8 - w^4 + 6w^2r^2 - 5r^4 + 6r^4w^4 + 4r^2w^6 + r^8 + 4r^6w^2)w_{,r}^4 + (16wr^3 - 8rw^3)w_{,r}^3 + (-6w^2r^2 - 5r^4 - w^4 + 4r^6w^2 + r^8 + 4r^2w^6 + w^8 + 6r^4w^4)w_{,r}^2 + (16wr^3 - 8rw^3)w_{,r} - 12w^2r^2, \quad (\text{A1a})$$

$$q_2 = (-8w^6 - 24r^2w^4 + 8w^2 - 8r^2 - 8r^6 - 24r^4w^2)w_{,r} + 16wr, \quad (\text{A1b})$$

$$q_3 = [r^7 + 3w^2r^5 + (3w^4 - 1)r^3 + (-w^2 + w^6)r]w_{,rr} + [r^6 + 3r^4w^2 + (3 + 3w^4)r^2 - w^2 + w^6]w_{,r}^3 - 4rw_{,r}w_{,r}^2 + [r^6 + 3r^4w^2 + (3 + 3w^4)r^2 - w^2 + w^6]w_{,r} - 4wr. \quad (\text{A1c})$$

- (b) Analytic solution for  $w(r)$  in the case of vanishing radial pressure:

$$w(r) = \pm \frac{\sqrt{2^{7/3}X^4 + 8cr^2 - 12 + 4r^4 + 4c^2 - 2^{14/3}r^2X^2 + 2^{8/3}cX^2}}{864^{1/6}X}, \quad (\text{A2a})$$

$$X^6 \equiv 3(24r^4 + 12r^8 - 60cr^2 + 36r^6c + 12c^3r^2 + 12 - 3c^2 + 36c^2r^4)^{1/2}2r^6 + 6cr^4 + (6c^2 + 18)r^2 + 2c^3 - 9c, \quad (\text{A2b})$$

where  $c$  is an arbitrary constant.

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