Loop quantum gravity phenomenology and the issue of Lorentz invariance

Martin Bojowald,[1,2,*] Hugo A. Morales-Técotl,[3,†] and Hanno Sahlmann[2,‡]

[1]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany*

[2]*Center for Gravitational Physics and Geometry, Pennsylvania State University, 104 Davey Lab, University Park, Pennsylvania 16802, USA*

[3]*Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa, A.P. 55-534 México D.F. 09340, México*

(Received 28 January 2005; published 13 April 2005)

A simple model is constructed which allows to compute modified dispersion relations with effects from loop quantum gravity. Different quantization choices can be realized and their effects on the order of corrections studied explicitly. A comparison with more involved semiclassical techniques shows that there is agreement even at a quantitative level. Furthermore, by contrasting Hamiltonian and Lagrangian descriptions we show that possible Lorentz symmetry violations may be blurred as an artifact of the approximation scheme. Whether this is the case in a purely Hamiltonian analysis can be resolved by an improvement in the effective semiclassical analysis.

DOI: 10.1103/PhysRevD.71.084012 PACS numbers: 04.60.Pp, 11.10.Ef, 11.30.Cp

I. INTRODUCTION

One of the cases where observations of quantum gravity effects have been imagined relies on modified dispersion relations for matter (such as photons or neutrinos) travelling on a quantum gravitational background [1]. Planck scale effects are expected to be negligible in standard circumstances, but observations of highly energetic particles travelling long distances or other high precision experiments may set bounds on possible effects. Such effects have been interpreted as indicating a violation of (standard) Lorentz symmetry [2] for which, as expected, the observational bounds are rather stringent [3]. The theoretical foundation, on the other hand, is still open and actively debated. For instance, while trying to link the breaking of Lorentz symmetry to a privileged frame has been argued to be in conflict with our current understanding of field theory [4], one can try to invoke a deformed, rather than broken, symmetry in the form of Doubly Special Relativity [5]. Independently of these field theoretical considerations, the task of candidate quantum theories of gravity is to provide reliable estimates on the magnitude of expected modifications to the standard dispersion relations to be compared with observations.

One such candidate is loop quantum gravity [6] which leads to a discrete structure of the geometry of space. This discreteness can be expected to lead to small-scale corrections of dispersion relations, just as the atomic structure of matter modifies continuum dispersion relations once the wavelength becomes comparable to the lattice size. There have been several studies already which derive modified dispersion relations motivated from particular properties of loop quantum gravity [7–9], but at this stage the control on the calculations is insufficient. The difficulty lies in the fact that loop quantum gravity is very successful in providing a completely nonperturbative and background independent quantization of general relativity which makes it harder to reintroduce a background such as Minkowski space over which a perturbation expansion could be performed. Techniques for constructing semiclassical states are available and still being developed further [10,11], but the calculations toward modified dispersion relations are very complicated. Moreover, the answer cannot be expected to be unique but to depend on several parameters as well as quantization choices in the full quantum theory.

In the first part of this paper we develop and study a simple model which allows us to introduce crucial properties of loop quantum gravity into the Hamiltonian of a matter field. As we will show, the model captures essentially all the effects that have been considered so far in loop motivated calculations of modified dispersion relations even at a quantitative level. We can also see how different quantizations would change the results, and which quantization choices should have the largest effect on the order at which corrections occur. Thus, we have the freedom to change basic objects according to the possibilities of loop quantum gravity, but a much simpler and more immediate way to check the consequences.

The essential idea in constructing the model is to consider space as being made of homogeneous patches defining a lattice on which the matter Hamiltonian, in particular, its space derivatives, will be discretized. This models the discrete structure of loop quantum gravity, but could also be used classically as an approximation (the metric field is then simply considered as a piecewise constant rather than continuous function). Such a classical approximation would become better and better if we choose smaller and smaller patch sizes. A second ingredient from quantum geometry then is that the patch geometry must be quantized (which is readily done for homogeneous or even isotropic patches [12]). This implies additional, quantum geometric corrections which grow with shrinking patches.

*Electronic address: mabo@aei.mpg.de
†Electronic address: hugo@xanum.uam.mx. Associate member of AS-ICTP Trieste, Italy
‡Electronic address: hanno@gravity.psu.edu

Thus, with effects from quantum geometry there is a nonzero patch size leading to a minimal deviation of the effective matter Hamiltonian from the classical one.

This model can be formulated at different levels of complexity which allows to consider more realistic situations and also to bring it closer to what one would have in full loop quantum gravity. In this paper, we only consider the simplest construction using isotropic patches of equal size, and only couple a scalar matter field with simple discretizations. Although there are still gaps between the model and full loop quantum gravity, it gives—because of its simplicity—a more direct link between quantum gravity phenomenology and the full theory.

In the second part of the paper, we study the possibility that the apparent Lorentz violation of dispersion relations obtained in our simple model, as well as in the more sophisticated treatments [7–9], be a result of the approximation scheme rather than of the theory itself. Common to all these computations is that they derive corrections to the matter *Hamiltonian*. This is natural in the setting of loop quantum gravity because the quantization of spatial geometry is readily available whereas 4d covariant quantities are harder to quantize. On the other hand, by way of examples we will demonstrate that a purely Hamiltonian analysis is much more subtle than a Lagrangian one, and discuss how a perturbative Hamiltonian analysis can be improved in order to draw reliable conclusions.

At first it might seem that Hamiltonian and Lagrangian descriptions are completely equivalent, which certainly is the case when theories are considered exactly. However, as we will demonstrate, the situation changes when approximation schemes, in particular, perturbation theory, are employed. When higher derivative corrections are involved, the Legendre transform does not commute with expansion in a perturbation series, and so the *perturbed* Hamiltonian and Lagrangian formulations are not necessarily equivalent. This is, for instance, the case for theories nonlocal in time, in particular, when time is not a continuous parameter but ‘‘discrete‘‘. This might well be the case in quantum gravity and specifically in loop quantum gravity [13]. We will show in examples that going over to an approximate continuum Lagrangian and then to the Hamiltonian description will lead to a Hamiltonian that one could not have obtained from a Hamiltonian model with continuous time in the spirit of what has been proposed for loop quantum gravity. Our conclusion will therefore be that many of the calculations done up to now (including the first part of this paper) can only yield preliminary results and that a definite answer to the question of Lorentz violation by loop quantum gravity will have to await a more complete treatment. The second part of the paper thus reinforces the need for a simplified setup, such as the one proposed in the first part, which allow one to do explicit calculations at the discrete level. We will conclude with an outlook on possible strategies in this direction.

II. QUANTUM CORRECTIONS TO THE SCALAR FIELD HAMILTONIAN

The simplest way to couple loop quantum gravity (LQG) to a free scalar field is via its Hamiltonian

$$H = \int_\Sigma d^3x\left(\frac{1}{2}N((\det q)^{-1/2}p_\phi^2 + \sqrt{\det q}\,q^{ab}\partial_a\phi\partial_b\phi)\right) \tag{1}$$

where Σ is a Cauchy surface of the space-time manifold M (assumed to be globally hyperbolic). The more complete treatments in the literature proceed by quantizing the gravitational part of this Hamiltonian with LQG methods [14] and then taking expectation values in a semiclassical state. That state comes with a discretization of the spacial slice Σ and as a consequence, the partial derivatives in the classical expression (1) are changed to lattice derivatives. Further differences to the classical expression result from quantum corrections to the classical values of $(\det q)^{\pm 1/2}$, q^{ab} upon taking expectation values of the corresponding operators in the semiclassical state.

To compare the corrected Hamiltonian obtained by this method with the standard expression (1), the lattice derivatives in the former are expanded in a Taylor series to obtain, by collecting the lower orders of this expansion and of the other quantum corrections, an effective Hamiltonian. Plane waves are solutions to the equations of motion generated by this effective Hamiltonian and the corresponding dispersion relations contain corrections compared to the standard one. We emphasize that in this section we follow the standard procedure [8,9].

To compute the corrections for the Hamiltonian in a simplified way, let us now propose a model which includes the expected properties (and which can always be made more complicated to be more realistic). Let us discretize space into patches on which we can assume the geometry to be approximately isotropic. Each patch α then carries two real numbers, one for the scalar ϕ_α and one for its momentum $p_{\phi,\alpha}$, and an isotropic semiclassical quantum state ψ_α for the geometry. (There is also a lapse function, a real number, per patch which is not so important for our purposes.) This corresponds to a scalar on a classical geometry made of patches of a size given by the expectation value of the volume operator in the semiclassical state.

Again, corrections in the Hamiltonian are of two kinds: Since it contains space derivatives of ϕ, which have to be replaced by finite differences, there is a discretization error. In addition, geometric quantities like $\sqrt{\det q}$ and its inverse have to be replaced by expectation values of the corresponding operators. Choosing each patch to have coordinate volume one (otherwise, there would be unnecessary coordinate factors), one obtains

$$H_{\rm disc} = \frac{1}{2}\sum_\alpha N_\alpha[p^2_{\phi,\alpha}((\det q)^{-1/2})_{\psi_\alpha} + (E^I_i E^J_i/\sqrt{\det q})_{\psi_\alpha} \times (\phi_{\alpha+e_I} - \phi_{\alpha-e_I})(\phi_{\alpha+e_J} - \phi_{\alpha-e_J})/4]. \quad (2)$$

Here, a subscript ψ_α means taking the expectation value in the state ψ_α, and $\alpha + e_I$ denotes the neighboring patch in direction I. For isotropic patches, the expression simplifies to

$$H_{\rm disc,iso} = \frac{1}{2}\sum_\alpha N_\alpha\left[p^2_{\phi,\alpha}(p^{-3/2})_{\psi_\alpha} + \frac{1}{4}(\sqrt{p})_{\psi_\alpha} \times \sum_I(\phi_{\alpha+e_I} - \phi_{\alpha-e_I})^2\right]. \quad (3)$$

where $p = a^2$ is the isotropic densitized triad component.

So far, there are many parameters to specify the background for the scalar: For each patch we have a state, which is characterized by its expectation value for p and its spread. At first, one can assume that all patches have the same values, which still leaves us with two scales in addition to the Planck length and a wavelength. Since the difference corrections increase with the size of the patches while the corrections for inverse powers increase with decreasing size, the first scale, the scale factor a of the isotropic patches, can be fixed by requiring a minimal sum of those corrections.

Specifically, we can relate the discrete scalar field values ϕ_α to a continuous field $\phi(x)$ by $\phi_\alpha = \phi(x_\alpha)$ where x_α is a point in the center of patch α. Expanding $\phi(x)$ we get

$$\begin{aligned}\phi_{\alpha+e_I} &= \phi(x_{\alpha+e_I}) \\ &= \phi(x_\alpha) + (x_{\alpha+e_I} - x_\alpha)^a\partial_a\phi(x_\alpha) \\ &\quad + \frac{1}{2}(x_{\alpha+e_I} - x_\alpha)^a(x_{\alpha+e_I} - x_\alpha)^b\partial_a\partial_b\phi(x_\alpha) + \ldots\end{aligned} \quad (4)$$

Now let us assume that the x_α are the vertices of a regular cubic lattice aligned with the coordinates on Σ. Then (4) simplifies to

$$\phi_{\alpha+e_I} = \phi(x_\alpha) + \partial_I\phi(x_\alpha) + \frac{1}{2}\partial_I^2\phi(x_\alpha) + \ldots. \quad (5)$$

The squared differences in the Hamiltonian (3) are thus approximated by

$$\frac{1}{4}\sum_I(\phi_{\alpha+e_I} - \phi_{\alpha-e_I})^2 = \sum_I\left[(\partial_I\phi)^2 + \frac{1}{3}\partial_I\phi\partial_I^3\phi + \ldots\right](x_\alpha), \quad (6)$$

i.e. we get the second derivative term we need plus higher derivative corrections. Let us write

$$B := \frac{1}{3}\sum_I\partial_I\phi\partial_I^3\phi \quad (7)$$

for the leading correction in a derivative expansion. It is certainly not rotation invariant, the symmetry having been broken by the introduction of the regular lattice of patches. In a more realistic calculation one would work with a random lattice or average over regular lattices with different orientations to define the semiclassical state. As we are only interested in an order of magnitude calculation, we disregard this issue here.

For the corrections of the momentum term in (3) we can use earlier calculations for the inverse scale factor resulting in $(p^{-3/2})_{\psi_\alpha} = a^{-3} + \Delta a^{-3}$ with a quantum correction Δa^{-3} which for larger a is perturbative in the Planck length. In a triad eigenstate [15], those corrections would be $p^2_\phi\Delta a^{-3} = cp^2_\phi a^{-3}\cdot\ell_{\rm P}^4/a^4 = ca^{-1}N^{-2}\dot\phi^2\ell_{\rm P}^4$ (since $p_\phi = Na^3\dot\phi$) with some constant c which can be computed once we make a choice on the explicit quantization, while a coherent state [16] would result in $d^2\ell_{\rm P}^2/a^4$ instead of $\ell_{\rm P}^4/a^4$. (Thus, the Planck length would be replaced by its geometric mean with the spreading scale d of the coherent state.) Using a coherent state thus makes the correction smaller, as expected, but we cannot yet tell the order in $\ell_{\rm P}$ since a is not fixed. To do this, we minimize the total correction to the Hamiltonian (3)

$$ca^{-1}N^{-2}\dot\phi^2\ell_{\rm P}^4 + aB \quad (8)$$

with respect to a. By this procedure, as in [10], we obtain the smallest correction to be expected rather than a precisely predicted correction term. This is sufficient for our purposes since we are mainly interested in order of magnitude estimates. We thus find

$$a = \left(\frac{c}{2}\ell_{\rm P}^4\frac{N^{-2}\dot\phi^2}{B}\right)^{1/2} \quad (9)$$

for a triad eigenstate and

$$a = \left(\frac{c}{2}d^2\ell_{\rm P}^2\frac{N^{-2}\dot\phi^2}{B}\right)^{1/2} \quad (10)$$

for a coherent state.

A classical wave solution with our notation (e.g. dimensionless coordinates) has the form

$$\phi = \exp(i(ak\cdot x + N\omega t)) \quad (11)$$

if all patches have the same size a (otherwise, we would have to sum to get the physical distance in the argument). Thus, we have an implicit expression for a

$$a = \sqrt{(c/2)\ell_{\rm P}^4\omega^2\lambda^4/(16\pi^4c'a^4)} \quad (12)$$

where we have expressed B as $c'(2\pi a/\lambda)^4\phi^2$ and c' is between $1/9$ and $1/3$, depending on the direction of propagation (this combines the factor $1/3$ in B with another factor of $|k|^{-4}\sum_I k_I^4$ which can be seen to be bound by $1/3|k|^4 \leq \sum_I k_I^4 \leq |k|^4$). Thus we get

$$a = ((c/(32\pi^4 c')\ell_P^4\omega^2\lambda^4)^{1/6} \approx ((c/(8\pi^2 c'))\ell_P^4\lambda^2)^{1/6} \tag{13}$$

where we used $\omega \approx 2\pi\lambda^{-1}$ which is only approximately true since we expect corrections to the dispersion relation (but corrections here would only affect higher order terms).

Thus, the patch size is a weighted mean of the Planck length and the wavelength, with a large weight on the Planck length. This means that we need rather small patches, and it strongly reduces the order of expected corrections: From the inverse powers of p we expect corrections of the order

$$\ell_P^4/a^4 \propto (\ell_P/\lambda)^{4/3} \quad \text{(triad eigenstate)}$$
$$\propto (\ell_P/\sqrt{d\lambda})^{8/3} \quad \text{(coh. state).} \tag{14}$$

The order of the corrections from the $\sqrt{p}$-term on the other hand is

$$a^2/\lambda^2 \propto (\ell_P/\lambda)^{4/3} \quad \text{(triad eigenstate)}$$
$$\propto (\sqrt{\ell_P d}/\lambda)^{4/3} \quad \text{(coh. state).} \tag{15}$$

These results square very nicely with [9]: First of all in both cases there is a spatial discretization, leading to discretized derivatives and, consequently, higher order derivative corrections in the effective Hamiltonian. The characteristic size of the discretization ϵ in [9] is (as a here) a weighted mean $\epsilon \approx \ell_P^\alpha\lambda^{1-\alpha}$. While α was not uniquely fixed there, $\alpha = 2/3$ would reproduce the result here. Also, the order of the first correction due to the higher derivative terms was, exactly as in our case, found to be ϵ^2/λ^2.

Next, since [9] also works with coherent states, there is a parameter (called a there) corresponding closely to our parameter d, distributing the width of the state between configuration and momentum degrees of freedom. There, this parameter is chosen macroscopic. One can probably understand this from our result here that the correction (14) decreases while the correction (15) increases with increasing d. However, larger d are favored because the decrease is governed by a higher power of d.

Finally, the relative order of the correction due to quantum effects found in [9] is $(\epsilon/\lambda)^{2/\alpha-1}$. Again, this corresponds precisely to our $(\ell_P/a)^4$ for $\alpha = 2/3$. As for comparison with the dispersion relations in [8] the analysis proceeds similarly.

III. THE ISSUE OF LORENTZ INVARIANCE

A case in which modified dispersion relations have good chance of being tested is when they break Lorentz invariance. This allows correction terms of the form $\ell_P E$ which can be high enough for sufficiently large energy E. Possible Lorentz invariant corrections, on the other hand, can at most be of the order $\ell_P m$ with the fixed and limited mass m. Accordingly, except for trivially modified dispersion relations that have been discussed for quantum gravity phenomenology, all break Lorentz invariance.

Loop quantum gravity, in particular, as used in the preceding section, is a Hamiltonian formalism where Lorentz invariance is not manifest. (The supposedly covariant twin of loop quantum gravity, spin foams, is under much less control currently and not yet suitable for explicit applications; moreover, anomaly free formulations may even lose manifest covariance [17].) If we first consider only the spatial aspects, rotational invariance is not manifestly broken by the discrete structure since one does not restrict the theory to a fixed lattice. Nevertheless, calculations of dispersion relations choose a fixed graph which cannot be rotationally invariant, but do the calculations in such a way that the result is rotationally symmetric. So *a priori* discreteness does not necessarily imply violations of symmetries in the approximate classical expressions. For Lorentz invariance, however, this is much more difficult to achieve since time does not appear directly in the theory.

The Hamiltonian formulation requires calculations to be based on a lattice in space such that only the spatial geometry is manifestly discrete. Nevertheless, dispersion relations are computed from classical field equations which involve coordinate time. This time parameter is introduced by computing the perturbative matter Hamiltonian on the lattice, and then treating it as the Hamiltonian of a classical field theory. Time then appears via the Hamiltonian equations of motion, but only at the classical level. In particular, time is always continuous in this setting unlike space, whose discrete structure is responsible for the very effects to be computed.

That the situation in loop quantum gravity is indeed such that the calculations done so far introduce Lorentz violations not coming from the theory is suggested by an additional complication for this kind of question caused by discrete theories. Discrete theories are nonlocal which implies that they have effective formulations of higher derivative type (when differences are to be expanded in a Taylor series to arrive at an effective Hamiltonian or action). For higher derivative theories, in turn, the Legendre transform does not commute with a perturbation expansion: If we start with a higher derivative Lagrangian in which the higher derivative terms can be treated as perturbations, and Legendre transform, then the resulting Hamiltonian will not be analytic in the perturbation parameters [18]. If, on the other hand, we first Legendre transform the full Lagrangian then the perturbation expansion of the resulting Hamiltonian must obviously be analytic in the perturbation parameter.

While a Lagrangian formulation would immediately show whether or not Lorentz invariance is broken by correction terms, the Hamiltonian formulation is more indirect. Since perturbation and Legendre transform do not commute, it is in general not viable to compute corrections to the Hamiltonian and then Legendre transform to find an effective Lagrangian to read off possible Lorentz violations. The corrected Hamiltonian itself would not be of higher order in time derivatives and so the corresponding Lagrangian would be analytic in the expansion parameters. But if higher derivatives for the full expressions have to be expected, the Legendre transform of the perturbed Hamiltonian would not coincide with the perturbed full Lagrangian. In particular if there are higher spatial derivatives in the Hamiltonian, as in any spatially discrete theory, there are two possibilities much more complicated to distinguish perturbatively: Either there are no higher time derivatives in the corresponding full Lagrangian, which would break Lorentz symmetry since there are higher space derivatives; or there are higher time derivatives, in which case the theory may or may not be Lorentz invariant.

We illustrate these issues with an example for a discrete theory with finitely many degrees of freedom. Let the action be $S = \sum_n \epsilon L(q_n, q_{n+1})$ with Lagrangian $L(q_n, q_{n+1}) = (q_{n+1} - q_n)^2/2\epsilon^2$, a discretization of a free particle with discrete time step ϵ. If we define the momentum by $p_n := \epsilon \partial L/\partial q_{n+1}$ and the Hamiltonian by $L(q_n, q_{n+1}) = p_n(q_{n+1} - q_n)/\epsilon - H(q_n, p_n)$, we obtain $p_n = (q_{n+1} - q_n)/\epsilon$ and $H(q_n, p_n) = p_n^2/2$.

We now assume that ϵ is small and approximate the discrete values q_n by a continuous function $q(t)$ such that $q_n = q(\epsilon n)$. Thus,

$$L(q) = \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{\epsilon^{k-1}}{k!} q^{(k)}\right)^2 = \frac{1}{2}(\dot{q}^2 + \epsilon \dot{q}\,\ddot{q} + \epsilon^2(\tfrac{1}{3}\dot{q}q^{(3)} + \tfrac{1}{4}\ddot{q}^2) + O(\epsilon^3))$$

which yields a higher derivative theory at second order in ϵ, which we will use in what follows. (To linear order in ϵ, however, the theory is not higher derivative since the only correction is a total derivative.) Only q itself and the first derivative $\dot{q}$ are independent variables since $q^{(3)}$ can be removed from the Lagrangian by integrating by parts. Removing all the total derivatives and higher orders in ϵ results in the Lagrangian

$$L(q) = \frac{1}{2}\dot{q}^2 - \frac{1}{24}\epsilon^2\ddot{q}^2$$

which after performing a (higher derivative) Legendre transform gives momenta

$$\pi_q := \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} = \dot{q} + \frac{1}{12}\epsilon^2 q^{(3)}$$

$$\pi_{\dot{q}} := \frac{\partial L}{\partial \ddot{q}} = -\frac{1}{12}\epsilon^2\ddot{q}$$

and the Hamiltonian

$$H(q, \pi_q, \dot{q}, \pi_{\dot{q}}) = \dot{q}\pi_q + \ddot{q}\pi_{\dot{q}} - L = \dot{q}\pi_q - \frac{1}{2}\dot{q}^2 - 6\epsilon^{-2}\pi_{\dot{q}}^2. \qquad (16)$$

As anticipated, the Hamiltonian is not analytic in ϵ.

Had we started in a Hamiltonian formulation and gotten our discrete formulation there, as in loop quantum gravity, we would have proceeded differently. First, expanding a Hamiltonian formulation does not introduce new degrees of freedom such as $\dot{q}$ above, which is independent of q and has its own momentum. Moreover, the symplectic structure would be left untouched since it is independent of the Hamiltonian (while a Lagrangian formulation mixes symplectic structure and dynamics). Thus, we would still work with the unperturbed momentum $\pi_q = \dot{q}$ for the only degree of freedom q. This fact can also be derived from the Hamiltonian (16) by solving the Hamiltonian equation of motion $\dot{q} = \partial H/\partial \pi_q$ for $\pi_q(\dot{q})$ which is now assumed not to be an independent variable. We obtain $\dot{q} = \dot{q} - (\dot{q} - \pi_q)\mathrm{d}\dot{q}/\mathrm{d}\pi_q$, which, since now $\dot{q}$ is by assumption no longer independent of π_q, immediately gives $\pi_q = \dot{q}$.

The second difference is that only perturbative corrections to the Hamiltonian could appear, and not the $1/\epsilon$-term above. If we remove this term and use the unperturbed momentum, we obtain $H = \frac{1}{2}\pi_q^2$, which coincides with the continuous approximation of the full discrete Hamiltonian. Thus, to this order of perturbation theory, the Hamiltonian would not show any corrections unlike the Lagrangian. The reason is that in this example the perturbative corrections are all of higher derivative form, which cannot be seen in the Hamiltonian.

The example can easily be extended in such a way that also the Hamiltonian receives perturbative corrections. We now use two discrete coordinates and Lagrangian $L = ((q_{m,n+1} - q_{m,n})^2 - (q_{m+1,n} - q_{m,n})^2)/2\epsilon^2$ where we interpret m as a discrete space coordinate and n as discrete time, as above. The Lagrangian is symmetric under the exchange of m with n, which mimics a space-time symmetry. The momentum of $q_{m,n}$ now is $p_{m,n} = (q_{m,n+1} - q_{m,n})/\epsilon$ and the Hamiltonian $H = \frac{1}{2}(p_{m,n}^2 + (q_{m+1,n} - q_{m,n})^2/\epsilon^2)$.

Perturbing the Lagrangian leads to $L = \frac{1}{2}(\dot{q}^2 - (q')^2 + \epsilon(\dot{q}\,\ddot{q} - q'q'') + \epsilon^2(\frac{1}{3}\dot{q}q^{(3)} + \frac{1}{4}\ddot{q}^2 - \frac{1}{3}q'q''' - \frac{1}{4}(q'')^2) + O(\epsilon^3))$. The perturbed momenta are as before, and the Legendre transform of the perturbed Lagrangian with total derivatives removed as before yields

$$H(q, \pi_q, \dot{q}, \pi_{\dot{q}}) = \dot{q}\pi_q - \frac{1}{2}\dot{q}^2 + \frac{1}{2}(q')^2 + \frac{1}{24}\epsilon^2(q'')^2 - 6\epsilon^{-2}\pi_{\dot{q}}^2.$$

With the prescription above, a perturbation of the Hamiltonian would have led to the analytic expression $H = \frac{1}{2}(\pi_q^2 + (q')^2 + \frac{1}{12}\epsilon^2(q'')^2)$, which only shows the higher order correction (which is of higher derivative form only in space but not in time), but not the nonanalytic correction coming from the higher derivative nature.

These examples have important hints for the calculation of corrected dispersion relations and the issue of Lorentz covariance. Since only higher order corrections will be seen when a Hamiltonian is perturbed, Lorentz violations are bound to appear as a consequence of this way of doing the calculation. Space and time derivatives of the classical fields have to be related in the Lagrangian in a way dictated by the symmetry. If those terms are torn apart, because one computes the Lagrangian from a perturbed Hamiltonian which only sees higher space derivatives but not higher time derivatives in its corrections, Lorentz invariance will be violated. This kind of violation of Lorentz symmetry is not a consequence of the theory but of the way to perform perturbative calculations.

We present one more example showing the role of higher derivatives in Lorentz invariant theories. We use the Lagrangian

$$L = -\frac{1}{2}\int(\psi(\Box + \epsilon\Box^2)\psi + m^2\psi^2)$$

for a scalar of mass m. It leads to the field equation

$$-(\Box + \epsilon\Box^2)\psi = m^2\psi$$

which is Lorentz invariant. The dispersion relation can be computed from the plane wave ansatz $\psi(x,t) = \exp(i(Et - kx))$ and takes the form

$$\epsilon E^4 + (1 - 2\epsilon k^2)E^2 + k^2(\epsilon k^2 - 1) = 0$$

such that

$$E^2 = k^2 - \frac{1}{2\epsilon} \pm \sqrt{1 + 4\epsilon m^2}/2\epsilon.$$

If $\epsilon \ll m^{-2}$, we obtain

$$E^2 = k^2 - \frac{1}{2\epsilon}(1 \mp (1 + 2\epsilon m^2 - 2\epsilon^2 m^4))$$

with two nonanalytic solutions, which have to be discarded in a perturbative situation, and the corrected relation

$$E^2 = k^2 + m^2 - \epsilon m^4 + O(\epsilon^2)$$

which is Lorentz invariant.

The Hamiltonian situation of this example is as follows. We have momenta $\pi_\psi = \dot\psi - 2\epsilon\Delta\dot\psi + \epsilon\dddot\psi$ and $\pi_{\dot\psi} = -\epsilon\ddot\psi$ leading to the Hamiltonian

$$H = -\frac{1}{2}\dot\psi^2 + \dot\psi\pi_\psi - \frac{1}{2}\psi\Delta\psi + \frac{1}{2}m^2\psi^2 + \epsilon(\dot\psi\Delta\dot\psi + \psi\Delta^2\psi/2) - \frac{1}{2}\epsilon^{-1}\pi_{\dot\psi}^2.$$

Again, this is nonanalytic in ϵ, but would lead to Lorentz invariant equations of motion. If, on the other hand, we had started with perturbing a Hamiltonian, we could only have seen the analytic part and would not have introduced new degrees of freedom and instead used $\pi_\psi = \dot\psi$ (this again also follows from the equations of motion under the assumption of having no additional degrees of freedom: $\dot\psi = \partial H/\partial\pi_\psi = -\dot\psi\partial\dot\psi/\partial\pi_\psi + \dot\psi + \pi_\psi\partial\dot\psi/\pi_\psi$ implies immediately $\pi_\psi = \dot\psi$ if, as per our assumption, $\partial\dot\psi/\partial\pi_\psi \neq 0$). Thus, we would have arrived at a Hamiltonian

$$H = \frac{1}{2}(\pi_\psi^2 - \psi\Delta\psi + m^2\psi^2 + \epsilon(2\pi_\psi\Delta\pi_\psi + \psi\Delta^2\psi))$$

and perturbed equation of motion

$$\ddot\psi = \Delta\psi - m^2\psi + \epsilon(\Delta^2\psi - 2m^2\Delta\psi).$$

The dispersion relation for this equation is

$$E^2 = k^2 + m^2 - \epsilon k^2(k^2 + 2m^2)$$

which does break Lorentz invariance.

IV. CONCLUSIONS

The observations presented here beg the question of what is the correct procedure to compute modified dispersion relations from a Hamiltonian point of view when higher derivative terms have to be expected. Such a procedure has to be amended incorporating the semiclassical dynamics in a more controlled way. This, in particular, has to take care of new degrees of freedom that emerge from higher derivative theories. One possibility is to derive a full, nonperturbative discrete Hamiltonian from the quantum theory, which is understood as a classical object but on a discrete space e.g. [19]. Before one expands and computes equations of motion, one has to transform to a Lagrangian, also on discrete space and time. From then on one can work with perturbative expansions and compute modified dispersion relations.

There are obvious difficulties in the way of implementing this procedure in loop quantum gravity since already calculations with a perturbed Hamiltonian are cumbersome. It should however be kept in mind that the calculations done up to now (including the model of the previous section) can only yield preliminary results and that a definite answer to the question of Lorentz violation by loop quantum gravity definitely has to await a more complete treatment, possibly along the lines sketched above. Alternatively, perturbative Hamiltonian techniques for effective actions, which also allow to see additional degrees of freedom coming from higher derivatives, can be developed. This approach, which is now under investigation, would allow to perform the perturbation expansion at the Hamiltonian level all the time.

We expect that the model presented in the first part of this paper can be used for a first step in applying those methods to the issue of dispersion relations. As we showed, it shares most qualitative and even some quantitative features with more elaborate calculations and thus is simple but reliable. It can therefore play a role in deriving modified dispersion relations that better take into account the higher derivative nature.

ACKNOWLEDGMENTS

We are grateful to Aureliano Skirzewski for discussions and comments on a draft. It is a pleasure to acknowledge the support and warm hospitality during visits to the Perimeter Institute and The Pennsylvania State University where this work was completed. HAMT has been partially supported by CONACYT-40745-F; this work was supported in part by the Eberly Research Funds of Penn State and by the NSF Grant No. PHY-00-90091.

[1] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature (London) **393**, 763 (1998); G. Amelino-Camelia, Nature (London) **398**, 216 (1999); D. V. Ahluwalia, Nature (London) **398**, 199 (1999); G. Amelino-Camelia, Lect. Notes Phys. **541**, 1 (2000).

[2] D. Colladay and A. Kostelecký, Phys. Rev. D **58**, 116002 (1998).

[3] S. D. Biller *et al.*, Phys. Rev. Lett. **83**, 2108 (1999); O. Bertolami and C. S. Carvalho, Phys. Rev. D **61**, 103002 (2000); C. Lämmerzahl and C. Bordé, Lect. Notes Phys. **562**, 463 (2001); M. P. Haugan and C. Lämmerzahl, Lect. Notes Phys. **562**, 562 (2001); R. J. Gleiser and C. N. Kozameh, Phys. Rev. D **64**, 083007 (2001); R. Brustein, D. Eichler, and S. Foffa, Phys. Rev. D **65**, 105006 (2002); T. J. Konopka and S. A. Major, New J. Phys. **4**, 57 (2002); D. Sudarsky, L. Urrutia, and H. Vucetich, Phys. Rev. Lett. **89**, 231301 (2002); J. Alfaro and G. Palma, Phys. Rev. D **67**, 083003 (2003); Phys. Rev. D **65**, 103516 (2002); T. Jacobson, S. Liberati and D. Mattingly, Nature (London) **424**, 1019 (2003); Phys. Rev. D **67**, 124011 (2003); G. Lambiase and P. Singh, Phys. Lett. B **565**, 27 (2003); F. W. Stecker, J. Phys. G **29**, R47 (2003); R. C. Myers and M. Pospelov, Phys. Rev. Lett. **90**, 211601 (2003); A. Perez and D. Sudarsky, Phys. Rev. Lett. **91**, 179101 (2003); J. Ellis *et al.*, Astron. Astrophys. **402**, 409 (2003); Astropart. Phys. **20**, 669 (2004); Nature (London) **428**, 386 (2004); R. Montemayor and L. F. Urrutia, Phys. Lett. B **606**, 86 (2005).

[4] J. Collins, A. Perez, D. Sudarsky, L. F. Urrutia, and H. Vucetich, Phys. Rev. Lett. **93**, 191301 (2004).

[5] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002); **11**, 1643 (2002); J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002);

[6] C. Rovelli, Living Rev. Relativity **1**, 1 (1998), http://www.livingreviews.org/Articles/Volume1/1998-1rovelli; T. Thiemann, gr-qc/0110034; A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).

[7] R. Gambini and J. Pullin, Phys. Rev. D **59**, 124021 (1999).

[8] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. Lett. **84**, 2318 (2000; J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. D **65**, 103509 (2002); J. Alfaro, M. Reyes, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. D **70**, 084002 (2004); J. Alfaro, M. Reyes, H. A. Morales-Técotl, and L. F. Urrutia, J. Phys. A **36**, 12097 (2003).

[9] H. Sahlmann and T. Thiemann, gr-qc/0207030; H. Sahlmann and T. Thiemann, gr-qc/0207031.

[10] H. Sahlmann, T. Thiemann, and O. Winkler, Nucl. Phys. **B606**, 401 (2001).

[11] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **18**, L117 (2001.

[12] M. Bojowald, Classical Quantum Gravity **19**, 2717 (2002); Classical Quantum Gravity **20**, 2595 (2003).

[13] M. Bojowald, Classical Quantum Gravity **18**, 1071 (2001).

[14] T. Thiemann, Classical Quantum Gravity **15**, 1281 (1998).

[15] M. Bojowald, Phys. Rev. D **64**, 084018 (2001); Classical Quantum Gravity **19**, 5113 (2002).

[16] A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003).

[17] M. Bojowald and A. Perez, gr-qc/0303026.

[18] J. Z. Simon, Phys. Rev. D **41**, 3720 (1990).

[19] C. Di Bartolo, R. Gambini, and J. Pullin, Classical Quantum Gravity **19**, 5275 (2002); R. Gambini and J. Pullin, Phys. Rev. Lett. **90**, 021301 (2003).