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Phantom energy traversable wormholes

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It has been suggested that a possible candidate for the present accelerated expansion of the Universe is "phantom energy". The latter possesses an equation of state of the form $\omega \equiv p/\rho < -1$, consequently violating the null energy condition. As this is the fundamental ingredient to sustain traversable wormholes, this cosmic fluid presents us with a natural scenario for the existence of these exotic geometries. 'Note, however, that the notion of phantom energy is that of a homogeneously distributed fluid. Nevertheless, it can be extended to inhomogeneous spherically symmetric spacetimes, and it is shown that traversable wormholes may be supported by phantom energy. Because of the fact of the accelerating Universe, macroscopic wormholes could naturally be grown from the submicroscopic constructions that originally pervaded the quantum foam. One could also imagine an advanced civilization mining the cosmic fluid for phantom energy necessary to construct and sustain a traversable wormhole. In this context, we investigate the physical properties and characteristics of traversable wormholes constructed using the equation of state $p = \omega\rho$, with $\omega < -1$. We analyze specific wormhole geometries, considering asymptotically flat spacetimes and imposing an isotropic pressure. We also construct a thin shell around the interior wormhole solution, by imposing the phantom energy equation of state on the surface stresses. Using the "volume integral quantifier" we verify that it is theoretically possible to construct these geometries with vanishing amounts of averaged null energy condition violating phantom energy. Specific wormhole dimensions and the traversal velocity and time are also deduced from the traversability conditions for a particular wormhole geometry. These phantom energy traversable wormholes have far-reaching physical and cosmological implications. For instance, an advanced civilization may use these geometries to induce closed timelike curves, consequently violating causality.

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I. INTRODUCTION

It is extraordinary that recent observations have confirmed that the Universe is undergoing a phase of accelerated expansion. Evidence of this cosmological expansion, coming from measurements of supernovae of type Ia (SNe Ia) [1,2] and independently from the cosmic microwave background radiation [3,4], shows that the Universe additionally consists of some sort of negative pressure "dark energy". The Wilkinson Microwave Anisotropy Probe (WMAP), designed to measure the CMB anisotropy with great precision and accuracy, has recently confirmed that the Universe is composed of approximately 70% of dark energy [3]. Several candidates representing dark energy have been proposed in the literature, namely, a positive cosmological constant, the quintessence fields, generalizations of the Chaplygin gas and so-called tachyon models. A simple way to parameterize the dark energy is by an equation of state of the form $\omega \equiv p/\rho$, where p is the spatially homogeneous pressure and ρ the energy density of the dark energy [5]. A value of $\omega < -1/3$ is required for cosmic expansion, and $\omega = -1$ corresponds to a cosmological constant [6]. The particular case of $\omega = -2/3$ is extensively analyzed in [7]. A possibility that has been widely explored, is that of quintessence, a cosmic scalar field ϕ which has not yet reached the minimum of its potential $V(\phi)$ [8,9]. A common example is the energy of a slowly evolving scalar field with positive potential energy, similar to the inflaton field used to describe the inflationary phase of the Universe. In quintessence models the parameter range is $-1 < \omega < -1/3$, and the dark energy decreases with a scale factor $a(t)$ as $\rho_Q \propto a^{-3(1+\omega)}$ [10,11].

However, a note on the choice of the imposition $\omega > -1$ is in order. This is considered to ensure that the null energy condition, $T_{\mu\nu}k^\mu k^\nu > 0$, is satisfied, where $T_{\mu\nu}$ is the stress-energy tensor and k^μ any null vector. If $\omega < -1$ [12–14], a case certainly not excluded by observation, then the null energy condition is violated, $\rho + p < 0$, and consequently all of the other energy conditions. Note that the dark energy density is positive, $\rho > 0$. Matter with the property $\omega < -1$ has been denoted "phantom energy". Apart from the null energy condition violation phantom energy possesses other strange properties, namely, phantom energy probably mediates a long-range repulsive force [15], phantom thermodynamics leads to a negative entropy (or negative temperature) [16,17], and the energy density increases to infinity in a finite time [11,18], at which point the size of the Universe blows up in a finite time. This is known as the Big Rip. To an observer on Earth this corresponds to observing the galaxies being stripped apart, the Earth being itself ripped from its gravitational attraction to the Sun, before being eventually ripped apart, followed by the dissociation of molecules and atoms, and finally of

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nuclei and nucleons [11]. However, it has been shown that in certain models the presence of phantom energy does not lead to the above-mentioned doomsday [19–24]. It was also shown that quantum corrections may prevent [25], or at least delay, the Big Rip by taking into account the back reaction of conformal quantum fields near the singularity [26–28], and the universe may end up in a deSitter phase before the scale factor blows up. (It has also been shown that quantum effects, without a negative kinetic term, could also lead to a superaccelerated phase of inflation, with a weak energy condition violating, on average and not just in fluctuations, dark energy equation of state $\omega < -1$ on cosmological scales [29,30]). Phantom cosmologies have also been studied in the context of braneworld scenarios [31] and research has been invested in black holes in an accelerating cosmic expansion with phantom energy. In an interesting paper it was shown that the masses of all black holes tend to zero as the phantom energy universe approaches the Big Rip [32]. An interesting candidate for phantom energy, considering a positive energy density so that the field is purely imaginary, is that of cosmic axions, described by a rank-3 tensor field predicted by supergravity and string theories [33]. It has also been shown, recently, that constraints from the current supernovae Ia Hubble diagram [34] favor a negative coupling between dark energy and dark matter, implying the existence of phantom energy, and an equation of state $w < -1$. In fact, recent fits to supernovae, cosmic microwave background radiation and weak gravitational lensing data indicate that an evolving equation of state ω which crosses -1 is mildly favored [35,36]. If confirmed in the future, this behavior has important implications for theoretical models of dark energy. For instance, this implies that dark energy is dynamical and, in addition, to excluding the cosmological constant as a possible candidate for dark energy, the models with a constant parameter, such as quintessence and phantom models, cannot de satisfied either. The evolving dark energy with an equation of state ω crossing -1 during its evolution was dubbed "Quintom" [35,37,38], as it is different from the quintessence or phantom fields in the determination of the evolution and fate of the universe. All of these models present an extremely fascinating aspect for future experiments focussing on supernovae, cosmic microwave background radiation and weak gravitational lensing and for future theoretical research.

As the possibility of phantom energy implies the violation of the null energy condition, this presents us with a natural scenario for the existence of traversable wormholes. The latter possess a peculiar property, namely, "exotic matter", involving a stress-energy tensor that violates the null energy condition [39,40], precisely the fundamental ingredient of phantom energy. (Recently, wormhole throats were also analyzed in a higher derivative gravity model governed by the Einstein-Hilbert Lagrangian, supplemented with $1/R$ and R^2 curvature scalar terms [41], and using the resulting equations of motion, it was found that the weak energy condition may be respected in the throat vicinity). Thus, one could imagine an absurdly advanced civilization, for instance, mining phantom energy necessary to construct and sustain a traversable wormhole. Another interesting scenario is that due to the fact of the accelerated expansion of the Universe, macroscopic wormholes could naturally be grown from the submicroscopic constructions that originally pervaded the gravitational vacuum, in a manner similar to the inflating wormholes analyzed by Roman [42]. In fact, González-Díaz analyzed the evolution of wormhole and ringhole spacetimes embedded in a background accelerating Universe [43] driven by dark energy. It was shown that the wormhole's size increases by a factor which is proportional to the scale factor of the Universe, and still increases significantly if the cosmic expansion is driven by phantom energy. González-Díaz further considered the accretion of dark and phantom energy onto Morris-Thorne wormholes [44,45]. It was shown that this accretion gradually increases the wormhole throat which eventually overtakes the accelerated expansion of the universe and becomes infinite at a time in the future before the big rip. As it continues accreting phantom energy, the wormhole becomes an Einstein-Rosen bridge, which pinches off rendering it nontraversal, and the respective mass decreases rapidly and vanishes at the big rip.

As the material with the properties necessary to sustain traversable wormholes, namely, null energy condition violating phantom energy, probably comprises of 70% of the constitution of the Universe, it is of particular interest to investigate the physical properties and characteristics of these specific wormhole geometries. In fact, one can trace back to the 1970s examples of wormholes with a negative kinetic energy scalar [46–49], which may be considered as the current phantom energy source [50]. In this paper, we will be interested in constructing wormhole solutions using the equation of state $p = \omega\rho$, with $\omega < -1$, that describes phantom energy in cosmology. Independent work was carried out by Sushkov [51] (which we became aware of after the completion of the present paper). Despite the fact that the notion of phantom energy is that of a homogeneously distributed fluid in the Universe, as emphasized in [51], it can be extended to inhomogeneous spherically symmetric spacetimes by regarding that the pressure in the equation of state $p = \omega\rho$ is now a negative radial pressure, and noting that the transverse pressure p_t may be determined from the Einstein field equations. This is fundamentally the analysis carried out by Sushkov and Kim [52], while constructing a time-dependent solution describing a spherically symmetric wormhole in a cosmological setting with a ghost scalar field. It was shown that the radial pressure is negative everywhere and far from the wormhole throat equals the transverse pressure, showing that the ghost scalar field behaves essentially as dark energy. Sushkov, in [51], considered specific choices for the distribution of the energy density threading the wormhole. However, we trace out a complementary approach by modelling an appropriate wormhole geometry imposing specific choices for the form function and/or the redshift function, and consequently determining the stress-energy components. We find that particularly interesting solutions exist, and by using the ''volume integral quantifier'', it is found that these wormhole geometries are, in principle, sustained by arbitrarily small amounts of averaged null energy condition (ANEC) violating phantom energy.

This paper is outlined in the following manner. In Sec. II, we present a general solution of a traversable wormhole comprising of phantom energy. We also present the case of a thin shell surrounding the interior wormhole geometry, in which the surface stresses obey the phantom energy equation of state. In Sec. III, we construct specific traversable wormhole geometries, namely, asymptotically flat spacetimes and the case of an isotropic pressure. We also show that using these specific constructions, and taking into account the ''volume integral quantifier'', one may theoretically construct these spacetimes with infinitesimal amounts of ANEC violating phantom energy. Specific wormhole dimensions and traversal velocities and time are also determined from the traversability conditions, by considering a particular wormhole geometry. Finally, in Sec. IV, we conclude.

II. TRAVERSABLE WORMHOLE SPACETIMES COMPRISING OF PHANTOM ENERGY

The spacetime metric representing a spherically symmetric and static wormhole is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, r. $\Phi(r)$ is denoted as the redshift function, for it is related to the gravitational redshift; $b(r)$ is called the form function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [39]. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat, to a, where the interior spacetime will be joined to an exterior vacuum solution.

Using the Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, in an orthonormal reference frame, (with $c = G = 1$) we obtain the following stress-energy scenario

$$\rho(r) = \frac{1}{8\pi}\frac{b'}{r^2}, \tag{2}$$

$$p_r(r) = \frac{1}{8\pi}\left[2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r} - \frac{b}{r^3}\right], \tag{3}$$

$$p_t(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r}\right], \tag{4}$$

in which $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. Using the conservation of the stress-energy tensor, $T^{\hat{\mu}\hat{\nu}}_{;\hat{\nu}} = 0$, we obtain the following equation

$$p_r' = \frac{2}{r}(p_t - p_r) - (\rho + p_r)\Phi', \tag{5}$$

which can be interpreted as the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole. Note that Eq. (5) can also be obtained from the field equations by eliminating the term Φ'' and taking into account the radial derivative of Eq. (3).

Now using the equation of state representing phantom energy, $p_r = \omega\rho$ with $\omega < -1$, and taking into account Eqs. (2) and (3), we have the following condition

$$\Phi'(r) = \frac{b + \omega r b'}{2r^2(1 - b/r)}. \tag{6}$$

We now have four equations, namely, the field equations, i.e., Eqs. (2)–(4) and (6), with five unknown functions of r, i.e., $\rho(r)$, $p_r(r)$, $p_t(r)$, $b(r)$ and $\Phi(r)$. To construct solutions with the properties and characteristics of wormholes, we consider restricted choices for $b(r)$ and/or $\Phi(r)$. In particular, by appropriately choosing the form function, one may integrate Eq. (6), to determine the redshift function, $\Phi(r)$. As an alternative, one may choose $\Phi(r)$, and through the following integral

$$b(r) = r_0\left(\frac{r_0}{r}\right)^{1/\omega} e^{-(2/\omega)[\Phi(r)-\Phi(r_0)]}\left[\frac{2}{\omega}\times\int_{r_0}^{r}\left(\frac{r}{r_0}\right)^{(1+\omega)/\omega}\Phi'(r)e^{(2/\omega)[\Phi(r)-\Phi(r_0)]}dr + 1\right], \tag{7}$$

obtained from Eq. (6), one may determine the form function, and consequently the stress-energy tensor components.

Now, in cosmology the energy density related to the phantom energy is considered positive, $\rho > 0$, so we shall maintain this condition. This implies that only form functions of the type $b'(r) > 0$ are allowed. At the throat we have the flaring out condition given by $(b - b'r)/b^2 > 0$ [39,40], which may be deduced from the mathematics of embedding. From this we verify that at the throat $b(r_0) = r = r_0$, the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with $e^{2\Phi} \to 0$, so that $\Phi(r)$ must be finite everywhere. Note that the condition $1 - b/r > 0$

is also imposed. We can construct asymptotically flat spacetimes, in which $b(r)/r \to 0$ and $\Phi \to 0$ as $r \to \infty$. However, one may also construct solutions with a cut-off of the stress-energy, by matching the interior solution to an exterior vacuum spacetime, at a junction interface. If the junction contains surface stresses, we have a thin shell, and if no surface stresses are present, the junction interface is denoted a boundary surface.

For instance, consider that the exterior solution is the Schwarzschild spacetime, given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$

In this case the spacetimes given by the metrics Eq. (1) and (8) are matched at a, and one has a thin shell surrounding the wormhole. Using the the Darmois-Israel formalism [53–57], the surface stresses are given by

$$\sigma = -\frac{1}{4\pi a}\left(\sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}}\right), \tag{9}$$

$$\mathcal{P} = \frac{1}{8\pi a}\left(\frac{1 - \frac{M}{a}}{\sqrt{1 - \frac{2M}{a}}} - \zeta\sqrt{1 - \frac{b(a)}{a}}\right), \tag{10}$$

where $\zeta = [1 + a\Phi'(a)]$ is the redshift parameter [58]; σ is the surface energy density and $\mathcal{P}$ the surface pressure. Construction of wormhole solutions by matching an interior wormhole spacetime to an exterior vacuum solution, at a junction surface, were also recently analyzed [58–60]. In particular, a thin shell around a traversable wormhole, with a zero surface energy density was analyzed in [59], and with generic surface stresses in [58]. A general class of wormhole geometries with a cosmological constant and junction conditions was explored in [61], and a linearized stability analysis of thin-shell wormholes with Λ was studied in [62]. A similar analysis for the plane symmetric case, with a negative cosmological constant, is done in [63].

The surface mass of the thin shell is given by $M_s = 4\pi a^2\sigma$. Note that by rearranging the terms in Eq. (9), one obtains the following relationship

$$M = \frac{b(a)}{2} + M_s\left(\sqrt{1 - \frac{b(a)}{a}} - \frac{M_s}{2a}\right), \tag{11}$$

where M may be interpreted as the total mass of the wormhole in one asymptotic region.

It is also of interest to impose the phantom energy equation of state on the surface stresses, i.e., $\mathcal{P} = \omega\sigma$, with $\omega < -1$. For this case, we have the following condition

$$(\zeta - 2|\omega|)\sqrt{1 - \frac{b(a)}{a}} = \frac{(2|\omega| - \frac{1}{2})\frac{2M}{a} - (2|\omega| - 1)}{\sqrt{1 - \frac{M}{a}}}. \tag{12}$$

One has several cases to analyze. If $\zeta \geq 2|\omega|$, then the thin shell lies in the following range

$$2M < a \leq \left(\frac{2|\omega| - \frac{1}{2}}{2|\omega| - 1}\right)2M. \tag{13}$$

If $\zeta < 2|\omega|$, then $a > 2M(2|\omega| - 1/2)/(2|\omega| - 1)$.

If one is tempted to construct a boundary surface, i.e., $\sigma = \mathcal{P} = 0$, then one needs to impose the conditions: $b(a) = 2M$ and $a = 2M(\zeta - 1/2)/(\zeta - 1)$. Note that from the latter condition, taking into account $a > 2M$, we have an additional restriction imposed on the redshift parameter, namely, $\zeta > 1$.

III. SPECIFIC WORMHOLE CONSTRUCTION

A. Asymptotically flat spacetimes

1. Specific choice for the form function: $b(r) = r_0(r/r_0)^\alpha$

Consider the particular choice of the form function $b(r) = r_0(r/r_0)^\alpha$, with $0 < \alpha < 1$. For this case we readily verify that $b'(r) = \alpha(r/r_0)^{\alpha-1}$, so that at the throat $b'(r_0) = \alpha < 1$, and that for $r \to \infty$ we have $b(r)/r = (r_0/r)^{1-\alpha} \to 0$. From Eq. (6), taking into account the above-mentioned form function, we find the following solution

$$\Phi(r) = \frac{1}{2}\left(\frac{1 + \alpha\omega}{1 - \alpha}\right)\ln\left[1 - \left(\frac{r_0}{r}\right)^{1-\alpha}\right]. \tag{14}$$

The spacetime metric in this case is given by

$$ds^2 = -[1 - (r_0/r)^{1-\alpha}]^{(1+\alpha)\omega/(1-\alpha)}dt^2 + \frac{dr^2}{1 - (r_0/r)^{1-\alpha}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{15}$$

Now, if $1 + \alpha\omega > 0$, then the spacetime is asymptotically flat, $e^{2\Phi} \to 1$ as $r \to \infty$ (recall that by construction $b/r \to 0$). However, an event horizon is located at $r = r_0$, implying the existence of a nontraversable wormhole.

If $1 + \alpha\omega = 0$, so that $e^{2\Phi} = 1$, then no event horizon is present. Thus, for this case the parameters α and ω are related by the condition $\alpha = -1/\omega$. Note that this solution can also be obtained in the following manner. Firstly, consider $\Phi(r) = \text{const}$, and from Eq. (7) one obtains $b(r) = r_0(r/r_0)^{-1/\omega}$. Then, by confronting this solution with the above-mentioned form function, one may identify the relationship $\alpha = -1/\omega$. Therefore, if one determines the parameter ω from observational cosmology, assuming the existence of phantom energy, then one may theoretically construct traversable wormholes, by imposing the condition $\alpha = -1/\omega$ in the above-mentioned form function and considering a constant redshift function.

Taking into account the notion of the "volume integral quantifier" one may quantify the "total amount" of energy condition violating matter. This notion amounts to calculating the definite integrals $\int T_{\mu\nu}U^\mu U^\nu dV$ and $\int T_{\mu\nu}k^\mu k^\nu dV$, and the amount of violation is defined as the extent to which these integrals become negative. (It is also interesting to note that recently, by using the "volume integral quantifier", fundamental limitations on "warp drive" spacetimes were found, for nonrelativistic warp velocities [64,65]). Visser *et al.* recently found that by considering specific examples of traversable wormhole geometries, one may theoretically construct these spacetimes with infinitesimal amounts of ANEC violating matter [66,67]. In this work we will also be interested in investigating whether this is the case for phantom energy. The integral which provides information about the "total amount" of ANEC violating matter in the spacetime is given by (see [66,67] for details)

$$I_V = \int [\rho(r) + p_r(r)]dV = \left[(r-b)\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]_{r_0}^{\infty} - \int_{r_0}^{\infty}(1-b')\left[\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]dr. \quad (16)$$

Note that the boundary term at the throat r_0 vanishes by construction of the wormhole, and the term at infinity only vanishes if we are assuming asymptotically flat spacetimes.

Considering the specific choices for the form function and redshift function for the traversable wormhole, Eq. (16) takes the form

$$I_V = \int_{r_0}^{\infty}\left[1-\alpha\left(\frac{r_0}{r}\right)^{1-\alpha}\right]\left\{\ln\left[1-\left(\frac{r_0}{r}\right)^{1-\alpha}\right]\right\}dr. \quad (17)$$

Now as in [66,67], suppose that we have a wormhole field deviating from the Schwarzschild solution from the throat out to a radius a. In particular, this amounts to matching the interior solution to an exterior spacetime at a. From Eqs. (9) and (10), and taking into account the choices of the form function and the redshift function considered above, we verify that the junction interface necessarily comprises of a thin shell. Consider, for simplicity, $\alpha = 1/2$, so that the volume integral assumes the value

$$I_V = r_0\left(1-\sqrt{\frac{a}{r_0}}\right) + a\left(1-\sqrt{\frac{r_0}{a}}\right)\left[\ln\left(1-\sqrt{\frac{r_0}{a}}\right)\right]. \quad (18)$$

Taking the limit $a \to r_0$, one verifies that $I_V = \int(\rho + p_r)dV \to 0$. Thus, as in the examples presented in [66,67], one may construct a traversable wormhole with arbitrarily small quantities of ANEC violating phantom energy. Although this result is not unexpected it is certainly a fascinating prospect that an advanced civilization may probably construct and sustain a wormhole with vanishing amounts of the material that comprises of approximately 70% of the constitution of the Universe.

It is also interesting to consider the traversability conditions required for a human being to journey through the wormhole. Firstly, the condition required that the acceleration felt by the traveller should not exceed Earth's gravity, $g_\oplus$, is given by (see [39] for details)

$$\left|\left(1-\frac{b}{r}\right)^{1/2} e^{-\Phi}(\gamma e^{\Phi})'\right| \leq g_\oplus. \quad (19)$$

Secondly, the condition required that the tidal accelerations should not exceed the Earth's gravitational acceleration yields the following restrictions

$$\left|\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi'\right]\right| |\eta^{\hat{1}'}| \leq g_\oplus, \quad (20)$$

$$\left|\frac{\gamma^2}{2r^2}\left[v^2\left(b'-\frac{b}{r}\right) + 2(r-b)\Phi'\right]\right| |\eta^{\hat{2}'}| \leq g_\oplus, \quad (21)$$

where $|\eta^{\hat{i}'}|$ is the separation between two arbitrary parts of the traveller's body, and as in [39] we shall assume $|\eta^{\hat{i}'}| \approx 2$ m along any spatial direction in the traveller's reference frame.

Finally, considering that the space stations are positioned just outside the junction radius, a, at $l = -l_1$ and $l = l_2$, respectively, where $dl = (1-b/r)^{-1/2}dr$ is the proper radial distance, the traversal time as measured by the traveller and for the observers that remain at rest at space stations are given by

$$\Delta\tau = \int_{-l_1}^{+l_2}\frac{dl}{v\gamma} \qquad \text{and} \qquad \Delta t = \int_{-l_1}^{+l_2}\frac{dl}{ve^{\Phi}}, \quad (22)$$

respectively,

We verify that for the wormhole constructed in this Section with the specific choice of $b(r) = r_0(r/r_0)^\alpha$ and $\Phi(r) = \text{const}$, considering for simplicity a constant nonrelativistic, $\gamma \approx 1$, traversal velocity, the restriction of the inequalities (19) and (20) are readily satisfied. From the restriction (21), evaluated at the throat one obtains the inequality

$$v \leq r_0\sqrt{\frac{2g_\oplus}{(1-\alpha)|\eta^{\hat{2}'}|}} \quad (23)$$

Considering the equality case, taking into account $\alpha = 1/2$, and imposing that the wormhole throat is given by $r_0 \approx 10^2$ m, then one obtains $v \approx 4 \times 10^2$ m/s for the traversal velocity. If one considers that the junction radius is given by $a \approx 10^4$ m, then from the traversal times $\Delta\tau \approx \Delta t \approx 2a/v$, one obtains $\Delta\tau \approx \Delta t \approx 50$ s.

2. Specific choice for the redshift function: $\Phi(r)=(r_0/r)$

Using the redshift function given by $\Phi(r) = (r_0/r)$, and solving the integral of Eq. (7), we have as a solution the form function given by

$$b(r) = 2r_0\left\{\left(-\frac{2r_0}{\omega r}\right)^{1/\omega} e^{-(2r_0/\omega r)} \times \left[C + \mathcal{F}\left(\frac{\omega-1}{\omega}, -\frac{2r_0}{\omega r}\right)\right] - 1\right\}, \qquad (24)$$

where C is a constant, given by

$$C = \frac{3}{2}\left(-\frac{2}{\omega}\right)^{-1/\omega} e^{2/\omega} - \mathcal{F}\left(\frac{\omega-1}{\omega}, -\frac{2}{\omega}\right) \qquad (25)$$

and the function $\mathcal{F}$ is defined as

$$\mathcal{F}(x, z) = \Gamma(x, z) - \Gamma(x), \qquad (26)$$

where $\Gamma(x)$ and $\Gamma(x, z)$ are the Gamma and the incomplete Gamma functions, respectively.

Although the form function is extremely messy, the message that one can extract from this analysis, for $\omega < -1$, is that one may prove that $b(r)/r \to 0$ as $r \to \infty$, and the condition $1 - b/r > 0$ is also obeyed. Thus, by construction, as $\Phi(r) \to 0$ when $r \to \infty$, the spacetime is asymptotically flat.

Using the "volume integral quantifier" provided by Eq. (16), and substituting the form and redshift functions chosen in this section, one ends up with an intractable integral. However, one may prove that for $\omega < -1$, the volume integral is vanishingly small, i.e., $I_V \to 0$ as $a \to r_0$, and the message is that once again one can theoretically construct traversable wormholes with vanishing amounts of ANEC violating phantom energy.

B. Isotropic pressure, $p_r = p_t = p$

It is of particular interest to consider an isotropic pressure, $p_r = p_t = p$, so that Eq. (5) with the equation of state $p = \omega\rho$, reduces to

$$p' = -\left(\frac{1+\omega}{\omega}\right)p\Phi', \qquad (27)$$

which is immediately integrated to provide the solution

$$p(r) = \omega\rho(r) = -\frac{1}{8\pi r_0^2} e^{-(1+\omega)/(\omega)[\Phi(r)-\Phi(r_0)]}. \qquad (28)$$

One may consider that $\Phi(r_0) = 0$ without a significant loss of generality. Note that the isotropic pressure is always negative, implying the presence of an isotropic tension.

Now, substituting Eq. (28) in Eq. (2), one deduces the following relationship

$$b'(r) = -\frac{1}{\omega}\left(\frac{r}{r_0}\right)^2 e^{-(1+\omega)/(\omega)\Phi(r)}, \qquad (29)$$

which may be rewritten as

$$\Phi(r) = -\left(\frac{\omega}{1+\omega}\right)\ln\left[-\omega b'(r)\left(\frac{r_0}{r}\right)^2\right]. \qquad (30)$$

From this relationship one verifies that for $\Phi(r)$ to be finite, then $b(r) \propto r^3$, so that generically one cannot construct asymptotically flat traversable wormholes with isotropic pressures. Nevertheless, one may match the interior wormhole solution to an exterior vacuum spacetime at a finite junction surface.

One may consider specific choices for the redshift function, then from Eq. (29), deduce $b(r)$. However, using the form function considered above, i.e., $b(r) = r_0(r/r_0)^\alpha$, with $0 < \alpha < 1$, one finds that the redshift function, Eq. (30), is given by

$$\Phi(r) = \ln\left[\left(-\frac{1}{\omega\alpha}\right)^{\omega/(1+\omega)}\left(\frac{r}{r_0}\right)^{\omega(3-\alpha)/(1+\omega)}\right]. \qquad (31)$$

As we have considered that $\Phi(r_0) = 0$, then the relationship $\alpha = -1/\omega$ is imposed. Thus, the spacetime metric is given by

$$ds^2 = -(r/r_0)^{2\omega(3-\alpha)/(1+\omega)}dt^2 + \frac{dr^2}{1-(r_0/r)^{1-\alpha}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (32)$$

which, as generically noted above, is not asymptotically flat, so we need to match this solution to an exterior vacuum spacetime. The stress-energy scenario is given by

$$p(r) = \omega\rho = -\frac{1}{8\pi r_0^2}\left(\frac{r_0}{r}\right)^{3-\alpha}. \qquad (33)$$

Using the "volume integral quantifier", Eq. (16), with a cut-off of the stress-energy at a, and taking into account the choices for the form function and redshift function considered above, we have

$$I_V = a\left[1 - \frac{b(a)}{a}\right]\left\{\ln\left[\frac{e^{2\Phi(a)}}{1-b(a)/a}\right]\right\} - \int_{r_0}^{a}(1-b')\left[\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]dr \qquad (34)$$

$$= a\left[1 - \left(\frac{r_0}{a}\right)^{1-\alpha}\right]\left\{\ln\left[\frac{(a/r_0)^{2\omega(3-\alpha)/(1+\omega)}}{1-(r_0/a)^{1-\alpha}}\right]\right\} - \int_{r_0}^{a}\left[1 - \alpha\left(\frac{r_0}{r}\right)^{1-\alpha}\right]\left\{\ln\left[\frac{(r/r_0)^{2\omega(3-\alpha)/(1+\omega)}}{1-(r_0/r)^{1-\alpha}}\right]\right\}dr. \qquad (35)$$

Considering the specific example of $\omega = -1/\alpha = -2$, the volume integral takes the following value

$$I_V = a(1 - \sqrt{r_0/a})\ln\left[\frac{(a/r_0)^{10}}{1-\sqrt{r_0/a}}\right] + (10a + 11r_0 - 21r_0\sqrt{a/r_0}) + a(\sqrt{r_0/a} - 1) \times \ln\left[\frac{(a/r_0)^{21/2}}{\sqrt{a/r_0} - 1}\right]. \qquad (36)$$

Once again taking the limit $a \to r_0$, one verifies that

$I_V \rightarrow 0$, and as before one may construct a traversable wormhole with arbitrarily small quantities of ANEC violating phantom energy.

IV. SUMMARY AND DISCUSSION

A possible candidate for the accelerated expansion of the Universe is ''phantom energy'', a cosmic fluid governed by an equation of state of the form $\omega = p/\rho < -1$, which consequently violates the null energy condition. This property is precisely the fundamental ingredient needed to sustain traversable wormholes. It was shown in [44] that as a (submicroscopic) wormhole accretes phantom energy, the radius of the respective throat will gradually increase to macroscopic dimensions. Thus, it seems that as the phantom energy dominates, a natural process for the formation and growth of macroscopic traversable wormholes exists. As the Universe probably constitutes of approximately 70% of phantom energy, one may also imagine an absurdly advanced civilization mining this cosmic fluid to construct and maintain traversable wormhole geometries with the characteristics described in this paper. We have analyzed the physical properties and characteristics of traversable wormholes using the specific phantom energy equation of state. We have constructed a thin shell around the interior wormhole solution, by imposing the phantom energy equation of state on the surface stresses. We have also analyzed specific wormhole geometries, by considering asymptotically flat spacetimes and by imposing an isotropic pressure. Using the ''volume integral quantifier'' we have verified that it is theoretically possible to construct these geometries with vanishing amounts of ANEC violating phantom energy. Specific wormhole dimensions and the traversal velocity and time were also deduced from the traversability conditions for a particular wormhole geometry.

These phantom energy traversable wormholes have far-reaching physical and cosmological implications. Apart from being used for interstellar shortcuts, an absurdly advanced civilization may convert them into time-machines [40,68,69]. This is a troubling issue, depending on one's point of view, as it probably implies the violation of causality. The cosmological implications are also extremely interesting. As the wormhole continues to accrete phantom energy, the throat will blow up in a finite time $\tilde{t}$ before the occurrence of the big rip singularity at t_* [44]. At $\tilde{t}$, the exotic energy density becomes zero, and the traversable wormhole is converted into a nontraversable Einstein-Rosen bridge, which pinches off producing a black hole/white hole pair, in which the mass tends to zero as the big rip singularity is approached [44]. In [43] it was shown that the solution of the scale factor derived from the Friedamnn equation shows two branches around the occurrence of the big rip, t_*. From the first, one verifies that the universe accelerates towards the big rip singularity at t_*, and the other solution, $t > t_*$, describes a universe which exponentially decelerates towards a zero size as $t \rightarrow \infty$. Therefore, in a rather speculative scenario, one may imagine a grown macroscopic wormhole with one mouth opening in the expanding universe and the other in the contracting universe. As the first mouth is expanding and the second contracting, a time-shift would be created between both mouths, transforming the wormhole into a time machine, so that a traveller journeying through the wormhole before the big rip would be transported into his future, and thus circumvent the big rip singularity. Finally, we point out that the confirmation of the existence of phantom energy, with an equation of state $w < -1$, from observational cosmology is an extremely fascinating aspect for future experiments focussing on supernovae, cosmic microwave background radiation and weak gravitational lensing, and consequently for theoretical research.

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