

Phase dynamics and particle production in preheatingT. Charters,^{1,*} A. Nunes,^{2,†} and J. P. Mimoso^{2,‡}¹*Departamento de Mecânica/Secção de Matemática, Instituto Superior de Engenharia de Lisboa, Rua Conselheiro Emídio Navarro, 1, P-1949-014 Lisbon, Portugal and Centro de Física Teórica e Computacional da Universidade de Lisboa, Avenida Professor Gama Pinto 2, P-1649-003 Lisbon, Portugal*²*Departamento de Física, Faculdade de Ciências da Universidade de Lisboa and Centro de Física Teórica e Computacional da Universidade de Lisboa, Avenida Professor Gama Pinto 2, P-1649-003 Lisbon, Portugal*

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We study a simple model of a massive inflaton field ϕ coupled to another scalar field χ with interaction term $g^2\phi^2\chi^2$. We use the theory developed by Kofman *et al.* [L. Kofman, A. D. Linde, and A. A. Starobinsky, *Phys. Rev. D* **56**, 3258 (1997).] for the first stage of preheating to give a full description of the dynamics of the χ field modes, including the behavior of the phase, in terms of the iteration of a simple family of circle maps. The parameters of this family of maps are a function of time when expansion of the Universe is taken into account. With this more detailed description, we obtain a systematic study of the efficiency of particle production as a function of the inflaton field and coupling parameters, and we find that for $g \lesssim 3 \times 10^{-4}$ the broad resonance ceases during the first stage of preheating.

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I. INTRODUCTION

The success of the reheating stage after inflation is crucial to most realizations of the inflationary paradigm. The Universe needs to recover a temperature high enough for primordial nucleosynthesis to take place in accordance with the usual pattern of the standard cosmological model. Excluding scenarios designed to avoid the extreme cooling produced by inflation (such as the warm inflation scenario), it is important that the necessary post-inflationary reheating be efficiently achieved.

The reheating mechanism was proposed as a period, immediately after inflation, during which the inflaton field ϕ oscillates coherently about its ground state and swiftly transfers its energy into ultrarelativistic matter and radiation, here modeled by another scalar field χ . This process depends on the coupling between ϕ and χ in the interaction Lagrangian. The classical theory of reheating was developed in [1–4]. The importance of broad resonance and approximations to deal with nonperturbative effects were introduced in [5–9]. The theory was put forward in [10–12], where the analysis included the effects of expansion of the Universe; see also [13,14]. This represented a shift away from the simple picture of static Mathieu resonant bands, due to large phase fluctuations, which behave irregularly in the nonperturbative regime in an expanding universe. Extensions in this setting include, among others, the study of metric perturbations and the application to string or supersymmetric theories [15–20].

The present understanding (see [21]) distinguishes the process in two parts. A preheating mechanism by which fluctuations of the inflaton couple to one (or more [22]) scalar fields, inducing the resonant amplification of perturbations in the latter. Depending on the coupling, efficient energy transfer requires the amplitude of the inflaton oscillation to be rather large, away from the narrow resonance regime where only perturbations with wave numbers in small intervals are unstable. As the amplitude of the perturbed field grows, backreaction effects may have to be considered, since the frequency of the inflaton oscillations is no longer given by its mass, but depends also on the total number density of the perturbed field particles through the coupling term. The first stage of preheating is the period when these backreaction effects are negligible, and the inflaton field dynamics is approximated by its uncoupled equations. Preheating ends when resonant amplification terminates, either because of the decreasing amplitude of the inflaton field or due to the backreaction and rescattering effects of the second stage. After the second stage of preheating, the reheating period corresponds to the decay of the perturbed fields as well as that of the inflaton field that comes out of the preheating period, leaving the Universe after thermalization with the temperature required by the subsequent processes, namely, nucleosynthesis.

We consider a basic model describing the inflaton field ϕ interacting with a scalar field χ in a flat Friedman-Robertson-Walker (FRW) universe,

$$L = \frac{1}{2} \phi_{,i} \phi^{,i} + \frac{1}{2} \chi_{,i} \chi^{,i} - V(\phi) - V_{\text{int}}(\phi, \chi). \quad (1)$$

This is the simplest model that still contains the basic

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features for the understanding of particle creation in the early Universe and one of the few models for which an analytical study can be performed; see also [7,11,23]. With this in mind we concentrate on the simplest chaotic model with the potential $V(\phi) = 1/2m_\phi^2\phi^2$ and interaction potential $V_{\text{int}}(\phi, \chi) = g^2\phi^2\chi^2$. The evolution of the flat FRW universe is given by

$$3H^2 = \frac{8\pi}{m_{\text{pl}}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\chi}^2 + g^2\phi^2\chi^2 \right) \quad (2)$$

where $H = \dot{R}/R$ and R is the FRW scalar factor. The equations of motion in a FRW universe for a homogeneous scalar field ϕ coupled to the k mode of the χ field are given by

$$\ddot{\phi} + 3H\dot{\phi} + (m_\phi^2 + g^2\chi_k^2)\phi = 0, \quad (3)$$

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \omega_k^2(t)\chi_k = 0, \quad (4)$$

where $\omega_k^2(t) = k^2/R^2 + g^2\phi^2$.

The rate of production of particles of a given momentum k is determined by the evolution of the perturbed field mode χ_k . The number density $n_k(t)$ of particles with momentum k can be evaluated as the energy of that mode divided by the energy of each particle,

$$n_k(t) = \frac{1}{2\omega_k(t)} (|\dot{\chi}_k|^2 + \omega_k^2(t)|\chi_k|^2) - \frac{1}{2}. \quad (5)$$

The exponential growth $\chi_k(t) \propto e^{\mu_k t}$ leads to exponential growth of the occupation numbers $n_k(t)$, and the total number density of χ particles is given by

$$n_\chi(t) = \frac{1}{4\pi^2 R^3} \int dk k^2 n_k(t). \quad (6)$$

The problem of determining the efficiency of particle production for a given model is thus reduced to the evaluation of μ_k as a function of the parameters of the model. In general, this is not an easy task, and most estimates are based on numerical integrations for typical parameter values [7].

However, the theory developed in [10] yields analytic results in Minkowski space-time, which provide an approximate simplified model that works rather well in the FRW scenario. The starting point of the theory is the fact that, for the model (3) and (4), preheating requires the amplitude of the inflaton field's oscillations, $\Phi(t)$, to verify $g\Phi(t) > m_\phi$. This means preheating is dominated by broad resonance, and the theory is based on the approximations that hold when $g\Phi(t)/m_\phi \gg 1$. The conclusions hold independently of the detailed form of the inflaton potential away from its minimum.

In this paper we extend the formalism of [10] to give a full description of the dynamics of the phase of the field modes χ_k , which in turn determines the evolution of the growth factor μ_k . In Sec. II, we consider the case of a

nonexpanding universe and show that the behavior of the phase can be described in terms of the iteration of a simple family of circle maps. The orbits of this family are of two possible types, depending on the value of the perturbation amplitude, and their asymptotic behavior, which determines the growth factor, is always independent of the initial condition.

In Sec. III we show that the results that hold for Minkowski space-time can be used to model the phase and growth factor evolution in a FRW universe. Expansion may be considered simply by taking the parameters of the Minkowski equations to be prescribed functions of time. Using this model as an alternative to numerical integration of the full equations, we check the estimates given in the literature for the total number density of χ particles created during preheating and for the duration of the first stage of preheating as a function of the interaction parameter g .

II. PHASE DYNAMICS IN MINKOWSKI SPACE-TIME

The important differences between the narrow and broad parametric resonance mechanisms in the context of cosmological models with post-inflationary reheating have first been noticed in the analysis of the evolution of the amplitudes of the χ_k modes in a static universe [6,8,24]. Although the equations in Minkowski space-time lack some of the fundamental ingredients to understand the overall efficiency of the reheating process, they can and indeed they should be considered as a toy model that sheds light on the mechanisms at play when the expansion of the Universe and other effects, such as backreaction and re-scattering, are taken into account.

This was the approach followed in Kofman *et al.* [10], where an analytic theory of broad resonance in preheating was established, relying on a detailed study of broad parametric resonance driven by the harmonic oscillations of an inflaton field without expansion of the Universe. This study is directed towards the computation of the k -mode growth factors μ_k , and the phase dynamics of the χ_k modes is not explicitly derived. However, Kofman *et al.* do mention the basic features of this phase dynamics, and they use them to explain the characteristics of mode amplification in an expanding universe, which they dubbed ‘‘stochastic resonance’’ in order to stress the difference with respect to the usual resonant bands scenario of the Mathieu equation.

In this section we explicitly compute the phase evolution equations in Minkowski space-time and show that phase stochasticity is already present in this model, as one of the two possible dynamical regimes, together with the fixed phase behavior identified in [10]. In this random phase regime, in the complement of the resonant bands, the growth factor μ_k is effectively zero, but a typical orbit undergoes random sequences of amplitude amplifications and reductions, much like in the case of stochastic reso-

nance. This phenomenon is pointed out frequently in the literature [25–27] and can be described analytically as one of the consequences of the theory when we consider the global phase dynamics of the χ_k modes [7,10].

We shall start by recalling the method of Kofman *et al.* to approximate in the broad resonance regime the solution of Eq. (4) in Minkowski space-time,

$$\ddot{\chi}_k + \omega_k^2(t)\chi_k = 0, \quad (7)$$

where $\omega_k^2(t) = a_k + b\sin^2(t)$ and the time variable is now $t \rightarrow m_\phi t$. The parameters a_k and b are given by $a_k = k^2/m_\phi^2$ and $b = g^2 A^2/m_\phi^2$, where A is the constant amplitude of the field ϕ . Typical values of the parameters are $g^2 \leq 10^{-6}$, $m = 10^{-6}m_{pl}$, $A = \alpha m_{pl}$, where $0 < \alpha < 1$, and thus $b \leq \alpha^2 \times 10^6$ [10,21,28]. In broad resonance, $\sqrt{b} \gg 1$.

Let the $\chi_k(t)$ be of the form

$$\begin{aligned} \chi_k(t) &= \frac{\alpha_k}{\sqrt{2\omega_k(t)}} \exp\left(-i \int_0^t \omega_k(s) ds\right) + \frac{\beta_k}{\sqrt{2\omega_k(t)}} \\ &\times \exp\left(i \int_0^t \omega_k(s) ds\right), \end{aligned} \quad (8)$$

where α_k and β_k are constants. Introducing (8) into (7) we obtain

$$\begin{aligned} \ddot{\chi}_k + \omega_k^2(t) \left[1 + \frac{1}{4} \left(\omega_k^{-1} \frac{d}{dt} \ln \omega_k \right)^2 \right. \\ \left. + \frac{1}{2\omega_k} \frac{d}{dt} \left(\omega_k^{-1} \frac{d}{dt} \ln \omega_k \right) \right] \chi_k = 0. \end{aligned} \quad (9)$$

So (8) approximates the solution of (7) provided that

$$\left| \omega_k^{-1} \frac{d}{dt} \ln \omega_k \right| \ll 1, \quad (10)$$

$$\left| \omega_k^{-1} \frac{d}{dt} \left(\omega_k^{-1} \frac{d}{dt} \ln \omega_k \right) \right| \ll 1 \quad (11)$$

hold. These are the adiabatic conditions identified in [29]. In the present case, $\sqrt{b} \gg 1$ and the adiabatic conditions are fulfilled except in the neighborhood of $t_j = j\pi$, $j = 0, 1, \dots$, when $\phi(t_j) = 0$, that is, every time the inflaton field crosses zero. Hence, the breakdown of the approximation given by (8) occurs periodically, and an approximate global solution of (7) can be constructed from a sequence of adiabatic solutions

$$\begin{aligned} \chi_k^j(t; \alpha_k^j, \beta_k^j) &= \frac{\alpha_k^j}{\sqrt{2\omega_k(t)}} \exp\left(-i \int_0^t \omega_k(s) ds\right) + \frac{\beta_k^j}{\sqrt{2\omega_k(t)}} \\ &\times \exp\left(i \int_0^t \omega_k(s) ds\right), \end{aligned} \quad (12)$$

where the parameters (α_k^j, β_k^j) for consecutive j will be determined by the behavior of the solution of (7) for t close to t_j . In a small neighborhood of $t_j = j\pi$, $j = 0, 1, \dots$

Eq. (7) can be approximated by

$$\ddot{\chi}_k + (a_k + b(t - t_j)^2)\chi_k = 0. \quad (13)$$

Equation (13) has an exact solution in the form of parabolic cylinder functions [30]. The asymptotic behavior for large t of this solution is of the form (8), and this provides the relation between the coefficients (α_k, β_k) of this adiabatic approximation on either side of t_j . Following Kofman *et al.* [10], this relation can be written as

$$\begin{bmatrix} \alpha_k^{j+1} e^{-i\theta_k^j} \\ \beta_k^{j+1} e^{i\theta_k^j} \end{bmatrix} = \begin{bmatrix} \frac{1}{D_\kappa} & \frac{R_\kappa^*}{D_\kappa} \\ \frac{R_\kappa}{D_\kappa} & \frac{1}{D_\kappa} \end{bmatrix} \begin{bmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \beta_k^j e^{i\theta_k^j} \end{bmatrix}, \quad (14)$$

where $\theta_k^j = \int_0^{t_j} \omega(s) ds$. The complex numbers R_κ and D_κ are given by $1/D_\kappa = \sqrt{1 + \rho_\kappa^2} \exp(i\varphi_\kappa)$ and $R_\kappa/D_\kappa = -i\rho_\kappa$ with $\rho_\kappa = \exp(-\pi\kappa^2/2)$, $\kappa^2 = a_k/\sqrt{b}$, and

$$\varphi_\kappa = \arg\left(\Gamma\left(\frac{1 + i\kappa^2}{2}\right)\right) + \frac{\kappa^2}{2} \left(1 + \ln\frac{2}{\kappa^2}\right). \quad (15)$$

The parameter $\kappa = k/\sqrt{Agm_\phi} \in [0, 1]$ is the normalized wave vector of the mode, and $\sqrt{Agm_\phi}$ the cutoff wave vector introduced in [10]. Since the number density of χ_k particles with momentum k is equal to $n_k = |\beta_k|^2$, one can use (14) and (15) to calculate the number density of particles $n_k^{j+1} = |\beta_k^{j+1}|^2$ after t_j in terms of $n_k^j = |\beta_k^j|^2$. The growth index μ_κ^j , defined by $n_k^{j+1} = n_k^j \exp(2\pi\mu_\kappa^j)$, is given by [10]

$$\begin{aligned} \mu_\kappa^j &= \frac{1}{2\pi} \ln\left(1 + 2\rho_\kappa^2 - 2\rho_\kappa \sqrt{1 + \rho_\kappa^2}\right) \\ &\times \sin\left(-\varphi_\kappa + \arg\beta_k^j - \arg\alpha_k^j + 2\theta_k^j\right), \end{aligned} \quad (16)$$

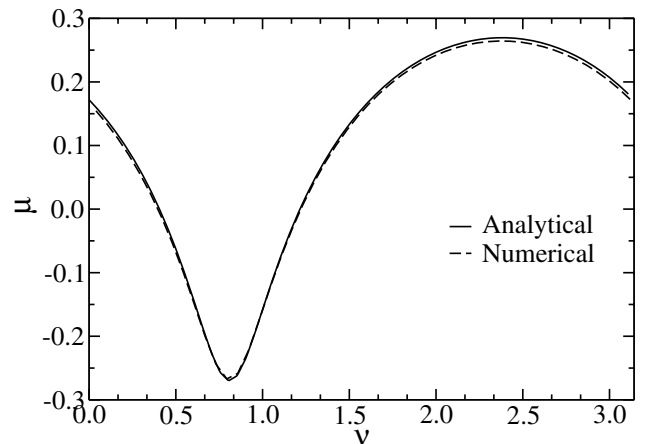


FIG. 1. For $b = 10^3$ and $a_k = 1$ (hence $\kappa^2 = 10^{-3/2}$) the analytical curve for the growth factor as a function of the phase given by Eq. (17) (solid line), and the numerical curve obtained from the integration of the equations of motion along half a period of the inflaton field (dashed line).

or, in terms of the phase $\nu_k^j = \arg \beta_k^j + \theta_k^j$ of the field χ_k when $t = t_j$,

$$\mu_k^j = \frac{1}{2\pi} \ln \left(1 + 2\rho_\kappa^2 - 2\rho_\kappa \sqrt{1 + \rho_\kappa^2} \sin(-\varphi_\kappa + 2\nu_k^j) \right). \quad (17)$$

In Fig. 1 we show, for $b = 10^3$ and $a_k = 1$, the analytic curve for the growth factor μ_k^1 as a function of the phase ν given by (17) and the numerical curve $\mu_\kappa = \mu_\kappa(\nu)$ obtained from the numerical integration of the full equation (7) along half a period of the inflaton field. We see that, as ν varies in $[0, \pi]$, μ takes positive and negative values. The phase interval for which μ is negative depends slightly on the value of κ .

As hinted by Kofman *et al.* [10], Eqs. (14) and (15) can be used to obtain the dynamics for the phase ν_k^j . In the remaining part of this section we explicitly compute the map $\nu_k^{j+1} = P_{b,\kappa}(\nu_k^j)$ and show that it can be approximated by a simple family of circle maps. From (14) one gets

$$\nu_k^{j+1} = \theta(b, \kappa) + \arg \left(\sqrt{1 + \rho_\kappa^2} e^{-i\varphi_\kappa} e^{i\nu_k^j} - i\rho_\kappa e^{-i\nu_k^j} \right), \quad (18)$$

where $\theta(b, \kappa) = \int_0^\pi \omega(s) ds$. An approximate expression for the phase map up to terms of order κ^2 is given by

$$\begin{aligned} \nu_k^{j+1} = P_{b,\kappa}(\nu_k^j) &= 2\sqrt{b} + \arctan \frac{\sqrt{2} \sin \nu_k^j - \cos \nu_k^j}{\sqrt{2} \cos \nu_k^j - \sin \nu_k^j} \\ &+ \kappa^2/2 \left(\log \frac{\sqrt{b}}{\kappa^2} + 4 \log 2 + 1 \right) \\ &- \kappa^2 \left(\frac{c_1 \cos \nu_k^{j2} + c_2 \cos \nu_k^j \sin \nu_k^j + c_3 \sin \nu_k^{j2}}{3 - 4\sqrt{2} \cos \nu_k^j \sin \nu_k^j} \right), \end{aligned} \quad (19)$$

where $c_1 = 2.074 - \log 2/\kappa^2$, $c_2 = -1.363 + 1.414 \log 2/\kappa^2$, and $c_3 = -0.147 - \log 2/\kappa^2$.

The maps (18) and (19) for $b = 10^3$ and $\kappa = 0.5$ are shown in Fig. 2, together with the phase map obtained from the numerical integration of the full equation (7) for the same values of the parameters over half a period of the inflaton field.

The properties of the family (18) and its approximation (19) are best understood by looking at the behavior of the family $P_{b,0}(\nu)$ parametrized by \sqrt{b} ,

$$P_{b,0}(\nu) = 2\sqrt{b} + \arctan \frac{\sqrt{2} \sin \nu - \cos \nu}{\sqrt{2} \cos \nu - \sin \nu}. \quad (20)$$

This family of circle maps is periodic with period π in the parameter \sqrt{b} , and its bifurcation diagram for $\sqrt{b} \in [0, \pi]$ is shown in Fig. 3. We see that the map has two different regimes. For $\tan 2\sqrt{b} \in [-1, 1]$, the map has a strongly attractive fixed point, and the phase converges rapidly to this fixed point (typically in a couple of iterations). For the remaining parameter values, the phase orbit

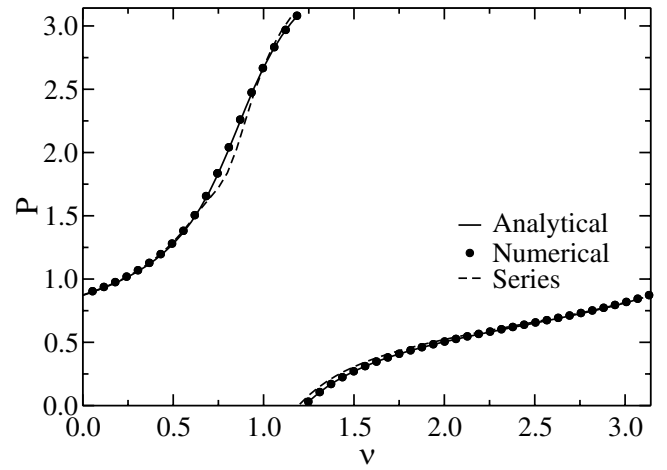


FIG. 2. For $b = 10^3$ and $\kappa = 0.5$ the maps (18) (solid line) and (19) (dashed line). Also shown is the phase map obtained from the numerical integration of the full equation (7) (full circle) for the same values of the parameters.

has random oscillations around the mean value that varies between $\pi/8$ and $\pi/2 - \pi/8$, or between $\pi/8 + \pi$ and $3\pi/2 - \pi/8$. The fixed point equation

$$\frac{\cos 2\nu}{\sqrt{2} - \sin 2\nu} = \tan 2\sqrt{b} \quad (21)$$

is satisfied for two values of ν for each $\sqrt{b} \in [-\pi/8, \pi/8] \cup [\pi/2 - \pi/8, \pi/2 + \pi/8]$, one of which corresponds to an unstable fixed point and the other to the stable fixed point shown in the bifurcation diagram of Fig. 3. The derivative of $P_{b,0}$ at the stable fixed point is

$$P'_{b,0}(\tilde{\nu}) = \frac{1}{3 - 2\sqrt{2} \sin 2\tilde{\nu}} \leq 1, \quad (22)$$

where the stable fixed point is $\tilde{\nu} \in [-\pi/2 - \pi/8, \pi/8] \cup$

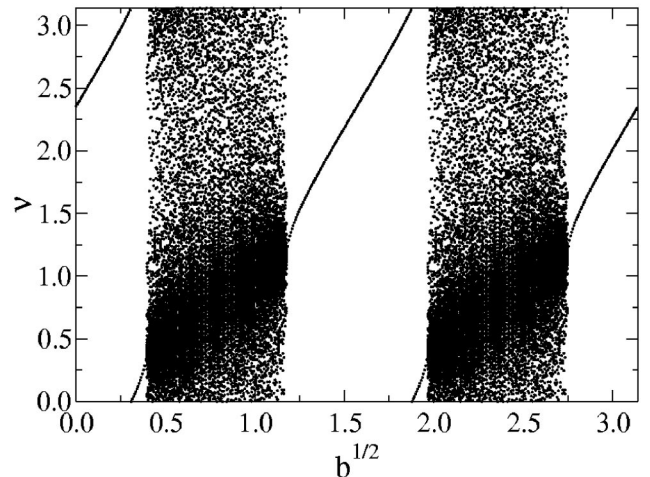


FIG. 3. Bifurcation diagram of the family of circle maps (20) for $\sqrt{b} \in [0, \pi]$.

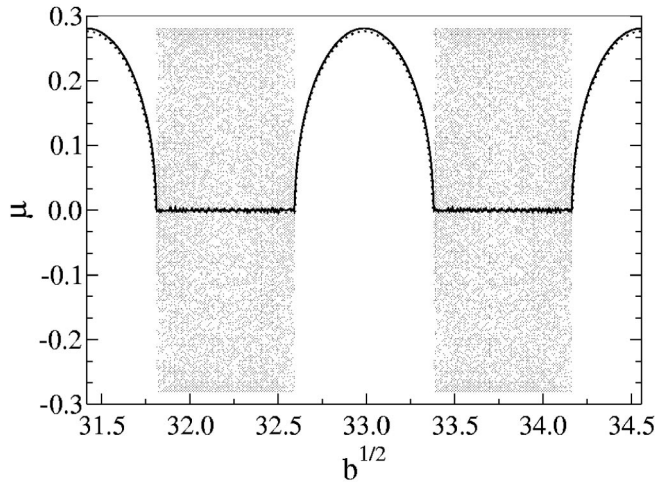


FIG. 4. The asymptotic value of μ_0 as a function of b for $\sqrt{b} \in [10\pi, 11\pi]$ computed analytically from Eqs. (17) and (20) (full line) and numerically from the integration of the full equation (7) (dotted line). We have taken $j_{\max} = 200$, and the two lines almost overlap. Also shown (in gray) are all the values of μ_0^j , $j = 100, 101, \dots, 200$.

$[\pi/2 - \pi/8, \pi + \pi/8]$. The equality $P'_{b,0}(\bar{\nu}) = 1$ is obtained at the boundaries of the random phase region, but the derivative decreases rather sharply into the stable fixed point region, where $P'_{b,0}(\bar{\nu}) < 0.5$ for most values.

The asymptotic value of growth factor μ_0 for the $k = 0$ mode as a function of b can then be computed from Eq. (17) evaluated at the fixed point or averaged over the random orbits, for values of $\sqrt{b} \in [-\pi/8, \pi/8] \cup [\pi/2 - \pi/8, \pi/2 + \pi/8]$ or in the complement of this interval, respectively. In Fig. 4 we show a plot of the asymptotic value of μ_0 as a function of b computed analytically from Eqs. (17) and (20), as described, and numerically from the integration of the full equation (7). We see that the stability regions identified in [10] correspond to the random wandering of the phase ν of the field χ at consecutive $t_j = j\pi$ around its average values. The random phase regimes

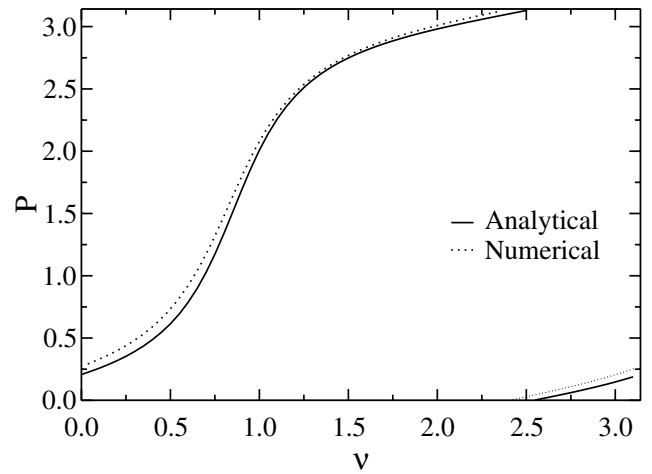


FIG. 6. The approximate phase map (solid line) and the phase map computed numerically from Eq. (7) (dotted line) for $b = 10$ and $\kappa = 1/2$.

correspond to different probability measures in the phase interval, for all of which the average value of μ is zero.

For other values of κ , the global dynamics shares the qualitative properties of the family $P_{b,0}$. In Fig. 5 we show the same information of Figs. 3 and 4 obtained for $P_{b,\kappa}$ with $300 \leq \sqrt{b} \leq 304$ and $\kappa = 1/2$.

Equation (18) provides a good approximation to the exact phase dynamics even for moderate values of b . In Fig. 6 we show the approximate phase map (18) and the phase map computed numerically from the full equation (7) for $b = 10$ and $\kappa = 1/2$. This, together with the rapid relaxation rates of the phase dynamics in the fixed point regime, is the reason why Eqs. (17) and (18) are still useful to obtain estimates for the growth number in FRW space-time. The existence of two simple regimes for the phase dynamics, one of them characterized by rapid relaxation to a fixed point and the other by random wandering of the phase with a well-defined probability density, shows that the system keeps no record of the initial phase and, in this

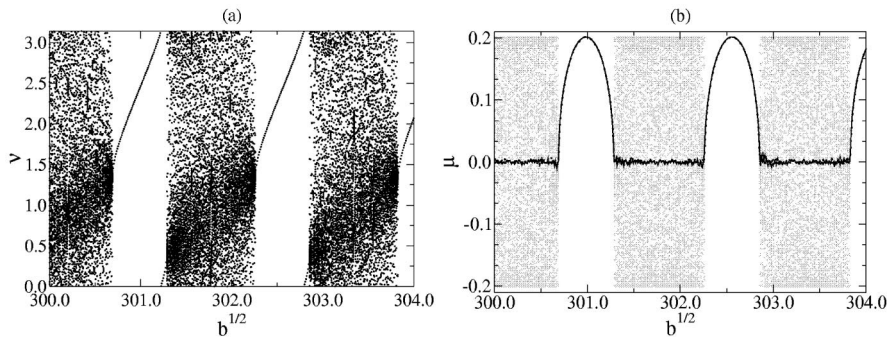


FIG. 5. (a) Bifurcation diagram for the map $P_{b,\kappa}$ with $\sqrt{b} \in [300, 304]$ and $\kappa = 1/2$. (b) The asymptotic value of μ_κ as a function of \sqrt{b} for the same values of the parameters computed analytically from Eqs. (17) and (18) (full line), and numerically from the integration of the full equation (7) (dotted line). We have taken $j_{\max} = 200$, and the two lines almost overlap. Also shown (in gray) are all the values of μ_κ^j , $j = 100, 101, \dots, 200$.

sense, has no memory. This explains why the evolution of the occupation number is independent of the initial phase, while the growth factor per period μ_k^j depends strongly on the phase ν_k^j . We shall come back to this point later.

III. DYNAMICS IN AN EXPANDING UNIVERSE

In this section we will show that Eqs. (17) and (18) can be used to compute the growth factor of the χ field modes during the first stage of the preheating period in a flat FRW.

It is well known [10] that the coherence of the oscillations of the ϕ field is not disturbed until the energy density of the χ field significantly contributes to (2) and (3) and backreaction and rescattering effects start to change the mechanism of the growth of the occupation number of the produced particles. More precisely, Kofman *et al.* show in [10] that these effects are negligible during the first preheating stage that ends when the total number density n_χ of χ particles satisfies

$$n_\chi(t) \approx \frac{m_\phi^2 \Phi(t)}{g}, \quad (23)$$

where $\Phi(t)$ is the varying amplitude of the inflaton field ϕ .

In the first stage of preheating, Eqs. (2) and (3) decouple from (4), and the evolution of the inflaton field and of the scale factor $R(t)$ is given in good approximation by [10]

$$\begin{aligned} \phi(t) &= \Phi(t) \sin t, & \Phi(t) &= \frac{m_{pl}}{3(\pi/2 + t)}, \\ R(t) &= \left(\frac{2t}{\pi}\right)^{2/3}. \end{aligned} \quad (24)$$

Equation (4) can be reduced to the form (7) through the change of variable $X_k = R^{3/2} \chi_k$, yielding

$$\ddot{X}_k + \varpi_k^2(t) X_k = 0, \quad (25)$$

where $\varpi^2 = (k^2/m_\phi^2 R(t)^2) + (g^2 \phi(t)^2/m_\phi^2) + (1/m_\phi^2)\delta$, and the last term is very small after inflation and will be disregarded.

We see that the χ_k modes are still governed by an equation of the form (7) like the one we considered in Minkowski space-time, but now the parameters a_k and b change with time. By definition, the preheating period ends when $g\Phi(t)/m_\phi \simeq 1$, and so, during preheating, the rate of variation of those parameters and the oscillations of the inflaton field are much slower than the oscillations of the χ_k modes. As pointed out in [10], the basic assumptions for the approximation developed for Minkowski space-time are thus still valid in preheating, and the changes in occupation numbers n_k will occur at $t = j\pi$ with an exponential growth rate given by (17), provided that the decreasing amplitude of the perturbations and the redshift of the wave numbers are taken into account. Hence, Kofman *et al.* model particle production in the first stage of preheating through

$$\mu_{\kappa_j}^j = \frac{1}{2\pi} \ln \left(1 + 2\rho_{\kappa_j}^2 - 2\rho_{\kappa_j} \sqrt{1 + \rho_{\kappa_j}^2} \sin(-\varphi_{\kappa_j} + 2\nu_{\kappa_j}^j) \right), \quad (26)$$

where $\kappa_j = k/(R(t_j)\sqrt{gm_\phi\Phi(t_j)})$.

We may also think of the phase dynamics as being essentially governed by the iteration of Eq. (18), but now the parameter b decreases in time, crossing the strips associated with the random phase and fixed point regimes, and giving rise to nontrivial phase dynamics. Kofman *et al.* describe the effects of the variation of the inflaton amplitude as implying random phase dynamics, and build their estimates for the total number density evolution $n_\chi(t)$ from (26), treating the phase ν_k^j as a random variable.

We shall extend the approach of [10] to study the phase dynamics in an expanding universe, taking the phase iteration map to be given by

$$\nu_k^{j+1} = \theta(b_j, \kappa_j) + \arg \left(\sqrt{1 + \rho_{\kappa_j}^2} e^{-i\varphi_{\kappa_j}} e^{i\nu_k^j} - i\rho_{\kappa_j} e^{-i\nu_k^j} \right), \quad (27)$$

where $\sqrt{b_j} = g\Phi(t_j)/m_\phi$.

Equations (26) and (27) provide an alternative to numerical integrations of the full equations that compute the occupation number of a given mode as a function of time. The phase, growth factor, and occupation number for a typical orbit (we have taken $b_0 = 5 \times 10^3$ and $\kappa_0 = 0.1$) as obtained from the iteration of Eqs. (26) and (27) are shown in Fig. 7, where the values given by numerical integration of Eq. (25) for the same initial conditions and parameter values are also plotted. Iteration and integration were carried out until preheating ends with $\sqrt{b(t_j)} \approx 1$. We see ‘‘reminiscences’’ of the phase dynamics of the Minkowski model. In particular, the fixed point regime interval is clearly visible after the first few ϕ oscillations. Also shown are the values of these same quantities averaged over the initial phase ν_k^0 . We see that, due to the characteristics of the phase dynamics, the efficiency of particle production is insensitive to the initial phase of the field mode χ_k , in spite of the strong dependence of the growth factor per period on the phase ν_k^j .

We shall now use Eqs. (26) and (27) above to look at the total number density $n_\chi(t)$ and check the estimates given in [10]. We consider the contributions to $n_\chi(t)$ of every mode such that $k \in [0, k_*(0) = \sqrt{g\Phi(0)m_\phi}]$ at $t = 0$, and compute for each j the leading mode’s growth factor, μ_j^j , given by

$$\mu_j^j = \max_{\kappa^2 \in [0,1]} \left\{ \frac{1}{2\pi} \sum_{\nu^0 \in [0,2\pi]} \mu_{b_j, \kappa_j}^j(P_{b_j, \kappa_j}^{(j)}(\nu^0)) \right\}, \quad (28)$$

with $b_j = 5 \times 10^3 (\Phi(t_j)/\Phi(0))^2$, $\kappa_j^2 = \kappa^2 \Phi(0)/(\Phi(t_j)R^2(t_j))$, and $P_{b_j, \kappa_j}^{(j)}$ the j th iteration of the map (27),

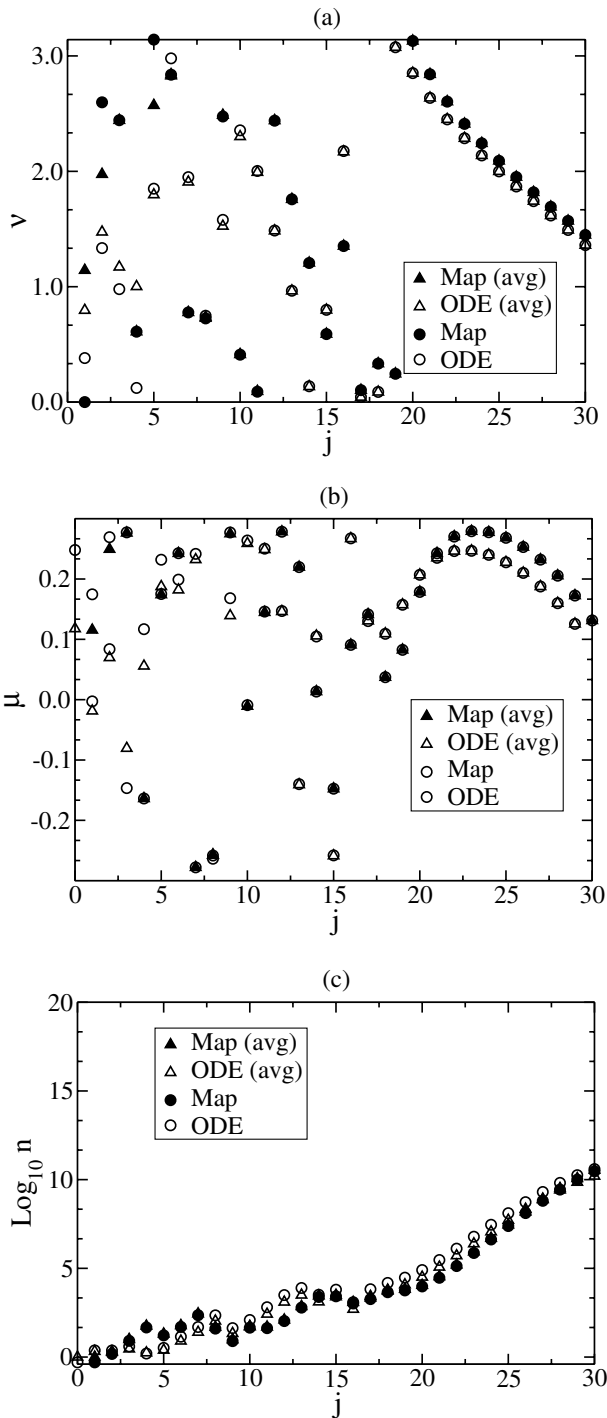


FIG. 7. For $b_0 = 5 \times 10^3$ and $\kappa_0 = 0.1$, the phase (a), growth factor (b), and occupation number (c) with initial conditions corresponding to $n_k^0 = 1/2$ and $\nu_k^0 = 0$. The values obtained from the iteration of Eqs. (26) and (27) are plotted as full circles, and the values given by numerical integration of Eq. (25) for the same initial conditions and parameter values are plotted as open circles. Iteration and integration were carried out until the end of preheating when $\sqrt{b(t_j)} \approx 1$. Also shown are the values of these same quantities averaged over the initial phase ν_k^0 [full triangles for the iterated maps (26) and (27) and open triangles for the numerical values].

where the parameters κ and b must be updated at each iteration. Then, whereas the estimate in [10] is

$$n_\chi(t) = \frac{(gm_\phi \Phi(0))^{3/2}}{64\pi^2 R^3(t) \sqrt{0.13\pi t}} \exp(2 \times 0.13t), \quad (29)$$

we have

$$n_\chi(t_j) = \frac{(gm_\phi \Phi(0))^{3/2}}{64\pi^2 R^3(t) \sqrt{\pi^2 \sum_{i=1}^j \mu_i^i}} \exp\left(2\pi \sum_{i=1}^j \mu_i^i\right), \quad (30)$$

or

$$n_\chi(t_j) = \frac{(gm_\phi \Phi(0))^{3/2}}{64\pi^2 R^3(t) \sqrt{\pi^2 \sum_{i=1}^j \mu_i^i}} \exp\left(2\pi \sum_{i=1}^j \mu_i^i\right), \quad (31)$$

if instead of actually determining the leading mode we assume that it corresponds to $k_*/2$. The curves corresponding to Eqs. (29)–(31) are shown in Fig. 8. We can still improve on the estimates given by (30) and (31). Since

$$n_\chi(t) = \frac{(gm_\phi \Phi(t))^{3/2}}{64\pi^2 R^3(t)} \int_0^1 \kappa^2 n_\kappa(t) d\kappa, \quad (32)$$

we may use (26) and (27) to compute this integral numerically from $n_\kappa(t_j) = 1/2 \exp(2\pi \sum_{i=1}^j \mu_{\kappa_i}^i)$. The result is also shown in Fig. 8, and we see that for this value of g the estimates of [10] are very accurate.

Finally, we have used again (26) and (27) together with (32) to obtain a systematic study both of the behavior of $n_\chi(t)$ as a function of the parameter g , and of the duration of the first stage of preheating as defined by (23), as a function of g . The results for g in the range $10^{-4} \leq g \leq 10^{-1}$ are shown in Fig. 9, where we have also plotted the result of the estimates of [10] for these quantities. We see that the duration of the first stage of preheating depends

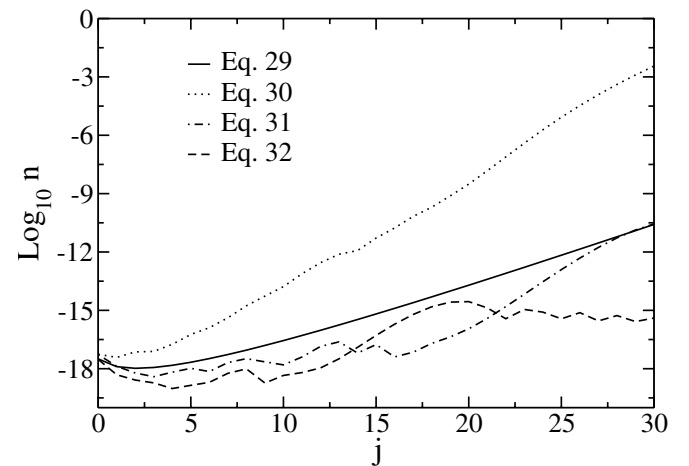


FIG. 8. For $b_0 = 5 \times 10^3$ (hence $g = 0.3 \times 10^{-3}$) the curves corresponding to Eq. (29) (full line), (30) (dotted line), (31) (slash-and-dot line), and (32) (slashed line). The evolution was computed until the end of preheating when $\sqrt{b_{30}} \approx 1$.

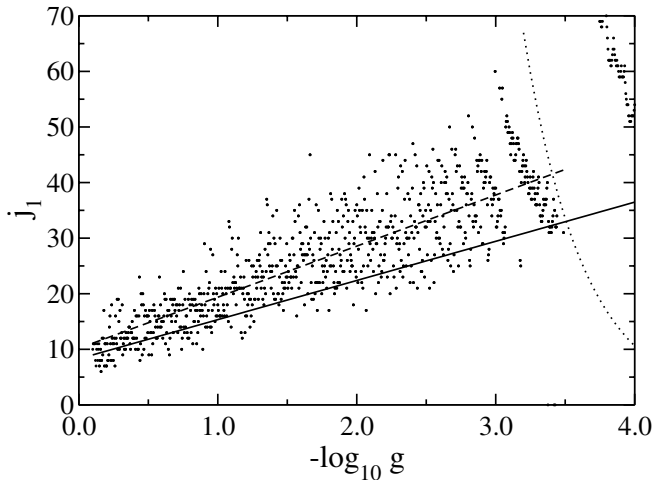


FIG. 9. Duration of the first stage of preheating j_1 as a function of g for $10^{-4} \leq g \leq 10^{-1}$ (dots). Also shown are the linear least square approximation (slashed line), the estimate given in [10] (full line), and the cutoff curve defined by $b_j = 1$ (dotted line). The values of j_1 that lie above the cutoff curve $b_j = 1$ were not considered in the least square approximation.

rather irregularly on the parameter g , and that the estimates of [10] (full line in the figure) provide a good lower bound for most values of g . The least squares linear fit (slashed line) yields a slightly larger value for the typical duration of the first stage. On the other hand, the value of g below which preheating always ceases during the first stage was found to be $g = 3 \times 10^{-4}$, as predicted in [10].

IV. CONCLUSION

In the simplest preheating scenario, where the coherent oscillations of the uncoupled inflaton field drive the amplification of the mode amplitudes of a field χ , we consider first the broad resonance regime in Minkowski space-time and use the theory of scattering in parabolic potentials developed in [10] to obtain the map whose iteration governs the phase dynamics of the modes χ_k . It is well known that the phase dynamics, the consecutive values of the phase of the χ_k fields at the times t_j when the inflaton amplitude goes through zero, determine the growth rates of the modes. In this work we show that the features of this

phase dynamics are given by the properties of a simple family of circle maps. The orbits of this family of maps are of two types, rapid convergence to fixed point solutions, and random oscillations around an average value. Hence, the “stochastic resonance” identified in [10] in the dynamics of an expanding universe is also present in the absence of expansion. The fixed phase and stochastic regimes occur in consecutive intervals of the value of the forcing amplitude. In the first case, the fixed point is always associated with a positive value of the growth factor $\mu_j = 1/(2\pi) \times \log(n^{j+1}/n^j)$ that controls the growth of the number of particles n^j in each mode for $t = t_j$. Thus, in this case, the average growth of the occupation numbers of the modes is exponential. In the second case, we show that the phase sampling is always such that the average growth factor is zero.

We then consider the case of an expanding universe, with the assumptions that hold in the first stage of preheating, and show that the equations for the phase dynamics and the growth number derived for Minkowski space-time still provide a good approximation of the true solutions, once the decay of the inflaton amplitude is taken into account. Moreover, the qualitative behavior of the phase and growth number evolution is reminiscent of the behavior found in the case without expansion, in the sense that it can be interpreted as a random phase regime followed by a slowly varying phase regime where occupation number growth is approximately exponential. These two regimes occur as the inflaton decay slows down and the perturbation amplitude crosses more and more slowly the intervals that give rise to fixed phase behavior.

We use this approximation to obtain a systematic study of the behavior of the total number density of created particles over time, and of the end of the first stage of preheating as a function of the $\phi - \chi$ coupling parameter g . Comparison with the estimates presented in [10] show an overall good agreement.

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