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Relativistic second-order perturbations of nonzero-Λ flat cosmological models and CMB anisotropies

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First the second-order perturbations of nonzero-Λ cosmological models are derived explicitly with an arbitrary potential function of spatial coordinates, using the nonlinear version of Lifshitz's method in the synchronous gauge. Their expression is the generalization (to the nonzero-Λ case) of second-order perturbations in the Einstein–de Sitter model which were derived previously by the present author. Next the second-order temperature anisotropies of cosmic microwave background radiation are derived using the gauge-invariant formula which was given by Mollerach and Matarrese. Moreover, the corresponding perturbations in the Poisson gauge are derived using the second-order gauge transformations formulated by Bruni *et al.* In the second order it is found in spite of gauges that tensor (gravitational-wave) perturbations and vector (shear) perturbations without vorticity are induced from the first-order scalar perturbations. These results will be useful to analyze the nonlinear effect of local inhomogeneities on cosmic microwave background anisotropies.

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I. INTRODUCTION

The anisotropies of the cosmic microwave background radiation (CMB) gives us an important information of the structure of our universe. So far the relation of CMB anisotropies to inhomogeneities along light paths has been analyzed using the linear theory of gravitational instability. Recently the measurements of the anisotropies have, however, become more and more precise, so that we can catch some information related to nonlinear effect of inhomogeneities such as in the form of non-Gaussianity [1].

The general-relativistic second-order nonlinear theory of gravitational instability was studied previously by the present author [2,3], in connection with structure formation, by extending Lifshitz's linear theory [4] to the second-order smallness. They were restricted to the vanishing Λ model (Einstein–de Sitter model) and the treatment in the synchronous gauge. Recently the nonlinear gauge transformations and the condition of second-order gauge invariantness have been studied by Bruni *et al.* [5] and the second-order transformation from the synchronous gauge to the Poisson gauge has been performed by Matarrese *et al.* [6], and the second-order temperature anisotropy of CMB radiation has been derived by Mollerach and Matarrese [7]. The second-order theory has, moreover, been extended to useful cases [8–11] of nonzero-Λ and nonvanishing pressure, to analyze non-Gaussianity in the CMB anisotropies.

In this paper we will derive the second-order perturbations corresponding to the first-order scalar perturbations, in the nonzero-Λ and pressureless case using the synchronous gauge, in which the perturbations are expressed explicitly with an arbitrary potential function (F) of spatial coordinates, similarly to our previous work [2].

Next we represent the second-order CMB anisotropies ($\delta_2 T/T$) using the derived metric perturbations in the synchronous gauge. The CMB anisotropies, which was derived by Mollerach and Matarrese [7] in arbitrary gauges, are gauge invariant, so that our expressions will be useful in the same way as those in the other gauges.

Last, we derive the second-order perturbations in the Poisson gauge from those in the synchronous gauge, using the nonlinear gauge transformation exploited by Bruni *et al.* [5] and Matarrese *et al.* [6]. The latter gauge may be familiar and comprehensive, because the line element in the Poisson gauge is similar to that in the Newtonian and post-Newtonian approximation [12].

It is found that, in the second order, tensor (gravitational-wave) perturbations and vector (shear) perturbations without vorticity are induced from the first-order scalar perturbations.

In the appendix the derivation of basic equations in the extended version of Lifshitz theory is reviewed.

II. SECOND-ORDER PERTURBATIONS IN THE SYNCHRONOUS GAUGE

First our background universe is assumed to be described by isotropic and homogeneous pressureless cosmological models which are spatially flat, and their spacetimes are expressed by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \quad (2.1)$$

where the greek and roman letters denote $0, 1, 2, 3$ and $1, 2, 3$, respectively, contrary to the notation in the previous paper [2], and $\delta_{ij}(= \delta^i_j = \delta^{ij})$ are the Kronecker delta. The conformal time $\eta(= x^0)$ is related to the cosmic time t by $dt = a(\eta)d\eta$.

In the comoving coordinates, the velocity vector and the energy-momentum tensor of the pressureless matter are expressed as

$$u^0 = 1/a, u^i = 0 \tag{2.2}$$

and

$$T^0_0 = -\rho, T^0_i = 0, T^j_i = 0, \tag{2.3}$$

where ρ is the matter density. From the Einstein equations, the equations for ρ and the scale factor a are obtained:

$$\rho a^2 = 3(a'/a)^2 - \Lambda a^2, \tag{2.4}$$

and

$$\rho a^3 = \rho_0, \tag{2.5}$$

where a prime denotes $\partial/\partial\eta$, Λ is the cosmological constant, and ρ_0 is an integration constant.

Next we consider first-order perturbations of the scalar type. The perturbations of metric, matter density, and velocity are represented by $\delta_1 g_{\mu\nu}(\equiv h_{\mu\nu})$, $\delta_1\rho$, and $\delta_1 u^\mu$.

When we adopt the synchronous coordinates, the metric perturbations satisfy the condition

$$h_{00} = 0 \quad \text{and} \quad h_{0i} = 0. \tag{2.6}$$

The scalar-type solutions of the perturbed Einstein equations are classified into the growing case and the decaying case, and the remaining components are expressed in both cases as follows:

(1) The growing case

$$h^j_i = \delta^j_i F + P(\eta) F^{|j}_{|i}, \qquad \delta_1 u^0 = 0,$$
$$\delta_1 u^i = 0, \qquad \delta_1\rho/\rho = \frac{1}{\rho a^2}\left(\frac{a'}{a}P' - 1\right)\Delta F, \tag{2.7}$$

where F is an arbitrary function of spatial coordinates x^1, x^2, and x^3, $\Delta \equiv \nabla^2$, $h^j_i = g^{jl}h_{li}$, and $P(\eta)$ satisfies

$$P'' + \frac{2a'}{a}P' - 1 = 0. \tag{2.8}$$

Its solution is expressed as

$$P = \int_0^\eta d\eta' a^{-2}(\eta') \int_0^{\eta'} d\eta'' a^2(\eta''). \tag{2.9}$$

The three-dimensional covariant derivatives $|i$ are defined in the space with metric $dl^2 = \delta_{ij}dx^i dx^j$ and their suffices are raised and lowered using δ_{ij}, so that their derivatives are equal to partial derivatives, i.e., $F^{|j}_{|i} = F_{,ij}$, where $F_{,i} \equiv \partial F/\partial x^i$.

(2) The decaying case

$$h^j_i = P(\eta)F_{,ij}, \qquad \delta_1 u^0 = 0, \qquad \delta_1 u^i = 0,$$
$$\delta_1\rho/\rho = \frac{a'/a}{\rho a^2}P'\Delta F, \tag{2.10}$$

where $P(\eta)$ satisfies

$$P'' + \frac{2a'}{a}P' = 0 \tag{2.11}$$

and the solution is

$$P = \int_0^\eta d\eta' a^{-2}(\eta'). \tag{2.12}$$

These first-order density perturbations are consistent with the gauge-invariant variable ϵ_m (defined by Bardeen [13]), which is described by the equation

$$\epsilon_m{}'' + \frac{a'}{a}\epsilon_m{}' - \frac{1}{2}\rho a^2 \epsilon_m = 0 \tag{2.13}$$

in the pressureless case without anisotropic stresses and entropy perturbations. In fact $\delta_1\rho/\rho$ in Eqs. (2.7) and (2.10) satisfy Eq. (2.13) for P in Eqs. (2.9) and (2.12). In both cases, the potential function F is determined by the last equation of Eq. (2.7) or Eq. (2.10) which is regarded as the cosmological Poisson equation, if the spatial distributions of $\delta_1\rho/\rho$ are given at a specified epoch.

Now let us derive the second-order perturbations $\delta_2 g_{\mu\nu}(\equiv \ell_{\mu\nu})$, $\delta_2\rho$, and $\delta_2 u^\mu$, where the total perturbations are

$$\delta g_{\mu\nu} = h_{\mu\nu} + \ell_{\mu\nu}, \qquad \delta u^\mu = \delta_1 u^\mu + \delta_2 u^\mu,$$
$$\delta\rho/\rho = \delta_1\rho/\rho + \delta_2\rho/\rho. \tag{2.14}$$

Here from the synchronous gauge condition, we have

$$\ell_{00} = 0 \quad \text{and} \quad \ell_{0i} = 0. \tag{2.15}$$

Then, by solving Eqs. (A16) and (A17) for ℓ_{ij} which are derived from the perturbed Einstein equations in the appendix, we obtain the following expressions of $\delta_2 g_{\mu\nu}(\equiv \ell_{\mu\nu})$ and using Eqs. (A17) and (A18) we obtain $\delta_2\rho$ and $\delta_2 u^\mu$.

(1) the growing case.

The metric perturbations reduce to

$$\ell^j_i = P(\eta)L^j_i + P^2(\eta)M^j_i + Q(\eta)N^{|j}_{|i} + C^j_i, \tag{2.16}$$

where $N^{|j}_{|i} = \delta^{jl}N_{|li} = N_{,ij}$ and $Q(\eta)$ satisfies

$$Q'' + \frac{2a'}{a}Q' = P - \frac{5}{2}(P')^2 \tag{2.17}$$

and the solution is expressed as

$$Q = \int_0^\eta d\eta' a^{-2}(\eta') \int_0^{\eta'} d\eta'' a^2(\eta'') \times \left[P(\eta'') - \frac{5}{2}(P'(\eta''))^2 \right]. \quad (2.18)$$

In the case $\Lambda = 0$, we have $P - (5/2)(P')^2 = 0$ because of $a \propto \eta^2$ and $P = \eta^2/10$, so that Q vanishes. The functions L_i^j and M_i^j are defined by

$$L_i^j = \frac{1}{2}\left[-3F_{,i}F_{,j} - 2F \cdot F_{,ij} + \frac{1}{2}\delta_{ij}F_{,l}F_{,l} \right],$$
$$M_i^j = \frac{1}{28}\{19F_{,il}F_{,jl} - 12F_{,ij}\Delta F - 3\delta_{ij}[F_{,kl}F_{,kl} - (\Delta F)^2]\} \quad (2.19)$$

and N is defined by

$$\Delta N = \frac{1}{28}[(\Delta F)^2 - F_{,kl}F_{,kl}]. \quad (2.20)$$

The velocity and density perturbations are found to be

$$\delta_2 u^0 = 0, \qquad \delta_2 u^i = 0, \quad (2.21)$$

and

$$\delta_2\rho/\rho = \frac{1}{2\rho a^2}\left\{ \frac{1}{2}\left(1 - \frac{a'}{a}P'\right)(3F_{,l}F_{,l} + 8F\Delta F) + \frac{1}{2}P[(\Delta F)^2 + F_{,kl}F_{,kl}] + \frac{1}{4}\left[(P')^2 - \frac{2}{7}\frac{a'}{a}Q'\right][(\Delta F)^2 - F_{,kl}F_{,kl}] - \frac{1}{7}\frac{a'}{a}PP'[4F_{,kl}F_{,kl} + 3(\Delta F)^2] \right\}. \quad (2.22)$$

The last term C_i^l satisfies the wave equation

$$\Box C_i^j = \frac{3}{14}(P/a)^2 G_i^j + \frac{1}{7}\left[P - \frac{5}{2}(P')^2 \right]\tilde{G}_i^j, \quad (2.23)$$

where the operator $\Box$ is defined by

$$\Box\phi \equiv g^{\mu\nu}\phi_{;\mu\nu} = -a^{-2}\left(\partial^2/\partial\eta^2 + \frac{2a'}{a}\partial/\partial\eta - \Delta \right)\phi \quad (2.24)$$

for an arbitrary function ϕ by use of the four-dimensional covariant derivative ;, and G_i^j and $\tilde{G}_i^j$ are expressed as

$$G_i^j \equiv \Delta(F_{,ij}\Delta F - F_{,il}F_{,jl}) + (F_{,ij}F_{,kl} - F_{,ik}F_{,jl})_{,kl} - \frac{1}{2}\delta_{ij}\Delta[(\Delta F)^2 - F_{,kl}F_{,kl}],$$
$$\tilde{G}_i^j \equiv F_{,ij}\Delta F - F_{,il}F_{,jl} - \frac{1}{4}\delta_{ij}[(\Delta F)^2 - F_{,kl}F_{,kl}] - 7N_{,ij}. \quad (2.25)$$

These functions satisfy the traceless and transverse relations

$$G_l^l = 0, \qquad G_{i,l}^l = 0, \qquad \tilde{G}_l^l = 0, \qquad \tilde{G}_{i,l}^l = 0, \quad (2.26)$$

so that C_i^j also satisfies

$$C_l^l = 0, \qquad C_{i,l}^l = 0. \quad (2.27)$$

This means that C_i^j represents the second-order gravitational radiation emitted by first-order density perturbations. The solution of the above inhomogeneous wave equation [Eq. (2.23)] can be represented in an explicit form using the retarded Green function for the operator $\Box$ [2,14,15].

(2) the decaying case.
The metric perturbations are

$$\ell_i^j = \frac{1}{4}P^2(\eta)\left\{ 2F_{,il}F_{,jl} - \Delta F \cdot F_{,ij} + \frac{1}{4}\delta_{ij}[(\Delta F)^2 - F_{,kl}F_{,kl}] \right\} + C_i^j, \quad (2.28)$$

where $P(\eta)$ in this case is given by Eq. (2.12). The last term C_i^j is described by the wave equation

$$\Box C_i^j = -\frac{1}{8}(P/a)^2 G_i^j, \quad (2.29)$$

where G_i^j satisfies Eq. (2.25).
The velocity and density perturbations are

$$\delta_2 u^0 = 0, \qquad \delta_2 u^i = 0 \quad (2.30)$$

and

$$\delta_2\rho/\rho = \frac{1}{8\rho a^2}P'\{P'[(\Delta F)^2 - F_{,kl}F_{,kl}] - \frac{a'}{a}P[(\Delta F)^2 + 3F_{,kl}F_{,kl}]\}. \quad (2.31)$$

Here $\delta_2\rho/\rho$ is the matter density perturbations observed by comoving observers and C_i^j represents the gravitational radiation emitted by the first-order density perturbations.

By the way we consider the rotational velocity ω^α and the corresponding scalar quantity ω defined by

$$\omega^\alpha \equiv \frac{1}{2}\eta^{\alpha\beta\mu\nu}u_\beta u_{\mu,\nu} \quad \text{and} \quad \omega \equiv (\omega^\alpha\omega_\alpha)^{1/2}. \quad (2.32)$$

In the above perturbations, it is clear that ω^α vanishes, because $\delta_1 u^\mu = \delta_2 u^\mu = 0$, and so ω also vanishes.

Next, for a later use we here express our solutions (in the growing case) using the notation which was employed by Matarrese, Mollerach, and Bruni [6,7] for the gauge-invariant treatment of second-order perturbations. In their notation our perturbations are expressed in the following form:

$$g_{00} = -a^2(\eta)[1 + 2\psi^{(1)} + \psi^{(2)}],$$
$$g_{0i} = a^2(\eta)\left[z_i^{(1)} + \frac{1}{2}z_i^{(2)}\right],$$
$$g_{ij} = a^2(\eta)\left\{[1 - 2\phi^{(1)} - \phi^{(2)}]\delta_{ij} + \chi_{ij}^{(1)} + \frac{1}{2}\chi_{ij}^{(2)}\right\}, \tag{2.33}$$

and

$$\chi_{ij}^{(r)} = \chi_{ij}^{(r)\|} + \chi_{ij}^{(r)\top}, \qquad (r = 1, 2), \tag{2.34}$$

where $\|$ and $\top$ denote the scalar and tensor perturbations, respectively, and

$$D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta. \tag{2.35}$$

The velocity and density perturbations are

$$u^\mu = \frac{1}{a}\left[\delta_0^\mu + v_{(1)}^\mu + \frac{1}{2}v_{(2)}^\mu\right] \tag{2.36}$$

and

$$\rho = \rho_{(0)} + \delta^{(1)}\rho + \frac{1}{2}\delta^{(2)}\rho. \tag{2.37}$$

For our perturbations in the synchronous gauge, we have

$$\psi^{(1)} = \psi^{(2)} = z_i^{(1)} = z_i^{(2)} = 0 \tag{2.38}$$

and the other components are expressed by use of our notation as

$$\phi^{(1)} = -h/6 = -\frac{1}{2}\left(F + \frac{1}{3}P\Delta F\right),$$
$$\phi^{(2)} = -\ell/3 = -\frac{1}{3}(PL_i^i + P^2M_i^i + Q\Delta N),$$
$$\chi_{ij}^{(1)\|} = D_{ij}\chi^{(1)\|} = h_i^j - \frac{1}{3}\delta_i^j h = P\left(F_{,ij} - \frac{1}{3}\delta_{ij}\Delta F\right)$$
$$\text{or} \quad \chi^{(1)\|} = PF,$$
$$\chi_{ij}^{(1)\top} = 0,$$
$$\frac{1}{2}\chi_{ij}^{(2)\|} = \ell_i^j - \frac{1}{3}\delta_i^j\ell = PL_i^j + P^2M_i^j + QN_{,ij} - \frac{1}{3}\delta_{ij}(PL_k^k + P^2M_k^k + Q\Delta N),$$
$$\frac{1}{2}\chi_{ij}^{(2)\top} = C_i^j, \qquad v_{(1)}^\mu = v_{(2)}^\mu = 0,$$
$$\delta^1\rho = \delta_1\rho, \qquad \frac{1}{2}\delta^{(2)}\rho = \delta_2\rho, \tag{2.39}$$

where $h = h_k^k$ and $\ell = \ell_k^k$. It is interesting that $\chi_{ij}^{(2)\|}$ includes not only scalar perturbations, but also vector (shear) perturbations (without vorticity [13]), because it does not reduce to the form of $D_{ij}\chi$ in general.

III. CMB ANISOTROPIES DUE TO FIRST-ORDER AND SECOND-ORDER PERTURBATIONS

In the unperturbed model universe, the observed temperature (T_0) of the CMB radiation is related to the emitted temperature (T_e) at the decoupling epoch (z_e) by $T_o = T_e/(1 + z_e)$, and represented also as

$$T_o = (\omega_o/\omega_e)T_e \tag{3.1}$$

using the observed and emitted frequencies ω_o and $\omega_e(= (1 + z_e)\omega_o)$.

In the perturbed universe, these temperatures depend on the motions of matter and observers and on light paths passing through the inhomogeneous matter, and are expressed as

$$T_o(\mathbf{x}_o, \mathbf{e}) = (\omega_o/\omega_e)T_e(\mathbf{p}, \mathbf{d}) \tag{3.2}$$

with $\omega = -g_{\mu\nu}u^\mu k^\nu$, where u^μ is the observer's and emitter's velocities, $k^\nu(= dx^\nu/d\lambda)$ is the wave vector of photons with affine parameter λ, $(\mathbf{x}_o, \mathbf{e})$ and $(\mathbf{p}, \mathbf{d})$ are (the position vectors and directional unit vectors) of the observer and emitter, respectively.

The wave vector k^μ satisfies the perturbed null geodesic equation and its solutions to the second order are expressed as

$$k^\mu = k_{(0)}^\mu + k_{(1)}^\mu + k_{(2)}^\mu, \qquad x^\mu = x_{(0)}^\mu + x_{(1)}^\mu + x_{(2)}^\mu, \tag{3.3}$$

where x^μ represents the light path and (r) denotes the r-order smallness.

The temperature at the decoupling epoch is expressed as

$$T_e(\mathbf{p}, \mathbf{d}) = T_e^{(0)}[1 + \tau(\mathbf{p}, \mathbf{d})] \tag{3.4}$$

and the frequencies to the second order are

$$\omega = \omega^{(0)}[1 + \tilde{\omega}^{(1)} + \tilde{\omega}^{(2)}]. \tag{3.5}$$

Then we have

$$T_o(\mathbf{x}_o, \mathbf{e}) = T_o^{(0)}[1 + \delta_1 T/T + \delta_2 T/T], \tag{3.6}$$

where

$$\delta_1 T/T = \tilde{\omega}_o^{(1)} - \tilde{\omega}_e^{(1)} + \tau,$$
$$\delta_2 T/T = \tilde{\omega}_o^{(2)} - \tilde{\omega}_e^{(2)} + (\tilde{\omega}_e^{(1)})^2 - \tilde{\omega}_o^{(1)}\tilde{\omega}_e^{(1)} + (\tilde{\omega}_o^{(1)} - \tilde{\omega}_e^{(1)})\tau + p^{(1)i}\partial\tau/\partial x^i + d^{(1)i}\partial\tau/\partial d^i. \tag{3.7}$$

The procedure for solving null geodesic equations in perturbed universe models was shown by Pyne and Carroll, in which the background null rays are given by $x^{(0)\mu} = (\lambda, (\lambda_o - \lambda)e^i)$ and $k^{(0)\mu} = (1, -e^i)$, and the boundary conditions at the origin are $x^{(1)\mu}(\lambda_o) = x^{(2)\mu}(\lambda_o) = 0$ and $k^{(1)i}(\lambda_o) = k^{(2)i}(\lambda_o) = 0$. The expressions for $\delta_1 T/T$ and $\delta_2 T/T$ were derived by Mollerach and Matarrese [7] in general gauges using Pyne and Carroll's procedure [16]. Their expressions are found to be gauge invariant, and so the values can be calculated in a specified gauge without loss of any generality. Here we describe them using our solutions in the synchronous gauge, under the condition that the observers and emitters are comoving, i.e. $v_o^{(r)} = v_e^{(r)} = 0$.

The following temperature perturbations are derived in the synchronous gauge from Eqs. (2.20)–(2.28) in Mollerach and Matarrese's paper [7]. First we have the first-order perturbations:

$$\delta_1 T/T = \tau - I_1(\lambda_e), \tag{3.8}$$

where

$$I_1(\lambda) = \int_{\lambda_o}^{\lambda} d\bar{\lambda} A^{(1)\prime}(\bar{\lambda}), \tag{3.9}$$

$$A^{(1)} = \phi^{(1)} - \frac{1}{2}\chi_{ij}^{(1)} e^i e^j, \tag{3.10}$$

and e^i denotes a component of the directional unit vector $\mathbf{e}$. The first-order wave vectors are

$$k^{(1)0}(\lambda) = -\phi_o^{(1)} + \frac{1}{2}\chi_o^{(1)ij} e_i e_j + I_1(\lambda) \tag{3.11}$$

and

$$k^{(1)i}(\lambda) = 2\phi_o^{(1)} e^i - \chi_o^{(1)ij} e_j - 2\phi^{(1)} e^i + \chi^{(1)ij} e_j - I_1^i(\lambda), \tag{3.12}$$

where

$$I_1^i(\lambda) = \int_{\lambda_o}^{\lambda} d\bar{\lambda} A^{(1)|i}(\bar{\lambda}) \tag{3.13}$$

and $|i = \delta^{il}\partial_l$ and $e_j = \delta_{ij} e^i$.

The first-order light paths are

$$x^{(1)0}(\lambda) = (\lambda - \lambda_o)\left[-\phi_o^{(1)} + \frac{1}{2}\chi_o^{(1)ij} e_i e_j\right] + \int_{\lambda_o}^{\lambda} d\bar{\lambda}(\lambda - \bar{\lambda}) A^{(1)\prime}(\bar{\lambda}), \tag{3.14}$$

$$x^{(1)i}(\lambda) = (\lambda - \lambda_o)[2\phi_o^{(1)} e^i - \chi_o^{(1)ij} e_j] - \int_{\lambda_o}^{\lambda} d\bar{\lambda}[2\phi^{(1)} e^i - \chi^{(1)ij} e_j + (\lambda - \bar{\lambda}) A^{(1)|i}(\bar{\lambda})]. \tag{3.15}$$

The second-order wave vectors satisfy the following relation

$$k_e^{(2)0} - k_o^{(2)0} = I_2(\lambda_e), \tag{3.16}$$

where

$$I_2(\lambda) = \int_{\lambda_o}^{\lambda} d\bar{\lambda}\Big[\frac{1}{2}A^{(2)\prime} + \chi_{ij}^{(1)\prime} e^j (k^{(1)i} + e^i k^{(1)0}) + 2k^{(1)0} A^{(1)\prime} + 2\phi^{(1)\prime} A^{(1)} + x^{(1)0} A^{(1)\prime\prime} + x^{(1)i} A_{,i}^{(1)\prime}\Big] \tag{3.17}$$

and

$$A^{(2)} \equiv \phi^{(2)} - \frac{1}{2}\chi_{ij}^{(2)} e^i e^j. \tag{3.18}$$

The second-order temperature perturbations are

$$\delta_2 T/T = -I_2(\lambda_e) + I_1(\lambda_e) \times \left[I_1(\lambda_e) - \tau - \phi_o^{(1)} + \frac{1}{2}\chi_o^{(1)ij} e_i e_j\right] + x_e^{(1)0} A_e^{(1)\prime} + (x_e^{(1)j} + x_e^{(1)0} e^j)\tau_{,j} + \frac{\partial \tau}{\partial d^i} d^{(1)i}, \tag{3.19}$$

where

$$d^{(1)i} = e^i - (e^i - k^{(1)i})/|e^i - k^{(1)i}|. \tag{3.20}$$

Moreover, if we substitute our metric perturbations into the above equations, we obtain

$$\delta_1 T/T = \tau + \frac{1}{2}\int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) F_{,ij} e^i e^j \tag{3.21}$$

for the first-order perturbation. Since $dP/d\lambda = P'$ and $dF/d\lambda = -F_{,i}e^i$, Eq. (3.21) can be expressed as

$$\delta_1 T/T = \Theta_1 + \Theta_2 \tag{3.22}$$

with

$$\Theta_1 \equiv \tau - \frac{1}{2}[(P'F_{,i})_e - (P'F_{,i})_o]e^i, \quad \Theta_2 \equiv \frac{1}{2}\int_{\lambda_o}^{\lambda_e} d\lambda P''(\eta) F_{,i} e^i. \tag{3.23}$$

The latter term Θ_2 represents the first-order Sachs-Wolfe effect.

Next, we notice that Eqs. (3.11), (3.12), (3.14), and (3.15) lead to

$$k^{(1)0} + k^{(1)i} e_i = -A^{(1)}, \quad x^{(1)0} + x^{(1)i} e_i = -\int_{\lambda_o}^{\lambda} d\lambda A^{(1)}, \tag{3.24}$$

and we have

$$\int_{\lambda_o}^{\lambda_e} d\lambda A^{(1)\prime}(\lambda) I_1(\lambda) = \frac{1}{2}[I_1(\lambda_e)]^2. \tag{3.25}$$

Then we obtain from Eq. (3.17)

$$I_2(\lambda_e) = \frac{1}{2}[I_1(\lambda_e)]^2 - (\lambda_e - \lambda_0)A_o^{(1)}A_e^{(1)\prime} + A_e^{(1)\prime}\int_{\lambda_o}^{\lambda_e} d\lambda[A^{(1)} + (\lambda_e - \lambda)A^{(1)\prime}] - A_o^{(1)}I_1(\lambda_e) + \int_{\lambda_o}^{\lambda_e} d\lambda\Big\{\frac{1}{2}A^{(2)\prime} + A^{(1)}A^{(1)\prime} - A^{(1)\prime\prime}\int_{\lambda_o}^{\lambda} d\bar\lambda A^{(1)}(\bar\lambda)\Big\}, \tag{3.26}$$

and therefore from Eq. (3.19)

$$\delta_2 T/T = I_1(\lambda_e)\Big[\frac{1}{2}I_1(\lambda_e) - \tau\Big] - [A_e^{(1)\prime} + \tau_{,i}e^i]\int_{\lambda_o}^{\lambda_e} d\lambda A^{(1)} - \int_{\lambda_o}^{\lambda_e} d\lambda\Big\{\frac{1}{2}A^{(2)\prime} + A^{(1)}A^{(1)\prime} - A^{(1)\prime\prime}\int_{\lambda_o}^{\lambda} d\bar\lambda A^{(1)}(\bar\lambda)\Big\} + \frac{\partial\tau}{\partial d^i}d^{(1)i}, \tag{3.27}$$

where $(\eta, x^i) = (\lambda, \lambda_o - \lambda)$ in the integrands and

$$I_1(\lambda_e) = -\frac{1}{2}\int_{\lambda_o}^{\lambda_e} d\lambda P' F_{,ij}e^ie^j, \qquad A^{(1)} = -\frac{1}{2}PF_{,ij}e^ie^j,$$
$$A^{(2)} = -\frac{1}{2}[PL_i^j + P^2M_i^j + QN_{,ij} + C_i^j]e^ie^j. \tag{3.28}$$

In Eq. (3.27) the terms with path integrations represent the second-order nonlinear Integral Sachs-Wolfe effect, which brings the observational coupling of two linearly independent inhomogeneities with different wavelengths. As can be seen from Eq. (3.28), the induced gravitational radiation contributes to the CMB anisotropies.

IV. PERTURBATIONS IN THE POISSON GAUGE

In this section we derive the perturbations in the Poisson gauge which is defined by the condition

$$z_i^{(r)|i} = 0 \quad \text{and} \quad \chi_{ij}^{(r)|j} = 0. \tag{4.1}$$

For this purpose, we use a gauge transformation from the perturbations in the synchronous gauge to those in Poisson gauge. The first-order gauge transformation has fully been studied by many authors (Bardeen [13], Kodama and Sasaki [17]). The second-order gauge transformation has more recently been derived by Bruni *et al.* [5] and the transformations of an arbitrary perturbed tensor $\mathcal{T} = \mathcal{T}_0 + \delta\mathcal{T}^{(1)} + \frac{1}{2}\delta\mathcal{T}^{(2)}$ are expressed in terms of generators $\xi_{(1)}$ and $\xi_{(2)}$ as

$$\delta\bar{\mathcal{T}}^{(1)} = \delta\mathcal{T}^{(1)} + \mathcal{L}_{\xi_{(1)}}\mathcal{T}_0,$$
$$\delta\bar{\mathcal{T}}^{(2)} = \delta\mathcal{T}^{(2)} + 2\mathcal{L}_{\xi_{(1)}}\delta\mathcal{T}^{(1)} + \mathcal{L}_{\xi_{(1)}}^2\mathcal{T}_0 + \mathcal{L}_{\xi_{(2)}}\mathcal{T}_0, \tag{4.2}$$

where $\mathcal{L}$ denotes the Lie derivative and the components of the generators are expressed as

$$\xi_{(r)}^0 = \alpha^{(r)} \tag{4.3}$$

and

$$\xi_{(r)}^i = \beta^{(r)|i} + d^{(r)i} \tag{4.4}$$

with $d_{,i}^{(r)i} = 0$.

This gauge transformation has been applied by Matarrese *et al.* to derive the second-order perturbations of the Einstein–de Sitter model in the Poisson gauge from those in the synchronous gauge. Here in the similar manner we will apply the transformation to our perturbations derived in Sect. II.

A. First-order transformation

For the transformation of our metric perturbations from the synchronous gauge (S) to the Poisson gauge (P), we have

$$\psi_P^{(1)} = \alpha'^{(1)} + \frac{a'}{a}\alpha^{(1)}, \qquad \alpha^{(1)} = \beta'^{(1)}, \qquad z_{Pi}^{(1)} = d_i^{(1)\prime},$$
$$\phi_P^{(1)} = \phi_S^{(1)} - \frac{1}{3}\Delta\beta^{(1)} - \frac{a'}{a}\alpha^{(1)}, \qquad \chi_S^{(1)\|} + 2\beta^{(1)} = 0,$$
$$d_i^{(1)} = 0. \tag{4.5}$$

Using Eq. (2.39) for $\phi_S^{(1)}$ and $\chi_S^{(1)\|}$, we obtain from the above equations

$$\alpha^{(1)} = -\frac{1}{2}P'F, \qquad \beta^{(1)} = -\frac{1}{2}PF,$$
$$\psi_P^{(1)} = \phi_P^{(1)} = \frac{1}{2}\Big(1 - \frac{a'}{a}P'\Big)F, \qquad z_{Pi}^{(1)} = 0. \tag{4.6}$$

The density and velocity perturbations satisfy the following relations:

$$\delta\rho_P^{(1)}/\rho = -\delta\rho_S^{(1)}/\rho + \frac{\rho'}{\rho}\alpha^{(1)},$$
$$v_P^{(1)0} = -\frac{a'}{a}\alpha^{(1)} - \alpha'^{(1)}, \qquad v_P^{(1)i} = -\beta'^{(1)|i} - d'^{(1)i}, \tag{4.7}$$

so that

$$\delta\rho_P^{(1)}/\rho = \frac{1}{2\rho a^2}\Big(\frac{a'}{a}P' - 1\Big)\Delta F + \frac{1}{2}\frac{a'}{a}P'F,$$
$$v_P^{(1)0} = -\frac{1}{2}\frac{a'}{a}\Big(\frac{a'}{a}P' - 1\Big)F, \qquad v_P^{(1)i} = \frac{1}{2}P'F_{,i}. \tag{4.8}$$

B. Second-order transformation

Similarly we use the transformations expressed as

$$\begin{aligned} \delta g^{(2)}_{\mu\nu P} &= \delta g^{(2)}_{\mu\nu S} + 2\mathcal{L}_{\xi_{(1)}} \delta g^{(1)}_{\mu\nu S} + \mathcal{L}^2_{\xi_{(1)}} g^{(0)}_{\mu\nu} + \mathcal{L}_{\xi_{(2)}} g^{(0)}_{\mu\nu}, \\ \delta \rho^{(2)}_P &= \delta \rho^{(2)}_S + (\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}})\rho_{(0)} + 2\mathcal{L}_{\xi_{(1)}} \delta \rho^{(1)}_S, \\ \delta u^\mu_{(2)P} &= \delta u^\mu_{(2)S} + (\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}}) u^\mu_{(0)} + 2\mathcal{L}_{\xi_{(1)}} \delta u^\mu_{(1)S}. \end{aligned} \tag{4.9}$$

More explicit expressions for these relations are given in Eqs. (5.8)–(5.14) of Matarrese *et al.*'s paper [6]. By analyzing these latter equations, we obtain the following generators:

$$\begin{aligned} \alpha^{(2)} &= P'\Big(\frac{50}{3}\Theta_0 + \frac{1}{2}F^2\Big) + \frac{50}{3}P'P''\Theta_0 - \frac{100}{21}\Big(PP' - \frac{1}{6}Q'\Big)\Psi_0, \\ \beta^{(2)} &= P\Big(\frac{50}{3}\Theta_0 + F^2\Big) + (P')^2\Big(\frac{25}{3}\Theta_0 + \frac{1}{8}F^2\Big) + P^2\Big(-\frac{50}{21}\Psi_0 + \frac{1}{8}F_{,l}F_{,l}\Big) + \frac{50}{63}Q\Psi_0, \\ \Delta d^{(2)}_i &= \Big[-\frac{200}{9}\Psi_{0,i} + \frac{1}{2}(F_{,l}F_{,l})_{,i} - F_{,i}\Delta F\Big]\Big[P + \frac{1}{2}(P')^2\Big], \end{aligned} \tag{4.10}$$

where

$$\Psi_0 \equiv \frac{9}{200}[F_{,kl}F_{,kl} - (\Delta F)^2], \qquad \Delta\Theta_0 \equiv \Psi_0 - \frac{3}{100}F_{,l}F_{,l}. \tag{4.11}$$

It is found from Eq. (4.10) that vector perturbations without vorticity [13] appear also in this gauge. The resulting metric perturbations are expressed as

$$\begin{aligned} \psi^{(2)}_P =& \frac{1}{4}\Big\{4 - 7\frac{a'}{a}P' + \Big[-\frac{a''}{a} + 5\Big(\frac{a'}{a}\Big)^2\Big](P')^2\Big\}F^2 + \frac{1}{4}P\Big(1 - \frac{a'}{a}P'\Big)F_{,l}F_{,l} \\ &+ \frac{50}{3}\Big\{2 - \frac{6a'}{a}P' + \Big[-\frac{2a''}{a} + 8\Big(\frac{a'}{a}\Big)^2\Big](P')^2\Big\}\Theta_0 - \frac{100}{21}\Big[(P')^2 + P\Big(1 - \frac{a'}{a}P'\Big) - \frac{1}{6}\Big(Q'' + \frac{a'}{a}Q'\Big)\Big]\Psi_0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \phi^{(2)}_P =& \frac{1}{4}P'\Big\{\frac{a'}{a} + \Big[-\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\Big]P'\Big\}F^2 + \frac{1}{4}P\Big(1 - \frac{a'}{a}P'\Big)F_{,l}F_{,l} - \frac{100}{3}\frac{a'}{a}P'\Big(1 - \frac{a'}{a}P'\Big)\Theta_0 \\ &+ \Big\{\frac{100}{21}\frac{a'}{a}\Big(PP' - \frac{1}{6}Q'\Big) - \frac{50}{9}\Big[P + \frac{1}{2}(P')^2\Big]\Big\}\Psi_0, \end{aligned} \tag{4.13}$$

$$(\Delta z^{(2)}_i)_P = P'(1 + P'')\Big[-\frac{200}{9}\Psi_{0,i} + \frac{1}{2}(F_{,l}F_{,l})_{,i} - F_{,i}\Delta F\Big], \tag{4.14}$$

and

$$(\chi^{(2)\parallel}_{ij})_P = 0 \qquad \text{and} \qquad \frac{1}{2}(\chi^{(2)\top}_{ij})_P = C_{ij}, \tag{4.15}$$

where we used Eq. (2.8) in the derivation of the above equations, and $\parallel$ and $\top$ denote the scalar perturbation and the transverse and traceless part representing induced gravitational radiation, respectively. In the limit of $\Lambda = 0$, these generators and solutions reduce to Eqs. (6.6)–(6.8) in Matarrese *et al.*'s paper [6], which were shown for the Einstein–de Sitter model.

The density and velocity perturbations are

$$(\delta^{(2)}\rho/\rho)_P = -100\frac{a'}{a}P'\Big(1-\frac{a'}{a}P'\Big)\Theta_0 + \frac{100}{7}\frac{a'}{a}\Big(PP'-\frac{1}{6}Q'\Big)\Psi_0 + \Big\{3\Big[\Big(\frac{a'}{a}\Big)^2 - \frac{1}{4}\frac{a''}{a}\Big](P')^2 - \frac{3}{4}\frac{a'}{a}P'(P''+2)\Big\}F^2 + \frac{3}{2}\Big[-2\frac{a'}{a}PP' + \Big(1-\frac{a'}{a}P'\Big)\Big/(\rho a^2)\Big]F_{,l}F_{,l} + \frac{1}{\rho a^2}\Big\{-\frac{1}{2}P'\Big[\Big(\frac{a''}{a}-5\Big(\frac{a'}{a}\Big)^2\Big)P' + 3\frac{a'}{a}\Big] + 4\Big(1-\frac{a'}{a}P'\Big)\Big\}F\Delta F + \frac{1}{\rho a^2}\Big\{\frac{1}{14}\frac{a'}{a}Q'[(\Delta F)^2 - F_{,kl}F_{,kl}] - \frac{1}{2}P\Big(\frac{a'}{a}P'-1\Big)F_{,l}\Delta F_{,l} - \frac{1}{7}\frac{a'}{a}PP'[3(\Delta F)^2 + 4F_{,kl}F_{,kl}] + \frac{1}{2}P[(\Delta F)^2 + F_{,kl}F_{,kl}] + \frac{1}{4}(P')^2[(\Delta F)^2 - F_{,kl}F_{,kl}]\Big\}, \tag{4.16}$$

$$(v^0_{(2)})_P = -\frac{100}{3}\Big[1 - 3\frac{a'}{a}P' + 2\Big(\frac{a'}{a}\Big)^2(P')^2\Big]\Theta_0 + \frac{100}{21}\Big[(P')^2 + P\Big(1-\frac{a'}{a}P'\Big) + \frac{1}{6}\Big(Q'' + \frac{a'}{a}Q'\Big)\Big]\Psi_0 + \frac{1}{4}\Big[-1 + \frac{a'}{a}P' + \Big(\frac{a''}{a} - 2\Big(\frac{a'}{a}\Big)^2\Big)(P')^2\Big]F^2 + \frac{1}{4}\Big[(P')^2 - P\Big(1-\frac{a'}{a}P'\Big)\Big]F_{,l}F_{,l}. \tag{4.17}$$

$$(v^i_{(2)})_P = -d^i_{(2)P} - \frac{50}{3}P'(1+P'')\Theta_{0,i} + \frac{100}{21}\Big(PP' - \frac{1}{6}Q'\Big)\Psi_{0,i} - \frac{1}{2}PP'F_{,l}F_{,li} - \frac{1}{2}P'\Big(5 - 3\frac{a'}{a}P'\Big)FF_{,i}. \tag{4.18}$$

In the limit of $\Lambda = 0$, these relations also are consistent with Eqs. (6.10)–(6.12) in Matarrese *et al.*'s paper [6], except for a few terms which may include some misprints with respect to $(\delta^{(2)}\rho)_P$ and $(v^i_{(2)})_P$.

Now we consider the second-order rotational velocity in the Poisson gauge. The rotational velocity vector $(\omega_P^{\alpha(r)})$ and the corresponding scalar quantity $(\omega_P^{(r)})$ in the Poisson gauge are related to those (S) in the synchronous gauge by Eq. (4.2) as $\delta\bar{\mathcal{T}}^{(r)} = \omega_P^{\alpha(r)}, \omega_P^{(r)}$ and $\delta\mathcal{T}^{(r)} = \omega_S^{\alpha(r)}, \omega_S^{(r)}$, respectively, for $r = 1, 2$. Here we have $\mathcal{T}^0 = 0$ and $\delta\mathcal{T}^{(r)} = 0$ for $r = 1, 2$, as was shown in Sec. II. It is found therefore from these relations that $\omega_P^{\alpha(r)}$ and $\omega_P^{(r)}$ vanish. That is, no rotational perturbations are induced also in the Poisson gauge. However vector (shear) perturbations without vorticity appear in the form of $\Delta d_i^{(2)}$.

V. CONCLUDING REMARKS

We have studied the second-order perturbations in spatially flat, pressureless cosmological models with nonzero cosmological constant, and could describe them explicitly using an arbitrary potential function F of spatial coordinates. It seems, however, to be difficult to derive similar results in the other cases with finite pressures.

It is found in the second order that tensor (gravitational-wave) perturbations and vector (shear) perturbations without vorticity are induced from the first-order scalar perturbations.

Next we have derived the second-order temperature anisotropies of CMB in terms of these perturbations. They will be useful to analyze the nonlinear influence of local inhomogeneities (at later stages) upon observed CMB anisotropies. Since nonlinearity brings the coupling of two linearly independent inhomogeneities with different wavelengths, it may appear as a small directional asymmetry and some deviations from the results which are expected in the standard linear theory.

APPENDIX: SECOND-ORDER PERTURBED RICCI TENSORS AND EINSTEIN EQUATIONS

We show here the expressions for perturbations of Ricci tensors to derive the perturbed Einstein equations. For the metric perturbations $\delta g_{\mu\nu} = h_{\mu\nu} + \ell_{\mu\nu}$, the contravariant metric perturbations $\delta g^{\mu\nu}$ are derived from the condition $(g_{\mu\nu} + \delta g_{\mu\nu})(g^{\nu\gamma} + \delta g^{\nu\gamma}) = \delta^\gamma_\mu$ as

$$\delta g^{\mu\nu} = -h^{\mu\nu} + (h^\mu_\gamma h^{\gamma\nu} - \ell^{\mu\nu}). \tag{A1}$$

The perturbed Christoffel symbols are

$$\delta_1\Gamma^\mu_{\nu\gamma} = \frac{1}{2}(h^\mu_{\nu;\gamma} + h^\mu_{\gamma;\nu} - h^{;\mu}_{\nu\gamma}),$$
$$\delta_2\Gamma^\mu_{\nu\gamma} = \frac{1}{2}(\ell^\mu_{\nu;\gamma} + \ell^\mu_{\gamma;\nu} - \ell^{;\mu}_{\nu\gamma}) - \frac{1}{2}h^\mu_\lambda(h^\lambda_{\nu;\gamma} + h^\lambda_{\gamma;\nu} - h^{;\lambda}_{\nu\gamma}), \tag{A2}$$

where a semicolon denotes the four-dimensional covariant derivative in the unperturbed universe. From Eqs. (A1) and (A2), we can derive the perturbed curvature tensors and Ricci tensors:

$$\delta_1 R^\mu_{\nu\lambda\gamma} = \frac{1}{2}(h^\mu_{\nu;\gamma\lambda} + h^\mu_{\gamma;\nu\lambda} - h^{;\mu}_{\nu\gamma;\lambda} - h^\mu_{\lambda;\nu\gamma} - h^\mu_{\nu;\lambda\gamma} + h^{;\mu}_{\nu\lambda;\gamma}),$$
$$\delta_1 R_{\nu\gamma} \equiv \delta_1 R^\mu_{\nu\mu\gamma} = \frac{1}{2}(h^\mu_{\nu;\gamma\mu} + h^\mu_{\gamma;\nu\mu} - h^{;\mu}_{\nu\gamma;\mu} - h_{;\nu\gamma}),$$
$$\delta_2 R^\mu_{\nu\lambda\gamma} = \frac{1}{2}(\ell^\mu_{\nu;\gamma\lambda} + \ell^\mu_{\gamma;\nu\lambda} - \ell^{;\mu}_{\nu\gamma;\lambda} - \ell^\mu_{\lambda;\nu\gamma} - \ell^\mu_{\nu;\lambda\gamma} + \ell^{;\mu}_{\nu\lambda;\gamma}) - \frac{1}{2}h^\mu_\alpha(h^\alpha_{\nu;\gamma\lambda} + h^\alpha_{\gamma;\nu\lambda} - h^{;\alpha}_{\nu\gamma;\lambda} - h^\alpha_{\lambda;\nu\gamma} - h^\alpha_{\nu;\lambda\gamma} + h^{;\alpha}_{\nu\lambda;\gamma})$$
$$+ \frac{1}{4}(h^\mu_{\alpha;\gamma} - h^\mu_{\gamma;\alpha} + h^{;\mu}_{\gamma\alpha})(h^\alpha_{\nu;\lambda} + h^\alpha_{\lambda;\nu} - h^{;\alpha}_{\nu\lambda}) - \frac{1}{4}(h^\mu_{\alpha;\lambda} - h^\mu_{\lambda;\alpha} + h^{;\mu}_{\alpha\lambda})(h^\alpha_{\nu;\gamma} + h^\alpha_{\gamma;\nu} - h^{;\alpha}_{\nu\gamma}),$$
$$\delta_2 R_{\nu\gamma} \equiv \delta_2 R^\mu_{\nu\mu\gamma} = \frac{1}{2}(\ell^\mu_{\nu;\gamma\mu} + \ell^\mu_{\gamma;\nu\mu} - \ell^{;\mu}_{\nu\gamma;\mu} - \ell_{;\nu\gamma}) - \frac{1}{2}h^\mu_\alpha(h^\alpha_{\nu;\gamma\mu} + h^\alpha_{\gamma;\nu\mu} - h^{;\alpha}_{\nu\gamma;\mu} - h^\alpha_{\mu;\nu\gamma})$$
$$+ \frac{1}{4}(h^\mu_{\alpha;\gamma} - h^\mu_{\gamma;\alpha} + h^{;\mu}_{\gamma\alpha})(h^\alpha_{\nu;\mu} + h^\alpha_{\mu;\nu} - h^{;\alpha}_{\nu\mu}) - \frac{1}{4}(2h^\mu_{\alpha;\mu} - h_{;\alpha})(h^\alpha_{\nu;\gamma} + h^\alpha_{\gamma;\nu} - h^{;\alpha}_{\nu\gamma}). \quad \text{(A3)}$$

Using the relation

$$R^\mu_\nu + \delta_1 R^\mu_\nu + \delta_2 R^\mu_\nu = (g^{\mu\gamma} + \delta g^{\mu\gamma})(R_{\nu\gamma} + \delta_1 R_{\nu\gamma} + \delta_2 R_{\nu\gamma}), \quad \text{(A4)}$$

we obtain the mixed components of the perturbed Ricci tensors:

$$\delta_1 R^\mu_\nu = g^{\mu\gamma}\delta_1 R_{\nu\gamma} - h^{\mu\gamma}R_{\nu\gamma}, \qquad \delta_2 R^\mu_\nu = g^{\mu\gamma}\delta_2 R_{\nu\gamma} - h^{\mu\gamma}\delta_1 R_{\nu\gamma} + (h^\mu_\lambda h^{\lambda\gamma} - \ell^{\mu\gamma})R_{\nu\gamma}. \quad \text{(A5)}$$

Here let us impose the synchronous gauge conditions (2.6) and (2.15) on $h_{\mu\nu}$ and $\ell_{\mu\nu}$. Then we obtain the following expressions for second-order perturbed Ricci tensors $\delta_2 R^\mu_\nu$.

$$2a^2\delta_2 R^j_i = \Phi^j_i + (\ell^j_i)'' + \frac{2a'}{a}(\ell^j_i)' + \frac{a'}{a}\ell'\delta^j_i - \left[h^j_k(h^k_i)'' + (h^k_i)'(h^j_k)' - \frac{1}{2}h'(h^j_i)' + \frac{a'}{a}\delta^j_i h^k_l(h^l_k)' + \frac{2a'}{a}h^j_k(h^k_i)'\right],$$
$$2a^2\delta_2 R^0_i = (\ell_{|i})' - (\ell^k_{i|k})' + h^k_l[(h^l_{i|k})' - (h^l_{k|i})'] - \frac{1}{2}(h^k_l)'h^l_{k|i} + \frac{1}{2}(h^k_i)'(2h^l_{k|l} - h_{|k}),$$
$$2a^2\delta_2 R^j_0 = (\ell^{|j})' - (\ell^{j|k}_k)' + h^{kl}(h^j_{k|l})' - h^k_l(h^{l|j}_k)' - \frac{1}{2}(h^k_l)'h^{l|j}_k + \frac{1}{2}(h^j_k)'(2h^{k|l}_l - h^{|k}) + h^j_k[(h^{|k})' - (h^{k|l}_l)'],$$
$$2a^2\delta_2 R^0_0 = \ell'' + \frac{a'}{a}\ell' - h^k_l\left[(h^l_k)'' + \frac{a'}{a}(h^l_k)'\right] - \frac{1}{2}(h^k_l)'(h^l_k)',$$
$$2a^2\delta_2 R \equiv 2a^2\delta_2(R^0_0 + R^k_k) = \Phi^k_k + 2\left(\ell'' + 3\frac{a'}{a}\ell'\right) - \left[2h^k_l(h^l_k)'' + \frac{3}{2}(h^k_l)'(h^l_k)' - \frac{1}{2}(h')^2 + \frac{6a'}{a}h^k_l(h^l_k)'\right], \quad \text{(A6)}$$

where $|i$ means the three-dimensional covariant derivative in the constant-curvature space with metric γ_{ij}, $h = h^k_k$, $\ell = \ell^k_k$, and Φ^j_i is defined as follows:

$$\Phi^j_i \equiv \ell^{k|j}_i + \ell^j_{k|i} - \ell^{j|k}_i - \ell^{|k}_{|k} - 4K\ell^j_i - h^k_l[h^{l|j}_i + h^j_{k|i} - h^{j|l}_i - h^l_{k|i}] - h^j_l[h^{k|l}_i + h^l_{k|i} - h^{l|k}_i - h^{|l}_{|i} - 4Kh^l_i]$$
$$+ \frac{1}{2}(h^{k|j}_l - \gamma^{km}h^j_{m|l} + h^{j|k}_l)(h^l_{i|k} + h^l_{k|i} - \gamma_{mk}h^{m|l}_i) - \frac{1}{2}(2h^k_{l|k} - k_{|l})(h^{l|j}_i + \gamma^{lm}h^j_{m|i} - h^{j|l}_i), \quad \text{(A7)}$$

where $K(= \pm 1, 0)$ is the three-dimensional spatial curvature. In the appendix, we take the spatial curvature and the pressure p into consideration.

The first-order and second-order perturbed Einstein equations are expressed as

$$\delta_1 R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\delta_1 R = \delta_1 T^\mu_\nu, \qquad \delta_2 R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\delta_2 R = \delta_2 T^\mu_\nu. \quad \text{(A8)}$$

The perturbations of the energy-momentum tensor are

$$\delta_1 T^\mu_\nu = \delta^\mu_\nu\delta_1 p + g_{\nu\alpha}[u^\alpha u^\mu(\delta_1 p + \delta_1\rho) + (\delta_1 u^\alpha u^\mu + u^\alpha\delta_1 u^\mu)(p + \rho)] + h_{\nu\alpha}u^\alpha u^\mu(p + \rho),$$
$$\delta_2 T^\mu_\nu = \delta^\mu_\nu\delta_2 p + g_{\nu\alpha}[u^\alpha u^\mu(\delta_2 p + \delta_2\rho) + (\delta_1 u^\alpha u^\mu + u^\alpha\delta_1 u^\mu)(\delta_1 p + \delta_1\rho) + (\delta_2 u^\alpha u^\mu + u^\alpha\delta_2 u^\mu + \delta_1 u^\alpha\delta_1 u^\mu)$$
$$\times (p + \rho)] + h_{\nu\alpha}[u^\alpha u^\mu(\delta_1 p + \delta_1\rho) + (\delta_1 u^\alpha u^\mu + u^\alpha\delta_1 u^\mu)(p + \rho)] + \ell_{\nu\alpha}u^\alpha u^\mu(p + \rho). \quad \text{(A9)}$$

For velocity perturbations, we obtain

$$\delta_1 u^0 = 0, \qquad \delta_2 u^0 = \frac{1}{2a} g_{kl} \delta_1 u^k \delta_1 u^l, \tag{A10}$$

using the identity

$$(g_{\mu\nu} + h_{\mu\nu} + \ell_{\mu\nu})(u^\mu + \delta_1 u^\mu + \delta_2 u^\mu) \times (u^\nu + \delta_1 u^\nu + \delta_2 u^\nu) = -1 \tag{A11}$$

and the synchronous gauge conditions. The components of the energy-momentum tensor are, therefore,

$$\delta_1 T_i^j = \delta_i^j \delta_1 p, \qquad \delta_1 T_0^j = -a(p+\rho)\delta_1 u^j, \qquad \delta_1 T_0^0 = -\delta_1 \rho, \tag{A12}$$

and

$$\begin{aligned}
\delta_2 T_i^j &= \delta_i^j \delta_2 p + (p+\rho) g_{ik} \delta_1 u^k \delta_1 u^j,\\
\delta_2 T_i^0 &= \frac{1}{a} g_{ik}[\delta_1 u^k(\delta_1 p + \delta_1 \rho)] + \delta_2 u^k (p+\rho) + \frac{1}{a} h_{ik} \delta_1 u^k (p+\rho),\\
\delta_2 T_0^j &= -a\delta_1 u^j(\delta_1 p + \delta_1 \rho) - a\delta_2 u^j (p+\rho),\\
\delta_2 T_0^0 &= -\delta_2 \rho - (p+\rho) g_{kl} \delta_1 u^k \delta_1 u^l.
\end{aligned} \tag{A13}$$

If we assume the equation of state $p = p(\rho)$, then we have

$$\delta_1 p = (dp/d\rho)\delta_1 \rho \quad \text{and} \quad \delta_2 p = (dp/d\rho)\delta_2\rho + \frac{1}{2}(d^2p/d\rho^2)(\delta_1\rho)^2, \tag{A14}$$

so that

$$\begin{aligned}
\delta_1 T_i^j &= -\delta_i^j (dp/d\rho)\delta_1 T_0^0,\\
\delta_2 T_i^j &= -\delta_i^j (dp/d\rho)\delta_2 T_0^0 + \frac{1}{2}\delta_i^j (d^2p/d\rho^2)(\delta_1\rho)^2 + (p+\rho)[g_{ik}\delta_1 u^j - \delta_i^j (dp/d\rho) g_{kl} \delta_1 u^l]\delta_1 u^k.
\end{aligned} \tag{A15}$$

When the first-order quantities h_i^j, δ_1, u^i, and $\delta_1\rho$ are known, the equations for determining the second-order metric perturbations are derived from Eqs. (A8) and (A15) as follows:

$$(\ell_i^j)'' + \frac{2a'}{a}(\ell_i^j)' + \Phi_i^j = h_k^j (h_i^k)'' + (h_i^k)'(h_k^j)' - \frac{1}{2}h'(h_i^j)' + \frac{2a'}{a} h_k^j (h_i^k)' + 2a^2(p+\rho) g_{ik} \delta_1 u^k \delta_1 u^j, \qquad (i \neq j), \tag{A16}$$

$$\begin{aligned}
&(\ell_i^i)'' + \frac{2a'}{a}(\ell_i^i)' - \ell'' - \frac{2a'}{a}\ell'(1 + dp/d\rho) + \Phi_i^i - \frac{1}{2}\Phi_k^k(1 + dp/d\rho)\\
&= h_k^i (h_i^k)'' + (h_i^k)'(h_k^i)' - \frac{1}{2}h'(h_i^i)' + \frac{2a'}{a} h_k^i (h_i^k)' - h_k^l (h_l^k)'' - \frac{3}{4}(h_l^k)'(h_k^l)' + \frac{1}{4}(h')^2\\
&\quad - \frac{2a'}{a} h_k^l (h_l^k)' + 2a^2(p+\rho)[g_{ik}\delta_1 u^i - (dp/d\rho) g_{kl}\delta_1 u^l]\delta_1 u^k - (dp/d\rho)\left[\frac{1}{4}(h_l^k)'(h_k^l)' + \frac{2a'}{a} h_k^i (h_i^k)' - (h')^2\right]\\
&\quad + a^2(d^2p/d\rho^2)(\delta_1\rho)^2,
\end{aligned} \tag{A17}$$

where in the latter equation the index i is nondummy but others are dummy, $h = h_k^k$, $\ell = \ell_k^k$, and Φ_i^j is defined in Eqs. (A7). Eqs. (A16) and (A17) consist of six equations, which corresponds to six independent components of ℓ_i^j.

The second-order density and velocity perturbations can be expressed in terms of metric perturbations as

$$\begin{aligned}
\delta_2\rho/\rho &= \frac{1}{\rho}\left[-\delta_2 R_0^0 + \frac{1}{2}\delta_2 R - (p+\rho) g_{kl}\delta_1 u^k \delta_1 u^l\right]\\
&= \frac{1}{2\rho a^2}\left[\frac{2a'}{a}\ell' - \frac{1}{4}(h_l^k)'(h_k^l)' - \frac{2a'}{a} h_l^k (h_k^l)' + \frac{1}{4}(h')^2 + \frac{1}{2}\Phi_k^k - 2a^2(p+\rho) g_{kl}\delta_1 u^k \delta_1 u^l\right],
\end{aligned} \tag{A18}$$

$$\begin{aligned}
a\delta_2 u^i &= -\frac{1}{p+\rho}[\delta_2 R_0^i + a\delta_1 u^i(\delta_1 p + \delta_1\rho)]\\
&= \frac{1}{2(p+\rho)a^2}\Big\{(\ell^{|i})' - (\ell_k^{i|k})' + h_l^k (h_k^{i|l})' - h_l^k (h_k^{l|i})' - \frac{1}{2}(h_l^k)' h_k^{l|i} + \frac{1}{2}(h_l^i)'(2h_k^{l|k} - h^{|l}) + h_k^i[(h^{|k})' - (h_l^{k|l})']\\
&\quad + 2a^3\delta_1 u^i(\delta_1 p + \delta_1\rho)\Big\}.
\end{aligned} \tag{A19}$$

[1] C. L. Bennet *et al.*, Astrophys. J. Suppl. Ser. **148**, 1 (2003).
[2] K. Tomita, Prog. Theor. Phys. **37**, 831 (1967).
[3] K. Tomita, Prog. Theor. Phys. **45**, 1747 (1971); **47**, 416 (1972).
[4] E. M. Lifshitz, J. Phys. (Moscow) **10**, 116 (1946); E. M. Lifshitz and I. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[5] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, Classical Quantum Gravity **14**, 2585 (1997).
[6] S. Matarrese, S. Mollerach, and M. Bruni, Phys. Rev. D **58**, 043504 (1998).
[7] S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (1997).
[8] N. Bartolo, S. Matarrese, and A. Riotto, J. Cosmol. Astropart. Phys. **01** (2004) 003.
[9] N. Bartolo, S. Matarrese, and A. Riotto, Phys. Rev. Lett. **93**, 231301 (2004).
[10] S. Mollerach, D. Harari, and S. Matarrese, Phys. Rev. D **69**, 063002 (2004).
[11] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, Phys. Rep. **402**, 103 (2004).
[12] K. Tomita, Prog. Theor. Phys. **85**, 1041 (1991); T. Futamase, Prog. Theor. Phys. **86**, 389 (1991); M. Shibata and H. Asada, Prog. Theor. Phys. **94**, 11 (1995); S. Matarrese and D. Terranova, Mon. Not. R. Astron. Soc. **283**, 400 (1996).
[13] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[14] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[15] H. Nariai and K. Kimura, Prog. Theor. Phys. **29**, 296 (1963).
[16] T. Pyne and S. M. Carroll, Phys. Rev. D **53**, 2920 (1996).
[17] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).