Measure of the path integral in lattice gauge theory

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(Received 14 March 2005; published 19 April 2005)

We show how to construct the measure of the path integral in lattice gauge theory. This measure contains a factor beyond the standard Haar measure. Such a factor becomes relevant for the calculation of a single transition amplitude (in contrast to the calculation of ratios of amplitudes). Single amplitudes are required for computation of the partition function and the free energy. For U(1) lattice gauge theory, we present a numerical simulation of the transition amplitude comparing the path integral with the evolution in terms of the Hamiltonian, showing good agreement.

DOI: 10.1103/PhysRevD.71.077502 PACS numbers: 11.15.Ha, 03.65.−w

I. INTRODUCTION

In lattice gauge theory, it is customary to compute the expectation value of an observable $\hat{O}[U]$, such as the Wilson loop, which is given by a ratio of Euclidean path integrals

$$\langle\Omega, t = +\infty|\hat{O}[U]|\Omega, t = -\infty\rangle = \frac{\int[dU]\hat{O}[U]]\exp[-S[U]/\hbar]}{\int[dU]\exp[-S[U]]/\hbar]}. \quad (1)$$

Here $[dU]$ denotes the Haar measure of the group of gauge symmetry, as for example, U(1), SU(2), SU(3), and U denotes the link variables, being elements of such a group. The state Ω denotes the vacuum. The reason why it is customary to compute such a ratio of path integrals is due to the Monte Carlo method with importance sampling (like Metropolis) which works only for such a ratio. In contrast, let us consider a single transition amplitude, like

$$\langle U_{\mathrm{fi}}, t = T|U_{\mathrm{in}}, t = 0\rangle = \int[dU]\exp[-S[U]/\hbar]|_{U_{\mathrm{in}},0}^{U_{\mathrm{fi}},T}. \quad (2)$$

The states $|U_{\mathrm{in}}\rangle$, $|U_{\mathrm{fi}}\rangle$ denote Bargmann link states, which are defined by assigning a value U_{ij} to each link ij on the lattice in a fixed time slice. Now the measure $[dU]$ is no longer given by the Haar measure only, but there is a factor Z^N involved for a lattice of $N-1$ intermediate time slices. It is the objective of this article to construct such a measure. Physical scenarios which require the calculation of single amplitudes are: amplitudes of decay reactions, the partition function $Z(\beta)$, the free energy $F(\beta)$ given in terms of the partition function, or matrix elements involved in the construction of the Monte Carlo Hamiltonian [1,2].

II. MEASURE IN 1D QUANTUM MECHANICS

Consider a Lagrangian of the form

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x). \quad (3)$$

The transition amplitude in imaginary time, expressed as a path integral over $N-1$ intermediate time slices is given by

$$\langle x_{\mathrm{fi}}|\exp[-HT/\hbar]|x_{\mathrm{in}}\rangle = \lim_{N\to\infty}\int_{-\infty}^{+\infty} dx_1\cdots dx_{N-1}\left(\sqrt{\frac{m}{2\pi\hbar a_0}}\right)^N \exp\left[-\frac{1}{\hbar}\sum_{j=0}^{N-1} a_0\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{a_0}\right)^2 - V(x_j)\right]\right],$$
$$x_{\mathrm{in}} = x_0, \qquad x_{\mathrm{fi}} = x_N, \qquad a_0 = T/N. \quad (4)$$

The measure

$$d\mu = dx_1\cdots dx_{N-1}\left(\sqrt{\frac{m}{2\pi\hbar a_0}}\right)^N \quad (5)$$

contains the physical parameters of mass m, Planck's constant $\hbar$, and the length of a time slice a_0, but is independent of the potential. It can be obtained by computing the propagator of the kinetic term for a single time slice

$$\left\langle x_{\mathrm{fi}}\left|\exp\left[-\frac{\hat{P}^2 a_0}{2m\hbar}\right]\right|x_{\mathrm{in}}\right\rangle = \left(\sqrt{\frac{m}{2\pi\hbar a_0}}\right) \times \exp\left[-\frac{m}{2a_0\hbar}(x_{\mathrm{fi}}-x_{\mathrm{in}})^2\right]. \quad (6)$$

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This propagator has the following properties. First, when a_0 goes to zero, the propagator goes over to $\delta(x_{\mathrm{fi}} - x_{\mathrm{in}})$. The right-hand side actually is a representation of the δ function by a Gaussian. Second, for any value of a_0 and x_{in} one has

$$\int_{-\infty}^{+\infty} dx_{\mathrm{fi}}\left(\sqrt{\frac{m}{2\pi\hbar a_0}}\right)\exp\left[-\frac{m}{2a_0\hbar}(x_{\mathrm{fi}} - x_{\mathrm{in}})^2\right] = 1. \quad (7)$$

Its interpretation is that the free propagator is the solution of a stochastic process (diffusion), and represents the probability for a random walker starting at x_{in} at $t = 0$ to arrive at x_{fi} at time $t = a_0$. Summed over all possible final destinations, the probability must be one. The solution of the diffusion equation has a probability density interpretation in the continuum case as well as in the discrete case [3]. More generally, in the presence of a potential, the Feynman-Kac theorem states that the Wiener path integral (path integral in imaginary time) has as solution $W(x_t, t|x_0, 0)$, which has an interpretation as probability density. This function is also a solution of the Bloch equation (analogue of Schrödinger equation in imaginary time) [4]. In the absence of a potential, the Bloch equation becomes the diffusion equation.

III. MEASURE IN LATTICE GAUGE THEORY

Let us consider QED on the lattice without fermions. The group of gauge symmetry is U(1). We keep in mind how to construct a lattice Hamiltonian (see, e.g, Creutz [5], Rothe [6]) via the transfer matrix, by splitting the lattice action into terms of timelike and spacelike plaquettes:

$$\begin{aligned} S[U] &= \frac{1}{g^2}\frac{a}{a_0}\sum_{\square_{\mathrm{timelike}}}[1 - \mathrm{Re}(U_\square)] \\ &\quad + \frac{1}{g^2}\frac{a_0}{a}\sum_{\square_{\mathrm{spacelike}}}[1 - \mathrm{Re}(U_\square)] \\ &\equiv S^{\mathrm{kin}}[U] + S^{\mathrm{pot}}[U]. \end{aligned} \quad (8)$$

As in quantum mechanics, the extra factor in the measure is determined solely by the kinetic term. Thus we consider the single transition amplitude

$$\langle U_{\mathrm{fi}}|\exp[-H^{\mathrm{kin}}T/\hbar]|U_{\mathrm{in}}\rangle, \quad (9)$$

where H^{kin} denotes the kinetic term of the Kogut-Susskind lattice Hamiltonian [6], given by

$$\begin{aligned} H &= \frac{g^2\hbar^2}{2a}\sum_{\langle ij\rangle}\hat{l}_{ij}^2 + \frac{1}{g^2 a}\sum_{\square_{\mathrm{spacelike}}}[1 - \mathrm{Re}(U_\square)] \\ &\equiv H^{\mathrm{kin}} + H^{\mathrm{pot}}. \end{aligned} \quad (10)$$

The Hamiltonian requires choosing a gauge and the Kogut-Susskind Hamiltonian has been obtained using the temporal gauge ($U_{\mathrm{timelike}} = 1$). The propagator, Eq. (9), expressed as a path integral reads

$$\begin{aligned} \langle U_{\mathrm{fi}}|\exp[-H^{\mathrm{kin}}T/\hbar]|U_{\mathrm{in}}\rangle &= \int Z[dU]\exp\left[-\frac{1}{\hbar}S^{\mathrm{kin}}[U]\right]\Bigg|_{U_{\mathrm{in}},t=0}^{U_{\mathrm{fi}},t=T} \\ &= \lim_{N\to\infty}\int\left[\prod_{\langle ij\rangle}Z_{ij}^N\prod_{k=1}^{N-1}dU_{ij}^{(k)}\right]\exp\left[-\frac{a}{\hbar g^2 a_0}\sum_{k=0}^{N-1}\sum_{\langle ij\rangle}[1 - \mathrm{Re}(U_{ij}^{(k)}(U_{ij}^{(k+1)})^\dagger)]\right]\Bigg|_{U_{\mathrm{in}},t=0}^{U_{\mathrm{fi}},t=T} \\ &= \prod_{\langle ij\rangle}\left(\lim_{N\to\infty}\int\left[Z_{ij}^N\prod_{k=1}^{N-1}dU_{ij}^{(k)}\right]\exp\left[-\frac{a}{\hbar g^2 a_0}\sum_{k=0}^{N-1}[1 - \mathrm{Re}(U_{ij}^{(k)}(U_{ij}^{(k+1)})^\dagger)]\right]\Bigg|_{U_{ij}^{\mathrm{in}},t=0}^{U_{ij}^{\mathrm{fi}},t=T}\right). \end{aligned} \quad (11)$$

The amplitude factorizes into independent amplitudes for each spatial link ij. In order to compute the factor Z, let us consider a single link and its time evolution for a single time step ($T = a_0$).

$$\langle U_{\mathrm{fi}}|\exp[-H^{\mathrm{kin}}a_0/\hbar]|U_{\mathrm{in}}\rangle = Z\exp\left[-\frac{a}{\hbar g^2 a_0}[1 - \cos(\alpha_{\mathrm{fi}} - \alpha_{\mathrm{in}})]\right]. \quad (12)$$

As in quantum mechanics and also in lattice gauge theory, the Euclidean propagator has a probabilistic interpretation, and the analogue of Eq. (7) holds:

$$\int dU_{\mathrm{fi}}\langle U_{\mathrm{fi}}|\exp[-H^{\mathrm{kin}}a_0/\hbar]|U_{\mathrm{in}}\rangle = \int_{-\pi}^{+\pi}\frac{d\alpha_{\mathrm{fi}}}{2\pi}Z\exp\left[-\frac{a}{\hbar g^2 a_0}[1 - \cos(\alpha_{\mathrm{fi}} - \alpha_{\mathrm{in}})]\right] = 1. \quad (13)$$

Defining $A = a/(\hbar g^2 a_0)$, and using the Bessel function of imaginary argument [7],

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} d\theta\exp[z\cos(\theta)], \quad (14)$$

Eq. (13) yields

$$Z(A) = \frac{\exp(A)}{I_0(A)}. \tag{15}$$

We keep T fixed, and let $\lim_{N\to\infty}$ which means $a_0 \to 0$ and $A \to \infty$, which is the continuum limit in time direction. However, in the space direction we have kept $a = 1$. The result for Z given by Eq. (15) holds for any a_0, while the asymptotic behavior when a_0 goes to zero is given by

$$Z(A) = \sqrt{2\pi A}\left[1 - \frac{1}{8}A^{-1} - \frac{7}{128}A^{-2} + O(A^{-3})\right]. \tag{16}$$

In the limit $a_0 \to 0$, the leading term $Z(A) = \sqrt{2\pi A}$ is sufficient to guarantee that the amplitude in Eq. (12) goes over to $\delta(U_{\rm fi} - U_{\rm in})$, as it should be. In quantum mechanics the leading term $Z(A) = \sqrt{A/(2\pi)}$ with $A = m/(\hbar a_0)$ is the exact result [see Eq. (5)]. However, in lattice gauge theory, the subleading terms are important and cannot be neglected. This has been confirmed by a numerical simulation discussed below.

IV. COMPARISON OF PROPAGATOR FROM PATH INTEGRAL AND HAMILTONIAN TIME EVOLUTION—A NUMERICAL SIMULATION

Let us consider a lattice which in the spatial direction has a single link ij and in the time direction has N time slices. We compare the propagator expressed via Hamiltonian time evolution with the path integral, using the measure of the group integral taking into account the previously calculated factor Z:

$$\langle U_{\rm fi}|\exp[-H^{\rm kin}T/\hbar]|U_{\rm in}\rangle = Z^N \int_{-\pi}^{+\pi}\left[\prod_{k=1}^{N-1}\frac{d\alpha^{(k)}}{2\pi}\right]\exp\left[-\frac{a}{\hbar g^2 a_0}\sum_{k=0}^{N}[1-\cos(\alpha^{(k)}-\alpha^{(k+1)})]\right]\Bigg|_{\alpha^{(0)}=\alpha_{\rm in}}^{\alpha^{(N)}=\alpha_{\rm fi}}. \tag{17}$$

The propagator expressed by Hamiltonian time evolution is given by

$$\langle U_{\rm fi}|\exp[-H^{\rm kin}T/\hbar]|U_{\rm in}\rangle = \sum_{n=0,\pm1,\pm2,\ldots}\exp\left[-\frac{g^2\hbar T}{2a}n^2\right]\cos[n(\alpha_{\rm in} - \alpha_{\rm fi})]. \tag{18}$$

This has been calculated using the basis of eigenstates of the electric field operator

$$\hat{l}_{ij}|\lambda_{ij}\rangle = \lambda_{ij}|\lambda_{ij}\rangle, \qquad \lambda_{ij} = 0, \pm1, \pm2, \ldots \tag{19}$$

and the connection from the Bargmann link basis to the electric field string basis given by the scalar product

$$\langle\lambda|U\rangle = (U)^\lambda. \tag{20}$$

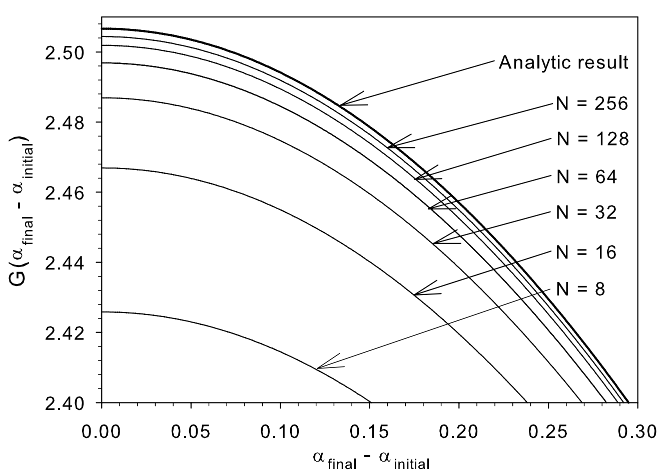

FIG. 1. Transition amplitude for the single link state as a function of initial and final links, $U_{\rm in} = \exp(i\alpha_{\rm in})$ and $U_{\rm fi} = \exp(i\alpha_{\rm fi})$. Transition time $T = 1$, lattice spacing $a = 1$. Comparison of Hamiltonian time evolution, Eq. (18) (bold line) with path integral, Eq. (17) (thin lines, number of time slices $N = 8, 16, 32, \ldots, 256$).

In the numerical simulations we kept $a = 1$ and varied a_0. Choosing the transition time $T = 1$ increasing the number N of time slices means $a_0 \to 0$. We computed the path integral given by Eq. (17) with Z given by Eq. (16). The path integral has been evaluated by numerical integration using 10000 mesh points in each time slice. The results as a function of the angle $\alpha_{\rm fi} - \alpha_{\rm in}$ (the curve is symmetric under global change $\alpha \to -\alpha$) and of the number of time slices N is shown in Fig. 1. One observes convergence when increasing $N = 8, 16, 32, \ldots, 256$. Figure 2 shows the relative error as function of $\alpha_{\rm fi} - \alpha_{\rm in}$. The analytic

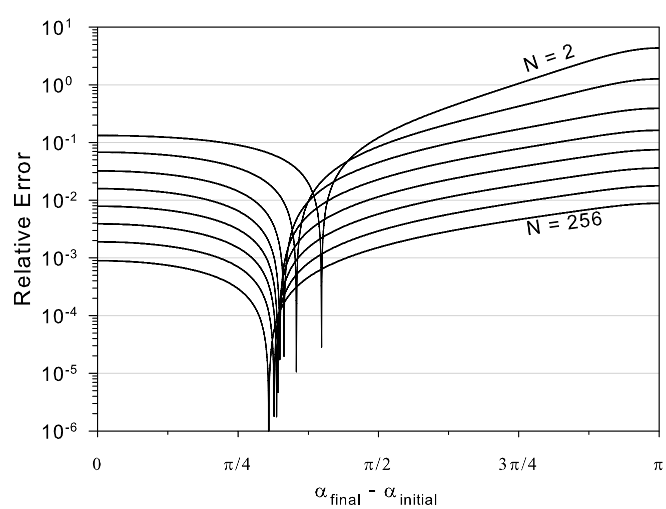

FIG. 2. Relative error of transition amplitude shown as a function of $N = 2, 4, 8, \ldots, 256$ and $\alpha_{\rm fi} - \alpha_{\rm in}$.

result, Eq. (18), gives a normalized curve, due to $\int dU' \langle U' | \exp[-H_{\rm kin} T/\hbar] | U \rangle = 1$. The numerical results from the path integral are also normalized. Hence, in the relative error occurs the subtraction of two normalized curves, which will cross at some point, which in a logarithmic plot gives (downward) spikes. The relative error varies between 10^{-2} and 10^{-3} depending on the angle and decreases monotonically when increasing $N = 2, 4, 8, \ldots, 256$. On the other hand, when taking into account only the leading term in Eq. (16) the relative error was found to be in the order of 10%–20%.

ACKNOWLEDGMENTS

H. Kröger and K. J. M. Moriarty have been supported by NSERC Canada. X. Q. Luo has been supported by Key Project of NSF China (10235040), and by National and Guangdong Ministries of Education.

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