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$U(1)$ noncommutative gauge fields and magnetogenesis

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The connection between the Lorentz invariance violation in the Lagrangian context and the quantum theory of noncommutative fields is established for the $U(1)$ gauge field. The modified Maxwell equations coincide with other derivations obtained using different procedures. These modified equations are interpreted as describing macroscopic ones in a polarized and magnetized medium. A tiny magnetic field (seed) emerges as particular static solution that gradually increases once the modified Maxwell equations are solved as a self-consistent equations system.

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I. INTRODUCTION

In the last few years several authors have suggested possible Lorentz invariance violations at quantum field theory and particle physics level [1]. If these theoretical suggestions are true, then our conception on Lorentz invariance and spacetime would be only approximate ideas coming from a more fundamental—still unknown—structure. From this point of view, these results could be another indication that the present relativistic quantum field theories descriptions would correspond to effective field theories.

Two important approaches going beyond the standard Lorentz symmetry are doubly special relativity [2] and the extended standard model [3], which are proposals that try to give an answer to largely unsolved problems in high energy physics, such as ultra high energy cosmic rays [4], matter-antimatter asymmetry [5], primordial magnetic field [6,7].

A third possibility is quantum theory with noncommutative fields, which has been proposed in [8], where the Lorentz symmetry is broken by modifying the canonical commutators including an ultraviolet and infrared scales. As we will consider an expanding universe surrounded by radiation, one could guess that a Bohr-Oppenheimer approach [9] naturally should generate a geometrical connection which produces a noncommutativity in the momenta space at the quantum level.

The goals of the present paper are two; firstly, we will investigate the connection between the Kosteleky *et al.* approach to quantum field theory and quantum theory with noncommutative fields for the particular context of the abelian gauge field and secondly, once the equivalence between both approaches is proven, we will explain some consequences for the primordial magnetic field.

More precisely, we will show that the modified Maxwell equations —that are formally the same equations found by Carroll *et al.* [10]—contain as a solution a universe filled with a tiny magnetic field. However, once this tiny magnetic field is given, the modified Maxwell equations generate *per se* a very natural self-interacting mechanism which is an alternative to the dynamo mechanism.

The paper is organized as follows: in Sec. II we will prove the equivalence between the Kostelecký *et al.* approach and quantum theory with noncommutative fields for an abelian gauge field. In Sec. III, we will reinterpret the modified Maxwell equations as macroscopic ones which suggests, in Sec. IV, the way in which a primordial magnetic field might appear. Finally in Sec. V, the conclusions and other possible physical implications are given.

II. $U(1)$ GAUGE FIELD AS A NONCOMMUTATIVE GAUGE FIELD

In order to discuss the $U(1)$ gauge field as a noncommutative one, let us recall the Hamiltonian formulation of the abelian gauge field.

The Lagrangian for an abelian gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

is invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda. \tag{2}$$

Thus, (1) has two symmetries, namely, the gauge and Lorentz symmetry.

The Hamiltonian analysis yields to the canonical momentum

$$\pi^\mu = F^{0\mu}, \tag{3}$$

and, therefore, one has the primary constraint

$$\pi^0 = 0. \tag{4}$$

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Thus, the canonical Hamiltonian is

$$H = \int d^3x \left(\frac{1}{2} \vec{\pi}^2 + \frac{1}{2} \vec{B}^2 + A_0 \vec{\nabla} \cdot \vec{\pi} \right), \tag{5}$$

and the preservation of (3) implies

$$\dot{\pi}^0(x) = [\pi^0(x), H] = \nabla.\vec{\pi}, \tag{6}$$

i.e., the secondary constraint is the Gauss' law.

The Gauss' law is a first order constraint and, from the Hamiltonian point of view, it generates the gauge symmetry (2).

Once the constraints are found, the gauge field is quantized changing the Poisson brackets

$$\begin{aligned} &[A_i(\vec{x}), A_j(\vec{y})]_{PB} = 0, \\ &[A_i(\vec{x}), \pi^j(\vec{y})]_{PB} = \delta_i^j \delta(\vec{x} - \vec{y}), \\ &[\pi_i(\vec{x}), \pi_j(\vec{y})]_{PB} = 0, \end{aligned} \tag{7}$$

by commutators according to the rule $[,]_{PB} \to [,]/i\hbar$.

The $U(1)$ noncommutative field is constructed by deforming the previous Poisson algebra as follows,

$$\begin{aligned} &[A_i(\vec{x}), A_j(\vec{y})]_{PB} = 0, \\ &[A_i(\vec{x}), \pi_j(\vec{y})]_{PB} = \delta_{ij} \delta(\vec{x} - \vec{y}), \\ &[\pi_i(\vec{x}), \pi_j(\vec{y})]_{PB} = \theta_{ij} \delta(\vec{x} - \vec{y}), \end{aligned} \tag{8}$$

where θ is the most general antisymmetric three dimensional matrix.

Although the Poisson brackets (8), of course, break Lorentz invariance, one can retain the gauge symmetry. Indeed, in order to do that, we must modify the Gauss' law appropriately.

Thus, the modified Gauss' law should be

$$\chi = \partial_i \pi_i + \text{something}, \tag{9}$$

where "something" represents the modified term, which is constrained to satisfy the relations,

$$\delta A_i(\vec{x}) = [A_i(\vec{x}), \Delta_\alpha]_{PB} = \partial_i \alpha(\vec{x}), \tag{10}$$

$$\delta \pi_i(\vec{x}) = [\pi_i(\vec{x}), \Delta_\alpha]_{PB} = 0, \tag{11}$$

where Δ_α is defined as

$$\Delta_\alpha \equiv - \int d^3x \alpha(x) \chi(x), \tag{12}$$

and where $\alpha(x)$ is an arbitrary real function.

Now, it is easy to see that the modified Gauss' law must be given by the constraint,

$$\chi = \nabla.\vec{\pi} - \vec{\theta}.\vec{B}, \tag{13}$$

where $\theta_{ij} A_j = \epsilon_{ijk} \theta_k A_j = -\vec{\theta} \times \vec{A}$, and therefore the gauge transform operator (12) can be written as,

$$\begin{aligned} \Delta_\alpha &= - \int d^3x \alpha(\vec{x}) \{ \vec{\nabla} \cdot \vec{\pi} - \vec{\theta} \cdot \vec{B} \} \\ &= \int d^3x \alpha(\vec{x}) \vec{\nabla} \cdot (\vec{\pi} + \vec{\theta} \times \vec{A}), \end{aligned} \tag{14}$$

and the modified total Hamiltonian which generalizes the $U(1)$ system should be,

$$H = \int d^3x \left[\frac{1}{2} \vec{\pi}^2 + \frac{1}{2} \vec{B}^2 + A_0 \nabla (\vec{\pi} + \vec{\theta}. \times \vec{A}) \right]. \tag{15}$$

Using (15) one finds that the equations of motion are

$$\dot{A}_i = [A_i, H]_{PB} = \pi_i - \partial_i A_0, \tag{16}$$

$$\dot{\pi}_i = [\pi_i, H]_{PB} = (\vec{\pi} \times \vec{\theta})_i - (\nabla \times \vec{B})_i. \tag{17}$$

The first equation, of course, is basically the standard definition of electric field, i.e.,

$$\pi_i = -E_i \equiv \dot{A}_i + \partial_i A_0, \tag{18}$$

and, hence, the second one

$$\frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} + \vec{E} \times \vec{\theta}, \tag{19}$$

is the modified Ampere's law.

The remaining equations, namely

$$\nabla.\vec{B} = 0, \tag{20}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{21}$$

have no changes. And as we said above, the Gauss' law is written as,

$$\nabla.\vec{E} + \vec{\theta}.\vec{B} = 0. \tag{22}$$

Now, let us find out the Lagrangian where these equations come from. In order to do that, we should find a set of canonical conjugated variables to A_i. This is, in fact, easy to find by taking in account (8).

We find that these new variables are just,

$$\tilde{\pi}_i \equiv \pi_i + \frac{1}{2} (\vec{\theta} \times \vec{A}). \tag{23}$$

From these results one gets the Lagrangian as follows; firstly, we write

$$\begin{aligned} L &= \int d^3x \tilde{\pi}_i \dot{A}_i - H \\ &= \int d^3x \left(\vec{E} - \frac{1}{2} \vec{\theta} \times \vec{A} \right) (\vec{E} + \vec{\nabla} A_0) - H \\ &= \int d^3x \left(\vec{E}^2 - \vec{B}^2 + \frac{1}{2} A_0 \vec{\theta}.\vec{B} - \frac{1}{2} \vec{A}.\vec{\theta} \times \vec{E} \right). \end{aligned} \tag{24}$$

Using the standard definition for $F_{\mu\nu}$ and $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$, one finds that the Lagrangian is

$$L = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \theta_\mu \tilde{F}^{\mu\nu} A_\nu \right), \tag{25}$$

where in our case the four-vector θ_μ is $(0, \vec{\theta})$.

The modified Maxwell equations obtained in this paper were derived from a completely different point of view to the used in [10]. Our calculation shows explicitly the connection between these two apparently nonrelated approaches.

A discussion on physical aspects related to the propagation of the light for these modified photons and other systems can be found in [11]. In particular, the dispersion relation in this spacelike approach is,

$$\omega_{\pm}^2 = \vec{k}^2 + \frac{1}{2}\vec{\theta}^2 \pm \sqrt{(\vec{k}\cdot\vec{\theta})^2 + \frac{1}{4}(\vec{\theta}^2)^2}. \tag{26}$$

III. INTERPRETING THE MODIFIED MAXWELL EQUATIONS

In this section we will give a physical interpretation of the modified Maxwell equations.

Let us start assuming that possible Lorentz invariance violation processes could have occurred in the early universe and some tiny relics could be observable presently. As photons are the most abundant particles in the present universe, one can think of that some relics could be accessible via electromagnetic processes.

It is interesting to note that the Modified Maxwell equations contain a "source" term $-\vec{\theta}.\vec{B}$ and $\vec{\theta}\times\vec{E}$ that can be interpreted as polarization charges and induced currents on a medium in a similar way to the standard electromagnetic theory.

Therefore, these modifications of the Maxwell equations suggest us to consider a sort of modified displacement vector ($\vec{D}$) and magnetic field vector ($\vec{H}$) where

$$\vec{D} = \vec{E} - \vec{\theta}\times\vec{A}, \tag{27}$$

$$\vec{H} = \vec{B} + \vec{\theta}A_0. \tag{28}$$

Using these definitions the modified Maxwell equations can be written as the standard Macroscopic Maxwell equations in a medium, i.e.,

$$\begin{aligned} \nabla.\vec{D} &= \rho, \\ \nabla\times\vec{H} &= \frac{\partial\vec{D}}{\partial t} + \vec{J}, \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}, \\ \nabla.\vec{B} &= 0, \end{aligned} \tag{29}$$

where ρ and $\vec{J}$ are possible external sources.

One should note that the polarization and magnetization vectors

$$\vec{P} = -\vec{\theta}\times\vec{A}, \tag{30}$$

$$\vec{M} = \vec{\theta}A_0, \tag{31}$$

are not gauge invariants, however this is not important because the physically relevant quantities are $\nabla.\vec{P}$ and $\nabla\times\vec{M} - \dot{\vec{P}}$ which are, in fact, gauge invariants [12].

It is interesting to note that in the static scenario, the electrostatic and magnetostatic effects appear mixed and, therefore, the presence of polarization implies a magnetization of a medium and vice versa.

This result is a consequence of the modified Maxwell theory and it is not true in the conventional electromagnetic theory.

IV. ORIGIN OF THE PRIMORDIAL MAGNETIC FIELD

The structure of the above modified Maxwell equations might give a guess on the origin of the intergalactic magnetic field as well as, it might provide of a simple alternative argument to dynamo mechanisms discussed in the literature (see e.g. [6,13,14,16]).

In order to explain this fact, let us suppose that it is generated a seed magnetic field ($\vec{B}^{(0)}$) parallel to $\vec{\theta}$. If this processes take place during a long time, we can suppose that only the stationary equations are the important ones, i.e.,

$$\vec{\nabla}\cdot\vec{E} + \vec{\theta}\cdot\vec{B} = 0 \tag{32}$$

$$\vec{\nabla}\times\vec{B} + \vec{\theta}\times\vec{E} = 0 \tag{33}$$

$$\vec{\nabla}\times\vec{E} = 0 \tag{34}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{35}$$

Then, a constant magnetic field is a solution of all equations at zeroth order in θ (or in the spatial scale r). However, if we consider the first order in θ (or we displace a distant r from the origin) Eq. (32) will demand an electric field,

$$\vec{E}^{(1)} = -\frac{1}{3}B^{(0)}(\theta r)\hat{r} \tag{36}$$

The next order in θ is given by the Eq. (32) (of course, in this process Eqs. (34) and (35) are always present).

This equation generates a second order correction in the magnetic field,

$$\vec{B}^{(2)} = -\frac{1}{30}B^{(0)}(\theta r)^2(7\cos\tilde{\theta}\hat{e}_r - 9\sin\tilde{\theta}\hat{e}_{\tilde{\theta}}) \tag{37}$$

where $(r, \tilde{\theta}, \tilde{\phi})$ and $(\hat{e}_r, \hat{e}_{\tilde{\theta}}, \hat{e}_{\tilde{\phi}})$ are the spherical coordinates and their unit vector fields. We can follow this expansion in order to get all orders in θ, and one should obtain a potential series for $\vec{B}$ and $\vec{E}$, i.e.,

$$\vec{B} = \vec{B}^{(0)} + \vec{B}^{(2)} + \vec{B}^{(4)} + \ldots + \vec{B}^{(2n)} + \ldots \tag{38}$$

$$\vec{E} = \vec{E}^{(1)} + \vec{E}^{(3)} + \vec{E}^{(5)} + \ldots + \vec{E}^{(2n+1)} + \ldots \tag{39}$$

where the superindices stand for the order in θ. It should be noted that $E\sim\theta B$, which means that the electric field is always a lower order of magnitude that the magnetic field, according to the experimental fact.

The possibility for these expansions to be divergent series suggests us that the system might evolve to a stable state with permanent magnetic and/or electric fields, in the similar way to ferromagnetic media. Then this mechanism would be a possible candidate for an alternative explanation to the dynamo mechanism of the primordial magnetic field observed in the universe.

V. CONCLUSIONS AND OTHER PHYSICAL IMPLICATIONS

In this paper we have shown that deformation of the canonical commutators for the electromagnetic field yields to a modification of the Maxwell electrodynamics where electrostatics and magnetostatics appears mixed [1], this is direct consequence of the Lorentz invariance violation.

Although the modified Maxwell equations are formally the standard macroscopic ones, the underlying physics is quite different to the conventional interpretation. Indeed, the mix between electrostatics and magnetostatics induces as a consequence polarizations and magnetizations and hence, physical electrical or magnetic fields.

From the physical point of view, this is a very interesting new effect because it could be the arena for the elusive primordial magnetic field. Indeed, as the universe expansion has spherical symmetry and as the universe is made mainly of photons, then one can see the universe as a sort of magnetized sphere. If we assume that the electromagnetic fields are—as a first approximation—static and the radius of the present universe is a, our universe should be filled with a magnetic field like

$$\vec{H} = -\frac{2}{3}\vec{M}, \tag{40}$$

however, as $\vec{M}$ is proportional to $|\vec{\theta}|$ it is a very tiny energy scale—like $\sqrt{\Lambda}$—then also $\vec{H}$ should be a tiny magnetic field filling our present universe.

In this sense, the modified electrodynamics—as a consequence of a tiny Lorentz invariance violation—might be a mechanism for the origin of the elusive seed field observed in galaxies.

Possibles implications with the Born-Oppenheimer approximations [9] and other nonperturbatives phenomena, will be discussed elsewhere.

[1]This fact has been noted recently in [15].

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