Quantum uncertainty in doubly special relativity

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A discussion of the modification of the phase-space commutators in a quantum mechanical relativistic theory with an invariant length scale (DSR) is presented. Two examples are discussed where a classical behavior is approached in one case when the energy approaches the inverse of the invariant length which appears as a cutoff in the energy and in the second case when the mass is much larger than the inverse of the invariant length.

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I. INTRODUCTION

One of the basic open problems in theoretical physics is to combine in a consistent way the classical description of the gravitational interaction (general relativity) with quantum mechanics (QM). The analysis of the problems that one finds in different attempts to combine these two theories can be used as a guiding principle to the search of a fundamental theory of quantum gravity.

On the other hand one has direct proposals such as string(M)-theory or loop quantum gravity. However, presently we are not able to establish if any of these or another future proposal is the correct theory. An alternative is to identify new basic ingredients which could serve as a criterion in the future to select among different alternatives. One idea—that has been discussed very often in this context [1]—is the possible appearance of a fundamental length, a scale associated to gravity not just as a dynamical scale but at the kinematical level.

Several arguments coming from string theory [2] or from gedanken experiments suggest that a minimal length [1] could be present as a quantum gravity consequence. Some attempts to identify a modification of the quantum mechanical commutators [3] which reproduce the generalized uncertainty principle have been considered as a way to find one of the ingredients of the quantum theory of gravity.

Another candidate for a signal of quantum gravity effects is the modification of Lorentz symmetry at very high energies, an idea explored intensively in the last years both from a theoretical as well as from a phenomenological perspective [4].

Alternatively the limitation due to gravity to explore beyond a minimal length has motivated to consider the possibility of a generalization of the relativity principle compatible with an invariant (but not necessarily a minimal) length scale which is called doubly special relativity (DSR) [5,6].

It is remarkable that one can have a generalized relativity principle which is continuously connected with standard Einstein special relativity (SR), compatible with all the tests of SR. The new corrections could be accessible to present or near future experiments or even already observed in the high energy tail of the cosmic ray spectrum [7,8]. Different proposals for DSR theories either based on a deformation of Poincare algebra or by considering directly a modification of the boosts that connect inertial frames has been considered recently from different points of view [5,6,9].

Instead of taking a generalization of the uncertainty principle as a starting point and looking for a modification of the QM commutators compatible with such a generalization, we follow a different path trying to calculate directly the modified commutation relations. In order to do that one would require a formulation of DSR in phase-space which is still an open problem. Different alternatives going from the introduction of a noncommutativity in the space-time realization of a generalized relativity principle [10] to a direct attempt to extend the nonlinear realization of Lorentz transformations to phase-space [11] have been considered recently. For a given realization of DSR in phase-space one will have a modification of QM which can be taken as a remanent of the fundamental theory of QG in the flat space-time limit. In this paper we illustrate this point in the simplest extension of DSR to phase-space by considering a linear action of Lorentz transformation in spacetime. Two examples of DSR are studied in detail. In both cases one finds that, in an appropriate limit, a classical behavior is approached. We conjecture that this result applies to the theory of QG.

II. MODIFIED QUANTUM MECHANICAL COMMUTATORS

A nonlinear realization of Lorentz transformations in energy-momentum (E, $\mathbf{p}$) space parametrized by an invari-

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ant length ℓ can be defined [12,13] by the relations

$$\epsilon = Ef(\ell E, \ell^2 \mathbf{p}^2) \quad \pi_i = p_i g(\ell E, \ell^2 \mathbf{p}^2) \tag{1}$$

where (ϵ, π) are auxiliary linearly transforming variables which define the nonlinear Lorentz transformation of the physical energy-momentum $(E, \mathbf{p})$. Then we have two functions of two variables (f, g) which parametrize the more general nonlinear realization of Lorentz transformations, with rotations realized linearly, depending on a dimensional scale. The condition to recover the special relativistic theory in the low energy limit reduces to the condition $f(0,0) = g(0,0) = 1$. Each choice of the two functions f, g will lead to a generalization of the relativity principle with an invariant length scale ℓ. Lorentz transformation laws connecting the energy-momentum of a particle in different inertial frames differ from the standard special relativistic linear transformation laws which are recovered when $\ell E \ll 1$, $\ell^2 \mathbf{p}^2 \ll 1$.

In order to have a quantum theory with such a deformed relativity principle one should find the appropriate deformation of relativistic quantum field theory (QFT). First attempts in this direction, based on the possible connection between a generalization of the relativity principle and a noncommutativity of spacetime, suggesting the formulation of QFT in a noncommutative space (κ-Minkowsky) as the appropriate deformation of QFT have been explored [14]. But there are general arguments that there will be difficulties to find a realization of a deformed relativity principle along these lines in the multiparticle sector [7,15]. Because of these problems we consider in this work a less ambitious program trying to give an implementation of DSR at the level of quantum mechanics. The simplest way to do this is to introduce space-time coordinates as the generators of translations in the auxiliary linearly transforming energy-momentum variables (ϵ, π) which then reduce to the usual space-time coordinates of special relativity in the limit $\ell \to 0$. In this case one does not have any signal of the modified relativity principle at the level of the space-time coordinate commutators which are still trivial but all the modifications appear at the level of the phase-space commutators which will be

$$[t, E] = i\hbar \frac{\partial E}{\partial \epsilon} \quad [t, p_i] = i\hbar \frac{\partial p_i}{\partial \epsilon} \tag{2}$$

$$[x_i, E] = i\hbar \frac{\partial E}{\partial \pi_i} \quad [x_i, p_j] = i\hbar \frac{\partial p_i}{\partial \pi_j}. \tag{3}$$

By considering the derivatives with respect to the auxiliary energy-momentum variables of Eqs. (1) defining the nonlinear realization of Lorentz transformations, one has a linear system of equations for the partial derivatives required to calculate the phase-space commutators (2) and (3). A straightforward algebra leads to the modified quantum mechanical commutators

$$[t, E] = i\hbar \frac{g + 2\ell^2 \mathbf{p}^2 \partial_2 g}{D} \quad [t, p_i] = -i\hbar \frac{\ell p_i \partial_1 g}{D} \tag{4}$$

$$[x_i, E] = -i\hbar \ell p_i \frac{2\ell E \partial_2 f}{D}, \tag{5}$$

$$[x_i, p_j] = \frac{i\hbar}{g}\left[\delta_{ij} - 2\ell^2 p_i p_j \frac{N}{D}\right] \tag{6}$$

where

$$D = [f + \ell E \partial_1 f][g + 2\ell^2 \mathbf{p}^2 \partial_2 g] - 2\ell^2 \mathbf{p}^2 \partial_1 g \ell E \partial_2 f, \tag{7}$$

$$N = f\partial_2 g + \ell E(\partial_1 f \partial_2 g - \partial_2 f \partial_1 g). \tag{8}$$

This result can be seen as an explicit realization of a general idea that in the presence of quantum gravity the quantum mechanical uncertainty principle, and then the phase-space commutators on which it is based, should be modified by terms depending on an invariant length [3]. Instead of guessing the general structure of the generalized uncertainty principle and the modification of the phase-space commutators leading to such a generalization of the uncertainty principle, a modification of the commutators (4)–(6) is obtained directly from a nonlinear realization of Lorentz transformations in momentum space parametrized by the functions f, g. In order to discuss the consequences of the modifications of the quantum mechanical commutators one has to specify the nonlinear transformations of energy-momentum. We discuss some cases in next two sections.

III. AN EXAMPLE WITH AN ENERGY CUTOFF: DSR2

When the two functions f, g parametrizing the nonlinear Lorentz transformations are independent of the momenta (i.e., when $\partial_2 f = \partial_2 g = 0$) the modified quantum mechanical commutators in (4)–(6) take a much simpler form

$$[t, E] = i\hbar \frac{1}{f + \ell E \partial_1 f} \tag{9}$$

$$[t, p_i] = -i\hbar \ell p_i \frac{\partial_1 g}{g[f + \ell E \partial_1 f]} \tag{10}$$

$$[x_i, E] = 0, \qquad [x_i, p_j] = i\hbar \delta_{ij} \frac{1}{g}. \tag{11}$$

A very simple choice for the functions f, g

$$f = g = (1 - \ell E)^{-1} \tag{12}$$

is what is known as DSR2 [6] and corresponds to the simplest realization of DSR with an energy cutoff ($E < 1/\ell$) identified as the inverse of the invariant length. The combinations of derivatives which appear in the phase-space commutators are given by

$$\partial_1 g = f + \ell E \partial_1 f = (1 - \ell E)^{-2} \tag{13}$$

and then one has

$$[t, E] = i\hbar(1 - \ell E)^2 \tag{14}$$

$$[t, p_i] = -i\hbar\ell p_i(1 - \ell E) \tag{15}$$

$$[x_i, E] = 0, \qquad [x_i, p_j] = i\hbar\delta_{ij}(1 - \ell E), \tag{16}$$

a result already anticipated in [13] where it was obtained by considering a possible realization of space-time coordinates as differential operators in momentum space. The conclusion that one gets from (14)–(16) is that there is a modification of the quantum mechanical commutators which becomes relevant when the energy approaches its maximum value. In the limit $E \to 1/\ell$ all commutators vanish and one has a classical phase-space. This is a result that one could have anticipated from the consistency of the quantum mechanical uncertainty principle with the possibility to explore an arbitrarily small region in space-time while having a cutoff on the available energies.

IV. AN EXAMPLE WITH A MOMENTUM CUTOFF: DSR1

Another example of a nonlinear realization of Lorentz transformations corresponds to the choice of functions

$$f = \frac{1}{2}\left[(1 + \ell^2\mathbf{p}^2)\frac{e^{\ell E}}{\ell E} - \frac{e^{-\ell E}}{\ell E}\right] \tag{17}$$

$$g = e^{\ell E}. \tag{18}$$

The relation between the energy and momentum for a particle of mass m is given by

$$(1 - \ell^2\mathbf{p}^2)e^{\ell E} + e^{-\ell E} = e^{\ell m} + e^{-\ell m}, \tag{19}$$

which is the dispersion relation of the model referred to as DSR1 [5]. One finds

$$e^{\ell E} = \frac{\cosh(\ell m) + \sqrt{\cosh^2(\ell m) - (1 - \ell^2\mathbf{p}^2)}}{1 - \ell^2\mathbf{p}^2} \tag{20}$$

for the energy as a function of momentum. One has in this case an upper bound on the momentum ($\mathbf{p}^2 < 1/\ell^2$) instead of the energy. If one replaces the function g and its derivatives

$$\partial_1 g = g = e^{\ell E} \qquad \partial_2 g = 0 \tag{21}$$

into the general expression (6) for the modified quantum mechanical commutators one has

$$[t, E] = i\hbar\frac{1}{D_1} \qquad [t, p_i] = -i\hbar\ell p_i\frac{1}{D_1} \tag{22}$$

$$[x_i, E] = -i\hbar\ell p_i\frac{2e^{-\ell E}\ell E\partial_2 f}{D_1} \tag{23}$$

$$[x_i, p_j] = i\hbar e^{-\ell E}\left[\delta_{ij} + \ell^2 p_i p_j\frac{2\ell E\partial_2 f}{D_1}\right] \tag{24}$$

where

$$D_1 = f + \ell E[\partial_1 f - 2\ell^2\mathbf{p}^2\partial_2 f]. \tag{25}$$

For the particular choice for f in (17)

$$2\ell E\partial_2 f = e^{\ell E} \tag{26}$$

and

$$D_1 = \frac{1}{2}[(1 - \ell^2\mathbf{p}^2)e^{\ell E} + e^{-\ell E}]. \tag{27}$$

If one uses the relation between energy and momentum (19), the right hand side in (27) reduces to $\cosh(\ell m)$ and the phase-space commutators become

$$[t, E] = \frac{i\hbar}{\cosh(\ell m)} \qquad [t, p_i] = -\ell p_i\frac{i\hbar}{\cosh(\ell m)} \tag{28}$$

$$[x_i, E] = -\ell p_i\frac{i\hbar}{\cosh(\ell m)} \tag{29}$$

$$[x_i, p_j] = i\hbar\left[e^{-\ell E}\delta_{ij} + \ell^2 p_i p_j\frac{1}{\cosh(\ell m)}\right]. \tag{30}$$

We see from these expressions that when the mass m is much larger than the inverse of the length scale ℓ all the commutators are (exponentially) small and a classical phase-space is approached. This result suggests the possibility to relate the transition from the quantum behavior at the microscopic level to the classical behavior at the macrosocopic level with the modification of quantum mechanics induced by a modification of the relativity principle.

As a final remark one can consider the massless case where

$$e^{\ell E} = \frac{1}{1 - \ell|\mathbf{p}|} \tag{31}$$

and the modified commutators are

$$[t, E] = i\hbar \qquad [t, p_i] = -i\hbar\ell p_i \tag{32}$$

$$[x_i, E] = -i\hbar\ell p_i \tag{33}$$

$$[x_i, p_j] = i\hbar[(1 - \ell|\mathbf{p}|)\delta_{ij} + \ell^2 p_i p_j]. \tag{34}$$

In contrast to the case of a cutoff in the energy, when the momentum approaches its maximum value one has a non trivial limit for the commutators which differs from the canonical commutation relations.

V. CONCLUSIONS

The standard arguments leading to a minimum physical length beyond which it is not possible to go in the presence of gravity assume that in the flat space-time limit one has the standard QM uncertainty principle.

If there is a remnant of gravity in the flat space-time limit –such as an invariant length and a modification of the QM commutators– then these arguments may not apply. In fact we have shown that in some cases, instead of a generalized uncertainty principle leading to a minimal length, one finds a modification of the QM commutators such that the system approaches to the classical limit with no uncertainties in the high energy limit and/or for large masses.

A discussion at the kinematical level, as the one presented in this work, has to be based on a set of implicit assumptions like the identification of physical energy-momentum variables as well as a simple realization of spacetime. The validity of this framework can only be established after a dynamical theory with these kinematical ingredients is formulated.

If the modifications of QM suggested by the examples analyzed in this work apply to the flat space-time limit of the QG theory then our understanding of different physical systems should be reconsidered. The qualitative description of physical systems on a macroscopic scale, based on standard QM, could be altered in some cases. Also the discussion of black hole evaporation could be modified when one approaches the invariant length scale where quantum black holes become classical. Even the quantum mechanical aspects of the evolution of the Universe could differ from standard physics expectations. A more systematic analysis of all possible QM commutators corresponding to the different nonlinear representations of Lorentz transformations in energy-momentum space and of the different ways to introduce the space-time sector is required in order to see whether the appearance of a new classical limit is a peculiarity of the examples considered in this work or a general property of a quantum relativistic theory with an invariant length scale.

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