Noncommutative theories and general coordinate transformations

C. D. Fosco[1] and G. Torroba[2]

[1]*Centro Atómico de Bariloche and Instituto Balseiro, Comisión Nacional de Energía Atómica, 8400 Bariloche, Argentina*
[2]*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855, USA*

(Received 11 January 2005; published 24 March 2005)

We study the class of noncommutative theories in d dimensions whose spatial coordinates $(x_i)_{i=1}^d$ can be obtained by performing a smooth change of variables on $(y_i)_{i=1}^d$, the coordinates of a standard noncommutative theory, which satisfy the relation $[y_i, y_j] = i\theta_{ij}$, with a constant θ_{ij} tensor. The x_i variables verify a commutation relation which is, in general, space dependent. We study the main properties of this special kind of noncommutative theory and show explicitly that, in two dimensions, any theory with a space-dependent commutation relation can be mapped to another where that θ_{ij} is constant.

DOI: 10.1103/PhysRevD.71.065012 PACS numbers: 11.10.Nx, 04.20.Gz, 11.10.Lm

I. INTRODUCTION

The usual starting point for the construction of noncommutative quantum field theories [1–3] is to assume the existence of nontrivial commutation relations between the spatial coordinates x_i, $i = 1, \ldots, d$. Those relations can be summarized by an expression with the general form

$$[x_i, x_j]_\star = i\Theta_{ij}(x), \tag{1}$$

where $\Theta_{ij}(x)$ is an antisymmetric real matrix, which naturally encodes the noncommutative structure of the space considered.

The usual situation corresponds to a constant Θ_{ij}, but the space-dependent case has also found some important applications (see, for example, [4–6]). Contrary to what happens when Θ_{ij} is a constant tensor, associativity of the $\star$ product requires some nontrivial conditions (which have the form of differential equations for Θ_{ij}) to be true [7]. Even when those conditions are met, the construction of quantum field theories on the resulting algebras can be rather difficult. Indeed, the $\star$ product, as well as the derivatives and integrals, essential in any quantum field theory application, have rather cumbersome expressions.

Closely related to the study of space-dependent $\star$ commutators is the consideration of general changes of variables in a noncommutative space which is defined by the fundamental commutation relation

$$[y_i, y_j]_\star = i\theta_{ij},$$

with $\theta_{ij} \equiv$ constant. The interest in considering those changes of variables is manifold. On the one hand, it is believed that quantum gravity may be at the origin of noncommutativity [8,9]. Therefore, one would expect that general coordinate transformations should play a role in rendering some curved-space effects more explicit.

Besides, and this is the focus of our interest in this article, it may be possible to use a change of variables to rewrite some particular space-dependent commutation relations as a constant-θ relation but in terms of new variables. This program was developed in [10], where some particular examples were developed and analyzed. In those examples, a simple study of the resulting noncommutative theory was possible since, for example, a closed expression for the $\star$ product can be derived in terms of the standard Moyal product. Moreover, derivatives and integrals can also be constructed, based on the existence of "canonical" coordinates, namely, those that have a constant commutator. A similar approach has been applied in [11] to κ-Minkowski noncommutative spacetime.

In this paper we consider the same problem in more generality, studying a space-dependent Θ_{ij}, obtained by performing a general change of coordinates (inspired by [12]) in a theory with a constant θ_{ij}. In Sec. II, we begin by analyzing these coordinate transformations in a planar theory, computing the resulting $\Theta(x)$. We also introduce an integral and derivatives, and use those constructs to write an explicit noncommutative field theory action, some features of which shall be interpreted as curved-space effects. Next, in Sec. III, we apply the results of the previous section to some examples in $d = 2$. We also present a generalization of the result of [10] to higher-dimensional spaces in Sec. IV. In Sec. V we apply the tools of Secs. II and III to show that a general space-dependent $\Theta(x)$ may be reduced to a constant θ by a suitable change of variables, which we construct explicitly. Finally, in Sec. VI we present our conclusions.

II. PLANAR THEORIES (d= 2)

To begin with, we introduce two noncommutative spatial coordinates in $d = 2$, (y_1, y_2), that verify the commutation relation:

$$[y_i, y_j]_\star = i\theta_{ij}, \tag{2}$$

where θ_{ij} is a constant. Since we are working in two dimensions, we may always write

$$\theta_{ij} = \theta\varepsilon_{ij} \tag{3}$$

where θ is a parameter with the dimensions of an area.

Furthermore, the time coordinate is assumed to commute with y_1 and y_2.

We then introduce two new coordinates, (x_1, x_2), by means of a nonsingular, continuous change of variables. Following [12] we may write it without any loss of generality as follows:

$$x_i = y_i + \theta_{ij}\tilde{A}_j(y), \qquad i = 1, 2. \tag{4}$$

The parametrization above is well suited for the analysis of coordinate transformations that are continuous deformations of the identity. Indeed, the field $\tilde{A}_j$ has the role of determining the nontrivial content of those transformations. Besides, the expression for the change of variables in terms of a vector field $\tilde{A}_j$ prepares the road for the introduction of some gauge theory concepts [12] which find a natural place in this context.

At this point we comment on the notation: a tilde on top of a function has been used to denote its functional dependence in terms of y variables. This will be useful later on, when we shall have to distinguish that from the corresponding expression of the same object as a function of x (where we shall omit the tilde), i.e., $A_j(x) \equiv \tilde{A}_j(y(x))$.

For the transformation (4) to be nonsingular, a necessary condition is that the Jacobian $\tilde{J}(y)$ be different from zero:

$$\tilde{J}(y) = \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| \neq 0. \tag{5}$$

We can easily see that *in two dimensions* $\tilde{J}$ may be written more explicitly as

$$\tilde{J}(y) = 1 + \frac{1}{2}\theta\varepsilon_{ij}\tilde{f}_{ij}(y) \tag{6}$$

with

$$\tilde{f}_{ij} = \partial_i\tilde{A}_j(y) - \partial_j\tilde{A}_i(y) + \theta\{\tilde{A}_i, \tilde{A}_j\}. \tag{7}$$

In the previous equation, the curly bracket is defined by

$$\{f, g\} \equiv \varepsilon_{ij}\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial y_j}. \tag{8}$$

We are interested in studying the effect of a general nonsingular change of variables (4) on the commutation relation (2). Namely, we want to find the form of the commutation relation satisfied by the x variables, which of course will necessarily fall under the general form

$$[x_i, x_j]_\star = i\Theta_{ij}(x), \tag{9}$$

where $\Theta_{ij}(x)$ is determined by (4). In what follows, we will find the relation between $\Theta_{ij}(x)$ and (4) for a general transformation.

A quite straightforward calculation allows one to find the commutation relation for the "new" variables (x_i), although in terms of the old ones (y_i):

$$[x_i, x_j]_\star = i\tilde{\Theta}_{ij}(y), \tag{10}$$

where

$$\tilde{\Theta}_{ij}(y) = \theta_{ij} + \frac{1}{2}(\theta_{ik}\theta_{jl} - \theta_{il}\theta_{jk})\tilde{F}_{kl}(y), \tag{11}$$

with

$$\tilde{F}_{ij}(y) = \frac{\partial\tilde{A}_j}{\partial y_i} - \frac{\partial\tilde{A}_i}{\partial y_j} - i[\tilde{A}_i, \tilde{A}_j]_\star. \tag{12}$$

Taking advantage of the fact that $d = 2$, the previous expressions (derived in [12] and valid for any d) can be further simplified. Indeed, we may write

$$\tilde{\Theta}_{ij}(y) = \tilde{\Theta}(y)\varepsilon_{ij} \tag{13}$$

where

$$\tilde{\Theta}(y) = \theta(1 + \theta\tilde{F}(y)) \tag{14}$$

with

$$\begin{aligned}\tilde{F}(y) &= \frac{1}{2}\varepsilon_{ij}\tilde{F}_{ij}(y) \\ &= \frac{\partial\tilde{A}_2}{\partial y_1}(y) - \frac{\partial\tilde{A}_1}{\partial y_2}(y) - i[\tilde{A}_1(y), \tilde{A}_2(y)]_\star.\end{aligned} \tag{15}$$

So far, we have dealt with an explicit expression for $\tilde{\Theta}_{ij}(y)$. Let us see now how to write, at least formally, the right-hand side (rhs) in (10) as a function of x. We may use now the coordinate transformation that is inverse to (4), to write the commutation relations for the x_i coordinates as a function of the same variables.

Using the chain rule in (15),

$$\tilde{F}(y) = \varepsilon_{ij}\frac{\partial A_i}{\partial x_k}\frac{\partial x_k}{\partial y_j} - i[A_1(x), A_2(x)]_\star. \tag{16}$$

Here, $[A_1(x), A_2(x)]_\star$ stands for the star product on x_j space (which we construct explicitly in Sec. II B) induced by the change of variables (4) between the functions $\tilde{A}_j(y(x))$.

By introducing (4) in this expression each time a derivative $\partial x/\partial y$ appears, we obtain a series expansion in powers of θ. Equivalently, the sum of that series can be found by deriving x_i with respect to y_k in (4) and then applying the chain rule, to obtain

$$\left(\delta_{il} - \theta_{ij}\frac{\partial A_j}{\partial x_l}\right)\frac{\partial x_l}{\partial y_k} = \delta_{ik}. \tag{17}$$

For a well-defined change of variables, this expression can be inverted to yield

$$\frac{\partial x_i}{\partial y_l} = \Delta^{-1}(x)\left(\delta_{il} + \theta_{ij}\frac{\partial A_l}{\partial x_j}\right), \tag{18}$$

where

$$\Delta(x) \equiv (1 - \theta\partial_1 A_2)(1 + \theta\partial_2 A_1) + \theta^2\partial_1 A_1\partial_2 A_2 \tag{19}$$

is the determinant of $\delta_{il} - \theta_{ij}(\partial A_j/\partial x_l)$.

Then, a straightforward calculation leads to

$$\begin{aligned}\Theta(x) &= \tilde{\Theta}(y(x)) \\ &= \theta + \theta^2 \varepsilon_{ij}\Big[\frac{1}{2}F_{ij}(x) + (\Delta^{-1} - 1)\frac{\partial A_j}{\partial x_i}(x) \\ &\quad + -\Delta^{-1}\theta\{A_i(x), A_j(x)\}\Big],\end{aligned} \tag{20}$$

which contains all the information about the effect of the change of variables on the commutation relation. Here,

$$F_{ij}(x) \equiv \frac{\partial A_j}{\partial x_i}(x) - \frac{\partial A_i}{\partial x_j}(x) - i[A_i(x), A_j(x)]_\star.$$

Expression (20) is very convenient when dealing with general, formal properties of the noncommutative theory in the new variables. However, its application to the derivation of the transformation between old and new variables that leads to a given $\Theta(x)$ can be quite involved. Indeed, the commutator $[x_1, x_2]_\star$ depends on F_{ij}, which in turn depends on $\star$ through $[A_1, A_2]_\star$. This leads to a highly nonlinear problem, whose solution we shall study for some particular cases. It is easy to get the leading term of (20) in an expansion in powers of θ:

$$\Theta(x) = \theta[1 + \theta(\partial_1 A_2 - \partial_2 A_1) + \mathcal{O}(\theta^2)], \tag{21}$$

which shows that the leading order is determined by the first term (linear in θ) of the Jacobian.

In the next subsection we consider the use of a change of variables in $d = 2$ under the light of deformation quantization.

Then, in Sec. II B, we derive some consequences and applications of (14) and (20), which summarize the effect of a coordinate transformation in $d = 2$, to the construction of noncommutative quantum field theories.

A. Gauge transformations

In deformation quantization, one is interested in constructing $\star$ products only up to gauge equivalence, with gauge transformations defined as [7]

$$f(y) \to D_\theta f(y) \equiv \Big(1 + \sum_{m\geq 1}\theta^m D_m\Big)f(y), \tag{22}$$

with D_m denoting differential operators. Under those transformations, the star product transforms as follows:

$$f \star' g = D(D^{-1}f \star D^{-1}g). \tag{23}$$

We will now show that these abstract gauge transformations do indeed have an interpretation in the context of the change of variables (4).

Let us concentrate in the set of infinitesimal transformations that leave invariant the Poisson structure (2). They can be written as

$$y'_i = y_i + \theta_{ij}\tilde{\xi}_j(y) \tag{24}$$

with $\tilde{\xi}_j$ infinitesimal and $\theta_{ij} = \theta\varepsilon_{ij}$. Expanding around y_i

$$f(y_i + \theta_{ij}\tilde{\xi}_j) = f(y) + \theta_{ij}\tilde{\xi}_j(y)\partial_i f(y) + \ldots, \tag{25}$$

which is a gauge transformation like (22). We have already calculated the effect of a transformation like (24) on θ_{ij}; we simply use Eqs. (4) and (11), identifying $\tilde{A}_j \equiv \tilde{\xi}_j$. It is clear that

$$\tilde{\Theta}_{ij}(y) = \theta_{ij} + \mathcal{O}(\tilde{\xi}^2) \Leftrightarrow \tilde{F} = 0 + \mathcal{O}(\tilde{\xi}^2).$$

In turn, this means that $\tilde{\xi}$ is a pure gauge $\tilde{\xi}_j = \partial_j\tilde{\phi}(y)$.

The set of infinitesimal gauge transformations [in the sense of Kontsevich's formula (22)], corresponds then to

$$y'_i = y_i + \theta_{ij}\frac{\partial\tilde{\phi}(y)}{\partial y_j}. \tag{26}$$

These transformations were discussed in [12], from the point of view of the mapping between fluids and noncommutative theories; there, they are identified with the diffeomorphisms preserving the fluid volume element.

As a simple calculation shows [12], the effect of (26) on $x_j(y)$ is

$$\delta x_j(y) = -i[x_j(y), \tilde{\phi}(y)]_\star; \tag{27}$$

it corresponds to an adjoint field in the noncommutative representation of $U(1)_\star$. From this, it follows that

$$\delta\tilde{A}_j(y) = \partial_j\tilde{\phi}(y) - i[\tilde{A}_j(y), \tilde{\phi}(y)]_\star \tag{28}$$

and consequently

$$\delta\tilde{F}_{ij}(y) = -i[\tilde{F}_{ij}(y), \tilde{\phi}(y)]_\star. \tag{29}$$

Therefore, the subgroup of Kontsevich's transformations preserving θ_{ij} are indeed gauge transformations with respect to $\tilde{A}_j$.

From (14) and (29) it follows that $\tilde{\Theta}_{ij}$ is not invariant under (26), but rather

$$\delta\tilde{\Theta}_{ij}(y) = -i[\tilde{\Theta}_{ij}, \tilde{\phi}(y)]_\star, \tag{30}$$

which is another representative in the class of gauge equivalent products of $\tilde{\Theta}_{ij}$. This agrees with the expected behavior of Θ:

$$\tilde{\Theta}_{ij}(y') = \tilde{\Theta}_{ij}(y) + \delta\tilde{\Theta}_{ij}(y). \tag{31}$$

We are interested in reducing a noncommutative space whose coordinates (x_j) have a space-dependent $\Theta(x)$ to the θ-constant case in terms of new variables (y_j), with the change of coordinates inverse to (4):

$$y_i = x_i - \theta_{ij}A_j(x). \tag{32}$$

Then the previous analysis reveals that, when this is possible, there exists an infinite set of coordinates y_j with a constant $\star$ commutator, all related through (26). In other words, (32) is defined up to transformations parametrized by ϕ:

$$y_i \to y_i' = x_i - \theta_{ij}\left\{A_j(x) - \Delta^{-1}(x)\left(\delta_{kj} + \theta_{jl}\frac{\partial A_k}{\partial x_l}\right)\frac{\partial \phi}{\partial x_k}(x)\right\}, \tag{33}$$

[where we used (18)]. For small θ,

$$y_i' = x_i - \theta_{ij}(A_j(x) - \partial_j \phi(x)) + \mathcal{O}(\theta^2), \tag{34}$$

which means that A and its transformed by ϕ are in the same gauge orbit (in this limit, for commutative Abelian gauge transformations).

B. Construction of the noncommutative field theory

We will now construct a field theory in the variables x_j. For this, we have to define integration and derivatives (refer to [13] for a general operatorial approach) and we have to find an explicit representation for the $\star$ product in the new variables x_j.

The first element we consider is the integration measure $d\mu$ in the x variables: we realize that it can be simply derived from the knowledge of the coordinate transformation and its Jacobian:

$$d\mu \equiv d^2y = d^2x|\Delta(x)|, \tag{35}$$

where we have assumed that the metric for the y coordinates is Euclidean and we used the relation

$$y_i = x_i - \theta_{ij}A_j(x). \tag{36}$$

We note that (35) is consistent with the known formula for the measure in general coordinates. Indeed, the metric tensor in the new coordinates, $g_{ij}(x)$, is given by

$$g_{ij}(x) = \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}\delta_{kl}, \tag{37}$$

and a simple calculation shows that

$$g(x) \equiv \det[g_{ij}(x)] = [\Delta(x)]^2. \tag{38}$$

Thus, the measure (35) is also identical to

$$d\mu = d^2x\sqrt{g(x)}, \tag{39}$$

which is the usual expression for the volume element in general coordinates. This shows that, indeed, there is a connection between noncommutative and gravitational effects; refer to [10] for further discussion on this issue.

Next we proceed to construct derivatives. From (18),

$$D_i \equiv \frac{\partial}{\partial y_i} = \frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j} = \Delta^{-1}(x)\left(\delta_{ij} + \theta_{jl}\frac{\partial A_i}{\partial x_l}\right)\frac{\partial}{\partial x_j}. \tag{40}$$

To interpret here the effect of $A_j(x)$, we rewrite this expression as an inner derivation, with the aid of (36):

$$D_i = i\theta^{-1}[\varepsilon_{ij}x_j,]_\star + i[A_i(x),]_\star. \tag{41}$$

This equation automatically verifies the Leibnitz rule for $\star$ and $\int d\mu D_i f(x) = 0 \forall f$ [13]. Besides, we see that A_j plays the role of a noncommutative connection.

Finally, we have to derive a representation for the commutation relation

$$[x_1, x_2]_\star = i\Theta(x). \tag{42}$$

Note that, up to this point, we always started from the canonical variables y_i and arrived at the new ones x_i with the change of variables (4). However, in most of the practical problems, the situation is inverse: we start from a space-dependent relation like (42) and we would like to find a change of variables in terms of which the commutator corresponds to the case of constant θ. What the deduction in the previous subsection shows is that every $\Theta(x)$ of the form (20) can be reduced[1] to a constant θ with the inverse change of variables of (4). Consequently, if we choose the usual Moyal product for the canonical variables:

$$\tilde{f}(y) \star \tilde{g}(y) \equiv \exp\left(\frac{i}{2}\theta\varepsilon_{jk}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k'}\right)\tilde{f}(y)\tilde{g}(y')\Big|_{y'=y}, \tag{43}$$

for such a $\Theta(x)$, a possible $\star$ product is

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\theta\Gamma(x, x')\right)f(x)g(x')\Big|_{x'=x}, \tag{44}$$

with

$$\Gamma \equiv \Delta^{-1}(x)\Delta^{-1}(x')\varepsilon_{jk}\left(\delta_{js} - \theta\varepsilon_{rs}\frac{\partial A_j}{\partial x_r}(x)\right) \times \frac{\partial}{\partial x_s}\left(\delta_{km} - \theta\varepsilon_{nm}\frac{\partial A_k}{\partial x_n'}(x')\right)\frac{\partial}{\partial x_m'}. \tag{45}$$

Now we have all the elements to study the effect of the coordinate transformation (4) on a noncommutative field theory defined on (y_1, y_2). For simplicity, we consider the case of a scalar field:

$$S[\tilde{\varphi}] = \int d\tau dy_1 dy_2\left[\frac{1}{2}(\partial_\tau\tilde{\varphi} \star \partial_\tau\tilde{\varphi} + \partial_j\tilde{\varphi} \star \partial_j\tilde{\varphi} + m^2\tilde{\varphi} \star \tilde{\varphi}) + V_\star(\tilde{\varphi})\right]. \tag{46}$$

Since the Moyal product between the same two functions may be written as the usual commutative product plus a total-derivate term, (46) can be simplified to yield

$$S[\tilde{\varphi}] = \int d\tau dy_1 dy_2\left[\frac{1}{2}((\partial_\tau\tilde{\varphi})^2 + (\partial_j\tilde{\varphi})^2 + m^2\tilde{\varphi}^2) + V_\star(\tilde{\varphi})\right]. \tag{47}$$

Then, using Eqs. (35), (40), and (47), the action in the new variables is

[1] of course, after solving the nonlinear problem. See Sec. III for examples.

$$S[\varphi] = \int d\tau dx_1 dx_2 |\Delta(x)|^2 \Big[-\frac{1}{2}\varphi(x)(\partial_\tau^2 + D_i^2 - m^2)\varphi(x) + V_\star(\varphi) \Big], \tag{48}$$

with the star product in $V_\star$ computed from (44). It is worth noting that D_i^2 induces a nondiagonal quadratic term in the momentum variables and a derivative coupling:

$$D_i^2 = \Delta^{-2}(x)(\delta_{ij} + \theta_{jl}\partial_l A_i)[(\delta_{in} + \theta_{nk}\partial_k A_i)\partial_j\partial_n + \theta_{nk}\partial_j\partial_k A_i \partial_n]. \tag{49}$$

The propagator is directly obtained by performing the change of variables in the simple expression for $\langle\tilde{\varphi}(y)\tilde{\varphi}(y')\rangle$:

$$\langle\varphi(x)\varphi(x')\rangle = \frac{1}{4\pi}\Big[(t-t')^2 + \{(x_i - x_i') - \theta\varepsilon_{ij}(A_j(x) - A_j(x'))\}^2\Big]^{-1/2}. \tag{50}$$

III. EXAMPLES

We now study some particular cases, which we define in terms of special properties of the coordinate transformation.

A. The case $\tilde{A}_2 = 0$

When one of the components of $\tilde{A}_j$ vanishes (the second, say) we of course have $[\tilde{A}_1, \tilde{A}_2]_\star = 0$ and the expression for $\tilde{\Theta}$ becomes

$$\tilde{\Theta}(y) = \theta[1 + \theta\tilde{f}(y)] \tag{51}$$

with $\tilde{f}(y) = -(\partial\tilde{A}_1/\partial y_2)$. Equivalently, in terms of x_i we have, from (20),

$$\Theta(x) = \theta\frac{1}{1 + \theta(\partial A_1/\partial x_2)}. \tag{52}$$

To find $\Theta(x)$, we need the change of variables that yields y_i in terms of x_j; this may be written as follows:

$$\begin{cases} y_1 = & x_1, \\ y_2 = & x_2 + \theta\tilde{A}_1(x_1, y_2). \end{cases} \tag{53}$$

The second line shows that, except for some particular cases, the explicit form of the inverse transformation for y_2 may not be found exactly. However, one can always use an expansion in powers of θ:

$$y_2 = \sum_{l=0}^{\infty} \theta_l y_2^{(l)} \tag{54}$$

where the first terms are given by

$$\begin{aligned} y_2^{(0)} &= x_2, \qquad y_2^{(1)} = \tilde{A}_1(x_1, x_2), \\ y_2^{(2)} &= \tilde{A}_1(x_1, x_2)\partial_2\tilde{A}_1(x_1, x_2), \\ y_2^{(3)} &= \tilde{A}_1(x_1, x_2)(\partial_2\tilde{A}_1(x_1, x_2))^2 \\ &\quad + \frac{1}{2}\tilde{A}_1^2(x_1, x_2)\partial_2^2\tilde{A}_1(x_1, x_2), \ldots \end{aligned} \tag{55}$$

Using the previous expansion we may also write an expansion for $(\partial A_1/\partial x_2)$, to be used in (52) to find $\Theta(x)$:

$$\begin{aligned} \partial_2 A_1 = {} & \partial_2\tilde{A}_1 + \theta[(\partial_2\tilde{A}_1)^2 + \tilde{A}_1\partial_2^2\tilde{A}_1] \\ & + \theta^2\Big[\frac{1}{2}\tilde{A}_1^2\partial_2^3\tilde{A}_1 + 3\tilde{A}_1\partial_2\tilde{A}_1\partial_2^2\tilde{A}_1 + (\partial_2\tilde{A}_1)^3\Big] \\ & + \theta^3\Big[\tilde{A}_1^3\Big(\frac{1}{3!}\partial_2^4\tilde{A}_1 + \partial_2\tilde{A}_1\partial_2^3\tilde{A}_1 + (\partial_2^2\tilde{A}_1)^2\Big) \\ & + \tilde{A}_1^2\Big(\frac{1}{2}(\partial_2^2\tilde{A}_1)^2 + 3\partial_2^2\tilde{A}_1(\partial_2\tilde{A}_1)^2\Big) \\ & + 4\tilde{A}_1(\partial_2\tilde{A}_1)^2\partial_2^2\tilde{A}_1 + (\partial_2\tilde{A}_1)^4\Big] + \mathcal{O}(\theta^4), \end{aligned} \tag{56}$$

where all the field arguments and the derivatives correspond to the x_1 and x_2 variables. For example, $\tilde{A}_1 \equiv \tilde{A}_1(x_1, x_2)$. Finally, expanding in the expression for $\Theta(x)$, we see that

$$\Theta(x) = \theta[1 - \theta\partial_2\tilde{A}_1 - \theta^2\tilde{A}_1\partial_2^2\tilde{A}_1 \ldots]. \tag{57}$$

The power series expansion cannot be summed exactly, except for some particular cases. We shall consider two of them, showing how $\Theta(x)$ may be found explicitly by solving exactly for y as a function of x, or by a summation of the previous series. The explicit examples we shall exhibit stem from an $\tilde{A}_1$ which is quadratic or linear in y_2, respectively. We shall, however, come back to the general case in the conclusions.

The quadratic case corresponds to

$$\tilde{A}_1(y_1, y_2) = \alpha(y_1)(y_2)^2 + \beta(y_1)y_2 + \gamma(y_1), \tag{58}$$

and produces a $\tilde{\Theta}(y)$ with the form

$$\tilde{\Theta}(y) = \theta[1 - \theta(2\alpha(y_1)y_2 + \beta(y_1))]. \tag{59}$$

On the other hand, we know that $y_1 = x_1$ and besides y_2 may be obtained from

$$x_2 = y_2 - \theta[\alpha(y_1)(y_2)^2 + \beta(y_1)y_2 + \gamma(y_1)], \tag{60}$$

which is a quadratic equation. Inserting its solution for y_2, and $y_1 = x_1$ into $\tilde{\Theta}$, we see that[2]

$$\Theta(x_1, x_2) = \theta\sqrt{(1 - \theta\beta(x_1))^2 - 4\theta\alpha(x_1)(\theta\gamma(x_1) - x_2)}, \tag{61}$$

[2]There is another solution of the quadratic equation, which yields a Θ with the opposite sign.

a result which depends on both variables, x_1 and x_2, but can still be described in terms of the canonical variables y_1 and y_2. Therefore, if we start from a space-dependent $\Theta(x)$, which can be written in the form (61) for an adequate choice of the functions α, β, γ, then the change of variables (60) will reduce the problem to the θ-constant case.

Note that $\Theta(x)$ may vanish on a region (a curve, in general) of the plane. That region is defined by the equation

$$\delta(x_1, x_2) = (1 - \theta\beta(x_1))^2 - 4\theta\alpha(x_1)(\theta\gamma(x_1) - x_2) = 0 \tag{62}$$

where δ is the discriminant of the quadratic equation (60). In [13] we have analyzed the physical effects of such a behavior in the noncommutativity parameter.

We conclude our analysis of the $\tilde{A}_2 = 0$ example by mentioning the linear case: $\tilde{A}_1 = y_2\beta(y_1)$, which leads to

$$\partial_2 A_1 = \frac{\alpha(x_1)}{1 - \theta\beta(x_1)} \tag{63}$$

and

$$\Theta(x) = \theta(1 - \theta\beta(x_1)), \tag{64}$$

in agreement with [10] [after making the identification $t(x_1) \equiv 1 - \theta\beta(x_1)$], and with the proper limit of the quadratic case. We will consider, in the next section, a generalization of this result to $d > 2$.

It is worth noting that this kind of change of variables can be extended to more general cases. Indeed, it is sufficient to have the possibility of solving explicitly the equation for x_2 in terms of y_i, and that can be done for many polynomial transformations. Besides, we note that any polynomial transformation may always be equivalently written as a polynomial (of the same degree) in the algebra, when that is required. This follows from the repeated application of the property

$$\alpha(y_1)(y_2)^n = \alpha(y_1) \star (y_2)^n - \sum_{l=1}^{n}\binom{n}{l}\left(i\frac{\theta}{2}\right)^l \alpha^{(l)}(y_1) \times (y_2)^{n-l}, \tag{65}$$

valid for all $n \in \mathbb{N}$. The resulting "$\star$ polynomial" shall be real, since it should be equivalent to a real function (the polynomial which only involves commutative products).

B. The case $\varepsilon_{ij}\partial_i\tilde{A}_j = 0$

We shall assume here that $\tilde{A}_j$ verifies

$$\varepsilon_{ij}\partial_i\tilde{A}_j(y) = 0 \tag{66}$$

for all the points in the plane, except for the origin $y_i = 0$. We have in mind an Abelian vortexlike configuration for the vector field $\tilde{A}_j$; then, for the change of variables we shall assume the domain of definition for the y variables to be contained in $\mathbb{R}^2 - \{(0, 0)\}$. We can write locally the gauge field as the gradient of a function $\tilde{\varphi}$, namely,

$$\tilde{A}_i = \partial_i\tilde{\varphi}(y), \tag{67}$$

where $\tilde{\varphi}$, to have a vortex configuration, has to be a multivalued function. The function $\tilde{\Theta}(y)$ will be given by the expression

$$\tilde{\Theta}(y) = \theta(1 - i\theta[\tilde{A}_1(y), \tilde{A}_2(y)]_\star). \tag{68}$$

Since the vortex is located at the origin, we fix the $\tilde{\varphi}$ function to be proportional to the polar angle:

$$\tilde{\varphi}(y) = \frac{q}{2\pi}\arctan\left(\frac{y_2}{y_1}\right), \tag{69}$$

where $q \in \mathbb{Z}$ is the "charge" (i.e., vorticity) of the configuration.

Let us now consider the equations for the change of variables under the previous assumptions

$$\begin{cases} x_1 = & y_1 + \theta\frac{\partial\tilde{\varphi}}{\partial y_2}(y), \\ x_2 = & y_2 - \theta\frac{\partial\tilde{\varphi}}{\partial y_1}(y), \end{cases} \tag{70}$$

or, by taking (69) into account,

$$\begin{cases} x_1 = & y_1 - \frac{q\theta}{2\pi}\frac{y_1}{(y_1)^2 + (y_2)^2}, \\ x_2 = & y_2 - \frac{q\theta}{2\pi}\frac{y_2}{(y_1)^2 + (y_2)^2}. \end{cases} \tag{71}$$

These can be more easily represented (and inverted) by introducing polar coordinates:

$$\begin{aligned} x_1 &= R\cos\phi, \qquad x_2 = R\sin\phi, \\ y_1 &= r\cos\alpha, \qquad y_2 = r\sin\alpha, \end{aligned} \tag{72}$$

since (71) yields

$$\phi = \alpha \quad (0 \le \alpha < 2\pi), \qquad R = r - \frac{q\theta}{2\pi}r^{-1}. \tag{73}$$

Regarding the range of the variables r and R, we can distinguish two different situations, depending on the sign of the product $q\theta$. If $q\theta > 0$, then from (73), we see that $R \ge 0$ requires the condition $r \ge \sqrt{(q\theta/2\pi)}$ to be satisfied:

$$q\theta > 0 \Rightarrow \sqrt{\frac{q\theta}{2\pi}} \le r < \infty, \qquad 0 \le R < \infty. \tag{74}$$

An identical condition is obtained for r when $q\theta \le 0$, in order to have a one-to-one transformation, i.e., to satisfy $\frac{dR}{dr} \ne 0$, $\forall r$:

$$q\theta < 0 \Rightarrow \sqrt{\frac{q\theta}{2\pi}} \le r < \infty, \qquad \sqrt{\frac{2q\theta}{\pi}} \le R < \infty. \tag{75}$$

The inverse transformation becomes, in both cases,

$$r = \left(R + \sqrt{R^2 + \frac{2q\theta}{\pi}} \right) \tag{76}$$

and, of course, $\alpha = \phi$.

Let us consider now the expression for $\tilde{\Theta}$ for the example at hand. From (68), we see that

$$\tilde{\Theta}(y) = \theta \left(1 + i \frac{q^2\theta}{(2\pi)^2} \left[\frac{y_2}{(y_1)^2 + (y_2)^2}, \frac{y_1}{(y_1)^2 + (y_2)^2} \right]_\star \right). \tag{77}$$

The leading term on the rhs is determined by the Poisson bracket of the corresponding elements. This yields, for small θ,

$$\tilde{\Theta}(y) = \theta \left[1 + \frac{1}{(2\pi)^2} \frac{q^2\theta^2}{((y_1)^2 + (y_2)^2)^2} + \mathcal{O}(\theta^4) \right], \tag{78}$$

or, by using (76),

$$\Theta(x) = \theta \left[1 + \frac{1}{2^4(2\pi)^2} \frac{q^2\theta^2}{((x_1)^2 + (x_2)^2)^2} + \mathcal{O}(\theta^4) \right]. \tag{79}$$

IV. A HIGHER-DIMENSIONAL EXAMPLE

We will now deal with $d > 2$, showing first some of the general features that survive from the $d = 2$ case, and then considering an example.

Our starting point is the formula for $\tilde{\Theta}_{ij}(y)$ in d dimensions:

$$\tilde{\Theta}_{ij}(y) = \theta_{ij} + \frac{1}{2}(\theta_{ik}\theta_{jl} - \theta_{il}\theta_{jk})\tilde{F}_{kl}(y). \tag{80}$$

In general, all the properties described in Sec. II are valid, except those relying on the explicit form $\theta_{ij} = \theta\varepsilon_{ij}$. In particular, the construction of the field theory in d dimensions follows the same steps as in Sec. II B.

As an example, we consider here the natural generalization to $d > 2$ of the case considered at the end of Sec. III A. The condition we impose on the gauge field configuration is now

$$\tilde{A}_i(y) = 0, \qquad \forall i = 2, \ldots, d, \qquad \tilde{A}_1(y) = \sum_{j=2}^{d} y_j \alpha_j(y_1). \tag{81}$$

This leads to an $\tilde{F}_{ij}$ tensor whose only nonvanishing components are

$$\tilde{F}_{k1} = -\tilde{F}_{1k} = \alpha_k(y_1). \tag{82}$$

Since, for the gauge field configuration defined above $y_1 = x_1$, we see that the answer for $\Theta_{ij}(x)$, the commutator between x_i and x_j, is

$$\Theta_{ij}(x) = \theta_{ij} + (\theta_{ik}\theta_{j1} - \theta_{i1}\theta_{jk})\alpha_k(x_1), \tag{83}$$

which is a function of x_1 only.

The measure for integration over the x variables can also be obtained explicitly, in terms of the corresponding Jacobian:

$$d\mu = d^d y = d^d x \frac{\partial(y_1, \ldots, y_d)}{\partial(x_1, \ldots, x_d)}, \tag{84}$$

where

$$\frac{\partial(y_1, \ldots, y_d)}{\partial(x_1, \ldots, x_d)} = \frac{1}{|1 - \theta^{1i}\alpha_i(x_1)|}, \tag{85}$$

as a little algebra easily shows.

V. REDUCTION OF THE GENERAL CASE TO CANONICAL VARIABLES

In the previous sections we considered a noncommutative theory with canonical Moyal variables $[y_1, y_2]_\star = i\theta$, and we studied the effect of a general change of variables $y_i \to x_i$ of the form (4). Now we have the necessary tools to address the inverse problem, namely, mapping a noncommutative theory with the general space-dependent parameter $\Theta(x)$ to a new theory with constant θ.

In the general case, the noncommutative space is a deformation of the classical Poisson structure:

$$\{x_1, x_2\} = \Theta(x) \to [x_1, x_2]_{\star_K} = i\Theta(x)$$

with Kontsevich's star product [7] over $C^\infty(\mathbf{R}^2)$ functions:

$$\begin{aligned} f \star_K g = fg &+ i\frac{\Theta}{2}\epsilon_{ij}\partial_i f \partial_j g - \frac{\Theta^2}{8}\epsilon_{ij}\epsilon_{kl}\partial_i\partial_k f \partial_j \partial_l g \\ &- \frac{i}{12}\Theta\partial_j\Theta\epsilon_{ij}\epsilon_{kl}(\partial_i\partial_k f\partial_l g - \partial_k f \partial_i \partial_l g) + \ldots. \end{aligned} \tag{86}$$

This satisfies the defining axioms of a star product and thus gives a well-defined noncommutative space [14].

We have to construct an explicit map $x_i \to y_i$ such that the star product $\star_K$, or more generally, a gauge equivalent product [see Eq. (22)] denoted simply by "$\star$," gives $[y_1, y_2]_\star = i\theta$, with θ a constant. From the approach described in Sec. II, it follows that this is equivalent to finding a solution $\tilde{A}_j$ to

$$\Theta(y_i + \theta_{ij}\tilde{A}_j(y)) = \theta(1 + \theta\tilde{F}(y)), \tag{87}$$

obtained from Eqs. (4) and (14).

Since $\tilde{F}$ contains the term $[\tilde{A}_1, \tilde{A}_2]_\star$, this is in fact an infinite-order nonlinear differential equation. Therefore, we do not expect to get a conclusive answer from this approach. However, as we learned from Sec. III A, a possible way to simplify this is to use the ansatz $\tilde{A}_2 = 0$. In this case, (87) becomes

$$\frac{\partial x_2}{\partial y_2}(y_1, y_2) = \theta^{-1}\Theta(y_1, x_2). \tag{88}$$

This should be regarded as a nonlinear first order differential equation for x_2 as a function of y_2, with y_1 playing the role of a parameter (no derivatives with respect to y_1 appear in the equation). Its solution may be found (formally) by one quadrature:

$$y_2 = \theta \int \frac{dx_2}{\Theta(y_1, x_2)} \tag{89}$$

where the integral is of course indefinite, and the result is not unique unless one imposes extra (initial) conditions.

Therefore, under adequate regularity conditions, every $\Theta(x)$ may in principle be mapped to a constant θ, by using Eq. (89). Among the regularity conditions is of course the nonvanishing of Θ as a function of its arguments, what is here clearly linked to the fact that the Jacobian is different from zero everywhere.

In this way, the geometry defined by $\Theta(x)$ and the field theories constructed on such a space can be traced back to the canonical case. Of course, it may not be possible to obtain an analytic expression for y_2 in the general case; however, in the context of deformation quantization, where $\theta \to 0$, (88) can always be solved by iterations.

The explicit map (89) connecting a general $\Theta(x)$ to the canonical constant case should not be a surprise. Indeed, we expect noncommutative theories to emerge as certain low energy limits of quantum gravity; besides, we know that in two dimensions, every metric is conformally flat, with no dynamical degrees of freedom. Only the Euler number, of topological nature, distinguishes different gravitational backgrounds. In fact, we have a similar situation at the level of the noncommutative theory: compute the integral of the difference between the commutators

$$\frac{1}{i\theta^2}\int d^2y([x_1, x_2]_\star - [y_1, y_2]_\star) = \int d^2y \tilde{F}(y), \tag{90}$$

where the commutator between x_1 and x_2 is written as a function of the y variables. We moved the constant factors in order to have dimensionless objects on both sides. Note that the object on the rhs is, for well-behaved changes of variables, a topological invariant. Indeed, the "non-Abelian" term vanishes for an $\tilde{A}_j$ which decreases sufficiently fast at infinity:

$$\int d^2y[\tilde{A}_1, \tilde{A}_2]_\star = 0 \tag{91}$$

(i.e., when the cyclicity of the trace is valid), and we then have

$$\frac{1}{i\theta^2}\int d^2y([x_1, x_2]_\star - [y_1, y_2]_\star) = \int d^2y \varepsilon_{ij}\partial_i\tilde{A}_j, \tag{92}$$

which, of course, can be written as a line integral at infinity, and thus it makes the topological invariance of the object more explicit. Note that the integral of each separate commutator is, in general, divergent, but the integral of their difference can indeed have a well-defined, finite value, at least for a class of $\tilde{A}_j$'s.

As a nontrivial example, the case

$$\Theta(x) = \theta[\Theta_0(x_1) + \Theta_1(x_1)x_2] \tag{93}$$

is particularly interesting, because (88) is exactly solvable yielding

$$x_1 = y_1, \qquad x_2 = -\frac{\Theta_0(y_1)}{\Theta_1(y_1)} + C(y_1)e^{\Theta_1(y_1)y_2}; \tag{94}$$

here $C(y_1)$ is an arbitrary smooth function. The same can be done in the even simpler case $\Theta_1 = 0$, obtaining the same result as in Sec. III A.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have examined the effect of a general coordinate transformation on a theory with constant $\star$ commutator, with the aim of mapping (for certain cases) a space-dependent $\Theta_{ij}(x)$ theory to another where that object is constant.

The method was first developed for $d = 2$, where we explicitly evaluated the effect of the change of variables on both the noncommutative space structure and on a scalar field theory action.

An interesting feature of the present approach is that, by defining the coordinate transformation in terms of a vector field A_j as in [12], one can interpret (a subgroup of) Kontsevich's equivalence relations between $\star$ products as noncommutative gauge transformations on A_j.

The examples constructed in Sec. III led to the central question of whether the reduction from the space-dependent case to the constant-θ one is always applicable or not. And indeed, in Sec. V we showed that this is true by constructing an explicit map. This agrees with Darboux's Theorem [15], which states that given a symplectic manifold $\mathcal{M}$ it is always possible to find local coordinates in the neighborhood of any point $x \in \mathcal{M}$ such that the symplectic 2-form is given by

$$\omega = dp_i \wedge dq^i.$$

For the case $\mathcal{M} = \mathbf{R}^2$, with $\Theta(x)$ positive definite, Darboux's Theorem holds *globally* because (a) the symplectic form is nonsingular everywhere, and (b) the domain where the symplectic form is defined allows for a global (not just local) application of the inverse Poincare Lema, as required by the proof of Darboux's Theorem. From the point of view of deformation quantization, this result may be regarded as a classical limit of our map Eq. (89).

ACKNOWLEDGMENTS

C. D. F. and G. T. have been supported by a Fundación Antorchas grant. G. T. is supported by the Department of Physics and Astronomy of Rutgers.

[1] M. Douglas and N. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[2] F. A. Schaposnik, hep-th/0408132.
[3] R. Szabo, Phys. Rep. **378**, 207 (2003).
[4] L. Cornalba and R. Schiappa, Commun. Math. Phys. **225**, 33 (2002).
[5] J. Madore, S. Schraml, P. Schupp, and J. Wess, Eur. Phys. J. C **16**, 161 (2000).
[6] W. Behr and A. Sykora, Nucl. Phys. **B698**, 473 (2004); A. Das and J. Frenkel, Phys. Rev. D **69**, 065017 (2004); X. Calmet and M. Wohlgenannt, Phys. Rev. D **68**, 025016 (2003).
[7] M. Kontsevich, Lett. Math. Phys. **66**, 157 (2003).
[8] V. Rivelles, Phys. Lett. B **558**, 191 (2003).
[9] J. Madore, gr-qc/9906059.
[10] D. H. Correa, C. D. Fosco, F. A. Schaposnik, and G. Torroba, J. High Energy Phys. 09 (2004) 064.
[11] A. Agostini, G. Amelino-Camelia, M. Arzano, and F. Andrea, hep-th/0407227.
[12] R. Jackiw, V. P. Nair, S. Y. Pi, and A. P. Polychronakos, J. Phys. A **37**, R327 (2004).
[13] C. D. Fosco and G. Torroba, hep-th/0405090.
[14] A. S. Cattaneo, math-qa/0403135.
[15] R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin/Cummings Publishing Company, Reading, MA, 1978), 2nd ed.