PHYSICAL REVIEW D **71,** 065009 (2005)

Thermal operator representation for Matsubara sums

Olivier Espinosa
Departamento de Física, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
(Received 28 January 2005; published 23 March 2005)

We prove in full generality the thermal operator representation for Matsubara sums in a relativistic field theory of scalar and fermionic particles. It states that the full result of performing the Matsubara sum associated to any given Feynman graph, in the imaginary-time formalism of finite-temperature field theory, can be directly obtained from its corresponding zero-temperature energy integral, by means of a simple linear operator, which is independent of the external Euclidean energies and whose form depends solely on the topology of the graph.

DOI: 10.1103/PhysRevD.71.065009 PACS numbers: 11.10.Wx

I. INTRODUCTION

Relativistic quantum field theory is the mathematical formalism that allows us to describe the interactions among elementary particles, according to the principles of quantum mechanics and special relativity. When considering the physics of very dense or very hot plasmas, such as the early universe or the quark-gluon plasma that should be produced in heavy ion collisions, it becomes necessary to formulate the relevant questions in terms of thermally averaged quantities, according to statistical mechanics. Most of the physical information, for systems both in and slightly out of thermal equilibrium, can be obtained from the study of the so-called thermal Green functions, defined as thermal expectation values of a time-ordered product of field operators. One fruitful and widely used approach to study these Green functions is perturbation theory, where these are computed by means of a systematic expansion in terms of Feynman diagrams [1].

The simplest diagrammatic analysis is obtained in the so-called imaginary-time formalism, in which the diagrams have the same topology and are computed according to basically the same Feynman rules as in the zero-temperature Euclidean theory, except for one very important difference: the energy component of both the external 4-momenta and the internal 4-momenta carried by the propagators is quantized (in slightly different ways, according to the nature—bosonic or fermionic—of the associated particle), that is, it must be a *Matsubara frequency* [2]. Accordingly, the calculation of a loop diagram in the imaginary-time formalism of quantum field theory at finite temperature necessarily involves sums over internal Matsubara frequencies [1], an operation that we shall generically call *the Matsubara sum* associated with the graph. Although this sum can be computed in a number of ways, usually in a systematic fashion, such computations can become quite tedious for higher loop diagrams [3,4]. Another related difficulty of the imaginary-time formalism is the separation of the vacuum contribution from the finite-temperature corrections, since the naive zero-temperature limit of any given loop graph is of the indeterminate type $0 \cdot \infty$ [5].

It was discovered some time ago [6] that, for some particular classes of diagrams in a scalar theory, the full result of performing the Matsubara sum associated with a Feynman graph could be completely determined from its zero-temperature counterpart, by means of a simple linear operator that depends on the topology of the diagram but is independent of the Euclidean energies carried by its external lines. Because of the simple and well-defined structure of this operator, dubbed *the thermal operator*, it was conjectured that this result should hold for all Feynman graphs.

In the recent paper [5] it has been partially shown that this conjecture is actually correct. The authors of Ref. [5] have rescued from oblivion an old, systematic and very elegant method to perform the Matsubara sum associated with an arbitrary graph, due to M. Gaudin [7], and have used it, as an illustration of the power of Gaudin's method, to show that the main part of our conjecture follows naturally from it. The proof given in Ref. [5] does address the relationship between the structure of the full thermal result and the zero-temperature result, but stops short of explicitly constructing the thermal operator that relates them. Also, it leaves undiscussed conjectures 2 and 3 presented in Ref. [6].

In this paper we restate all the conjectures of Ref. [6] as full-blown theorems concerning general properties of the Feynman graphs in the imaginary-time formalism of thermal field theory, extending their validity to theories containing Dirac fields as well. Following Ref. [5], we make full use of Gaudin's method to prove these theorems, although in a slightly different and more explicit form.

The structure of the rest of the paper is as follows: In Sec. II we present, in the form of theorems, the general properties of the finite-temperature imaginary-time Feynman graphs that were previously presented as conjectures, in terms of the thermal operator, suitably extended to incorporate possible fermionic lines. In Sec. III we summarize Gaudin's method to perform the sums over Matsubara frequencies, which will be central to the proofs of the theorems, presented in Secs. IV (scalars) and V (fermions). Our conclusions are presented in Sec. VI.

*Electronic address: olivier.espinosa@usm.cl

II. THE THERMAL OPERATOR

In a scalar field theory, the mathematical expression corresponding to an amputated graph with V vertices, I internal lines, and external 4-momenta $P_\alpha = (p_\alpha, \mathbf{p}_\alpha)$ has the form

$$\frac{(-\lambda)^V}{S}\int\left[\prod_{i=1}^{I}\frac{d^3k_i}{(2\pi)^3 2E_i}\prod_{v=1}^{V-1}(2\pi)^3\delta^{(3)}(\mathbf{k}_v)\right]D(p,E,T), \tag{1}$$

where λ represents the coupling constant and S is the symmetry factor of the graph; $\mathbf{k}_i$ is the spatial 3-momentum of the i-th internal line and $E_i := (\mathbf{k}_i^2 + m_i^2)^{1/2}$ is its associated kinematic energy; $\mathbf{k}_v$ denotes the total 3-momentum entering vertex v; the unsubscripted symbols p and E denote, respectively, the full set of Euclidean external and kinematic internal energies, $p := \{p_1, p_2, \ldots, p_V\}$ (with $\sum_{i=1}^{V} p_i = 0$) and $E := \{E_1, E_2, \ldots, E_I\}$; and T is the temperature. The delta functions ensure conservation of spatial 3-momentum at each vertex, so that the integration measure reduces essentially to an integration over the 3-momenta of the $L = I - V + 1$ independent loops. In the finite-temperature Euclidean formalism all scalar lines, external and internal, carry discrete Euclidean energies which are integer multiples of $2\pi T$. Each internal line has an associated Matsubara frequency, denoted by $k_i = \omega_{n_i} := 2\pi T n_i$. The D-function is given by the normalized Matsubara sum

$$D(p,E,T) = \gamma_E T^L \sum_{\{n_i\}}\prod_{i=1}^{I}\Delta(k_i,E_i)\delta(p,k), \tag{2}$$

where

$$\gamma_E := \prod_{i=1}^{I} 2E_i, \tag{3}$$

L is the number of independent loops in the graph, and $\Delta(k_i, E_i)$ is the scalar propagator associated with the i-th internal line, with

$$\Delta(k,E) := \frac{1}{k^2+E^2}. \tag{4}$$

The sums over each n_i ($i = 1, \ldots, I$) run from $-\infty$ to $+\infty$. The δ-function, with $k = \{k_1, \ldots, k_I\}$, is a generalized Kronecker delta which ensures conservation of energy at each vertex. The topology of the diagram is totally contained in this generalized delta.

In a theory containing both fermions and scalars, the structure of (1) is basically unchanged, except for extra spin indices carried by the external fermionic lines and a possible extra sign associated with fermionic loops. The D-function is still given by (2), except that each fermionic line carries a Matsubara frequency consistent with antiperiodic boundary conditions on the fermionic fields,

$$k = \tilde{\omega}_n := 2\pi T(n + 1/2), \tag{5}$$

entering through the fermionic propagator, which now has a matrix structure and depends explicitly on the spatial momentum $\mathbf{k}$,

$$\tilde{\Delta}(k,\mathbf{k}) = \frac{m + ik\gamma^0 - \mathbf{k}\cdot\gamma}{k^2 + E_k^2}, \tag{6}$$

where the $\gamma^\mu = (\gamma^0, \gamma)$ are the usual Minkowski space gamma matrices, obeying the standard anticommutation relations, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

In order to state our general theorems in a way that is applicable to a general theory containing both scalars and fermions, it will be convenient to define the following generalized *signed* thermal occupation number function, which takes into account the statistics corresponding to both kinds of internal lines:

$$N(E) := \begin{cases} n(E), & \text{for a scalar line;} \\ -\tilde{n}(E), & \text{for a fermionic line,} \end{cases} \tag{7}$$

where $n(E) = (e^{\beta E} - 1)^{-1}$ and $\tilde{n}(E) = (e^{\beta E} + 1)^{-1}$ are, respectively, the Bose-Einstein and Fermi-Dirac thermal occupation factors, for the case of vanishing chemical potential.

Additionally, the theorems will be formulated in terms of a simple reflection operator on the space of functions of several variables,

$$\hat{S}_E f(E,x) := f(-E,x), \tag{8}$$

where E stands for any one variable and x for all the others.

The main result presented in this paper is enunciated in the next theorem. It basically states that the "energy part" of any Feynman graph in the finite-temperature imaginary-time formalism, represented here in terms of the D-function introduced in (1), can be obtained directly from the corresponding zero-temperature energy integral.

Theorem 1. (Thermal operator representation)—The D-function defined in (2) for an amputated Feynman graph can be expressed in the form

$$D(p,E,T) = \hat{\mathcal{O}}(E,T)D_0(\omega,E)|_{\omega=p}, \tag{9}$$

where $D_0(\omega, E)$ is the D-function of the Euclidean zero-temperature graph and $\hat{\mathcal{O}}(E,T)$, the thermal operator, is the following linear operator:

$$\hat{\mathcal{O}}(E,T) := 1 + \sum_{i=1}^{I} N_i(1+\hat{S}_i) + \sum_{\langle i_1,i_2\rangle}{}' N_{i_1}N_{i_2}(1+\hat{S}_{i_1})(1+\hat{S}_{i_2}) + \ldots + \sum_{\langle i_1,\ldots,i_L\rangle}{}' \prod_{l=1}^{L} N_{i_l}(1+\hat{S}_{i_l}). \tag{10}$$

Here $N_i \equiv N(E_i)$, where $N(E)$ is the generalized signed thermal occupation factor defined in (7); $\hat{S}_i := \hat{S}_{E_i}$, where $\hat{S}_E$ is the reflection operator defined in (8); the indices $i_1, i_2, \ldots$ run from 1 to I (the number of internal propagators) and the symbol $\langle i_1, \ldots, i_k \rangle$ stands for an unordered k-tuple with no repeated indices, representing a particular set of internal lines. The primes on the summation symbols imply that certain tuples $\langle i_1, \ldots, i_k \rangle$ are to be excluded from the sums: those such that if we snip all the lines $i_1, \ldots, i_k$ then the graph becomes disconnected.

Note that the operator $\hat{\mathcal{O}}(E, T)$ contains products of at most L thermal occupation factors $N(E_i)$, since for a L-loop graph the maximum number of lines that can be snipped without disconnecting the graph is precisely L. This generic feature of the thermal graph in the imaginary-time formalism is of course well known. However, based on a general property of the zero-temperature D-function, formulated in theorem II below, it is also possible to use a modified thermal operator, which has a simpler algebraic form:

Theorem 2. (Simpler form of the thermal operator)—When acting on the zero-temperature D-function, $D_0(p, E)$, the thermal operator $\hat{\mathcal{O}}(E, T)$ can be replaced by the simpler

$$\hat{\mathcal{O}}_\star(E, T) = \prod_{i=1}^{I} [1 + N_i(1 + \hat{S}_i)]. \tag{11}$$

Note that the operator $\hat{\mathcal{O}}_\star(E, T)$ in (10) can be expanded as in (10), with the only difference that the summation symbols will carry no primes, that is, all tuples $\langle i_1, \ldots, i_k \rangle$ $(1 \le k \le I)$ will be allowed in the sum. Clearly, forms (10) and (11) of the thermal operator will be equivalent if we can prove that tuples associated with disconnected graphs [the ones excluded from the summations in (10)] give rise to operators that produce a vanishing contribution to the D-function in (9). This is the content of our last theorem:

Theorem 3. (Cut sets do not contribute)—The zero-temperature D-function, $D_0(\omega, E)$, is annihilated by the operators

$$\hat{\mathcal{A}}(C) := \prod_{i_l \in C} (1 + \hat{S}_{i_l}), \tag{12}$$

where C stands for a *cut set* of the graph, that is, any set of indices $i_1, \ldots, i_k$ such that the graph becomes disconnected if the corresponding lines are snipped.

The concept of *cut set*, as used here, bears no connection to the concepts of cut and cut diagrams as they are usually understood in diagrammatic quantum field theory. Cut sets are determined solely by the topology of the diagram, and have no further mathematical or physical meaning.

III. GAUDIN METHOD

Here we present a summary of Gaudin's method and its use in the computation of the Matsubara D-function, in the purely scalar case. For all the details see Refs. [5,7]. The changes in the presence of fermionic lines are commented upon at the end of this section.

Gaudin's method to perform the Matsubara sum associated to a given graph is based on two main ideas. The first is to make use of the spectral representation of the propagator, which puts the dependence on the Matsubara frequency linearly in the denominator. For the scalar case it reads

$$\Delta(\omega_n, \mathbf{k}) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{\rho(k^0, \mathbf{k})}{k^0 - i\omega_n}, \tag{13}$$

with the spectral function given by

$$\rho(k^0, \mathbf{k}) = 2\pi\epsilon(k^0)\delta(k^2 - m^2) \tag{14}$$

$$= \frac{2\pi}{2E_k}[\delta(k^0 - E_k) - \delta(k^0 + E_k)]. \tag{15}$$

$\epsilon(k^0)$ is the sign of k^0.

Once all the propagators have been represented by means of (13), the Matsubara sum to be performed takes the form

$$T^L \sum_{\{n_i\}} \prod_{i=1}^{I} \frac{1}{k_i^0 - ik_i} \delta(p, k), \tag{16}$$

where the $k_i = \omega_{n_i}$ are the Matsubara frequencies and the k_i^0 are external real variables. The generalized Kronecker delta $\delta(p, k)$ enforces $V - 1$ independent linear relations satisfied by the Matsubara frequencies k, also involving the external Euclidean energies p, which we shall write (following Ref. [5]) as

$$R_v(p, k) = 0, \quad \text{for} \quad v = 1, \ldots, V - 1. \tag{17}$$

This system of linear equations allows us to solve for $V - 1$ of the I Matsubara frequencies in terms of a set of $L = I - V + 1$ independent ones. In general, there will be several distinct ways of choosing this set of independent Matsubara frequencies. As shown by Gaudin, there is a one-to-one correspondence between the collection of all possible sets of independent Matsubara frequencies and the set of all *trees* associated to the given (connected) diagram Γ.

A tree is a set of lines of Γ joining all vertices and making a connected graph with no loops. Every tree $\mathcal{T}$ will contain $V - 1$ lines and its complement $\bar{\mathcal{T}}$ (the set of lines which do not belong to $\mathcal{T}$) will have L lines. The Matsubara frequencies corresponding to the L lines in $\bar{\mathcal{T}}$, denoted by k_l, will constitute a set of independent Matsubara frequencies in terms of which the system (17) can be solved. The Matsubara frequencies associated with the lines of the tree, k_j, with $j \in \mathcal{T}$, will be linear combinations of the independent Matsubara frequencies and the external Euclidean energy variables,

$$k_j = \Omega_j^{\mathcal{T}}(p, k_l), \qquad j \in \mathcal{T}, l \in \bar{\mathcal{T}}. \tag{18}$$

As a simple example, in Fig. 1 we show the three possible trees for the two-loop two-vertex graph shown. In this case, each tree $\mathcal{T}$ is composed by a single internal line (heavy line), whose Matsubara frequency can be expressed, after using energy conservation at one of the vertices, in terms of the two independent Matsubara frequencies associated with the (thin) lines that do not belong to the tree (these are the lines in $\bar{\mathcal{T}}$) and the external energy p. For instance, for the first tree we have $k_3 = p - k_1 - k_2$, etc.

Gaudin's main insight is the following identity for the rational function appearing in (16):

$$\sum_{\{n_i\}} \prod_{i=1}^{I} \frac{1}{k_i^0 - ik_i} \delta(p, k) = \sum_{\mathcal{T}} \prod_{j \in \mathcal{T}} \frac{1}{k_j^0 - i\Omega_j^{\mathcal{T}}(p, -ik_l^0)} \times \sum_{\{n_l\}} \prod_{l \in \bar{\mathcal{T}}} \frac{1}{k_l^0 - ik_l}. \tag{19}$$

In our example in Fig. 1, this identity takes the form

$$\begin{aligned}\sum_{n_1,n_2,n_3} &\frac{\delta(p - k_1 - k_2 - k_3)}{(k_1^0 - ik_1)(k_0^2 - ik_2)(k_3^0 - ik_3)} \\ &= \frac{1}{(k_1^0 + k_2^0 + k_3^0 - ip)} \bigg\{ \sum_{n_1,n_2} \frac{1}{(k_1^0 - ik_1)(k_2^0 - ik_2)} \\ &\quad + \sum_{n_1,n_3} \frac{1}{(k_1^0 - ik_1)(k_3^0 - ik_3)} \\ &\quad + \sum_{n_2,n_3} \frac{1}{(k_2^0 - ik_2)(k_3^0 - ik_3)} \bigg\}.\end{aligned}$$

So, using Gaudin's decomposition (19), for any given graph the full Matsubara sum is split into a number of other sums, one for each tree $\mathcal{T}$, where each one of these sums is actually simply a product of independent sums over each of the Matsubara frequencies in $\bar{\mathcal{T}}$. Each independent sum has to be regulated. Gaudin assigns to each internal line a regulator $e^{ik_i\tau_i}$, where the time τ_i is taken to zero at the end. The sum that results is well defined,

$$T \sum_{n_l} \frac{e^{i\omega_l T_l}}{k_l^0 - ik_l} = \epsilon_l n(\epsilon_l k_l^0) e^{k_l^0 T_l}, \tag{20}$$

where $n(k^0)$ is the Bose-Einstein occupation factor. T_l is some linear combination of the times τ_i associated to the lines that belong to the loop defined by $l \in \bar{\mathcal{T}}$, and ϵ_l is the sign of T_l. Only ϵ_l matters when the regulators are removed.

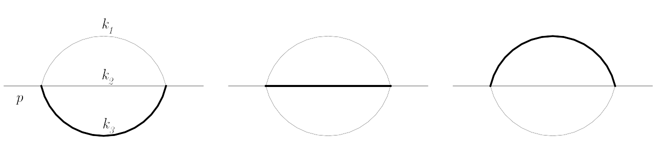

FIG. 1. The trees of a simple two-loop graph, shown as dark lines.

Consequently, Gaudin's result for the D-function in the purely scalar case is

$$D(p, E, T) = \prod_{i=1}^{I} \int_{-\infty}^{\infty} dk_i^0 \bar{\rho}(k_i^0, E_i) \times \sum_{\mathcal{T}} \bigg(\prod_{j \in \mathcal{T}} \frac{1}{k_j^0 - i\Omega_j^{\mathcal{T}}} \prod_{l \in \bar{\mathcal{T}}} \epsilon_l n(\epsilon_l k_l^0) \bigg), \tag{21}$$

where $\Omega_j^{\mathcal{T}} = \Omega_j^{\mathcal{T}}(p, -ik_l^0)$ and $\bar{\rho}$ is the reduced spectral function,

$$\bar{\rho}(k^0, E_k) = \frac{2E_k}{2\pi} \rho(k^0, \mathbf{k}) = \delta(k^0 - E_k) - \delta(k^0 + E_k). \tag{22}$$

Gaudin's method applies equally well in the case the graph contains fermionic lines. As for the scalar propagator, the fermionic propagator (6) admits a spectral representation,

$$\tilde{\Delta}(\tilde{\omega}_n, \mathbf{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\tilde{\rho}(k_0, \mathbf{k})}{k_0 - i\tilde{\omega}_n}, \tag{23}$$

with

$$\tilde{\rho}(k_0, \mathbf{k}) = 2\pi\varepsilon(k_0)(\not{k} + m)\delta(k^2 - m^2), \tag{24}$$

where the metric signature is that of Minkowski space, $\not{k} = k_\mu \gamma^\mu = k_0\gamma^0 - \mathbf{k} \cdot \gamma$ and $k^2 = k_0^2 - \mathbf{k}^2$. Note that the spectral representation of the fermionic propagator (6) "hides" the dependence on the Matsubara frequency k appearing in its numerator, so that the only difference with the scalar case is the extra matrix structure $(\not{k} + m)$.

Gaudin's method to perform the Matsubara sum in the form (16), through the decomposition in terms of trees, makes no reference to the bosonic or fermionic nature of the frequencies to be summed over. If an independent frequency $\tilde{k}_l$ ($l \in \bar{\mathcal{T}}$) is fermionic, we shall need the sum

$$T \sum_{n_l} \frac{e^{i\tilde{k}_l T_l}}{k_l^0 - i\tilde{k}_l} = -\epsilon_l \tilde{n}(\epsilon_l k_l^0) e^{k_l^0 T_l}, \tag{25}$$

which can be obtained from the bosonic analog by writing $k_l^0 - i\tilde{k}_l = (k_l^0 - i\pi T) - ik_l$. $\tilde{n}(k^0)$ is the Fermi-Dirac function,

$$\tilde{n}(k^0) = \frac{1}{e^{\beta k^0} + 1}. \tag{26}$$

We see that the fermionic result (25) can be simply obtained from its bosonic analog (20) through the replacement

$$n \rightarrow -\tilde{n}.$$

Therefore, the result for the D-function in the presence of fermionic propagators will be identical in form to (21), except that for the fermionic lines the reduced spectral function $\bar{\rho}$ will have an extra matrix factor, and the thermal factor $\epsilon_l n(\epsilon_l k_l^0)$ will be replaced by $-\epsilon_l \tilde{n}(\epsilon_l k_l^0)$.

IV. PROOFS OF THE THEOREMS FOR THE SCALAR CASE

In order to prove the theorems enunciated in Sec. II, we shall need to identify the vacuum ($T = 0$) limit of Gaudin's result (22) for the Matsubara D-function $D(p, E, T)$. The temperature dependence is solely contained in the thermal factors $n(\epsilon_l k_l^0)$. Following Ref. [5], we use the identity

$$n(k^0) = -\theta(-k^0) + \epsilon(k^0) n(|k^0|), \tag{27}$$

from which it follows that

$$\epsilon_l n(\epsilon_l k_l^0) = -\epsilon_l \theta(-\epsilon_l k_l^0) + \epsilon(k_l^0) n(|k_l^0|).$$

Writing the reduced spectral function defined in (22) in terms of the reflection operator $\hat{S}_E$ as

$$\bar{\rho}(k^0, E) = (1 - \hat{S}_E)\delta(k^0 - E), \tag{28}$$

and using the notation $\hat{S}_l \equiv \hat{S}_{E_l}$, we can perform the integrations over each of the variables k_l^0 ($l \in \bar{\mathcal{T}}$) in (21) as

$$\begin{aligned}\int_{-\infty}^{\infty} dk_l^0 \bar{\rho}(k_l^0, E_l)\epsilon_l n(\epsilon_l k_l^0) f(k_l^0) &= (1 - \hat{S}_l)\int_{-\infty}^{\infty} dk_l^0 \delta(k_l^0 - E_l)\epsilon_l n(\epsilon_l k_l^0) f(k_l^0)\\ &= (1 - \hat{S}_l)\epsilon_l n(\epsilon_l E_l) f(E_l)\\ &= (1 - \hat{S}_l)[-\epsilon_l \theta(-\epsilon_l E_l) + \epsilon(E_l) n(|E_l|)] f(E_l),\end{aligned}$$

where E_l is for the moment considered as an arbitrary real variable. We now apply the reflection operator $\hat{S}_l$ explicitly and then use the fact that E_l is actually a positive quantity. The vacuum part is given by

$$\begin{aligned}(1 - \hat{S}_l)[-\epsilon_l \theta(-\epsilon_l E_l) f(E_l)] &= -\epsilon_l \theta(-\epsilon_l E_l) f(E_l) + \epsilon_l \theta(\epsilon_l E_l) f(-E_l)\\ &= f(-\epsilon_l E_l),\end{aligned}$$

whereas the thermal part is given by

$$\begin{aligned}(1 - \hat{S}_l)[\epsilon(E_l) n(|E_l|) f(E_l)] &= \epsilon(E_l) n(|E_l|) f(E_l) - \epsilon(-E_l) n(|-E_l|) f(-E_l)\\ &= n_l f(E_l) + n_l f(-E_l)\\ &= n_l (1 + \hat{S}_l) f(E_l)\\ &= n_l (1 + \hat{S}_l) f(-\epsilon_l E_l),\end{aligned}$$

since $(1 + \hat{S}_l) f(E_l) = (1 + \hat{S}_l) f(-E_l)$, and where we have denoted $n_l \equiv n(E_l)$.

Therefore,

$$D(p, E, T) = \sum_{\mathcal{T}} \prod_{l \in \bar{\mathcal{T}}} [1 + n_l(1 + \hat{S}_l)] D_0^{\mathcal{T}}(p, E) \tag{29}$$

where $D_0^{\mathcal{T}}(p, E)$ is the contribution to the vacuum D-function associated with the tree $\mathcal{T}$:

$$\begin{aligned}D_0^{\mathcal{T}}(p, E) &= \prod_{j \in \mathcal{T}} \int_{-\infty}^{\infty} dp_j^0 \bar{\rho}(k_j^0, E_j) \frac{1}{k_j^0 - i\Omega_j^{\mathcal{T}}(p, i\epsilon_l E_l)}\\ &= \prod_{j \in \mathcal{T}} (1 - \hat{S}_j) \frac{1}{E_j - i\Omega_j^{\mathcal{T}}(p, i\epsilon_l E_l)}.\end{aligned} \tag{30}$$

Eqs. (29) and (30) are our starting points for the proofs of our three theorems. To start with, we notice that the function $D_0^{\mathcal{T}}(p, E)$ is annihilated by each of the operators $(1 + \hat{S}_j)$ with $j \in \mathcal{T}$, due to the identity

$$(1 + \hat{S}_E)(1 - \hat{S}_E) = 1 - \hat{S}_E^2 = 0. \tag{31}$$

This allows us to extend the index of the product $\prod_{l \in \bar{\mathcal{T}}}$ in (29) to all possible values, for any tree $\mathcal{T}$:

$$\prod_{l \in \bar{\mathcal{T}}} [1 + n_l(1 + \hat{S}_l)] D_0^{\mathcal{T}}(p, E) = \prod_{j=1}^{I} [1 + n_j(1 + \hat{S}_j)] D_0^{\mathcal{T}}(p, E).$$

In this form, the operator acting on $D_0^{\mathcal{T}}(p, E)$ becomes $\mathcal{T}$-independent, which allows us to move the sum over all trees in (29) through it:

$$\begin{aligned}D(p, E, T) &= \prod_{j=1}^{I} [1 + n_j(1 + \hat{S}_j)] \sum_{\mathcal{T}} D_0^{\mathcal{T}}(p, E)\\ &= \hat{O}_*(E, T) D_0(p, E),\end{aligned}$$

and this proves theorem 2.

Theorem 3 is readily proven by noticing that if C is a cut set of the graph Γ, then the operator

$$\hat{\mathcal{A}}(C) = \prod_{k \in C} (1 + \hat{S}_k)$$

will contain at least one factor $(1 + \hat{S}_{j_{\mathcal{T}}})$ with $j_{\mathcal{T}} \in \mathcal{T}$ for *every* tree $\mathcal{T}$. As we have already pointed out, this factor will annihilate the corresponding $D_0^{\mathcal{T}}(p, E)$, and therefore $D_0(p, E)$ will be annihilated by $\hat{\mathcal{A}}(C)$. This proves theorem 3.

Finally, as explained in Sec. II, theorem 1 follows directly from theorems 2 and 3.

V. EXTENSION TO FERMIONS

As it was shown in Sec. III, in the presence of fermionic lines the Matsubara D-function has essentially the same structure as in the scalar case. Now we use the identity

$$\tilde{n}(k^0) = \theta(-k^0) + \epsilon(k^0)\tilde{n}(|k^0|), \tag{32}$$

which has the same contents as the well-known $\tilde{n}(-E) = 1 - \tilde{n}(E)$. We note that both identities (27) and (32) can be written in terms of the generalized occupation number function defined by (7) as

$$N(k^0) = -\theta(-k^0) + \epsilon(k^0)N(|k^0|), \tag{33}$$

and therefore the manipulation of Sec. IV are valid in general, with n_i replaced by N_i.

The only other difference with the scalar case is the spectral function. The reduced spectral function for fermions is

$$\begin{aligned}\tilde{\tilde{\rho}}(k^0, \mathbf{k}, E_k) &= \frac{2E_k}{2\pi}\tilde{\rho}(k_0, \mathbf{k}) \\ &= (\not{k} + m)[\delta(k^0 - E_k) - \delta(k^0 + E_k)] \\ &= (1 - \hat{S}_{E_k})\delta(k^0 - E_k)(\not{k} + m). \end{aligned}\tag{34}$$

The propagator for a fermionic line i will be represented in the form

$$\begin{aligned}\tilde{\Delta}(\tilde{\omega}_i, \mathbf{k}_i) &= \frac{1}{2E_i}(1 - \hat{S}_i) \\ &\quad \times \int_{-\infty}^{\infty} dk_i^0 \delta(k_i^0 - E_i)(\not{k} + m)\frac{1}{k_i^0 - i\tilde{\omega}_i}, \end{aligned}\tag{35}$$

which has the necessary structure for the derivations presented in the previous section to hold. Hence, in the presence of fermionic lines the Matsubara D-function will still have the form (29), apart from the change $n \longrightarrow -\tilde{n}$ to account for the Fermi-Dirac statistics for fermions and some extra structure in the numerator of the right-hand side of (30). Therefore, the proofs presented at the end of the previous section are unaffected.

VI. CONCLUSIONS

In this paper we have proven rigorously, extending the approach presented in Ref. [5], a very general property of Feynman diagrams in the imaginary-time formalism for finite-temperature relativistic field theories (including scalar and fermionic fields), previously put forward as a conjecture in Ref. [6]. This property states that the full result of performing the Matsubara sum associated to any given Feynman graph can be obtained from its zero-temperature counterpart, by means of a simple linear operator, given by (10), whose form depends solely on the topology of the graph.

The thermal operator (10) has the important feature of being independent of the discrete Euclidean energies carried by the external lines of the graph. It follows from this that all issues related to analytic continuations of imaginary-time formalism Green's functions [8–11] can be completely settled at the zero-temperature level. The implications of this fact as well as the connections of the thermal operator representation with the real-time formalism are under investigation and will be presented elsewhere.

We have not studied the validity of the thermal operator representation in the case of gauge theories. It would be interesting to determine the classes of gauge fixings under which it holds, clarifying the role of ghost fields in the formalism.

ACKNOWLEDGMENTS

The diagrams presented in this paper were produced with JaxoDraw [12]. This work was supported by CONICYT, under Grant Fondecyt No. 1030363.

[1] For a review of the early literature on the subject see, for example, N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987). For more modern introductions to the subject see, for instance, J. Kapusta, *Finite-temperature field theory* (Cambridge University, Cambridge, England, 1993); M. Le Bellac, *Thermal field theory* (Cambridge University, Cambridge, England, 1996); A. Das, *Finite temperature field theory* (World Scientific, Singapore, 1997).

[2] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).

[3] R. Balian and C. de Dominicus, Nucl. Phys. **16**, 502 (1960); G. Baym and A. M. Sessler, Phys. Rev. **131**, 2345 (1963); I. E. Dzyaloshinski, Sov. Phys. JEPT **15**, 778 (1962); R. P. Pisarski, Nucl. Phys. **B309**, 476 (1988); E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990).

[4] F. Guerin, Phys. Rev. D **49**, 4182 (1994).

[5] J-P. Blaizot and U. Reinosa, hep-ph/0406109.

[6] O. Espinosa and E. Stockmeyer, Phys. Rev. D **69**, 065004 (2004).

[7] M. Gaudin, Nuovo Cimento **38**, 844 (1965).

[8] F. T. Brandt, A. Das, J. Frenkel, and A. J. da Silva, Phys. Rev. D **59**, 065004 (1999).

[9] R. Kobes, Phys. Rev. D **42**, 562 (1990); Phys. Rev. Lett. **67**, 1384 (1991); T. S. Evans, Phys. Lett. B **249**, 286 (1990); Phys. Lett. B **252**, 108 (1990); Nucl. Phys. **B374**, 340 (1992); P. Aurenche and T. Becherrawy, Nucl. Phys **B379**, 259 (1992); R. Baier and A. Niégawa,

Phys. Rev. D **49**, 4107 (1994); M. A. van Eijck, R. Kobes, and Ch. G. van Weert, Phys. Rev. D **50**, 4097 (1994).

[10] R. L. Kobes and G. W. Semenoff, Nucl. Phys **B260**, 714 (1985); **B272**, 329 (1986); R. Kobes, Phys. Rev. D **43**, 1269 (1991); P. V. Landshoff, Phys. Lett. B **386**, 291 (1996); P. F. Bedaque, A. Das, and S. Naik, Mod. Phys. Lett. A **12**, 2481 (1997); A. Niégawa, Phys. Rev. D **57**, 1379 (1998); M. E. Carrington, H. Defu, and R. Kobes, Phys. Rev. D **67**, 025021 (2003);

[11] H. A. Weldon Phys. Rev. D **28**, 2007 (1983).

[12] D. Binosi and L. Theußl, Comput. Phys. Commun. **161**, 76 (2004).