

Dark energy, wormholes, and the big ripV. Faraoni¹ and W. Israel^{2,3}¹*Physics Department, Bishop's University, Lennoxville, Quebec, Canada J1M 1Z7*²*Department of Physics and Astronomy, University of Victoria, P.O. Box 3055, Victoria, British Columbia, Canada V8W 3P6*³*Canadian Institute for Advanced Research Cosmology Program*

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The time evolution of a wormhole in a Friedmann universe approaching the big rip is studied. The wormhole is modeled by a thin spherical shell accreting the superquintessence fluid—two different models are presented. Contrary to recent claims that the wormhole overtakes the expansion of the universe and engulfs it before the big rip is reached, it is found that the wormhole becomes asymptotically comoving with the cosmic fluid and the future evolution of the universe is fully causal.

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I. INTRODUCTION

The 1998 discovery that the expansion of the universe is accelerated [1], confirmed by the analysis of the cosmic microwave background power spectrum [2,3], has led to postulate, as a possible explanation within the context of general relativity, the existence of a form of *dark energy* or *quintessence* with negative pressure $P < -\rho/3$ (where ρ is the dark energy density). The dark energy is dynamically irrelevant in the earlier stages of the evolution of the universe and starts dominating the cosmic dynamics only at recent times. At present the observational data favor an even more exotic dark energy with effective equation of state $P < -\rho$ violating the weak energy condition [4]. This violation is rather disturbing because it allows, in principle, for exotic solutions of general relativity such as wormholes and warp drives and the possibility of time travel associated with them. Another concern is that a universe dominated by dark energy with effective equation of state parameter $w \equiv P/\rho < -1$ may end its existence at a big rip singularity at which the scale factor and the energy density and pressure of quintessence diverge at a finite time in the future [5]. Dark energy with $w < -1$ is called *superquintessence* or *phantom energy*.

Recently, the evolution of a wormhole embedded in a Friedmann-Lemaître-Robertson-Walker (hereafter “FLRW”) universe approaching the big rip was studied by Gonzalez-Diaz [6], with the conclusion that the wormhole accreting superquintessence expands faster than the background FLRW universe and that the radius of the wormhole throat diverges before the big rip is reached—thus the wormhole engulfs the entire universe, which will reappear from the other wormhole throat. The resulting spacetime is not globally hyperbolic and is acausal with closed timelike curves threading the wormhole throat. Such bizarre scenarios are potentially of interest as constraints: if it can be established that phantom energy leads in principle to unacceptable consequences, this may be sufficient to rule out its existence.

The analysis of Ref. [6] is based on a qualitative estimate of the accretion rate of phantom energy onto the wormhole

and the rate of variation of the throat radius, which borrows from a recent study of a similar problem of accretion of dark energy onto black holes [7]. The purpose of this work is to present exact solution models of a wormhole immersed in a spatially flat FLRW universe and to compare the expansion of the wormhole throat with that of the cosmic fluid as the big rip is approached. Two different and rather general wormhole models are studied. In the first model it turns out that, in the simplifying approximation of stationary accretion advocated in Ref. [6], the wormhole is asymptotically comoving with the FLRW background as the big rip is approached—the size of the wormhole with respect to a comoving ruler does not increase and the universe cannot disappear within the wormhole. This conclusion holds also in the second wormhole model (presented in Sec. V), which is more general.

II. A THIN SHELL WORMHOLE EMBEDDED IN A SPATIALLY FLAT FLRW UNIVERSE

We study an exact solution of the Einstein equations describing a spherically symmetric thin shell wormhole embedded in a spatially flat FLRW universe described by the metric [8]

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (2.1)$$

in comoving polar coordinates (t, r, θ, φ) . The thin shell is located on a surface Σ of constant comoving radius. Two FLRW regions joining smoothly on the surface Σ constitute the regions “above” and “below” Σ . The assumption that the line element is given by Eq. (2.1) implies that the wormhole shell does not perturb the surrounding universe (see the discussion in Sec. IV). The equation of the shell Σ is

$$r = \frac{R(t)}{a(t)} \equiv e^{-\alpha(t)}, \quad (2.2)$$

where $R(t)$ is the comoving radius of the shell and the function $\alpha(t)$ is introduced for later convenience. The normal vector to the shell, obtained by differentiating Eq. (2.2), is

$$N^\alpha = (-\alpha_t e^{-\alpha}, a^{-2}, 0, 0) \quad (2.3)$$

where a subscript t denotes differentiation with respect to the comoving time t of the FLRW background. The norm squared of N^α is

$$g_{\alpha\beta} N^\alpha N^\beta = -\alpha_t^2 e^{-2\alpha} + \frac{1}{a^2} = \left(\frac{\beta}{a}\right)^2, \quad (2.4)$$

where

$$\beta \equiv \sqrt{1 - \alpha_t^2 a^2 e^{-2\alpha}} \equiv \sqrt{1 - v^2} \quad (2.5)$$

can be interpreted as the inverse of the Lorentz factor constructed with the velocity of the shell relative to the background

$$v \equiv -\alpha_t R. \quad (2.6)$$

By normalizing N^α one obtains the unit normal to the surface Σ

$$n^\mu = \left(\frac{-\alpha_t R}{\beta}, \frac{1}{a\beta}, 0, 0\right) \quad (2.7)$$

and the three-metric on the surface Σ is given by

$$\begin{aligned} ds^2|_\Sigma &= -\beta^2 dt^2 + R^2(t)(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= -d\tau^2 + R^2(\tau)d\Omega^2 \end{aligned} \quad (2.8)$$

in coordinates (t, θ, φ) on Σ , where τ is the proper time in the shell's frame (defined by $d\tau = \beta dt$) and $d\Omega^2$ is the line element on the unit two-sphere. By using the tetrad

$$\{e_{(t)}^\alpha, e_{(\theta)}^\alpha, e_{(\varphi)}^\alpha\} = \{\beta\delta_t^\alpha, \delta_\theta^\alpha, \delta_\varphi^\alpha\} \quad (2.9)$$

($a, b = t, \theta, \varphi$), one computes the extrinsic curvature of Σ

$$K_{\alpha\beta} = e_\alpha^{(a)} e_\beta^{(b)} \nabla_a n_b = e_\alpha^{(a)} e_\beta^{(b)} (\partial_a n_b - \Gamma_{ab}^c n_c). \quad (2.10)$$

Its components are found to be

$$K_t^t = -\frac{1}{\beta^2} \partial_t \left(\frac{\alpha_t R}{\beta}\right), \quad (2.11)$$

$$K_\theta^\theta = K_\varphi^\varphi = \frac{\beta}{R} - \frac{\alpha_t R_t}{\beta} \quad (2.12)$$

and its trace is

$$K = K_t^t + K_\theta^\theta + K_\varphi^\varphi = -\frac{1}{\beta^2} \partial_t \left(\frac{\alpha_t R}{\beta}\right) + \frac{2\beta}{R} - \frac{2\alpha_t R_t}{\beta}. \quad (2.13)$$

Since the shell has no interior and two FLRW regions match on Σ , the jumps of K^a_b and of K at Σ , which appear in the Einstein equations at the shell, are given by

$$[K^a_b] = 2K^a_b, \quad [K] = 2K \quad (2.14)$$

The Einstein equations at Σ are [9]

$$[K^a_b - K\delta^a_b] = -8\pi S^a_b, \quad (2.15)$$

where

$$S_{ab} = (\sigma + P_{(\Sigma)})u_a^{(\Sigma)}u_b^{(\Sigma)} + P_{(\Sigma)}g_{ab}|_\Sigma \quad (2.16)$$

is the stress-energy tensor of the (exotic) matter living on the shell and $u_{(\Sigma)}^\mu$ is the four-velocity of the shell. σ and $P_{(\Sigma)}$ are, respectively, the (surface) energy density and the pressure on the shell. The time-time component of the Einstein equations yields

$$\sigma = -\frac{1}{2\pi} \left(\frac{\beta}{R} - \frac{\alpha_t R_t}{\beta}\right), \quad (2.17)$$

while the $\theta\theta$ or the $\varphi\varphi$ component yields

$$\begin{aligned} P_{(\Sigma)} &= \frac{1}{4\pi} \left[-\frac{1}{\beta^2} \partial_t \left(\frac{\alpha_t R}{\beta}\right) + \frac{\beta}{R} - \frac{\alpha_t R_t}{\beta} \right] \\ &= -\frac{\sigma}{2} - \frac{\partial_t(u^\alpha n_\alpha)}{4\pi\beta^2}, \end{aligned} \quad (2.18)$$

where u^μ is the four-velocity field of the cosmic fluid. The effective equation of state of the exotic matter on the shell is given by

$$\sigma + 2P_{(\Sigma)} = -\frac{1}{2\pi\beta^2} \partial_t \left(\frac{\alpha_t R}{\beta}\right) = -\frac{1}{2\pi\beta^2} \partial_t(u^\alpha n_\alpha), \quad (2.19)$$

while the material energy residing on the shell is

$$M \equiv 4\pi R^2 \sigma = -2 \left(\beta R - \frac{\alpha_t R_t R^2}{\beta} \right). \quad (2.20)$$

The relative speed $v = -\alpha_t a e^{-\alpha}$ between the shell and the background fluid, $\beta = \sqrt{1 - v^2}$, and $u^\alpha n_\alpha = -v/\beta$, are constant when the matter on the shell satisfies the equation of state $P_{(\Sigma)} = -\sigma/2$.

The Einstein equation

$$\tilde{K}_{ab} S^{ab} = [T_{\alpha\beta} n^\alpha n^\beta], \quad (2.21)$$

where

$$\tilde{K}_{ab} = \frac{K_{ab}^{(+)} + K_{ab}^{(-)}}{2} = 0 \quad (2.22)$$

is the mean of the extrinsic curvature on both sides of the shell, provides the rather obvious matching condition for the energy density and pressure of the FLRW cosmic fluid on both sides of Σ :

$$\rho^{(+)} = \rho^{(-)}, \quad P^{(+)} = P^{(-)}. \quad (2.23)$$

It is sometimes convenient to use the parameter χ defined by

$$v = -\alpha_t R = -\dot{\alpha} \beta R = \tanh \chi, \quad (2.24)$$

or by $\cosh \chi = \beta^{-1}$, where an overdot denotes differentiation with respect to the comoving time of the shell τ and $\dot{f} = f_t/\beta$ for any differentiable function f .

The four-velocity of the shell

$$u_{(\Sigma)}^\alpha = \frac{dx^\alpha}{d\tau} = \frac{1}{\beta} \frac{dx^\alpha}{dt} = \frac{1}{\beta} (1, -\alpha_t e^{-\alpha}, 0, 0) \quad (2.25)$$

has the properties

$$u_{(\Sigma)}^\alpha n_\alpha = 0, \quad (2.26)$$

$$u_{(\Sigma)}^\alpha u_\alpha^{(\Sigma)} = -1. \quad (2.27)$$

The conservation equation (eq. (5) of Ref. [9]) projected along the four-velocity of the shell $u_{(\Sigma)}^\alpha$ yields

$$S_{a;b}{}^b u_{(\Sigma)}^a = -[u_{(\Sigma)}^\alpha T_\alpha^\beta n_\beta]. \quad (2.28)$$

The left-hand side $S_{a;b}{}^b u_{(\Sigma)}^a = (S_a{}^b u_{(\Sigma)}^a)_{;b} - S_a{}^b u_{(\Sigma);b}^a$ of Eq. (2.28) reduces to $-(\sigma u_{(\Sigma)}^b)_{;b} - P_{(\Sigma)} u_{(\Sigma);b}^b$, where $(\sigma u_{(\Sigma)}^b)_{;b} = \dot{M}/\mathcal{A}$, $\mathcal{A} \equiv 4\pi R^2(t)$ is the (proper) surface area of the shell, and $M \equiv \sigma \mathcal{A}$ is the material energy located on the shell. Similarly, it is found that $u_{(\Sigma);b}^b = \dot{\mathcal{A}}/\mathcal{A}$. The stress-energy tensor of the cosmic perfect fluid in the FLRW background is

$$T_{\alpha\beta} = (P + \rho)u_\alpha u_\beta + P g_{\alpha\beta}, \quad (2.29)$$

where u^α is the four-velocity of the FLRW comoving observers. Then

$$u_{(\Sigma)}^\alpha T_\beta^\alpha n^\beta = (P + \rho)(u_{(\Sigma)}^\alpha u_\alpha)(u^\beta n_\beta) + P u_{(\Sigma)}^\alpha n_\alpha. \quad (2.30)$$

By using the relations $u_{(\Sigma)}^\alpha u_\alpha = -\beta^{-1}$, $u^\beta n_\beta = \dot{\alpha}R$, and $u_{(\Sigma)}^\alpha n_\alpha = 0$ one obtains

$$u_{(\Sigma)}^\alpha T_\beta^\alpha n^\beta = -\frac{(P + \rho)}{\beta} \dot{\alpha}R. \quad (2.31)$$

Since the unit normal n^μ to Σ has opposite sign above and below Σ , the jump in this quantity is

$$[u_{(\Sigma)}^\alpha T_\beta^\alpha n^\beta] = -\frac{2}{\beta} (P + \rho) \dot{\alpha}R \quad (2.32)$$

and the conservation Eq. (2.28) yields, using $\dot{\alpha}R = -v/\beta$,

$$\dot{M} + P_{(\Sigma)} \dot{\mathcal{A}} = \frac{2}{\beta^2} (P + \rho) \mathcal{A} v. \quad (2.33)$$

This equation describes the rate of accretion of the cosmic fluid by the wormhole and can be interpreted as follows. The quantity $(P + \rho)v$ on the right-hand side is the flux density of the cosmic fluid crossing the shell Σ radially, the factor two arises because the outflow is from both faces of the shell, one factor β^{-1} comes from the relativistic mass dilation, while another factor β^{-1} comes from Lorentz contraction in the radial direction due to the relative motion of the shell and the FLRW background. Note that in a de Sitter background enjoying the equation of state $P = -\rho$ there is no accretion on the shell and static solutions with

both \dot{M} and \dot{R} vanishing (such as those considered in Ref. [10] or their generalizations) become possible.

If the strong energy condition is satisfied by the cosmic fluid and $P + \rho > 0$ then a wormhole shell expanding relative to the cosmic substratum ($v > 0$) accretes positive cosmic fluid energy while a shell contracting ($v < 0$) relative to the cosmic substratum experiences an energy outflow through Σ .

III. $v = \text{CONSTANT}$ SOLUTIONS

In the special case in which the relative radial velocity of the cosmic fluid and the wormhole shell is constant,

$$v = -\alpha_t R = \tanh\chi = v_0, \quad (3.1)$$

the equation of state of the exotic matter on the shell is $P_{(\Sigma)} = -\sigma/2$ and one can eliminate $R(t)$ between Eqs. (3.1) and (3.2) obtaining

$$-a(t)\partial_t(e^{-\alpha(t)}) + v_0 = 0. \quad (3.2)$$

We are interested in a FLRW universe approaching the big rip and therefore, as done in Ref. [6], we assume for simplicity that the equation of state of the cosmic fluid is constant, $P = w\rho$ with $w < -1$. This guarantees the occurrence of a big rip described by the form of the scale factor

$$a(t) = a_0(t_{rip} - t)^{2/3(w+1)}, \quad (3.3)$$

which diverges together with the energy density $\rho = \rho_0 a^{3(w+1)}$ and the pressure $P = w\rho$ as $t \rightarrow t_{rip}$. Under this assumption the constant v solution of Eq. (3.2) for the motion of the wormhole shell is given by

$$e^{-\alpha(t)} = C v_0 (t_{rip} - t)^{(3w+1)/3(w+1)} + e^{-\alpha_0}, \quad (3.4)$$

where C and α_0 are constants. The comoving radius of the shell is

$$R(t) = a(t)e^{-\alpha(t)} \simeq a(t)e^{-\alpha_0}. \quad (3.5)$$

It scales asymptotically like the scale factor $a(t)$ and hence the wormhole does not overtake the expansion of the universe, contrary to what is suggested in Ref. [6].

The fact that the shell cannot expand faster than the universe indefinitely can also be seen by differentiating Eq. (2.2) with respect to t , which yields

$$\frac{R_t}{R} = H + \frac{v}{R}, \quad (3.6)$$

where $H \equiv a_t/a$ is the Hubble parameter of the FLRW universe. When v is constant and the radius of the shell expands to infinity the term v/R in Eq. (3.6) becomes negligible and the expansion rate of the shell (with respect to comoving time) coincides with the expansion rate of the background universe, even if the shell starts out expanding at a faster rate than the background. The final state is indistinguishable from one with $v = 0$ and it is the same

irrespective of the initial conditions. (Strictly speaking, this argument does not assume that the big rip is approached, but only that the universe expands forever and $v = \text{const.}$, which in turn implies that the wormhole shell expands to infinity, $R(t) \rightarrow +\infty$).

Perhaps a more physical way of looking at this aspect is the following: if the shell were to expand faster than the cosmological background in the approach to the big rip, and given that $H \rightarrow +\infty$ near the big rip, the relative speed v between the shell and the cosmic fluid would increase its magnitude without bound. This would contradict the fact that $|v|$ must be bound by unity, or else the wormhole shell becomes tachyonic. Hence the asymptote $|v| \simeq 1$ must be approached before the big rip is reached, with $\partial_t v \rightarrow 0$. A comparison of these results with those of Ref. [6] is given in Sec. VI.

IV. GRAVITATING MASS OF THE WORMHOLE

We briefly discuss our assumption that the wormhole considered here does not perturb the FLRW surroundings. This assumption must correspond to a zero gravitational mass for the wormhole. In the static limit $a \equiv 1$ the mass-energy on the shell is the only mass-energy in the entire spacetime. Because the latter is asymptotically flat, the total gravitational mass is given by the Tolman mass [11]; for a shell with surface stress-energy tensor S_{ab} in an asymptotically flat spacetime the Tolman mass is given by the expression [12]

$$\begin{aligned} M_T &= \int_{\Sigma} d\Sigma \sqrt{-g_{00}} (-S_0^0 + S_2^2 + S_3^3) \\ &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta R^2 \beta \sin\theta (\sigma + 2P_{(\Sigma)}) \\ &= -\frac{2R^2}{\beta} \partial_t \left(\frac{\alpha_t R}{\beta} \right). \end{aligned} \quad (4.1)$$

If $\alpha_t R$ is constant, corresponding to constant v and β , the Tolman mass M_T vanishes identically and the wormhole does not affect the Minkowski background both above and below Σ . Note that not only the material mass $4\pi R^2 \sigma$, but also the pressure $P_{(\Sigma)}$ contributes to the Tolman mass, and that the pressure is adjusted to compensate the contribution of the material mass in such a way that the Minkowski background is not altered.

For a wormhole embedded in a FLRW space the Tolman mass is replaced by the Hawking quasilocal mass [13,14]: since the metric (2.1) outside the wormhole shell is exactly a FLRW one, the quasilocal mass reduces to that of a FLRW space, $E = 4\pi r^3 \rho/3$, with no contribution from the shell. One concludes that the gravitational mass of the wormhole is zero. If instead a spherical black hole is embedded in a FLRW background, as described, e.g., by the McVittie solution [15], the quasilocal mass coincides with the black hole mass times the scale factor $a(t)$ [16]. As an astrophysical object, a zero-mass wormhole construct

located in a galaxy would not perturb the orbits of stars that do not fall directly into its throat, and it would not cause gravitational lensing (lensing by negative mass wormholes is studied in Refs. [17]).

V. MORE GENERAL WORMHOLES

The wormhole model considered in Secs. III and IV suffers from the intrinsic limitation that the wormhole shell is adjusted so that it does not perturb the surrounding cosmological background. It is interesting to ask whether removing this assumption has an effect on the results of Sec. III about the relative motion of the shell and the background fluid as the big rip singularity is approached. To this purpose, consider the McVittie metric [15]

$$ds^2 = -\left[\frac{B(t, r)}{A(t, r)} \right]^2 dt^2 + a^2(t) A^4(t, r) (dr^2 + r^2 d\Omega^2) \quad (5.1)$$

in isotropic coordinates (t, r, θ, φ) , where

$$A(t, r) \equiv 1 + \frac{m(t)}{2r}, \quad B(t, r) \equiv 1 - \frac{m(t)}{2r}. \quad (5.2)$$

The modification of the FLRW metric introduced by a nonvanishing function $m(t)$ is caused by a spherical wormhole shell Σ located at the radius

$$r = r_{\Sigma}(t) = \frac{R(t)}{a(t)A^2(t, r_{\Sigma})}. \quad (5.3)$$

The three-dimensional metric on Σ is given by

$$ds^2|_{\Sigma} = -\left(\frac{B}{A} \operatorname{sech}\chi \right)^2 dt^2 + R^2(t) d\Omega^2 \quad (5.4)$$

in coordinates (t, θ, φ) on Σ and with the function χ defined by

$$\tanh\chi(t) \equiv a(t) \frac{A^3(t, r_{\Sigma})}{B(t, r_{\Sigma})} \frac{dr_{\Sigma}}{dt}. \quad (5.5)$$

The unit normal to Σ is

$$n_{\mu} = \lambda \left(-\frac{dr_{\Sigma}}{dt}, 1, 0, 0 \right), \quad \lambda = aA^2 \cosh\chi, \quad (5.6)$$

the four-velocity of the shell is

$$u_{(\Sigma)}^{\mu} = \frac{A}{B} \cosh\chi \left(1, \frac{dr_{\Sigma}}{dt}, 0, 0 \right), \quad (5.7)$$

and the four-velocity of the cosmic fluid is

$$u^{\mu} = \left(\frac{A}{B}, 0, 0, 0 \right). \quad (5.8)$$

The projection of the shell four-velocity on the cosmic fluid four-velocity is

$$u_{(\Sigma)\alpha}^{\alpha} u_{\alpha} = -\cosh\chi = -\frac{1}{\sqrt{1-v^2}}, \quad (5.9)$$

where the three-dimensional velocity of the shell relative to the cosmic fluid is defined as

$$v \equiv \tanh\chi(t). \quad (5.10)$$

By differentiating (R/a) and using the expressions (5.2) of $A(t, r)$ and $B(t, r)$ one obtains an equation regulating the dynamics of the wormhole shell:

$$\frac{dr_\Sigma}{dt} = \frac{1}{AB} \left[\frac{d}{dt} \left(\frac{R}{a} \right) - A \frac{dm}{dt} \right]. \quad (5.11)$$

The extrinsic curvature of the shell Σ given by Eq. (2.10) has the only nonzero components

$$K^t_t = \lambda \left[\frac{A^2}{B^2} \left(\frac{d^2 r_\Sigma}{dt^2} + \frac{1}{ABr_\Sigma} \frac{dr_\Sigma}{dt} \frac{dm}{dt} \right) + \frac{m}{a^2 A^5 B r_\Sigma^2} \right], \quad (5.12)$$

$$K^\theta_\theta = K^\varphi_\varphi = \lambda \left[\frac{B}{a^2 A^5 r_\Sigma} + \frac{A^2}{B^2} \frac{dr_\Sigma}{dt} \left(H + \frac{1}{Ar_\Sigma} \frac{dm}{dt} \right) \right] \quad (5.13)$$

where, again, $H \equiv a_t/a$. The Einstein equations (2.15) at Σ then yield the surface density σ and the pressure $P_{(\Sigma)}$ of the shell material

$$\begin{aligned} \sigma &= -\frac{K^\theta_\theta}{2\pi} \\ &= -\frac{1}{2\pi} \left[\frac{B}{aA^3 r_\Sigma} \cosh\chi + \frac{A}{B} \left(H + \frac{1}{Ar_\Sigma} \frac{dm}{dt} \right) \sinh\chi \right], \end{aligned} \quad (5.14)$$

$$\begin{aligned} \sigma + 2P_{(\Sigma)} &= \frac{K^t_t}{2\pi} \\ &= \frac{\lambda}{2\pi} \left[\frac{A^2}{B^2} \left(\frac{d^2 r_\Sigma}{dt^2} + \frac{1}{ABr_\Sigma} \frac{dr_\Sigma}{dt} \frac{dm}{dt} \right) + \frac{m}{a^2 r_\Sigma^2 A^5 B} \right] \end{aligned} \quad (5.15)$$

and the mass of exotic matter on the wormhole shell can be written, using Eq. (5.3),

$$\begin{aligned} M &\equiv 4\pi R^2(t)\sigma \\ &= -2R \left[\frac{B}{A} \cosh\chi + \frac{AR}{B} \left(H + \frac{1}{Ar_\Sigma} \frac{dm}{dt} \right) \sinh\chi \right]. \end{aligned} \quad (5.16)$$

The cosmic imperfect fluid is described by the stress-energy tensor

$$T_{\alpha\beta} = (P + \rho)u_\alpha u_\beta + P g_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha, \quad (5.17)$$

where the purely spatial vector q^μ (with $q^\mu u_\mu = 0$) is the radial energy flux density. The covariant conservation equation of mass-energy for the shell, Eq. (2.28), yields

$$\begin{aligned} \frac{1}{\mathcal{A}} \left(\frac{dM}{d\tau_\Sigma} + P_{(\Sigma)} \frac{d\mathcal{A}}{d\tau_\Sigma} \right) &= 2\lambda \cosh\chi \left\{ (P + \rho) \frac{A}{B} \frac{dr_\Sigma}{dt} \right. \\ &\quad \left. - q^r \left[\frac{A^2}{B^2} \left(\frac{dr_\Sigma}{dt} \right)^2 a^2 A^4 + 1 \right] \right\}, \end{aligned} \quad (5.18)$$

where $\mathcal{A} \equiv 4\pi R^2(t)$ and τ_Σ is the proper time of the shell defined by the three-metric (5.4) as

$$d\tau_\Sigma = \frac{B}{A} \operatorname{sech}\chi dt. \quad (5.19)$$

The Einstein equation $G^1_0 = 8\pi T^1_0 = 8\pi q^r u_t$ determines the radial flux density q^r as

$$\frac{2m}{ABr_\Sigma^2} \left(H + \frac{1}{m} \frac{dm}{dt} \right) = 8\pi q^r u_t, \quad (5.20)$$

where $r_S(t, r) \equiv a(t)A^2(t, r)r$. Further use of Eqs. (5.6) and (5.11) yields

$$\frac{dM}{dt} + P_{(\Sigma)} \frac{d\mathcal{A}}{dt} = \frac{2}{\beta} \left[\frac{B}{A} (P + \rho) \mathcal{A} v + \frac{A}{B} \frac{(1 + v^2)}{\beta} \frac{d(ma)}{dt} \right], \quad (5.21)$$

or

$$\frac{dM}{d\tau_\Sigma} + P_{(\Sigma)} \frac{d\mathcal{A}}{d\tau_\Sigma} = \frac{2}{\beta} \left[(P + \rho) \mathcal{A} \frac{v}{\beta} + \frac{A}{B} (1 + v^2) \frac{d(ma)}{d\tau_\Sigma} \right], \quad (5.22)$$

in terms of τ_Σ , where $\beta = \sqrt{1 - v^2}$ again. Equation (5.22) reduces to Eq. (2.33) when $m(t) \equiv 0$. The second term on the right-hand side of Eq. (5.21) is present even when $v = 0$ and describes a contribution to accretion onto the shell due to the radial energy flux of cosmic fluid—this term vanishes if

$$\frac{1}{m} \frac{dm}{dt} = -H, \quad (5.23)$$

equivalent to $m(t) = \text{const.}/a(t)$, or $q^r = 0$ and $G^1_0 = 0$. The quantity $m(t)a(t)$ appearing in the second term on the right-hand side of Eq. (5.21) coincides with the Hawking quasiloal mass [18]. The ‘‘Schwarzschild mass function’’ $m_S(t, r)$ instead is defined by

$$1 - \frac{2m_S(t, r)}{r_S(t, r)} \equiv g^{\alpha\beta}(\nabla_\alpha r_S)(\nabla_\beta r_S). \quad (5.24)$$

The definitions of A , B , and r_S and the time-time component of the Einstein equations

$$3\left(\frac{A}{B}\right)^2\left(H + \frac{1}{rA} \frac{dm}{dt}\right)^2 = 8\pi\rho \quad (5.25)$$

yield

$$m_S(t, r) = m(t)a(t) + \frac{4\pi}{3}r_S^3\rho, \quad (5.26)$$

which reduces to the usual Schwarzschild mass in the absence of cosmic fluid.

We are left with the problem of solving for the dynamics of the wormhole shell, which is determined by Eq. (5.11). The latter can be written as

$$\frac{d}{dt}\left(\frac{R}{a}\right) - A\frac{dm}{dt} = \frac{v}{a}\left(\frac{B}{A}\right)^2. \quad (5.27)$$

During the approach to a big rip singularity in which the scale factor $a(t)$ has the form (3.3), the metric component $g_{00} = -(B/A)^2$ stays finite even if $m(t)$ diverges. It seems reasonable to require that both A and B be finite and that the divergence be contained in $a(t)$, keeping in mind that the mass of the wormhole shell (which is allowed to diverge) is not $m(t)$, but is obtained by integrating Eq. (5.21). On the other hand, if v is finite ($v \leq 1$) Eq. (5.27) reduces to the asymptotic equation

$$\frac{d}{dt}\left(\frac{R}{a}\right) - A\frac{dm}{dt} = 0 \quad (5.28)$$

By using the implicit definition (5.3) of r_Σ one obtains $B(t, r_\Sigma)dr_\Sigma/dt = 0$, which has the constant solution $r_\Sigma = C$, or $R(t) = a(t)\{C + m(t) + [m^2(t)]/4C\}$. Again, approaching the big rip, the wormhole shell becomes comoving with the cosmic substratum.

VI. DISCUSSION AND CONCLUSIONS

We are now ready to discuss the difference between our results and those of Gonzalez-Diaz [6]. Gonzalez-Diaz begins by considering a static wormhole solution appearing in the appendix of Ref. [19] (Eqs. (A.28) of [19]) and consisting of a spherical thin shell of exotic matter with throat radius b_0 and mass μ related by [20]

$$\mu = -\frac{\pi b_0}{2}. \quad (6.1)$$

Gonzalez-Diaz extrapolates Eq. (6.1) to a time-dependent wormhole embedded in a FLRW universe and proceeds to estimate the accretion rate on such a wormhole by adopting the formula derived for stationary accretion onto a black

hole embedded in a FLRW universe [7]

$$\dot{\mu} = 4\pi D\mu^2(P + \rho), \quad (6.2)$$

where D is a constant. By combining Eqs. (6.1) and (6.2), Gonzalez-Diaz obtains an equation for the rate of change of the wormhole throat \dot{b}_0 and proceeds to solve it in a FLRW universe approaching the big rip, concluding that the wormhole shell of radius $b_0(t)$ expands faster than the universe and ends up engulfing it and destroying global hyperbolicity. Although treating the wormhole as a black hole with negative mass seems reasonable, unfortunately Eq. (6.1) does not hold for a time-dependent wormhole embedded in a FLRW universe—cf. Equation (2.17) or Eq. (2.20) for the first wormhole model, or Eq. (5.16) for the second wormhole model presented. The time evolution of the wormhole shell cannot be guessed *a priori* but needs to be derived from detailed models like the ones presented here.

The first wormhole model studied in this paper is appropriate for elucidating the physical situation considered in Ref. [6]. In fact, by comparing Eq. (2) of Ref. [6] and our Eq. (2.33), one sees that the regime of stationary accretion onto the wormhole considered in Ref. [6] corresponds to the situation $v = \text{constant}$ in our formalism, for which the dynamics of the shell has been explicitly solved (the static limit of such a solution would correspond to $a \equiv 1$ and $\dot{M} = -8\pi P_{(\Sigma)}R\dot{R}$). In this case the wormhole shell ends up expanding at the same rate as the universe in which it is embedded. However, this wormhole model has one limitation, namely, the assumption that the wormhole does not perturb the surrounding FLRW universe. More general exact solutions describing a wormhole embedded in, and modifying, the surrounding cosmological background are presented in Sec. V. This more general class of exact solutions contains three free functions $a(t)$, $m(t)$, and $R(t)$. By imposing the form (3.3) of the scale factor $a(t)$ appropriate to the description of a big rip and by noting that the metric component $g_{00} = -(B/A)^2$ is bounded even if $m(t)$ diverges, one deduces again that the wormhole shell becomes comoving with the cosmological background as the big rip is approached.

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Note added in proof.—After this work was completed we learned about Ref. [22] which contains a spherically symmetric wormhole solution joining two FLRW universes approaching the big rip. Although less general than the solutions presented here, also this wormhole is comoving with the background and constitutes another counterexample to the acausal evolution suggested in Ref. [6].

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