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Semiclassical limit for Dirac particles interacting with a gravitational field

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The behavior of a spin-$1/2$ particle in a weak static gravitational field is considered. The Dirac Hamiltonian is diagonalized by the Foldy-Wouthuysen transformation providing also the simple form for the momentum and spin polarization operators. The operator equations of momentum and spin motion are derived for a first time. Their semiclassical limit is analyzed. The dipole spin-gravity coupling in the previously found (another) Hamiltonian does not lead to any observable effects. The general agreement between the quantum and classical approaches is established, contrary to several recent claims. The expression for the gravitational Stern-Gerlach force is derived. The helicity evolution in the gravitational field and corresponding accelerated frame coincides, being the manifestation of the equivalence principle.

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I. INTRODUCTION

Interaction of elementary particles with gravitational fields poses an interesting problem with important astrophysical applications. One of the approaches to this problem is provided by a corresponding Dirac equation in an external gravitational field. It was recently solved [1] using the exact diagonalization by an appropriate unitary transformation for the wide class of static gravitational fields. However, the presence of a dipole spin-gravity coupling in the final results of Refs. [1,2] is controversial [3,4]. For accelerated frames, there is not any similar coupling (see [5–7]).

There is also a related problem of disagreement between the classical formula for the angle of particle deflection by a gravitational field and the corresponding expression for Dirac particles claimed recently by another author [8].

In the present article we resolve these contradictions. The diagonalization of the Dirac equation is still insufficient to get the semiclassical equations of spin motion formerly obtained in [9,10]. The problem is that the derivation of the equations of motion requires also the knowledge of respective dynamical operators, in particular, that of momentum and spin. We investigate this problem and show that these operators have a rather complicated form in the representation used in [1,2], which is because that representation, although diagonal, does not possess all the properties of the Foldy-Wouthuysen (FW) one. As a result, the dipole spin-gravity coupling appearing in [1,2] does not lead to new observable effects.

To bypass this difficulty, we construct the "standard" FW representation where the dynamical operators take the simple form. We derive (for the first time, up to our knowledge) the operator equations of momentum and spin motion in a weak spherically symmetric gravitational field and uniformly accelerated frame. We study the semiclassical limit of these equations to get the momentum, spin polarization, and helicity evolution. The results fully agree with the classical gravity (so that the disagreement found in [8] is not confirmed) and contain quantum corrections. In particular, the expression for the gravitational Stern-Gerlach (SG) force acting on relativistic particles is found.

II. DIRAC EQUATION FOR PARTICLES IN A STATIC GRAVITATIONAL FIELD

An interaction of a spin-$1/2$ particle with a gravitational field is described by the covariant Dirac equation:

$$(i\gamma^\alpha D_\alpha - m)\psi = 0, \qquad \alpha = 0, 1, 2, 3, \tag{1}$$

where γ^α are the Dirac matrices. The system of units $\hbar = c = 1$ is used. The spinor covariant derivatives are defined by

$$D_\alpha = h^i_\alpha D_i, \qquad D_i = \partial_i + \frac{i}{4}\sigma_{\alpha\beta}\Gamma_i^{\alpha\beta}, \tag{2}$$

where h^i_α and $\Gamma_i^{\alpha\beta} = -\Gamma_i^{\beta\alpha}$ are the coframe and Lorentz connection coefficients, $\sigma^{\alpha\beta} = i(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)/2$ (see Refs. [1,2] and references therein). Following these references we limit ourselves to the case of the static spacetime

$$ds^2 = V^2(dx^0)^2 - W^2(d\boldsymbol{r}\cdot d\boldsymbol{r}). \tag{3}$$

Here V, W are arbitrary functions of $\boldsymbol{r}$. Particular cases belonging to this family are pointed out in [1,2] and include the following:

(i) the flat Minkowski spacetime in an accelerated frame

$$V = 1 + \boldsymbol{a}\cdot\boldsymbol{r}, \qquad W = 1, \tag{4}$$

and (ii) Schwarzschild spacetime in the isotropic coordinates

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$$V = \left(1 - \frac{GM}{2r}\right)\left(1 + \frac{GM}{2r}\right)^{-1}, \qquad W = \left(1 + \frac{GM}{2r}\right)^2 \tag{5}$$

with $r = |\boldsymbol{r}|$. For metric (3), the Dirac equation can be brought to the Hamilton form [1,2]

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}\psi, \qquad \mathcal{H} = \beta m V + \frac{1}{2}\{\mathcal{F}, \boldsymbol{\alpha}\cdot\boldsymbol{p}\}, \tag{6}$$

where $\mathcal{F} = V/W$ and $\{\ldots,\ldots\}$ denotes the anticommutator. This equation is the starting point of our analysis.

III. CONNECTION BETWEEN THE FOLDY-WOUTHUYSEN AND ERIKSEN-KORLSRUD REPRESENTATIONS

The FW transformation [11] provides the correct physical interpretation of Dirac Hamiltonians. The important advantage of the FW representation is the simple form [12] of polarization operator $\boldsymbol{O}_{FW}$ being equal to the matrix

$$\boldsymbol{O}_{FW} = \Pi = \beta\Sigma. \tag{7}$$

In principle, this form of polarization operator may be considered as a definition of the FW representation.

In Refs. [1,2], an exact block diagonalization of Hamiltonian (6) by the Eriksen-Korlsrud (EK) method [13] has been performed. However, a block diagonalization of the Hamiltonian may be nonequivalent to the FW transformation. There exists an infinite set of representations where all the operators are block diagonal. Therefore, the equivalence of any representation to the FW one should be verified. For example, the transformation performed in Ref. [14] for particles in a uniform magnetic field has led to a block-diagonal Hamiltonian. However, this Hamiltonian differs from the corresponding Hamiltonian in the FW representation [15].

It is easy to prove the FW and EK representations are not equivalent even for free particles. The unitary operator of transformation from the Dirac representation to the EK one is given by [1,2,13]

$$U_{D\to EK} = \frac{1}{2}(1 + \beta J)(1 + J\Lambda), \qquad J = i\gamma_5\beta,$$
$$\Lambda = \frac{\mathcal{H}}{\sqrt{\mathcal{H}^2}} = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m}{\epsilon}, \qquad \epsilon = \sqrt{m^2 + \boldsymbol{p}^2}. \tag{8}$$

The unitary operator of transformation from the Dirac representation to the FW one is equal to [11,16]

$$U_{D\to FW} = \frac{\epsilon + m + \beta\boldsymbol{\alpha}\cdot\boldsymbol{p}}{\sqrt{2\epsilon(\epsilon + m)}}.$$

Therefore, the operator providing the transformation from the FW representation to the EK one is

$$U_{FW\to EK} = U_{EK}U_{FW}^{-1} = \frac{\epsilon + m + i(\Pi\cdot\boldsymbol{p})}{\sqrt{2\epsilon(\epsilon + m)}}. \tag{9}$$

For free particles, this operator does not change the form of the Hamiltonian. However, operator (9) is not equal to the unit matrix and therefore changes the wave eigenfunctions. Consequently, the FW and EK representations are nonequivalent.

It is easy to see that the polarization operator in the EK representation is very different from the corresponding operator in the FW representation even for free particles:

$$\boldsymbol{O}_{EK} = U_{FW\to EK}\Pi U_{FW\to EK}^{-1} = \Pi + \frac{\boldsymbol{p}\times\Sigma}{\epsilon} + \frac{\boldsymbol{p}\times(\boldsymbol{p}\times\Pi)}{\epsilon(\epsilon + m)}. \tag{10}$$

For particles in external fields, this circumstance brings a difference between Hamiltonians, especially for the terms proportional to the polarization operator. Thus, the block diagonalization of the Hamiltonian needs to be fulfilled carefully.

It is important that the forms of the position operator in two representations also differ. In the FW representation, this operator is just the radius vector $\boldsymbol{r}$ [17,18]. In the EK representation, it is given by

$$\boldsymbol{r}_{EK} = U_{FW\to EK}\boldsymbol{r}U_{FW\to EK}^{-1} = \boldsymbol{r} + \frac{\boldsymbol{p}\times\Sigma}{2\epsilon(\epsilon + m)} + \frac{\Pi}{2\epsilon} - \frac{\boldsymbol{p}\cdot(\boldsymbol{p}\cdot\Pi)}{2\epsilon^2(\epsilon + m)}.$$

Thus, the EK transformation does not lead to the FW representation.

IV. FOLDY-WOUTHUYSEN TRANSFORMATION FOR SPIN-1/2 PARTICLES IN A STATIC GRAVITATIONAL FIELD

Let us transform Hamiltonian (6) to the FW representation. For this purpose, we apply the method of relativistic FW transformation elaborated in Ref. [16]. The validity of this method is confirmed by the consistency of results obtained by different methods for the electromagnetic interaction of particles (see Ref. [16]). So, we expect it to be valid and provide the simple expression for dynamical operators also for the gravitational field.

Unfortunately, we are unable to perform the exact FW transformation. Therefore, we use the weak-field approximation which makes it possible to obtain the FW Hamiltonian as a power series in parameters of an external field. In our case this requires that $|V - 1|, |W - 1| \ll 1$.

Hamilton operator (6) can be written in the form

$$\mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \qquad \beta\mathcal{E} = \mathcal{E}\beta, \qquad \beta\mathcal{O} = -\mathcal{O}\beta,$$

where

$$\mathcal{E} = \beta m(V - 1), \qquad \mathcal{O} = \frac{1}{2}\{\mathcal{F}, \boldsymbol{\alpha}\cdot\boldsymbol{p}\}$$

mean terms commuting and anticommuting with the matrix β, respectively.

Other notations $\boldsymbol{\phi} = \nabla V$, $\boldsymbol{f} = \nabla \mathcal{F}$ follow Refs. [1,2].

Let us perform the FW transformation for relativistic particles with an allowance for first-order terms in the metric tensor and its derivatives up to the second order.

After the first transformation with the operator (see Ref. [16])

$$U = \frac{\epsilon' + m + \beta\mathcal{O}}{\sqrt{2\epsilon'(\epsilon' + m)}}, \qquad \epsilon' = \sqrt{m^2 + \mathcal{O}^2},$$

the Hamilton operator takes the form:

$$\mathcal{H} = \beta\epsilon' + \mathcal{E}' + \mathcal{O}', \qquad \beta\mathcal{E}' = \mathcal{E}'\beta,$$
$$\beta\mathcal{O}' = -\mathcal{O}'\beta,$$

where

$$\epsilon' = \sqrt{m^2 + \mathcal{O}^2} = \sqrt{m^2 + \boldsymbol{p}^2 + \{\boldsymbol{p}^2, \mathcal{F} - 1\} + \frac{1}{2}[\Sigma \cdot (\boldsymbol{f} \times \boldsymbol{p}) - \Sigma \cdot (\boldsymbol{p} \times \boldsymbol{f}) + \nabla \cdot \boldsymbol{f}]},$$
$$\mathcal{E}' = \frac{\beta}{2}\left\{\frac{m^2}{\epsilon}, V - 1\right\} - \frac{\beta m}{4\epsilon(\epsilon + m)}[\Sigma \cdot (\boldsymbol{\phi} \times \boldsymbol{p}) - \Sigma \cdot (\boldsymbol{p} \times \boldsymbol{\phi}) + \nabla \cdot \boldsymbol{\phi}]$$
$$+ \frac{\beta}{8} \cdot \frac{(2\epsilon^3 + 2\epsilon^2 m + 2\epsilon m^2 + m^3)m}{\epsilon^5(\epsilon + m)^2}(\boldsymbol{p} \cdot \nabla)(\boldsymbol{p} \cdot \boldsymbol{\phi}),$$
$$\epsilon = \sqrt{m^2 + \boldsymbol{p}^2}.$$

We neglect the noncommutativity of operators in small terms proportional to derivatives of the metric tensor. The calculation of $\mathcal{O}'$ is unnecessary because its contribution to the final FW Hamiltonian is of order of $(V - 1)^2$.

The quantity ϵ' can be represented as

$$\epsilon' = \epsilon + \frac{1}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon}, \mathcal{F} - 1\right\} + T + \frac{1}{4\epsilon}[\Sigma \cdot (\boldsymbol{f} \times \boldsymbol{p}) - \Sigma \cdot (\boldsymbol{p} \times \boldsymbol{f}) + \nabla \cdot \boldsymbol{f}]. \qquad (11)$$

To determine the operator T, it is necessary to square both parts of Eq. (11). As a result of the calculation,

$$T = -\frac{\epsilon^2 + m^2}{4\epsilon^5}(\boldsymbol{p} \cdot \nabla)(\boldsymbol{p} \cdot \boldsymbol{f}),$$

and the final expression for the FW Hamiltonian takes the form

$$\mathcal{H}_{FW} = \beta\epsilon + \frac{\beta}{2}\left\{\frac{m^2}{\epsilon}, V - 1\right\} + \frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon}, \mathcal{F} - 1\right\} - \frac{\beta m}{4\epsilon(\epsilon + m)}[\Sigma \cdot (\boldsymbol{\phi} \times \boldsymbol{p}) - \Sigma \cdot (\boldsymbol{p} \times \boldsymbol{\phi}) + \nabla \cdot \boldsymbol{\phi}] + \frac{\beta m(2\epsilon^3 + 2\epsilon^2 m + 2\epsilon m^2 + m^3)}{8\epsilon^5(\epsilon + m)^2} \times (\boldsymbol{p} \cdot \nabla)(\boldsymbol{p} \cdot \boldsymbol{\phi}) + \frac{\beta}{4\epsilon}[\Sigma \cdot (\boldsymbol{f} \times \boldsymbol{p}) - \Sigma \cdot (\boldsymbol{p} \times \boldsymbol{f}) + \nabla \cdot \boldsymbol{f}] - \frac{\beta(\epsilon^2 + m^2)}{4\epsilon^5}(\boldsymbol{p} \cdot \nabla)(\boldsymbol{p} \cdot \boldsymbol{f}). \qquad (12)$$

It is obvious that this expression differs from the corresponding one derived in Refs. [1,2]. To perform a more detailed analysis, we can rewrite Eq. (13) from Ref. [2] in the weak-field approximation [19]:

$$\mathcal{H}_{EK} = \beta\left(mV + \frac{\boldsymbol{p}^2}{2m}\right) - \frac{\beta}{4m}\{\boldsymbol{p}^2, V - 1\} + \frac{\beta}{2m}\{\boldsymbol{p}^2, \mathcal{F} - 1\} + \frac{\beta}{4m}[2\Sigma \cdot (\boldsymbol{f} \times \boldsymbol{p}) + \nabla \cdot \boldsymbol{f}] + \frac{1}{2}(\Sigma \cdot \boldsymbol{\phi}), \qquad (13)$$

and compare it with Eq. (12) of the present work in the nonrelativistic approximation:

$$\mathcal{H}_{FW} = \beta\left(mV + \frac{\boldsymbol{p}^2}{2m}\right) - \frac{\beta}{4m}\{\boldsymbol{p}^2, V - 1\} + \frac{\beta}{2m}\{\boldsymbol{p}^2, \mathcal{F} - 1\} + \frac{\beta}{4m}[2\Sigma \cdot (\boldsymbol{f} \times \boldsymbol{p}) + \nabla \cdot \boldsymbol{f}] - \frac{\beta}{8m}[2\Sigma \cdot (\boldsymbol{\phi} \times \boldsymbol{p}) + \nabla \cdot \boldsymbol{\phi}]. \qquad (14)$$

FW Hamiltonians (12) and (14), contrary to the EK Hamiltonian (13), do not contain the term $(\Sigma \cdot \boldsymbol{\phi})/2$ but contain additional terms proportional to derivatives of V. These additional terms describe both the spin-orbit and contact interactions. To check the compatibility with [1,2], the semirelativistic transformation (with an accuracy up to v/c) of Hamiltonian (13) to the FW representation can be performed. With this accuracy, transformation operator (9) takes the form

$$U_{EK\to FW} = U_{FW\to EK}^{-1} = 1 - \frac{\boldsymbol{p}^2}{8m^2} - \frac{i(\Pi \cdot \boldsymbol{p})}{2m}. \qquad (15)$$

As a result, Hamiltonian (13) is transformed by operator (15) to form (14).

This transformation shows that the calculation fulfilled in [1,2] was correct. However, the Hamiltonian itself is insufficient for an analysis of observable spin effects. One needs to know the spin operator as well. As the Hamiltonian was obtained in the EK representation, the spin operator (10) is rather complicated. At the same time, this operator acquires the simple form (7) in the FW representation. Let us stress that only for such a simple form of the spin operator the terms of the Hamiltonian may be simply interpreted in terms of observable physical effects. However, this is not true when a spin operator is complicated. In particular, the term $(\Sigma \cdot \boldsymbol{\phi})/2$ in (13) describing the dipole spin-gravity coupling disappears after the transformation to the FW representation. Therefore, this term does not lead to new observable effects.

V. EQUATIONS OF PARTICLE MOMENTUM AND SPIN MOTION

The problem of quantum description of particle and spin motion is very important. However, quantum equations of momentum and spin motion in a gravitational field were never derived.

The FW representation dramatically simplifies the derivation of quantum equations. The operator equations of motion obtained via commutators of the Hamiltonian with the momentum and polarization operators take the form

$$\frac{d\boldsymbol{p}}{dt} = i[\mathcal{H}_{FW}, \boldsymbol{p}] = -\frac{\beta}{2}\left\{\frac{m^2}{\epsilon}, \boldsymbol{\phi}\right\} - \frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon}, \boldsymbol{f}\right\} + \frac{m}{2\epsilon(\epsilon+m)} \times \nabla(\Pi \cdot (\boldsymbol{\phi}\times\boldsymbol{p})) - \frac{1}{2\epsilon}\nabla(\Pi\cdot(\boldsymbol{f}\times\boldsymbol{p})) \quad (16)$$

and

$$\frac{d\Pi}{dt} = i[\mathcal{H}_{FW}, \Pi] = \frac{m}{\epsilon(\epsilon+m)}\Sigma\times(\boldsymbol{\phi}\times\boldsymbol{p}) - \frac{1}{\epsilon}\Sigma\times(\boldsymbol{f}\times\boldsymbol{p}), \quad (17)$$

respectively. These equations constitute our principal new result.

It is possible to prove that EK Hamiltonian (13) leads to the spin motion equation consistent with Eq. (17). Within the semirelativistic approximation, the polarization operator in the EK representation takes the form

$$\boldsymbol{O}_{EK} = \Pi + \frac{\boldsymbol{p}\times\Sigma}{m} + \frac{\boldsymbol{p}\times(\boldsymbol{p}\times\Pi)}{2m^2}.$$

Commuting Hamiltonian (13) with the polarization operator $\boldsymbol{O}_{EK}$ leads to the approximate equation of spin motion

$$\frac{d\boldsymbol{O}_{EK}}{dt} = i[\mathcal{H}_{EK}, \boldsymbol{O}_{EK}] = \frac{\beta}{2m}\boldsymbol{O}_{EK}\times(\boldsymbol{\phi}\times\boldsymbol{p}) - \frac{\beta}{m}\boldsymbol{O}_{EK}\times(\boldsymbol{f}\times\boldsymbol{p}) \quad (18)$$

that agrees with Eq. (17). This explicitly shows that dipole spin-gravity coupling cancels with the extra terms in the spin operator in the EK representation and does not affect observable quantities.

Let us pass to the studies of the semiclassical limit of these equations. The contribution of the lower spinor is negligible and the transition to the semiclassical description is performed by averaging the operators in the equations for the upper spinor [16]. It is usually possible to neglect the commutators between the coordinate and the momentum operators. As a result, the operators $\boldsymbol{\sigma}$ and $\boldsymbol{p}$ should be substituted by the corresponding classical quantities: the polarization vector (doubled average spin), $\boldsymbol{\xi}$, and the momentum. For the latter quantity, we retain the notation $\boldsymbol{p}$. The semiclassical equations of motion are

$$\frac{d\boldsymbol{p}}{dt} = -\frac{m^2}{\epsilon}\boldsymbol{\phi} - \frac{\boldsymbol{p}^2}{\epsilon}\boldsymbol{f} + \frac{m}{2\epsilon(\epsilon+m)}\nabla(\boldsymbol{\xi}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})) - \frac{1}{2\epsilon}\nabla(\boldsymbol{\xi}\cdot(\boldsymbol{f}\times\boldsymbol{p})) \quad (19)$$

and

$$\frac{d\boldsymbol{\xi}}{dt} = \frac{m}{\epsilon(\epsilon+m)}\boldsymbol{\xi}\times(\boldsymbol{\phi}\times\boldsymbol{p}) - \frac{1}{\epsilon}\boldsymbol{\xi}\times(\boldsymbol{f}\times\boldsymbol{p}), \quad (20)$$

respectively. In Eq. (19), two latter terms describe a force dependent on the spin. This force is similar to the electromagnetic Stern-Gerlach force (see Ref. [16]). Because it is weak, the approximate semiclassical equation of particle motion takes the form

$$\frac{d\boldsymbol{p}}{dt} = -\frac{m^2}{\epsilon}\boldsymbol{\phi} - \frac{\boldsymbol{p}^2}{\epsilon}\boldsymbol{f}. \quad (21)$$

Equation (20) can be represented as

$$\frac{d\boldsymbol{\xi}}{dt} = \Omega\times\boldsymbol{\xi}, \quad (22)$$

where the angular velocity of spin rotation is given by

$$\Omega = -\frac{m}{\epsilon(\epsilon+m)}(\boldsymbol{\phi}\times\boldsymbol{p}) + \frac{1}{\epsilon}(\boldsymbol{f}\times\boldsymbol{p}). \quad (23)$$

We can find similar equations describing a change of the direction of particle momentum, $\boldsymbol{n} = \boldsymbol{p}/p$:

$$\frac{d\boldsymbol{n}}{dt} = \boldsymbol{\omega}\times\boldsymbol{n}, \qquad \boldsymbol{\omega} = \frac{m^2}{\epsilon p}(\boldsymbol{\phi}\times\boldsymbol{n}) + \frac{p}{\epsilon}(\boldsymbol{f}\times\boldsymbol{n}). \quad (24)$$

Therefore, the spin rotates with respect to the momentum direction and the angular velocity of this rotation is

$$\boldsymbol{o} = \Omega - \boldsymbol{\omega} = -\frac{m}{p}(\boldsymbol{\phi}\times\boldsymbol{n}). \quad (25)$$

The quantity $\boldsymbol{o}$ does not depend on $\boldsymbol{f}$ and vanishes for massless particles. Therefore, the gravitational field cannot change the helicity of massless Dirac particles. The evolution of the helicity $\zeta \equiv |\boldsymbol{\xi}_\parallel| = \boldsymbol{\xi}\cdot\boldsymbol{n}$ of massive particles is defined by the formula

$$\frac{d\zeta}{dt} = (\Omega - \boldsymbol{\omega}) \cdot (\boldsymbol{\xi}_\perp \times \boldsymbol{n}) = -\frac{m}{p}(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\phi}), \qquad (26)$$

where $\boldsymbol{\xi}_\perp = \boldsymbol{\xi} - \boldsymbol{\xi}_\parallel$.

VI. PARTICLE IN A SPHERICALLY SYMMETRIC FIELD

Let us consider the interaction of particles with a spherically symmetric gravitational field and compare the obtained formulas with previous results. This field is a weak limit of the Schwarzschild one which yields

$$V = 1 - \frac{GM}{r}, \qquad W = 1 + \frac{GM}{r}. \qquad (27)$$

Correspondingly,

$$\mathcal{F} = 1 - \frac{2GM}{r}, \qquad \boldsymbol{f} = 2\boldsymbol{\phi} = \frac{2GM}{r^3}\boldsymbol{r} = -2\boldsymbol{g},$$

where $\boldsymbol{g}$ is the Newtonian acceleration.

When we neglect the terms of the order of $(\boldsymbol{p} \cdot \nabla) \times (\boldsymbol{p} \cdot \boldsymbol{g})/\epsilon^2$, Hamiltonian (12) takes the form

$$\mathcal{H}_{FW} = \beta\epsilon - \frac{\beta}{2}\left\{\frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon}, \frac{GM}{r}\right\} - \frac{\beta(2\epsilon + m)}{4\epsilon(\epsilon + m)}[2\Sigma \cdot (\boldsymbol{g} \times \boldsymbol{p}) + \nabla \cdot \boldsymbol{g}]. \qquad (28)$$

In this case, the operator equations of momentum and spin motion are given by

$$\frac{d\boldsymbol{p}}{dt} = -\beta\frac{GM}{2}\left\{\frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon}, \frac{\boldsymbol{r}}{r^3}\right\} - GM \cdot \frac{2\epsilon + m}{\epsilon(\epsilon + m)} \cdot \nabla\left(\frac{\Pi \cdot (\boldsymbol{r} \times \boldsymbol{p})}{r^3}\right), \qquad (29)$$

$$\frac{d\Pi}{dt} = -\frac{GM}{r^3} \cdot \frac{2\epsilon + m}{\epsilon(\epsilon + m)}\Sigma \times (\boldsymbol{r} \times \boldsymbol{p}). \qquad (30)$$

In Eq. (29), the last term determines the gravitational SG force. The semiclassical formula for this post-Newtonian force is [20]

$$\boldsymbol{F}_{SG} = -GM \cdot \frac{2\epsilon + m}{\epsilon(\epsilon + m)} \cdot \nabla\left(\frac{\boldsymbol{\xi} \cdot (\boldsymbol{r} \times \boldsymbol{p})}{r^3}\right). \qquad (31)$$

This formula can be transformed to a more convenient form where the quantities $\hbar$ and c are kept explicit for a moment:

$$\boldsymbol{F}_{SG} = -\frac{GM\hbar}{cr^3} \cdot \frac{2\gamma + 1}{\gamma + 1}\left[\boldsymbol{\beta} \times \boldsymbol{\xi} - \frac{3\boldsymbol{r}(\boldsymbol{r} \cdot [\boldsymbol{\beta} \times \boldsymbol{\xi}])}{r^2}\right], \qquad (32)$$

$\boldsymbol{\beta} = \boldsymbol{v}/c$ and γ is the Lorentz factor. The SG force is of the order of $\frac{\hbar\beta}{mcr}$ with respect to the Newtonian one.

Neglecting the SG force, one gets the semiclassical equations of momentum and spin motion:

$$\frac{d\boldsymbol{p}}{dt} = \frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon}\boldsymbol{g}, \qquad (33)$$

$$\frac{d\boldsymbol{\xi}}{dt} = \frac{2\epsilon + m}{\epsilon(\epsilon + m)}\boldsymbol{\xi} \times (\boldsymbol{g} \times \boldsymbol{p}). \qquad (34)$$

The semiclassical expressions for the angular velocities of the rotation of the unit momentum vector, $\boldsymbol{n} = \boldsymbol{p}/p$, and spin are

$$\boldsymbol{\omega} = -\frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon\boldsymbol{p}^2}\boldsymbol{g} \times \boldsymbol{p} = \frac{GM}{r^3} \cdot \frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon\boldsymbol{p}^2}\boldsymbol{l}, \qquad (35)$$

$$\Omega = -\frac{2\epsilon + m}{\epsilon(\epsilon + m)}\boldsymbol{g} \times \boldsymbol{p} = \frac{GM}{r^3} \cdot \frac{2\epsilon + m}{\epsilon(\epsilon + m)}\boldsymbol{l}, \qquad (36)$$

where $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$ is the angular moment.

Equations (35) and (36) agree with the classical gravity. Equation (35) leads to the expression for the angle of particle deflection by a gravitational field

$$\theta = \frac{2GM}{\rho}\left(2 + \frac{m^2}{\boldsymbol{p}^2}\right) = \frac{2GM}{\rho\boldsymbol{v}^2}(1 + \boldsymbol{v}^2) \qquad (37)$$

coinciding with Eq. (13) of Problem 15.9 from Ref. [21] (see also Ref. [22]). This directly proves the full compatibility of quantum and classical consideration and disagrees with the results obtained in [8].

Equation (36) and the corresponding equation obtained in Ref. [9] by the very different method coincide, up to the sign due to the different definition of angular velocity [23]. For a nonrelativistic particle, the angular velocity of spin rotation is described by the same formula as the de Sitter one for a classical gyroscope [24]. Such a similarity [25] of classical and quantum rotators is a manifestation of the equivalence principle (see e.g. [26,27] and references therein). In the nonrelativistic approximation, the last term in Hamiltonian (28) describing the spin-orbit and contact (Darwin) interactions coincides with the corresponding term in Ref. [28].

The momentum and spin rotate in the same direction. Formula (25) for the angular velocity of spin rotation with respect to the momentum direction, defining the evolution of particle helicity, takes the form

$$\boldsymbol{o} = \Omega - \boldsymbol{\omega} = \frac{m}{\boldsymbol{p}^2}(\boldsymbol{g} \times \boldsymbol{p}). \qquad (38)$$

The ratio of particle momentum and spin deflection angles (θ and Θ, respectively) is constant and equal to

$$\frac{\Theta}{\theta} = \frac{(2\epsilon + m)(\epsilon - m)}{2\epsilon^2 - m^2}.$$

If these angles are small, the helicity of particle, whose helicity is originally $+1$, is given by

$$\zeta = 1 - \frac{(\Theta - \theta)^2}{2}. \qquad (39)$$

Therefore, the evolution of the helicity is described by the equation

$$\zeta = 1 - \frac{\theta^2}{2(2\gamma - \gamma^{-1})^2}, \tag{40}$$

where $\gamma = \epsilon/m$ is the Lorentz factor. This equation agrees with [26,27] [see Eqs. (17) and (19) from [27] obtained by the full quantum treatment]. At the same time, the expression obtained earlier by the similar method [29] contains the dependence on the graviton source mass M and looks much more complicated. We found that the large M behavior of numerical values as presented in Fig. 3 of that reference is at reasonable agreement with (40), while their asymptotic formula (12) is at variance with us. Note also that in [29] the disagreement with the semiclassical treatment [30] was stated, while we observe the full agreement between the semiclassical and quantum approaches.

We may conclude for three of the most important problems formulas (35), (36), and (40) are in the best agreement with previous results. We also have established a consent between the classical and quantum theories and found the new quantum corrections to the Newtonian force.

VII. PARTICLE IN A UNIFORMLY ACCELERATED FRAME AND THE EQUIVALENCE PRINCIPLE

Consideration of the particle motion in an accelerated frame permits one to relate the gravity and acceleration. The simplest case is the flat Minkowski spacetime in a uniformly accelerated frame [see (i) of Sec. II]. For this problem, the exact Dirac Hamiltonian derived by Hehl and Ni [5] is given by

$$\mathcal{H} = (1 + \boldsymbol{a}\cdot\boldsymbol{r})\beta m + \frac{1}{2}\{(1 + \boldsymbol{a}\cdot\boldsymbol{r}), \boldsymbol{\alpha}\cdot\boldsymbol{p}\}, \tag{41}$$

where $\boldsymbol{a}$ is the particle acceleration. In this case, the metric tensor corresponds to the choice (4). Metric (4) corresponds to the following form of the FW Hamiltonian (12):

$$\mathcal{H}_{FW} = \beta\left(\epsilon + \frac{1}{2}\{\epsilon, \boldsymbol{a}\cdot\boldsymbol{r}\}\right) + \frac{\Pi\cdot(\boldsymbol{a}\times\boldsymbol{p})}{2(\epsilon + m)}, \tag{42}$$

where $\epsilon = \sqrt{m^2 + \boldsymbol{p}^2}$. The contact (Darwin) interaction does not appear because the effective field $\boldsymbol{a}$ is uniform. Equation (42) shows that the particle energy is multiplied by the factor V except for the last term that is of a purely quantum origin. An appearance of this term describing the inertial spin-orbit coupling has been discovered by Hehl and Ni [5]. In the present work, generalizing the result of this reference, the relativistic expression for the Hamiltonian has been derived. This expression happens to agree with the nonrelativistic ones from [5,6].

Equation (42) for the Hamiltonian of relativistic particle in a uniformly accelerated frame agrees with the corresponding nonrelativistic expressions from [5,6].

The equations of particle and spin motion are given by

$$\frac{d\boldsymbol{p}}{dt} = -\beta\epsilon\boldsymbol{a}, \qquad \frac{d\Pi}{dt} = -\frac{\Sigma\times(\boldsymbol{a}\times\boldsymbol{p})}{\epsilon + m}. \tag{43}$$

In the uniformly accelerated frame, the SG force does not exist.

The semiclassical transition brings Eq. (43) to the form

$$\frac{d\boldsymbol{p}}{dt} = -\epsilon\boldsymbol{a}, \qquad \frac{d\boldsymbol{\xi}}{dt} = -\frac{\boldsymbol{\xi}\times(\boldsymbol{a}\times\boldsymbol{p})}{\epsilon + m}. \tag{44}$$

The angular velocities of rotation of the unit momentum vector and spin are equal to

$$\boldsymbol{\omega} = \frac{\epsilon}{\boldsymbol{p}^2}(\boldsymbol{a}\times\boldsymbol{p}), \qquad \Omega = \frac{\boldsymbol{a}\times\boldsymbol{p}}{\epsilon + m}. \tag{45}$$

The relative angular velocity defining the helicity evolution is given by

$$\boldsymbol{o} = \Omega - \boldsymbol{\omega} = -\frac{m}{\boldsymbol{p}^2}(\boldsymbol{a}\times\boldsymbol{p}). \tag{46}$$

When $\boldsymbol{a} = -\boldsymbol{g}$, values of $\boldsymbol{o}$ in Eqs. (38) and (46) are the same. It is the manifestation of the equivalence principle which was discussed with respect to the helicity evolution in [26,27].

At the same time, the manifestation of the equivalence principle for the spin rotation is not so trivial. In particular, the spin of nonrelativistic particles in the spherically symmetric gravitational field rotates 3 times more rapidly in comparison to the accelerated frame.

To trace the origin of this difference, let us compare the rotation of the momentum direction in these cases. Although it is the same in the nonrelativistic limit, the expressions for the relativistic particles differ. To understand this from the point of view of the equivalence principle, the approach of [26,27] is convenient. Let us consider [26,27] the matrix element $\mathcal{M}$ of particle scattering in the external gravitational field

$$\mathcal{M} = \frac{1}{2}\langle p'|T^{\mu\nu}|p\rangle h_{\mu\nu}(q), \qquad q = p - p', \tag{47}$$

where $T^{\mu\nu}$ is the Belinfante energy-momentum tensor and $h_{\mu\nu}$ is a Fourier component of a deviation of the metric tensor from its Minkowski value. The particle momentum evolution is fully determined by the forward matrix element fixed by the momentum conservation

$$\langle p|T^{\mu\nu}|p\rangle = 2p^\mu p^\nu. \tag{48}$$

The matrix element for the particle at rest takes the form

$$\mathcal{M} = m^2 h_{00}(q). \tag{49}$$

Coincidence of the (00) components of the metric in the gravitational field and accelerated frame proves the equivalence principle, appearing in such an approach as a low-energy theorem rather than a postulate.

At the same time, for the moving particle, the space components of metric $h_{zz} = h_{xx} = h_{yy} = h_{00}$ (see e.g. [24]) also contribute. As a result, the matrix elements in the gravitational field ($\mathcal{M}_g$) and in the accelerated frame ($\mathcal{M}_a$) differ by the obvious kinematical factor:

$$\mathcal{M}_g = (\epsilon^2 + \boldsymbol{p}^2)h_{00}(q), \qquad \mathcal{M}_a = \epsilon^2 h_{00}(q). \tag{50}$$

The ratio of these matrix elements

$$R = \frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon^2} \tag{51}$$

is exactly equal to the ratio of the right-hand side (r.h.s.) of the equations of particle momentum motion, and, consequently, to the ratio of the angular velocities of rotation of their directions. It is now clearly seen that this difference is a direct kinematical consequence of the equivalence principle.

Note that the general expression (26) for the helicity evolution is insensitive to the space components of the metric which is the entire origin of the kinematical factor differing in the gravitational field and accelerating frame. This provides the additional argument for the simple form of the equivalence principle when helicity is considered. Namely, the helicity evolution in any static gravitational field and corresponding accelerating frame merely coincides.

Let us consider the effect of the mentioned kinematical factor for the spin motion. Equations (35) and (45) describing the angular velocity of momentum motion can be written in the form

$$\begin{aligned}\boldsymbol{\omega}_g &= -\left[\frac{m}{\boldsymbol{p}^2} + \frac{2\epsilon + m}{\epsilon(\epsilon + m)}\right](\boldsymbol{g}\times\boldsymbol{p}) \\ &= -\boldsymbol{o} - \frac{2\epsilon + m}{\epsilon(\epsilon + m)}(\boldsymbol{g}\times\boldsymbol{p})\end{aligned} \tag{52}$$

for the spherically symmetric gravitational field and

$$\boldsymbol{\omega}_a = \left(\frac{m}{\boldsymbol{p}^2} + \frac{1}{\epsilon + m}\right)(\boldsymbol{a}\times\boldsymbol{p}) = -\boldsymbol{o} + \frac{1}{\epsilon + m}(\boldsymbol{a}\times\boldsymbol{p}) \tag{53}$$

for the uniformly accelerated frame. Here the relative angular velocity $\boldsymbol{o}$, common for two cases, is extracted. The remaining terms in the r.h.s. are just the angular velocities of spin rotation. The current derivation explicitly shows that their difference is the consequence of the equivalence principle and kinematical factors in (50).

It is interesting that in the nonrelativistic limit both $\boldsymbol{\omega}$ and $\boldsymbol{o}$ diverge as $1/\boldsymbol{p}^2$. Their finite differences in Eqs. (52) and (53) provide the nonrelativistic limit of the angular velocities of spin rotation. In this limit the momentum rotation in the gravitational field and accelerated frame coincides, as it is seen from (51). However, the mentioned divergence of the angular velocities "compensates" the infinitesimal deviation of R from unity. In more detail, one is dealing with the low $\boldsymbol{p}^2$ expansion of two expressions

$$\boldsymbol{\omega}_g = -\frac{\epsilon^2 + \boldsymbol{p}^2}{\epsilon\boldsymbol{p}^2}(\boldsymbol{g}\times\boldsymbol{p}) \approx -\left(\frac{m}{\boldsymbol{p}^2} + \frac{3}{2m}\right)(\boldsymbol{g}\times\boldsymbol{p}) \tag{54}$$

and

$$\boldsymbol{\omega}_a = \frac{\epsilon}{\boldsymbol{p}^2}(\boldsymbol{a}\times\boldsymbol{p}) \approx \left(\frac{m}{\boldsymbol{p}^2} + \frac{1}{2m}\right)(\boldsymbol{a}\times\boldsymbol{p}). \tag{55}$$

While the left-hand side (l.h.s.) of these expressions has the same nonrelativistic limit, the mentioned effect provides the ratio 3 for finite terms in the r.h.s.

For completeness let us also consider the operator equation for the particle acceleration:

$$\begin{aligned}\ddot{\boldsymbol{r}} &= -[\mathcal{H}, (\mathcal{H}, \boldsymbol{r})] \\ &= -\frac{1}{2}\left\{(1 + \boldsymbol{a}\cdot\boldsymbol{r}), \left[\boldsymbol{a} - \frac{2\boldsymbol{p}(\boldsymbol{a}\cdot\boldsymbol{p})}{\epsilon^2}\right]\right\}.\end{aligned} \tag{56}$$

In this equation, small terms depending on the spin matrix are omitted. In the semiclassical approximation,

$$\begin{aligned}\ddot{\boldsymbol{r}} &= -(1 + \boldsymbol{a}\cdot\boldsymbol{r})\left[\boldsymbol{a} - \frac{2\boldsymbol{p}(\boldsymbol{a}\cdot\boldsymbol{p})}{\epsilon^2}\right] \\ &= -(1 + \boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{a} - \frac{2\boldsymbol{v}(\boldsymbol{a}\cdot\boldsymbol{v})}{1 + \boldsymbol{a}\cdot\boldsymbol{r}}.\end{aligned} \tag{57}$$

After substitution of standard nonrelativistic expressions for $\boldsymbol{r}$ and $\boldsymbol{v}$ we reach the full agreement with the approximate result of Huang and Ni [[7], Eq. (82)].

VIII. DISCUSSION AND SUMMARY

We showed that the elegant exact EK transformation [1,2] does not provide a simple form for dynamical operators and therefore does not allow for a straightforward derivation of quantum and semiclassical equations of motion. We constructed the FW transformation leading to simple dynamical operators and derived the quantum (which is our main new result) and semiclassical equations of momentum and spin motion. For the case of a weak spherically symmetric field the semiclassical limit reproduces all the known results for the momentum, spin, and helicity evolution and resolves the existing contradictions. The new quantum corrections provide, in particular, the post-Newtonian gravitational SG force. We found that semiclassical equations are in full agreement with classical gravity.

We checked that the derived equations of motion are compatible with those obtained from the Hamiltonian of [1,2] and the respective (complicated) dynamical operators. However, the difference between the FW and the EK representations means that the physical interpretation of the approach [1,2] should be made carefully. Say, the term in the Hamiltonian [1,2] describing the dipole spin-gravity coupling does not appear in the FW Hamiltonian. As soon as physical effects are dependent on both the Hamiltonian and the dynamical operators, the correspondent term in the EK representation is canceled, when the complicated spin operator (10) is used. Consequently, there is no reason for the precession of spin of particles being at rest, which is explicitly seen from Eq. (17).

The equivalence principle, understood as minimal coupling of fermions to gravity (1) and (47), is always valid. However, its specific manifestations depend on the observable. From this point of view, the simplest observable for the Dirac particle is *helicity*. The helicity evolution in the gravitational field and accelerated frame is *the same*. The manifestation of the equivalence principle for momentum and spin motion in these two cases is affected by kinematical corrections due to the space components of metric tensor. In particular, this leads to the enhancement by the factor 3 of the frequency of spin precession in the gravitational field with respect to the accelerating frame.

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