Numerical examination of an evolving black string horizon

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We use the numerical solution describing the evolution of a perturbed black string presented by M. Choptuik, L. Lehner, I. Olabarrieta, R. Petryk, F. Pretorius, and H. Villegas [Phys. Rev. D 68, 044001 (2003)] to elucidate the intrinsic behavior of the horizon. It is found that by the end of the simulation, the affine parameter on the horizon has become very large and the expansion and shear of the horizon in turn very small. This suggests the possibility that the horizon might pinch off in infinite affine parameter.

I. INTRODUCTION

While in four-dimensional spacetimes black holes are stable, it has been shown by Gregory and Laflamme [2] that at least some of their higher dimensional analogs, in particular, black strings, are *linearly* unstable. As opposed to the situation in four-dimensions, at the linear level perturbations of these black strings grow exponentially above some critical length of the string. Although this picture has been well understood through a variety of analysis [2–7], it is not yet known what an unstable black string evolves to. In [2], using the linear instability of the string coupled to entropy considerations, it was argued that an unstable black string would "pinch off" and form a set of black holes which would later merge. This scenario, at the classical level, must give rise to a violation of cosmic censorship. However, this possibility was made doubtful by more recent work of Horowitz and Maeda (HM) [8]. There, a theorem is proved using the properties of expansion, shear and affine parameter of the geodesic generators of the black string horizon. This theorem asserts that a pinch off in finite affine parameter could only occur if a singularity formed on or outside the horizon. Under the assumption that this, perhaps too drastic scenario, is unlikely, the only other option left open for the event horizon pinching off is that it does so in *infinite affine parameter*. This possibility, although mathematically possible, was argued against by HM. Their argument is based on one main observation together with a plausibility argument. The observation is that although their theorem does indeed allow for a pinch off in infinite time, the rate, with respect to affine parameter, at which circles at $r = $ const can decrease in size is extremely slow. HM then argue that if

the system decays so slowly, its dynamics must go through a series of essentially static nonuniform strings, and it then might just stay at one of these solutions. Therefore, instead of the pinch off scenario, HM conjectured that the black string would evolve into a black string that is static but not uniform in the z direction: a static wiggly black string. Families of static wiggly black strings were found numerically by Wiseman [9], however, all the solutions of [9] had too large a mass to be the endpoint of the evolution of the unstable black string as conjectured in [10]. Note however the recent work by Sorkin [5] (and related discussions in [11]) where it is discussed how for large enough spherical dimensions the static solutions found do indeed have lower mass, and could in principle be those conjectured by [8].

In an attempt to resolve the issue of the fate of the linearly unstable black string, in [1] a numerical discretization of the Einstein equations in five-dimensions was presented. An SO(3) symmetry was assumed, effectively making the problem $2 + 1$ dimensional, and hence one that could be solved on contemporary computer systems. This code was employed to study the evolution of slight perturbations of a static, uniform black string. The simulation tracked the behavior of the system well into the nonlinear regime, though unfortunately it could not be extended far enough to elucidate the final fate of the string. Nevertheless, the numerical solution revealed significant and rich dynamics, illustrating, in particular, that the string evolves toward a shape resembling large spherical black holes connected by thin strings. In terms of areal radius, the ratio of the maximum to minimum radius was about 10 by the end of the simulation.

In this paper, we re-examine the data of [1], paying closer attention to the analysis of [8] and examine relevant features of the solution. In particular, we numerically find the expansion, shear and affine parameter of the black string horizon and discuss their behavior in connection to the conjecture of [8] where a scenario is ruled out as

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improbable, though as we argue here, it might very well be possible. The methods used are presented in Sec. II, results in Sec. III and conclusions in Sec. IV.

II. METHODS

Our starting point is to consider a spacetime described by the metric provided by the numerical simulation of [1]. For simplicity we adopt the same gauge conditions used there (to avoid performing a numerical coordinate transformation, which in general would involve interpolating functions in space and time). The metric used in [1] takes the form

$$ds^2 = (-\alpha^2 + \gamma_{AB}\beta^A\beta^B)dt^2 + 2\gamma_{AB}\beta^A dx^B dt$$
$$+ \gamma_{AB}dx^A dx^B + \gamma_\Omega d\Omega^2 \qquad (1)$$

where $x^A = (r, z)$ and $d\Omega^2$ is the unit two-sphere metric. Our goal is to extract intrinsic properties of the event horizon and its generators. The horizon can be described by a surface $r = R(t, z)$. To extract the sought-after information, the first step is to find the geodesic generators of the horizon. To this end, we construct the vector k_a normal to the horizon defined as $k_a = (k_t, k_r, k_z) = (-\dot{R}, 1, -R')$. Here an overdot denotes differentiation with respect to t and a prime denotes differentiation with respect to z. Since k^a is null, it is tangent to the generators of the horizon; these are given by

$$\frac{dz}{dt} = \alpha^2\left(\frac{\gamma^{rz} - \gamma^{zz}R'}{\beta^r + \dot{R}}\right). \qquad (2)$$

Note, however, that these geodesics are not affinely parametrized. To obtain the affinely parametrized geodesics one can exploit the fact that their tangent vectors l^a obey the simple relation $l^a = e^{-\nu}k^a$ where ν is defined by

$$\frac{d\nu}{dt} = \frac{\alpha^2 h}{\beta^r + \dot{R}} \qquad (3)$$

and h is given by

$$h = \frac{1}{2}\frac{\partial}{\partial r}[\gamma^{rr} + \gamma^{zz}(R')^2 - 2\gamma^{rz}R' - \alpha^{-2}(\dot{R} + \beta^r)^2]. \qquad (4)$$

Thus, the affine parameter λ of these geodesics is found by integrating

$$\frac{d\lambda}{dt} = \frac{\alpha^2 e^\nu}{\beta^r + \dot{R}}. \qquad (5)$$

Next we turn our attention to obtaining the expansion and shear of these affinely parametrized generators. Since the spacetime is spherically symmetric, these quantities are given by just two components of $\nabla_a l_b$: Let x^a be a unit vector in the two-sphere direction and let y^a be a unit vector orthogonal to k^a and to the two-sphere directions. Define the quantities A and B by

$$A = 2x^a x^b \nabla_a l_b \qquad (6)$$

$$B = -y^a y^b \nabla_a l_b, \qquad (7)$$

which measure the distortion along spherical and longitudinal directions, respectively. Then, the expansion and squared shear are given in terms of A and B by

$$\theta = A - B \qquad (8)$$

$$\sigma^{ab}\sigma_{ab} = \frac{1}{6}(A + 2B)^2. \qquad (9)$$

The equations above determine the affinely parametrized generators of the horizon and its expansion and shear. As we will see later, due to the exponential behavior of these quantities, it will turn out to be convenient to consider the logarithm of the affine parameter $s \equiv \ln\lambda$ and rescaled quantities defined as follows

$$(\tilde{\theta}, \tilde{\sigma}_{ab}, \tilde{A}, \tilde{B}) \equiv \lambda(\theta, \sigma_{ab}, A, B) \qquad (10)$$

It then follows that

$$\tilde{\theta} = \tilde{A} - \tilde{B} \qquad (11)$$

$$\tilde{\sigma}^{ab}\tilde{\sigma}_{ab} = \frac{1}{6}(\tilde{A} + 2\tilde{B})^2 \qquad (12)$$

$$\frac{ds}{dt} = \frac{\alpha^2 e^{(\nu-s)}}{\beta^r + \dot{R}}. \qquad (13)$$

The quantity $\tilde{A}$ is found by

$$\tilde{A} = e^{(s-\nu)}k^a\nabla_a(\ln\gamma_\Omega). \qquad (14)$$

To find $\tilde{B}$ we note that there is freedom to add a multiple of k^a to y^a and use that freedom to make y^t vanish. It then follows that the components of y^a are

$$y^z = [\gamma_{zz} + 2\gamma_{rz}R' + \gamma_{rr}(R')^2]^{-1/2} \qquad (15)$$

and $y^r = y^z R'$, from which

$$\tilde{B} = e^{(s-\nu)}[(y^z)^2 R'' + k_b\Gamma^b_{ac}y^a y^c] \qquad (16)$$

Finally, we point out the following observation, which will be crucial in the numerical evaluations to be carried out later. As we have seen above, θ and σ_{ab} (and their scaled counterparts) can be calculated independently. However, they are related by Raychaudhuri's equation, which for the five-dimensional vacuum null case is

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 + \sigma^{ab}\sigma_{ab} = 0, \qquad (17)$$

while the corresponding equation for the rescaled quantities is

$$\frac{d\tilde{\theta}}{ds} - \tilde{\theta} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} = 0. \qquad (18)$$
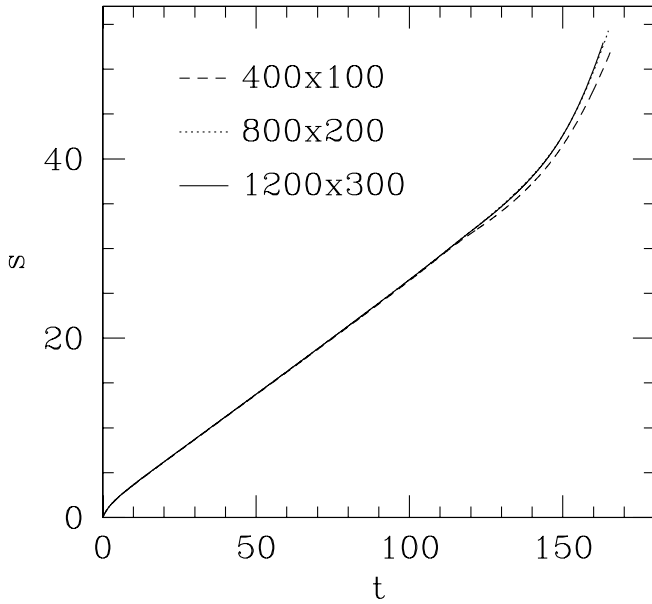
FIG. 1. s vs t for the horizon generator of minimum radius, from three simulations of identical initial data though differing resolution (the curves are labeled by $N_r x N_z$, where $N_r(N_z)$ is the number of grid points in the $r(z)$ direction). We show results from three different resolutions to demonstrate that we are in the convergent regime. Recall that the affine parameter $\lambda = e^s$, and therefore λ is growing very rapidly with simulation time t.

III. RESULTS

The simulations performed in [1] provide a (partial) description of the spacetime describing a slightly perturbed black string that evolves to a state where the black string is quite deformed as judged by the apparent horizon. The surface that we use here as the event horizon is the boundary of the causal past of the region exterior to the apparent horizon at the latest time in the simulation (calculated using a generalization of the technique described in [1]), and it turns out that this surface is very well approximated by the apparent horizon for the entire length of the simulation. We initialize the affine parameter to 1 (hence $s = 0$) at the beginning of the simulation. Among all generators, we are particularly interested in the one corresponding to the minimum radius, since it is there where any pinch off would occur. Because of the reflection symmetry of the initial data, the set of points on the horizon that at each time correspond to the minimal radius form a geodesic generator, and that generator stays at constant z.

Figure 1 shows plots of s as a function of the simulation time coordinate t for the generator of minimum radius, from simulations with three different resolutions (to give some measure of the error in the results). Note that s has reached 50 near the end of the simulation. This corresponds to an affine parameter of about 10^{21}. Thus, though the simulation has only taken a moderate amount of simulation time, it has gone for an enormous amount of affine parameter.
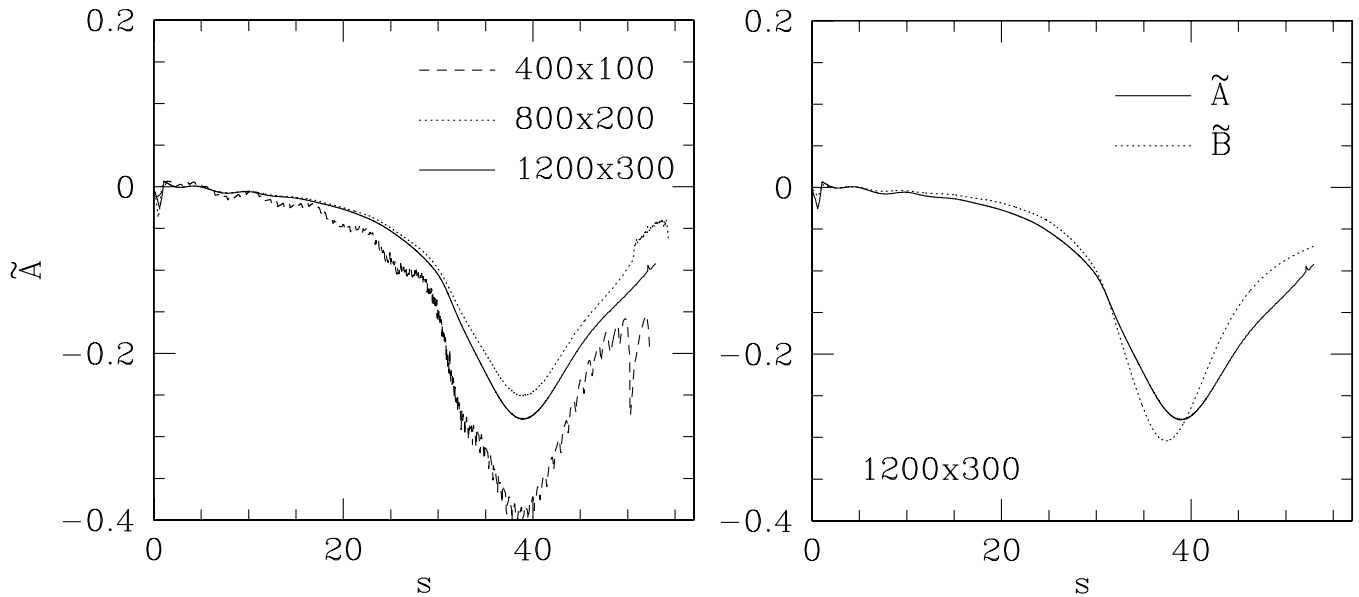

FIG. 2. The plot to the left depicts $\tilde{A}$ as a function of s, while the plot to the right shows $\tilde{A}$ and $\tilde{B}$ vs s from the highest resolution simulation, both for the horizon generator of minimum radius. What the latter plot demonstrates is that $\tilde{A}$ and $\tilde{B}$ are very similar in magnitude; in particular, the difference $\tilde{A} - \tilde{B}$ (which is just $\tilde{\theta}$) at late times is completely dominated by numerical error, the magnitude of which can be estimated by using the data from the three simulations. Specifically, the error in $\tilde{A}$ from the highest resolution simulation is quite small initially, is near 5% near the minimum of $\tilde{A}$, and has grown to around 40% by the end of the simulation ($\tilde{B}$ has similar error).
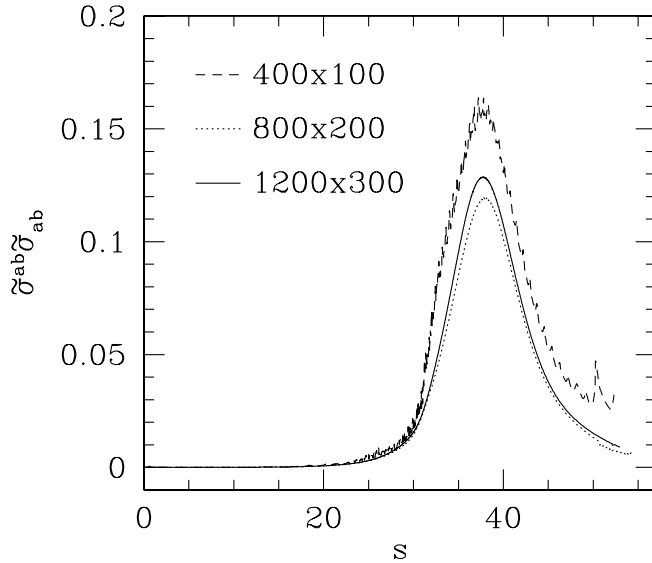
FIG. 3. $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ vs s for the horizon generator of minimum radius.

Figure 2 shows plots of $\tilde{A}$ and $\tilde{B}$ vs s for the generator of minimum radius. Note that due to the rescaling this means that by the end of the simulation, the unrescaled expansion θ and shear σ_{ab} are of order 10^{-22}. Note too, that though the behavior of $\tilde{A}$ and $\tilde{B}$ as a function of s is quite dynamic throughout the simulation, since $d/d\lambda = \lambda^{-1}d/ds$ it follows that viewed as functions of λ even the rescaled expansion and shear would be regarded as slowly varying.

We now turn to the behavior of the rescaled quantities $\tilde{\theta}$ and $\tilde{\sigma}_{ab}$. We can calculate the shear by simply evaluating (12) along the generator of minimum radius: Fig. 3 shows the result of this as a function of s for the three simulations. In principle we can also calculate $\tilde{\theta}$ in a similar manner using (11), which says $\tilde{\theta}$ is the *difference* between $\tilde{A}$ and $\tilde{B}$. However, the problem with this is that $\tilde{A}$ and $\tilde{B}$ are very close to one another in magnitude, as can be seen from Fig. 2, and so the difference will be dominated by numerical error. In fact, from the plot of $\tilde{A}$ in Fig. 2, and the assumed second order convergence of the numerical scheme, one can estimate that the error in $\tilde{A}$ (and similarly for $\tilde{B}$), is on the order of 5% at intermediate times (near the minimum of $\tilde{A}$) and grows to as much as 40% at late times. Therefore at late times the difference $\tilde{A} - \tilde{B}$ will be completely swamped by numerical error.

In order to alleviate this problem, we use Raychaudhuri's equation to integrate $\tilde{\theta}$. However, in the form of (18) there is in general no stable direction of integration due to the linear and quadratic terms in $\tilde{\theta}$; in other words, numerical errors in $\tilde{\theta}$ could cause a blow up integrating in either direction in s. Therefore, we eliminate the $\tilde{\theta}^2$ term using (11), and $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ using (12), though we keep the linear term in $\tilde{\theta}$ to give

$$\frac{d\tilde{\theta}}{ds} - \tilde{\theta} + \frac{1}{2}\tilde{A}^2 + \tilde{B}^2 = 0. \qquad (19)$$

In this form there is a stable direction of integration from large to small s, and the only "sources" in $\tilde{A}$ and $\tilde{B}$ are quadratic and of the same sign, hence we are not as
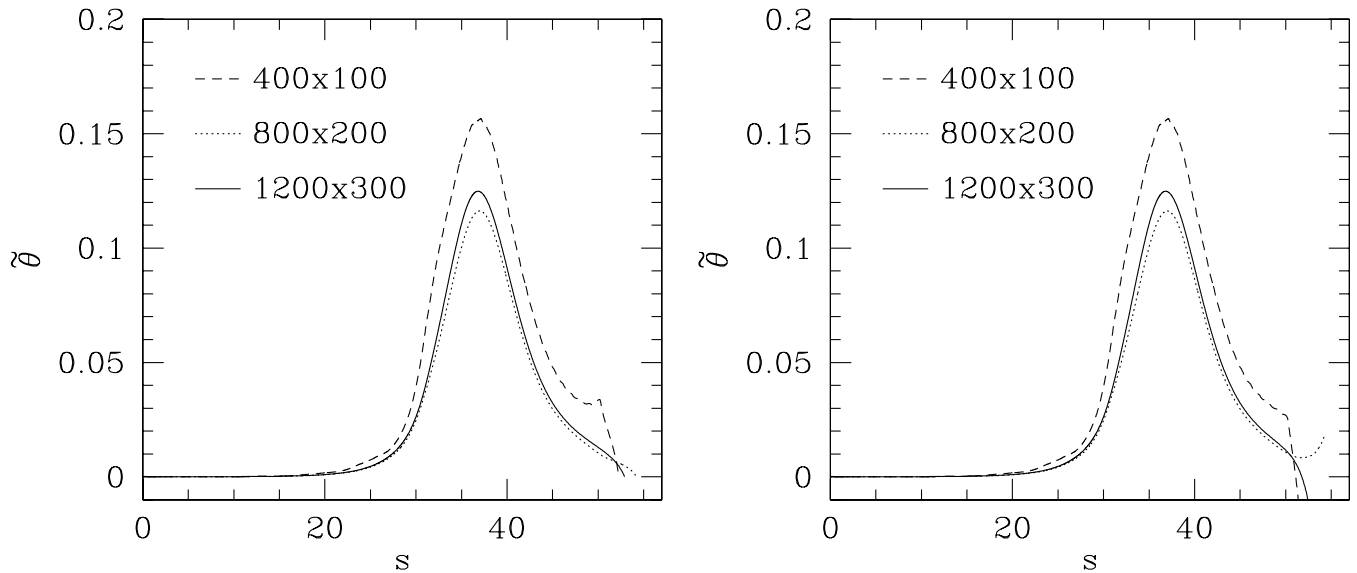

FIG. 4. $\tilde{\theta}$ vs s for the horizon generator of minimum radius, calculated by integrating (19). For the integrations shown in the left figure, the initial value of $\tilde{\theta}$ (which is at the *largest s*, as we integrate from large to small s) is 0, while in the right figure the initial $\tilde{\theta}$ was calculated using (11). We show curves calculated using two different sets of initial conditions to demonstrate that the integration is fairly insensitive to the initial value of $\tilde{\theta}$; hence, except near the end of the simulation, the error in the integrated $\tilde{\theta}$ is comparable to that of $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ shown in Fig. 3—specifically around 10% near the maxima for the highest resolution simulation, assuming second order convergence.

susceptible to numerical errors arising from the difference of similar numbers. One potential difficulty is in specifying initial conditions for the integration, which we need to do at the end of the simulation where $\tilde{\theta}$ obtained from (11) has the largest error. Fortunately the integration seems to be fairly insensitive to the initial condition in $\tilde{\theta}$, as illustrated in Fig. 4, where we show the results of integration of (19) starting from both $\tilde{\theta}$ calculated using (11), and $\tilde{\theta} = 0$ (which is within the numerical margin of error at late times).

IV. CONCLUSIONS

We have found the behavior of the horizon of a black string in terms of the affine parameter, expansion and shear of the generators. The most significant aspects of our results are the extremely large values of the affine parameter, and correspondingly the extremely small values of the (unrescaled) expansion and shear. In hindsight, this result is not too surprising as it is well known that for a Schwarzschild black hole there is an exponential relation between the Killing time coordinate and the affine parameter of the horizon generators. Since the straight black string is just the Schwarzschild spacetime crossed with a circle, and since the spacetime treated here is initially just a small perturbation of the straight string, it appears natural that very large values of the affine parameter are reached. The immediate conclusion is that in some sense the logarithm of the affine parameter is a more natural "dynamical time" than the affine parameter.

What do our results say about the final fate of the unstable black string? They certainly do not yet resolve the issue. The same three possible alternatives are still there: the string can (i) evolve to a static wiggly string, (ii) "terminate" in a naked singularity in finite affine parameter or (iii) pinch off in infinite affine parameter. It is clear both from [1] and from our work that the simulation has not gone far enough to read off the asymptotic behavior of the string. For that, a more robust simulation is needed. However, our results would seem to push the evidence more in favor of alternative (iii). Recall that in [8] it was correctly pointed out that alternative (iii) requires very small (unrescaled) expansion and shear and very slow dependence on affine parameter. Our results here show that precisely this is the sort of behavior that occurs, at least in some regime of the evolution of the black string. In fact, one might argue that alternative (iii) is the most plausible for lower spherical dimensions of the spacetime *if the only nonuniform static solutions are those found in* [9]. This fact would make alternative (i) unlikely as these solutions are too massive for spacetimes with dimension lower than 13. Option (ii) requires a singularity forming *on or outside* the horizon, and no hints of such behavior developing were seen in the simulations of [1]. Such a singularity would also invalidate the picture of a black string bifurcating into a sequence of black holes (as sug-

gested by entropy arguments [2])—in fact it is unclear what the nature of such a singularity would be, or how the spacetime could be extended beyond it. In contrast, even though option (iii) may result in a naked singularity, the singularity will occur at the moment the string reaches zero radius, and thus one can plausibly argue that the horizon will then bifurcate into a sequence of spherical black holes (of course, a theory of quantum gravity would be needed to resolve this bifurcation singularity).

Under alternative (iii) one might still have different scenarios for evolution of the string at late times. One possibility is that pinch off is reached through a cascading sequence of instabilities on ever smaller scales. This would happen if first, the relationship between affine time λ and the local dynamical time s continues to be logarithmic ($s \propto \ln\lambda$), as then the evolution will take an infinite amount of "time" relative to local physical processes; and second, if the stringlike portion of the solution (near the minimum radius) at late times can be described as a perturbed uniform string. For then an instability will develop around this portion of the horizon, as the local ratio of length to radius is above the critical value for a stable string, and this part of the spacetime will then also evolve toward a solution resembling (smaller) spherical black holes connected by thin black strings, and so on. Note that in this scenario the pinch off would probably occur in finite asymptotic time t, as the time-scale for the growth of the instability, measured in units of t, is proportional to the local radius of the string, which is tending to zero.

Still considering option (iii), if the conditions for a cascading sequence of instabilities are not met (namely that either the stringlike portion of the solution has a geometry sufficiently different from a uniform black string that it does not suffer from a Gregory-Laflamme instability, or that there is insufficient time for a second instability to grow), then the pinch off may simply occur as a continuation of the trend observed at the late stages of the simulation of [1]. Here, the pinch off could also occur in finite simulation time though infinite affine time, if the affine parameter grows exponentially with simulation time with the factor in the exponential inversely proportional to the radius (as is does for the uniform black string solution). Thus if the radius goes to zero rapidly enough, this sort of behavior could lead to infinite affine parameter in finite simulation time. Indeed, from Fig. 1 it is clear that the slope of log affine parameter vs simulation time is increasing in the regime where the minimal radius is decreasing.

In a more quantitative vein, we can consider the possible asymptotic behavior of the (rescaled) expansion and shear. Let p and c be positive constants. Then it is consistent with Eq. (18) for both $\tilde{\theta}$ and $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ to asymptotically approach $c^2 s^{-2p}$. This would result from both $\tilde{A}$ and $\tilde{B}$ approaching $-cs^{-p}$ asymptotically, but their difference approaching $c^2 s^{-2p}$. That this sort of behavior is possible was recognized in [8] but there it was thought that such behavior was

somehow unnaturally slow and indicative of an approximately static solution. If $p \leq 1$ then this behavior results in pinch off in infinite affine parameter. If in addition, $p > 1/2$ then this pinch off occurs with finite $\int \theta d\lambda$ as assumed in [8].

If such a power law is the asymptotic behavior of the unstable black string, then the simulations have not yet run for long enough to be sufficiently far into the asymptotic regime to try to calculate the exponents. Nonetheless, in Figs. 3 and 4 the quantities $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ and $\tilde{\theta}$ peak and then decrease. This decrease may be the beginning of a power law tail to be found in an improved black string simulation.

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[1] M. Choptuik, L. Lehner, I. Olabarrieta, R. Petryk, F. Pretorius, and H. Villegas, Phys. Rev. D **68**, 044001 (2003)

[2] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993).

[3] R. Gregory and R. Laflamme, Nucl. Phys. **B428**, 399 (1994).

[4] S. S. Gubser, Class. Quant. Grav. **19**, 4825 (2002).

[5] E. Sorkin, Phys. Rev. Lett. **93**, 031601 (2004).

[6] H. Reall, Phys. Rev. D **64**, 044005 (2001).

[7] O. Sarbach and L. Lehner, Phys. Rev. D **71**, 026002 (2005).

[8] G. Horowitz and K. Maeda, Phys. Rev. Lett. **87**, 131301 (2001).

[9] T. Wiseman, Classical Quantum Gravity **20**, 1137 (2003).

[10] B. Kol, hep-th/0206220.

[11] B. Kol and E. Sorkin, Class. Quant. Grav. **21**, 4793 (2004).