

Lagrangian brane dynamics in general relativity and in Einstein-Gauss-Bonnet gravity

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(Received 10 January 2005; published 8 March 2005)

This paper gives a new, simple and concise derivation of brane actions and brane dynamics in general relativity and in Einstein-Gauss-Bonnet gravity. We present a unified treatment, applicable to timelike surface layers and spacelike transition layers, and including consideration of the more difficult lightlike case.

DOI: 10.1103/PhysRevD.71.064008

PACS numbers: 04.50.+h, 04.20.Fy, 04.70.-s

I. INTRODUCTION

Thin walls and shells of matter have surfaced increasingly in a variety of situations in astrophysics, cosmology and quantum gravity. Highly compressed expanding shells of material emerge from supernova explosions [1], as false-vacuum bubble walls in inflationary phase transitions [2], and in hypothetical scenarios of “new-universe” creation [3]. In a sudden global phase transition, the transition region can sometimes be idealized as an infinitesimally thin spacelike surface layer [4]. Theoretical exploration of basic issues of principle, such as the possible outcomes of a classical or quantum gravitational collapse [5], are often simplified, for purposes of a first reconnaissance, by idealizing the collapsing matter as a thin shell, thus reducing the complex differential field equations to simple algebraic junction conditions.

With the advent of brane-world scenarios [6], the scope of thin-shell dynamics has broadened to embrace higher dimensions and string-inspired extensions of Einsteinian gravity, in particular, Einstein-Gauss-Bonnet (EGB) theory [7]. Recently, a class of Weyl-conformally invariant p -brane theories, which includes lightlike branes, has been proposed [8].

For purely Einsteinian shells the classical dynamics is straightforward [9] (though the variational and Hamiltonian aspects can be subtle [10]). Appending Gauss-Bonnet terms to the usual Einstein-Hilbert action, however, is attended by a considerable step-up in complexity. Initially, there were even doubts whether EGB shells admit a distributional description at all: the EGB bulk field equations develop ill-defined products of delta- and step-functions in the thin-layer limit unless the terms are arranged with care [11]. The key is to express the field equations in canonical form, with distance from the layer in the role of “time”; this segregates the most singular terms into the canonical momentum. The canonical mo-

mentum is the dynamical variable that “jumps” cleanly at a thin layer.

The key role of canonical momentum suggests that the EGB junction conditions are most easily derived from the action. This derivation was carried through by Davis [12], so that the basic equations of EGB shell dynamics are now well established and widely employed [13]. Still to be desired is a systematic, self-contained exposition which draws together general-relativistic and EGB shell dynamics within a unified Lagrangian framework, and includes consideration of the lightlike limit [14]. We hope this paper will go some way toward filling this gap.

II. TOY MODEL

To illustrate the essential ideas, we take a simple example from one-dimensional particle mechanics. We choose a “bulk” action functional of the path $q = q(t)$,

$$S_{bulk}[q] = \int_{t_i}^{t_f} (L - V_{ext}) dt, \quad (2.1)$$

with an acceleration-dependent Lagrangian of the form

$$L = -b(q)\ddot{q} - \frac{1}{2}b'(q)\dot{q}^2 - V(q), \quad (2.2)$$

where the functions b, V are arbitrary and $V_{ext}(q, t)$ is an arbitrary external potential.

We are considering q as an analogue of the metric, t as an analogue of distance normal to a boundary surface or layer, and \dot{q} as an analogue of extrinsic-curvature $K \sim \partial g / \partial n$. The two terms of (2.1) simulate the geometrical and matter actions; the external force $F = -\partial V_{ext} / \partial q$ is the analogue of material stress-energy $T_{\alpha\beta}$. The particular, quasilinear functional form (2.2) is patterned after the Einstein-Hilbert (EH) and Gauss-Bonnet (GB) Lagrangians. (The EH Lagrangian is quasilinear in the narrow sense that the coefficients of the second-derivative terms are functions of the metric only, not its first derivatives. But the corresponding coefficients do depend on first derivatives in the case of the GB Lagrangian).

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Although L involves second derivatives, its quasilinear-ity ensures that the Euler-Lagrange equation

$$\frac{\delta L}{\delta q} + F = 0, \quad (2.3)$$

for the classical path is no higher than second order. The classical path is thus uniquely determined by fixing its two endpoints q_i, q_f . But this path does not extremize the bulk action (2.1), because the endpoint velocities \dot{q}_i, \dot{q}_f can still be varied freely: we have

$$\delta S_{bulk}[q] = \int_{t_i}^{t_f} \left(\frac{\delta L}{\delta q} + F \right) \delta q(t) dt + [p \delta q - \delta B(q, \dot{q})]_{t_i}^{t_f}, \quad (2.4)$$

where

$$p = b'(q)\dot{q}; \quad B(q, \dot{q}) = b(q)\dot{q}. \quad (2.5)$$

To have an action that is extremized by the classical path, one must add a boundary term [15] to the bulk action (2.1):

$$S[q] = S_{bulk}[q] + B(q, \dot{q})|_{t_i}^{t_f}, \quad (2.6)$$

The extremal of the action (2.6) now depends solely on the endpoints q_i, q_f of the classical path, in accordance with the Hamilton-Jacobi equation

$$\delta S_{extrem}(q_i, t_i; q_f, t_f) = p \delta q|_{t_i}^{t_f}, \quad (2.7)$$

More fundamentally, the boundary term B is needed to preserve the composition law

$$S[1 \rightarrow 2 \rightarrow 3] = S[1 \rightarrow 2] + S[2 \rightarrow 3], \quad (2.8)$$

for an arbitrary continuous path joining the points 1, 2, 3 with a sharp bend at 2 (as used, for example, in the ‘‘zig-zag’’ definition of the path integral [16]).

The Euler-Lagrange Eq. (2.3) can be expressed as

$$\frac{dp}{dt} + \frac{\partial V}{\partial q} = F(q, t). \quad (2.9)$$

This is the analogue of the gravitational field equations, with the right-hand side representing the stress-energy of matter. To simulate a thin surface layer, we consider an impulsive force acting at time t_0 :

$$F(q, t) = \sigma(q)\delta(t - t_0). \quad (2.10)$$

This will produce a discontinuity in the momentum p . Since all delta-function contributions to the left-hand side are gathered into dp/dt , it is straightforward to integrate the equation of motion (2.9) to obtain the jump across the discontinuity:

$$[p] = \sigma(q), \quad (2.11)$$

where $[p] = \lim_{\epsilon \rightarrow 0} \{p(t_0 + \epsilon) - p(t_0 - \epsilon)\}$. This is the analogue of the geometrical junction conditions at a surface layer. The key to deriving it is just the identification of

the canonical momentum p —most easily from the Hamilton-Jacobi equation (2.7), or from a suitable adaptation of Lagrange’s definition applied to an equivalent first-order Lagrangian, see (2.16) below.

The jump condition (2.11) would follow as

$$\frac{\partial S_{imp}(q_0, \dot{q}_0^\pm)}{\partial q_0} = 0, \quad (2.12)$$

from an ‘‘impulsive action’’ S_{imp} , as a function of position $q(t_0) = q_0$ and pre- and post-shock velocities $\dot{q}(t_0 \pm \epsilon) = \dot{q}_0^\pm$. This is given by

$$S_{imp} = -[B(q, \dot{q})]_{t_0-\epsilon}^{t_0+\epsilon} - \int_{t_0-\epsilon}^{t_0+\epsilon} V_{ext}(q, t) dt, \quad (2.13)$$

and it coincides with the bulk action (2.1) if we choose $t_f = t_0 + \epsilon, t_i = t_0 - \epsilon$. Alternatively, (2.13) is obtainable without integrating L through the shock, instead considering $t_0 - \epsilon$ and $t_0 + \epsilon$ as future and past endpoints, respectively, each with its own boundary action B , so that the bulk action (2.6) becomes

$$S[q] = \left(\int_{t_i}^{t_0-\epsilon} + \int_{t_0+\epsilon}^{t_f} \right) L dt - \int_{t_i}^{t_f} V_{ext} dt + [B]_{t_i}^{t_0-\epsilon} + [B]_{t_0+\epsilon}^{t_f}, \quad (2.14)$$

when an impulse acts at t_0 .

It should be noted that the close relationship (2.13) and (2.14) of the boundary action B to an impulsive action hinges on the special form (2.2) of the action, and does not extend to an arbitrary quasilinear Lagrangian. It does, nevertheless carry over to the EH action, and also to the GB action modulo removable $K^2 \partial K / \partial n$ terms. The origins of this peculiar circumstances will emerge in Sec. III.

The bulk + boundary action (2.6) is really a thinly disguised first-order action:

$$S[q] = \int_{t_i}^{t_f} (L_1 - V_{ext}) dt; \quad L_1 = \frac{1}{2} b'(q) \dot{q}^2 - V(q), \quad (2.15)$$

with the standard definition

$$p = \frac{\partial L_1}{\partial \dot{q}}. \quad (2.16)$$

Therewith everything relating to this mechanical model takes on a trivial appearance. Not so, however, for its gravitational counterparts: there, only the original, second-order Lagrangian is a geometrical object and a scalar; the split into a first-order Lagrangian and a pure divergence cannot be made in a coordinate invariant and boundary-independent way. One is essentially forced to retain the bulk + boundary formulation.

Let us finally note that, because of the freedom to redefine the bulk part of the total action by adding a total derivative, the definitions of canonical momentum and

boundary action are (trivially) arbitrary to the extent

$$\begin{aligned} L_1 &\rightarrow L_1 + \frac{d}{dt}f(q); & B &\rightarrow B - f(q); \\ p &\rightarrow p + f'(q). \end{aligned} \quad (2.17)$$

This has no effect on the impulsive jump conditions (2.11), because the arbitrary function $f(q)$ is continuous at the shock.

III. BRANE DYNAMICS: EINSTEIN-HILBERT ACTION

We begin by focussing on general relativity and on higher-dimensional gravitational theories governed by the Einstein-Hilbert action. Our purpose is to derive the well-known junction conditions [9,14] which determine the motion of a surface layer in such theories from the action. We shall present a unified treatment, applicable to timelike, spacelike or lightlike layers.

In an $(n+1)$ -dimensional spacetime, the Einstein-Hilbert bulk action is

$$S_{bulk} = \frac{1}{2\kappa} \int \mathcal{L}_{EH}(g, \Gamma) d^{n+1}x, \quad (3.1)$$

where

$$\mathcal{L}_{EH} = \sqrt{-g}R = \sqrt{-g}g^{\mu\nu}(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}) - \mathcal{L}_1, \quad (3.2)$$

$$\mathcal{L}_1 = \sqrt{-g}g^{\mu\nu}(\Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta}). \quad (3.3)$$

The second-order Lagrangian (3.2) is degenerate, and can be reduced to first-order form by extracting a pure divergence:

$$\mathcal{L}_{EH} = -\partial_\lambda \sigma^\lambda + \mathcal{L}_1, \quad (3.4)$$

where

$$\sigma^\lambda(g, \Gamma) \equiv \sqrt{-g}(g^{\lambda\mu} \Gamma^\alpha_{\mu\alpha} - g^{\mu\nu} \Gamma^\lambda_{\mu\nu}), \quad (3.5)$$

which is reducible to

$$\sigma^\lambda = \frac{1}{\sqrt{-g}} \partial_\mu (-g g^{\lambda\mu}). \quad (3.6)$$

The first-order Lagrangian \mathcal{L}_1 dates back to Lorentz, Hilbert, Einstein, Weyl and Felix Klein, and was employed by Einstein [17] to define his pseudotensor for gravitational energy and radiation.

The complete Einstein-Hilbert action S_{EH} complements the bulk action (3.1) with a term S_{bdy} coming from the boundary, which soaks up the pure divergence in (3.4) [17]. Including also the matter contribution, the complete action reads

$$S_{EH} = S_{bulk} + S_{bdy} + S_{mat} = \frac{1}{2\kappa} \int \mathcal{L}_1 d^{n+1}x + S_{mat}. \quad (3.7)$$

Here

$$S_{bdy}[g, \Gamma] = \frac{1}{2\kappa} \int \partial_\lambda \sigma^\lambda d^{n+1}x, \quad (3.8)$$

is convertible to a surface integral over the boundary Σ , and S_{mat} yields the material stress-energy tensor

$$\frac{\delta S_{mat}}{\delta g^{\mu\nu}} = -\frac{1}{2} \sqrt{-g} T_{\mu\nu}. \quad (3.9)$$

Computing the variation of the bulk action (3.1) is facilitated by the Palatini identity

$$\delta R_{\mu\nu} = \delta \Gamma^\lambda_{\mu\nu|\lambda} - \delta \Gamma^\alpha_{\mu\alpha|\nu}, \quad (3.10)$$

where $|$ stands for the covariant derivative associated with the metric $g_{\mu\nu}$. Recalling (3.5), and momentarily treating g and Γ as independent, this gives

$$\begin{aligned} \delta \mathcal{L}_{EH}(g, \Gamma) &= (\delta_g + \delta_\Gamma) \{ \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma) \} \\ &= \delta g^{\mu\nu} \sqrt{-g} G_{\mu\nu} - \partial_\lambda (\delta \Gamma^\alpha \sigma^\lambda). \end{aligned} \quad (3.11)$$

From (3.7), (3.8), and (3.9) we thus obtain

$$\begin{aligned} \delta S_{EH} &= \frac{1}{2\kappa} \int (G_{\mu\nu} - \kappa T_{\mu\nu}) \delta g^{\mu\nu} \sqrt{-g} d^{n+1}x \\ &\quad + \delta_g S_{bdy}[g, \Gamma]. \end{aligned} \quad (3.12)$$

The last term depends only on the metric variation $\delta g_{\alpha\beta}$ at the boundary, not $\delta \Gamma$. Thus, for metric variations which vanish at the boundary, we find

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (3.13)$$

which are the bulk equations.

While the second-order Lagrangian (3.2) is a geometrical invariant, its split (3.4) into a first-order part and a pure divergence is coordinate-dependent. This split can, however, be endowed with geometrical significance by a special choice of the bulk coordinates x^μ which anchors them to the boundary Σ . In general, Σ will be characterized by parametric equations

$$\Sigma: x^\mu = x^\mu(\xi^a, z); \quad z(x^\mu) = 0, \quad (3.14)$$

where $\xi^a (a = 1, \dots, n)$ are arbitrary intrinsic coordinates. Then (3.8) integrates to

$$S_{bdy} = \frac{1}{2\kappa} \int_\Sigma \sigma^\lambda \partial_\lambda z dA, \quad (3.15)$$

where dA is a (noninvariant) element of n -area. We now impose the anchoring condition on x^μ :

$$\Sigma: x^0 \equiv z = 0, \quad x^a = \xi^a. \quad (3.16)$$

Then (3.15) becomes

$$S_{bdy} = \frac{1}{2\kappa} \int_{\Sigma} \sigma^0(g, \Gamma) d^n \xi, \quad (3.17)$$

with

$$\sigma^0 = \sigma^\lambda \partial_\lambda z = (-g)^{-1/2} \partial_\lambda [(-g) g^{\lambda\mu} \partial_\mu z]. \quad (3.18)$$

Its noncovariant appearance notwithstanding, (3.18) is actually (twice) the mean extrinsic-curvature density of Σ , up to a dynamically irrelevant term. Moreover, it is regular even for a lightlike Σ , so that (3.18) may be considered a valid extension of the notion of mean extrinsic-curvature density to the lightlike case.

To verify these statements, suppose first that Σ is non-lightlike, and thus has a unit normal n^μ , which is transverse to Σ , say in the direction of increasing z , ($n^0 > 0$); $n \cdot n = \epsilon$ with $\epsilon = +(-)1$ whenever Σ is time(space)like. Then (3.18) can be shown to reduce to

$$\sigma^0 = 2\epsilon \mathcal{K} + \sqrt{-g} \epsilon N^{-2} \partial_a s^a; \quad \mathcal{K} = |^{(n)}g|^{1/2} K. \quad (3.19)$$

where $K = n^\alpha |_{\alpha}$ is the mean extrinsic-curvature, and N, s^a the lapse and shift which appear in the standard ADM form of the bulk metric

$$ds^2 = g_{ab}(d\xi^a + s^a dz)(d\xi^b + s^b dz) + \epsilon N^2 dz^2, \quad (3.20)$$

so that

$$\begin{aligned} g^{00} &= \epsilon N^{-2}; & g^{0a} &= -\epsilon N^{-2} s^a; \\ |^{(n)}g|^{1/2} &= \sqrt{-g} N^{-1}. \end{aligned} \quad (3.21)$$

In the anchored coordinates (3.16), n^μ has components

$$n_\mu = \epsilon N \partial_\mu z, \quad N n^0 = \epsilon, \quad N n^a = -s^a. \quad (3.22)$$

It is now straightforward to derive (3.19): from (3.18), (3.21), and (3.22),

$$\begin{aligned} \sigma^0 - 2\epsilon \mathcal{K} &= (-g)^{-1/2} \partial_\mu [(-g) \epsilon N^{-1} n^\mu] \\ &\quad - 2\epsilon N^{-1} \partial_\mu (\sqrt{-g} n^\mu) \\ &= -\sqrt{-g} \epsilon N^{-2} \partial_\mu (N n^\mu) = \sqrt{-g} \epsilon N^{-2} \partial_a s^a. \end{aligned} \quad (3.23)$$

This extra contribution (3.23) to the boundary action (3.17) and (3.19) is ‘‘velocity-independent’’ (i.e., independent of transverse derivatives $\partial_0 g_{\mu\nu}$), and has the same role as $f(q)$ in (2.17). It could be reconverted to a volume integral and included with \mathcal{L}_1 in the first-order bulk action in (3.7). Its contribution to the momentum is trivial. Moreover, it contributes nothing at all to the *jump* of momentum across an extrinsic-curvature discontinuity, since its jump across Σ is zero if x^μ and $g_{\mu\nu}$ are continuous. This will continue to hold in the lightlike limit ($N \rightarrow \infty$).

Recalling (3.5), we conclude that

$$\begin{aligned} \epsilon[\mathcal{K}(g, \Gamma)] &= C_\lambda^{\mu\nu}(g) [\Gamma^\lambda_{\mu\nu}]; \\ C_\lambda^{\mu\nu}(g) &= \frac{\sqrt{-g}}{2} (g^{\sigma(\mu} \delta_\lambda^{\nu)}) - g^{\mu\nu} \delta_\lambda^\sigma \partial_\sigma z, \end{aligned} \quad (3.24)$$

represents precisely the jump of mean extrinsic-curvature density at an *arbitrary* (lightlike or nonlightlike) extrinsic-curvature discontinuity Σ . (For lightlike Σ , ϵ is defined by continuity with the timelike case.)

A surface layer can be characterized by supposing the bulk divided into two subdomains $z < 0$ and $z > 0$ with edges $\Sigma_-(z = -0)$ and $\Sigma_+(z = +0)$, glued together to form a common boundary Σ , loaded with a surface distribution of stress-energy; Σ_+ and Σ_- are supposed intrinsically isometric, but $\partial_\sigma g_{\mu\nu}$ undergoes a jump at Σ in smooth (e.g., skew-Gaussian) coordinates, i.e., Σ is an extrinsic-curvature discontinuity. Dynamically, the surface layer is accounted for by adding to the Einstein-Hilbert action (3.7) a shell contribution equal to the sum of the actions (3.17) for the boundaries Σ_- and Σ_+ of the subdomains, taking into account the opposite directions of their outward normals:

$$S_{EH,shell} = \frac{1}{2\kappa} \int_{\Sigma} 2\epsilon[\mathcal{K}] d^n \xi + S_{mat,shell}. \quad (3.25)$$

The second (matter) term $S_{mat,shell}$ generates the surface stress-energy density $S^{\alpha\beta}$ via

$$\left. \frac{\delta S_{mat,shell}}{\delta g_{\alpha\beta}} \right|_{\Sigma} = \frac{1}{2} S^{\alpha\beta}, \quad (3.26)$$

in analogy with (3.9).

Variation of the total action (3.7)+(3.25) then yields the jump conditions

$$[\pi^{\alpha\beta}] = \frac{1}{2} S^{\alpha\beta}, \quad (3.27)$$

in addition to the bulk field Eqs. (3.13).

The canonical field momentum density $\pi^{\alpha\beta}$ associated with any boundary Σ is defined in anchored coordinates (3.14) by

$$\pi^{\alpha\beta} = \frac{1}{2\kappa} \left. \frac{\partial \mathcal{L}_1}{\partial g_{\alpha\beta,0}} \right|_{\Sigma}. \quad (3.28)$$

It is more conveniently extracted from the Hamilton-Jacobi variational formula

$$\delta S_{EH} = \int_{\Sigma} \pi^{\alpha\beta} \delta g_{\alpha\beta} d^n \xi = \delta_g S_{bdy}(g, \Gamma), \quad (3.29)$$

modulo the bulk field Eqs. (3.13), and recalling (3.12). For the shell (3.25), we obtain from (3.29), (3.17), and (3.19) the jump

$$\kappa[\pi^{\alpha\beta}] = \frac{\partial[\epsilon \mathcal{K}(g, \Gamma)]}{\partial g_{\alpha\beta}}, \quad (3.30)$$

with g, Γ treated as independent, in analogy with the bulk

identity

$$\begin{aligned}\sqrt{-g}G_{\alpha\beta} &= \frac{\delta}{\delta g^{\alpha\beta}} \int \sqrt{-g}Rd^{n+1}x \\ &= \frac{\partial}{\partial g^{\alpha\beta}} [\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)].\end{aligned}\quad (3.31)$$

Explicit evaluation of (3.30) with the help of (3.24) yields

$$[\pi^{\alpha 0}] = 0, \quad (3.32)$$

$$\begin{aligned}\kappa[\pi^{ab}] &= \frac{1}{4}\sqrt{-g}[g_{cd,0}]\{g^{00}(g^{ab}g^{cd} - g^{ac}g^{bd}) \\ &\quad + 2g^{0(a}g^{b)(c}g^{d)0} - g^{a0}g^{b0}g^{cd} - g^{ab}g^{c0}g^{d0}\}.\end{aligned}\quad (3.33)$$

This is equivalent to a result (Eq. (17) of Barrabès-Israel [14]) previously derived by integration of the field equations through the layer.

Like the Bianchi identity $G^{\alpha\beta}{}_{|\beta} = 0$ for the bulk field Eqs. (3.13), the transversality condition (3.32),

$$[\pi^{\alpha\beta}](\partial_\beta z) = 0, \quad (3.34)$$

may be regarded as a consequence of the coordinate-invariance of the action. The boundary action (3.17) is invariant under the infinitesimal anchored coordinate transformation

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu + \frac{1}{2}z^2\eta^\mu(x). \quad (3.35)$$

This gives

$$\delta_L g_{\alpha\beta} = g_{\alpha\beta}(\bar{x}) - \bar{g}_{\alpha\beta}(\bar{x}) = 2\eta_{(\alpha}\partial_{\beta)}z + O(z), \quad (z \rightarrow 0). \quad (3.36)$$

Hence from (3.29),

$$\delta_L S_{EH} = \int_\Sigma \pi^{\alpha\beta} 2\eta_\alpha \partial_\beta z d^n \xi, \quad (3.37)$$

which is required to vanish by coordinate invariance of the total action, leading to (3.33).

For a nonlightlike layer ($g^{00}|_\Sigma \neq 0$), (3.33) simplifies to

$$\kappa[\pi^{ab}] = \frac{1}{4}\sqrt{-g}[g_{cd,0}]g^{00}(\Delta^{ab}\Delta^{cd} - \Delta^{ac}\Delta^{bd}) \quad (3.38)$$

where Δ^{ab} projects onto Σ and coincides with the inverse intrinsic metric in anchored coordinates

$$\Delta^{ab} = g^{ab} - \frac{g^{a0}g^{b0}}{g^{00}} = g^{ab} - \epsilon n^a n^b = {}^{(n)}g^{ab}. \quad (3.39)$$

The jump conditions (3.27) can then be reduced to their standard nonlightlike form [9]

$$-\frac{\epsilon}{2}|{}^{(n)}g|^{1/2}[K^{ab} - {}^{(n)}g^{ab}K] = \kappa[\pi^{ab}] = \frac{\kappa}{2}|{}^{(n)}g|^{1/2}S^{ab}, \quad (3.40)$$

in terms of the jump of extrinsic-curvature

$$[K_{ab}] = \frac{1}{2}\left[\frac{\partial g_{ab}}{\partial n}\right] = \frac{1}{2}|g^{00}|^{1/2}[\partial_0 g_{ab}], \quad (3.41)$$

and of the surface stress-energy tensor S^{ab} of the shell

$$S^{ab} = |{}^{(n)}g|^{-1/2}S^{ab}. \quad (3.42)$$

IV. GAUSS-BONNET ACTION

When the Einstein-Hilbert action (3.1) is augmented with term quadratic in the curvature the simple form (3.40) of the junction conditions is no longer valid. In fact, a distributional brane dynamics is no longer even possible in general, because the bulk field equations now involve inadmissible products of distributions in the thin-layer limit. The exception is the case when the quadratic terms have the Gauss-Bonnet form. In this case the bulk field equations are quasilinear, and a distributional description of thin layers remains viable. In this section, we shall examine how the junction conditions (3.40) are modified.

The Gauss-Bonnet action is (see the Appendix for further details)

$$S_{bulk} = \frac{\alpha}{2\kappa} \int \mathcal{L}_{GB}(g, \Gamma) d^{n+1}x, \quad (4.1)$$

where α is the Gauss-Bonnet coupling constant, and

$$\begin{aligned}\mathcal{L}_{GB}(g, \Gamma) &= \sqrt{-g}\mathcal{R} \\ &= \sqrt{-g}\frac{1}{4}\delta_{121'2'}^{343'4'}g^{25}g^{2'5'}R^1_{534}R^{1'}_{5'3'4'}.\end{aligned}\quad (4.2)$$

Because of the plethora of indices in such expressions, we have found it convenient in many instances to let numerical indices $1, 2, \dots, 1', 2', \dots$ do duty for literal indices $\alpha_1, \alpha_2, \dots, \alpha'_1, \alpha'_2, \dots$. They are understood to run from 0 to n , and repeated indices are to be summed. In contrast, we reserve the index 0 to stand just for its numerical self, and, as in (3.16), $x^0 = z = 0$ will represent the boundary Σ .

There is no holonomic split, analogous to (3.4), of \mathcal{L}_{GB} into a first-order piece and a pure divergence. We must therefore proceed more indirectly to find the supplementary boundary action which effectively removes second derivatives from the GB bulk action (4.1).

Varying the affine connection Γ in (4.2), and noting the Palatini and Bianchi identities, $\delta R^1_{534} = -2\delta\Gamma^1_{53|4}$ and $\delta^{43'4'}_{5'3'4'|4}R^{1'}_{5'3'4'} = 0$, we see that the Γ -variation of S_{bulk} involves a pure divergence, convertible to a surface integral:

$$\frac{2\kappa}{\alpha}\delta_\Gamma S_{bulk} = \int \sqrt{-g}\sigma^\lambda{}_{|\lambda} d^{n+1}x = \int_\Sigma \sqrt{-g}\sigma^\lambda \partial_\lambda z d^n x, \quad (4.3)$$

where

$$\sigma^\lambda = -\delta_{121'2'}^{3\lambda 3'4'}g^{25}R^{1'2'}_{3'4'}\delta\Gamma^1_{53}. \quad (4.4)$$

This is in complete analogy with (3.11) in the Einstein-Hilbert case, and it means that the bulk field equations

$$\frac{1}{2\kappa} \frac{\partial \mathcal{L}(g, \Gamma)}{\partial g^{\mu\nu}} + \frac{1}{2} \frac{\delta S_{mat}}{\delta g^{\mu\nu}} = 0, \quad (4.5)$$

with $\mathcal{L} = \mathcal{L}_{EH} + \alpha \mathcal{L}_{GB}$, are obtainable by simply differentiating the bulk Lagrangians (4.2) and (3.2) partially with respect to g .

When the bulk field Eqs. (4.5) are satisfied, the boundary term (4.3) gives the total variation of the bulk action. This involves $\delta\Gamma$, from which the variations $\delta g_{\alpha\beta,0}$ of transverse derivatives—i.e., of extrinsic-curvature in the non-lightlike case—must be removed by compensating variation of a suitable boundary action.

To isolate these extrinsic-curvature variations, we assume for simplicity that Σ is nonlightlike and introduce Gaussian coordinates based on Σ as in (3.20). Then

$$\begin{aligned} {}^{(n+1)}\Gamma_{ab}^c &= {}^{(n)}\Gamma_{ab}^c & {}^{(n+1)}\Gamma_{ab}^0 &= \epsilon K_{ab}, & {}^{(n+1)}\Gamma_{0b}^a & \\ &= K_b^a, & & & & \end{aligned} \quad (4.6)$$

where the extrinsic-curvature $K_{ab} = \frac{1}{2} \partial_0 g_{ab}$, and Latin indices run from 1 to n . We note also the Gauss-Codazzi relations

$${}^{(n+1)}R_{abcd} = {}^{(n)}R_{abcd} - 2\epsilon K_{a[c} K_{d]b}, \quad (4.7)$$

$${}^{(n+1)}R_{\mu bcd} n^\mu = 2K_{b[c;d]}, \quad (4.8)$$

where ∇ represents the covariant derivative associated with the n -dimensional metric g_{ab} . Retaining only the variations δK_{ab} in (4.4)—i.e., assuming $\delta g_{\alpha\beta}|_\Sigma = 0$ —we find

$$\sigma^0 = -\delta_{101'2'}^{303'4'} R^{1'2'}_{3'4'} 2\epsilon \delta K^1_3, \quad (4.9)$$

so that (4.3) becomes

$$\begin{aligned} \frac{2\kappa}{\alpha} \delta_K S_{bulk} &= -2\epsilon \delta_{011'2'}^{033'4'} \int_\Sigma ({}^{(n)}R^{1'2'}_{3'4'}) \\ &\quad - 2\epsilon K^1_{3'} K^{2'}_{4'} \delta K^1_3 \sqrt{-g} d^n x, \end{aligned} \quad (4.10)$$

where we have made use of (4.7).

The boundary action S_{bdy} must be chosen so that its K -variation cancels (4.10):

$$\delta_K (S_{bulk} + S_{bdy}) = 0, \quad (4.11)$$

Since the intrinsic Riemann tensor $({}^{(n)}R)$ is independent of K_{ab} , the choice

$$\begin{aligned} \frac{2\kappa}{\alpha} S_{bdy} &= 2\epsilon \delta_{011'2'}^{033'4'} \int_\Sigma ({}^{(n)}R^{1'2'}_{3'4'}) \\ &\quad - \frac{2\epsilon}{3} K^1_{3'} K^{2'}_{4'} K^1_3 \sqrt{-g} d^n x, \end{aligned} \quad (4.12)$$

meets this requirement, a result originally due to Myers [18].

We are now ready to derive the Gauss-Bonnet field momentum π^{ab} associated with Σ from the Hamilton-Jacobi equation

$$\delta(S_{bulk} + S_{bdy}) = \int_\Sigma \pi^{ab} \delta g_{ab} d\Sigma, \quad (4.13)$$

modulo the field Eqs. (4.5).

Evaluation of the left-hand side of (4.13) is simplified by noting that there is no K -contribution because of (4.11), and none from the metric factors in (4.2) because of (4.5). So $\delta S_{bulk} = \delta_\Gamma S_{bulk}$ is again given by (4.3) and (4.4), with $\delta\Gamma$ now effectively determined solely by the metric variation δg_{ab} , with K_{ab} fixed. We re-express (4.4) as

$$\begin{aligned} \sigma^0 &= -2\delta_{121'0}^{303'4'} g^{25} R^{1'0}_{3'4'} \delta\Gamma^1_{53} \\ &= 4\epsilon \delta_{011'2'}^{033'4'} K^1_{3;4'} g^{2'5'} \delta\Gamma^1_{5'3'}, \end{aligned} \quad (4.14)$$

where we have interchanged primed and unprimed indices in the second line and made use of the Gauss-Codazzi relation (4.8).

Turning now to δS_{bdy} in (4.13), one piece of this arises from variation of the intrinsic curvature $({}^{(n)}R)$ in (4.12):

$$\frac{2\kappa}{\alpha} \delta S_{bdy} \rightarrow 2\epsilon \delta_{011'2'}^{033'4'} \int_\Sigma g^{2'5'} \delta({}^{(n)}R^{1'}_{5'3'4'}) K^1_3 |{}^{(n)}g|^{1/2} d^n x. \quad (4.15)$$

Applying the intrinsic Palatini identity $\delta({}^{(n)}R^{1'}_{5'3'4'}) = -2\delta\Gamma^1_{5'[3';4']}$, one sees that the integrands (4.15) and (4.14) add up to an intrinsic divergence, which may be discarded.

All that remains to account for is the metric variations arising from $\sqrt{-g}$, the raised indices ($2'$ on $({}^{(n)}R)_{\dots}$, $1'2'1$ on the K -factors) in (4.12), and from $\delta\Gamma^1_{03} = \delta g^{1b} K_{b3}$ in the variation of the bulk action (4.3) and (4.4). The result of a straightforward calculation is

$$\begin{aligned} \frac{2\kappa}{\alpha} \pi_{ab} &= |{}^{(n)}g|^{1/2} \{ 6K_{a[m} K^n_{b]} + 6K_{b[m} K^n_{a]} \\ &\quad - 4g_{ab} K^l_{[l} K^m_{n]} + 4\epsilon K^{cd*(n)} R^*_{abcd} \}. \end{aligned} \quad (4.16)$$

where $*({}^{(n)}R^*_{ac}{}^{bd})$ —see the Eq. (A7) of the Appendix for its definition—is the left and right dual of the intrinsic curvature tensor of Σ . This is equivalent to results previously obtained by Davis and others [12].

Following the argument leading to (3.27), we conclude that the dynamics of a *nonlightlike* shell in Einstein-Gauss-Bonnet theory, with bulk action

$$S_{bulk} = \frac{1}{2\kappa} \int \sqrt{-g} (R + \alpha \mathcal{R}) d^{n+1}x + S_{mat}, \quad (4.17)$$

is governed by the junction conditions

$$[\pi^{ab}] = \frac{|{}^{(n)}g|^{1/2}}{2} S^{ab}; \quad \pi^{ab} = \pi_{EH}^{ab} + \pi_{GB}^{ab}, \quad (4.18)$$

where S^{ab} is the surface stress-energy tensor of the shell,

and is defined by (3.26) and (3.42). The momenta π_{EH}^{ab} and π_{GB}^{ab} are given by the first of (3.40) and (4.16) respectively, and jump together with the extrinsic-curvature across the shell. The action due to the shell augments (4.17) with a surface term—cf. (3.25) and (4.12)—and is equal to

$$S_{shell} = \frac{1}{2\kappa} \int_{\Sigma} ([B_{EH}] + \alpha[B_{GB}])d\Sigma + S_{mat,shell}, \quad (4.19)$$

where

$$B_{EH} = 2\epsilon\kappa K \quad (4.20)$$

$$B_{GB} = 2\epsilon\delta_{abe}^{cdf} \binom{(n)}{R}{}^{ab}{}_{cd} - \frac{2\epsilon}{3} K^a{}_c K^b{}_d K^e{}_f. \quad (4.21)$$

These results hold for non-lightlike shells, for which extrinsic-curvature is well defined. We now add some remarks on the lightlike case. Since this is a special limit of the timelike case, one might at first sight expect the resulting junction conditions to be simpler. Actually, however, this is far from being the case. Lightlike discontinuities propagate along characteristics of the field equations. It is a nontrivial matter to disentangle the lightlike discontinuities due to the shell from the accompanying gravitational shock waves. For pure Einstein theory this is still quite manageable, and we have presented the results in Sec. III. But we have not yet succeeded in reducing the lightlike junction conditions for Einstein-Gauss-Bonnet theory to a form that we consider worth publishing. The nature of the characteristics themselves is made more complicated by the fact that the field equations are now quasilinear only in the broad sense (linear in second derivatives but with coefficients depending on first derivatives).

It is, however, straightforward to obtain the Gauss-Bonnet boundary and shell actions in the lightlike case. Under an arbitrary metric variation, the total variation of the bulk action, is still given (modulo the bulk field equations) by (4.3) and (4.4), which we can express in the form

$$\sigma^0 = -4^*R^{*3012}\delta\Gamma_{1,23}, \quad (4.22)$$

where we have introduced the left and right dual of the curvature tensor, defined by (A3) and (A6) and in the appendix. In (4.22) we must separate out (and neutralize with a boundary action) the contribution of variations $\delta g_{\alpha\beta,0}$ of transverse derivatives:

$$\delta\Gamma_{[1,2]3} = -\partial_{[1}\delta g_{2]3} \rightarrow -(\partial_{[1}z)\delta g_{2]3,0}. \quad (4.23)$$

Hence the boundary action must have the compensating variation

$$\frac{2\kappa}{\alpha} \delta S_{bdy} = 4 \int {}^*R^{*0203} \delta g_{23,0} \sqrt{-g} d^n x. \quad (4.24)$$

Now, when the surfaces, $x^0 = const.$ are lightlike, the components ${}^*R^{*0203}$ do not contain second transverse derivatives $g_{\alpha\beta,00}$ and they are linear in first derivatives, i.e.,

$${}^*R^{*0a0b} = K_{(0)}^{ab} + K_{(1)}^{abcd} g_{cd,0}; \quad K_{(1)}^{abcd} = K_{(1)}^{cdab}, \quad (4.25)$$

where $K_{(0)}$, $K_{(1)}$ are independent of $g_{ab,0}$ ($K_{(1)}$ has a fairly complicated linear dependence, not reproduced here, on the ‘‘nominal extrinsic-curvature’’ $\mathcal{K}_{ab} = -\Gamma_{ab}^0$, which depends only on the intrinsic geometry for a lightlike Σ). From (4.24) and (4.25) we infer

$$\frac{2\kappa}{\alpha} S_{bdy} = 4 \int \left(K_{(0)}^{ab} + \frac{1}{2} K_{(1)}^{abcd} g_{cd,0} \right) g_{ab,0} \sqrt{-g} d^n x, \quad (4.26)$$

as the form of the Gauss-Bonnet boundary action.

V. CONCLUDING REMARKS

We have given an elementary, self-contained derivation of the action (useful for calculation of quantum tunneling amplitudes) and dynamical equations (i.e., junction conditions) for thin shells and branes in Einstein-Gauss-Bonnet theory. Our exposition has attempted to integrate as far as possible the treatment of timelike, spacelike and lightlike layers. For lightlike shells, the dynamics is complicated (especially in the Gauss-Bonnet case) by the fact that gravitational shock waves will in general accompany the shell. However, this problem should be ameliorated in situations of high symmetry, and this is currently under investigation.

ACKNOWLEDGMENTS

The present work was partly supported by the NATO Collaborative Linkage Grant No. 979723, by the Centre National de la Recherche Scientifique (France), and by NSERC of Canada.

APPENDIX A: NOTATIONS FOR GB

In 1932, Lanczos [19] noted that the Lagrangian

$$\mathcal{R} = \frac{1}{4} \delta_{121'2'}^{343'4'} R_{34}^{12} R_{3'4'}^{1'2'}, \quad (A1)$$

leads, like the Einstein-Hilbert Lagrangian

$$R = \frac{1}{2} \delta_{12}^{34} R_{34}^{12}, \quad (A2)$$

to field equations which involve no higher than second derivatives of the metric. R and \mathcal{R} are the first two members of a family of Lagrangians having the same property found by Lovelock [20]. The n^{th} member involves a product of n curvature factors formed by an obvious generalization of (A1) and (A2). It has the ‘‘Gauss-Bonnet’’ property of being a pure divergence in a space of dimension $2n$, and it vanishes identically in spaces of lower dimension. Properties of the Lovelock family are reviewed by Meissner and Olechowski [7] and Deruelle and Madore [7].

By defining the left and right dual of the curvature tensor

$${}^*R^*_{12}{}^{34} = \frac{1}{4}\delta_{121'2'}^{343'4'}R^{1'2'}_{3'4'}, \quad (\text{A3})$$

and using the identity

$$\begin{aligned} \delta_{121'2'}^{343'4'} &= 4!\delta_1^{[3}\delta_2^4\delta_{1'}^{3'}\delta_{2'}^{4']} \\ &= \delta_{12}^{34}\delta_{1'2'}^{3'4'} + \delta_{12}^{3'4'}\delta_{1'2'}^{34} - 2\delta_{12}^{3[3'}\delta_{1'2'}^{4]4'} - 2\delta_{12}^{4[4'}\delta_{1'2'}^{3]3'}, \end{aligned} \quad (\text{A4})$$

where $\delta_{12}^{34} = \delta_1^3\delta_2^4 - \delta_1^4\delta_2^3$, (A1) and (A3) can be recast as

$$\mathcal{R} = R^{12}{}_{34}{}^*, \quad R^*_{12}{}^{34} = (R_{1234})^2 - 4(R_{12})^2 + R^2, \quad (\text{A5})$$

(which was the form originally given by Lanczos). The tensor ${}^*R^*_{12}{}^{34}$ generalizes to spacetime with dimension higher than four the left and right dual of the curvature tensor of general relativity. It is equal to

$$\begin{aligned} {}^*R^*_{12}{}^{34} &= R_{12}{}^{34} - 4\delta_{[1}^{[3}R^4]_{2]} + \frac{1}{2}\delta_{12}^{34}R \\ &= R_{12}{}^{34} - 4\delta_{[1}^{[3}G^4]_{2]} - \frac{1}{2}\delta_{12}^{34}R. \end{aligned} \quad (\text{A6})$$

A similar quantity ${}^{(n)}R^*_{ab}{}^{cd}$ can be defined for the intrinsic curvature tensor

$$\begin{aligned} {}^{(n)}R^*_{ab}{}^{cd} &= \frac{1}{4}\delta_{aba'b'}^{cdc'd'}{}^{(n)}R^{a'b'}_{c'd'} \\ &= {}^{(n)}R_{ab}{}^{cd} - 4\delta_{[a}^{[c}{}^{(n)}R^d]_{b]} + \frac{1}{2}\delta_{ab}^{cd}{}^{(n)}R, \end{aligned} \quad (\text{A7})$$

where the latin indices run from 1 to n .

Because, as noted in Sec. IV, the Γ -variations of R and \mathcal{R} are pure divergences, the bulk field equations

$$G_{\mu\nu} + \alpha\mathcal{G}_{\mu\nu} = \kappa T_{\mu\nu}, \quad (\text{A8})$$

can be obtained from the bulk action

$$S_{bulk} = \frac{1}{2\kappa} \int \sqrt{-g}(R + \alpha\mathcal{R}) + S_{mat}, \quad (\text{A9})$$

simply by partially differentiating the Lagrangian, holding $R^1{}_{234}(\Gamma)$ fixed:

$$\mathcal{G}_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{R}) = 2\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}, \quad (\text{A10})$$

where

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}; \quad \mathcal{R}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\mathcal{R}_{\alpha\mu\beta\nu} + \mathcal{R}_{\alpha\nu\beta\mu}), \quad (\text{A11})$$

$$\mathcal{R}_{\alpha\beta\mu\nu} = \frac{1}{4}\delta_{1\alpha 3'\beta}^{1'2'34}R^1{}_{\mu 34}R^{3'}{}_{\nu 1'2'}. \quad (\text{A12})$$

(A10) confirms that the field equations follow the action in containing no higher than second derivatives of the metric. Noether's theorem and coordinate invariance of R and \mathcal{R} imply the contracted Bianchi identities

$$G^{\alpha\beta}{}_{|\beta} = \mathcal{G}^{\alpha\beta}{}_{|\beta} = 0, \quad (\text{A13})$$

which ensure compatibility of (A8) with the conservation law $T^{\alpha\beta}{}_{|\beta} = 0$. All the above considerations extend straightforwardly to higher members of the Lovelock family.

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