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Inflation from superstring and M-theory compactification with higher order corrections

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We study time-dependent solutions in M and superstring theories with higher-order corrections. We first present general field equations for theories of Lovelock type with stringy corrections in arbitrary dimensions. We then exhaust all exact and asymptotic solutions of exponential and power-law expansions in the theory with Gauss-Bonnet terms relevant to heterotic strings and in the theories with quartic corrections corresponding to the M theory and type II superstrings. We discuss interesting inflationary solutions that can generate enough *e* foldings in the early universe.

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I. INTRODUCTION

There are two major questions that confront the current cosmology. One is the horizon problem which asks why the early universe is highly homogeneous beyond causally disconnected regions. The other is the flatness problem: why does the present universe appear so extremely flat? It is widely believed that these can be resolved once we accept that our universe underwent an inflationary evolution in the early epoch. The recent cosmological observation confirmed the existence of the early inflationary cosmological epoch as well as the accelerated expansion of the present universe [1].

Though it is not difficult to construct cosmological models with these features if one introduces scalar fields with suitable potentials, it is desirable to derive such a model from the fundamental theories of particle physics that incorporate gravity without making special assumptions in the theories. The most promising candidates for such theories are the 10-dimensional superstrings or 11-dimensional M theory, which are hoped to give models of accelerated expansion of the universe upon compactification to four dimensions.

However, it was believed that there was a "no-go" theorem which forbids such solutions if the six- or seven-dimensional internal space is a time-independent nonsingular compact manifold without boundary [2]. Progress has recently been made by the discovery that this no-go theorem can be evaded if the size of the internal space depends on time. In fact, it has been shown that a model with a certain period of accelerated expansion can be obtained from the higher-dimensional vacuum Einstein equation if one takes the internal space hyperbolic and its size depending on time [3]. It has been shown [4] that this class of models is obtained from what are known as S branes [5,6] in the limit of vanishing flux of three-form fields (see also [7]). It is found that this wider class of solutions give accelerating universes for internal spaces with zero and positive curvature as well if the flux is introduced. Further discussion of this class of models has been given in Refs. [8–12].

It turns out, however, that the models thus obtained do not give enough *e* foldings necessary to explain the cosmological problems mentioned above [4,8]. The reason for this can be understood from the viewpoint of the effective four-dimensional theory, where one gets gravitational theory coupled to scalar fields which characterize the sizes of the internal spaces. Typically one finds exponential potential, and the slope for the scalar fields in this potential is too steep to produce large enough expansion [10]. Some efforts to overcome this problem were made in the present framework in Ref. [11].

The scale when the acceleration occurs in this type of model is basically governed by the Planck scale in the higher ten or 11 dimensions. With phenomena at such high energy, it is expected that we cannot ignore higher-order corrections such as higher derivative terms in the theories at least in the early universe. It is known that there are terms of higher orders in the curvature to the lowest effective supergravity action coming from superstrings or M theory [13–15]. With such corrections, they will significantly affect the inflation at the early stage of the evolution of our universe.

The cosmological models in higher dimensions were studied intensively in the 1980's by many authors [16–19]. It was shown that inflation is indeed possible with higher-order curvature corrections [17,18]. (The no-go theorem does not apply to theories with higher derivatives.) In particular the model with the Gauss-Bonnet (GB) terms is interesting because they are the special combination without ghost [20] and they exist as higher-order corrections in the heterotic string theories [13]. It was shown that there are two exponentially expanding solutions, which may be called generalized de Sitter solutions since the size of the internal space also depends on time (otherwise

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there is no solution of this type) [18]. In both solutions, the external space inflates, while the internal space shrinks exponentially. (There are also two time-reversed solutions, i.e., the external space shrinks exponentially but the internal space inflates.) One solution is stable and the other is unstable. Since the present universe is not in the phase of de Sitter expansion with this energy scale, we cannot use the stable solution for a realistic universe. If we adopt the unstable solution, on the other hand, we may not find sufficient inflation unless we fine-tune the initial values. The higher-order curvature terms called Lovelock gravity [21] and other types [22] were also considered in higher-dimensional cosmology.

A good news is that large e-folding number was obtained in these models. However, most of the work considered powers of scalar curvature or Lovelock gravity, which are not the types of corrections known to arise in superstring theories or M theory. It is thus important to examine if the above result of small e folding is modified with higher-order corrections expected in these fundamental theories. In our previous paper [23], we have presented brief report of our results on this problem for M theory. Here we give the details of our results in M theory as well as superstrings. We focus on the solutions to the vacuum Einstein equations with higher-order corrections since the basic feature can be obtained in this simple setting. In this paper, we exhaust exact solutions as well as past and future asymptotic solutions and discuss inflationary solutions among them. The past and future asymptotic solutions are useful in describing the inflation at the early universe and the present accelerating cosmology, respectively. In a forthcoming paper [24], we shall discuss more detailed properties of these solutions including stability and possible scenario for the history of our universe.

In the next section, we present our actions and field equations to be solved. We write down these for $D = 1 + p + q$ dimensions with p external and q internal space dimensions. Though we are mainly interested in $p = 3$ in this paper, there may be interesting applications if we keep the dimension p arbitrary. Also we give the equations for maximally symmetric spaces with nonvanishing curvatures. Their explicit forms are given in Appendixes A, B, and C. The Lovelock part of the field equations generalizes those in Ref. [21] and should be useful for examining various other nontrivial solutions. We also discuss the relation of the solutions to those in the Einstein frame in $(p + 1)$ dimensions.

In Sec. III, we examine solutions to the vacuum Einstein equations with GB corrections, corresponding to the heterotic strings [13,14]. We exhaust possible generalized de Sitter and power-law solutions, and find inflationary models for several types of internal spaces with positive, zero, and negative curvatures. We find that exponential type solutions are possible for flat external and internal spaces, corresponding to those solutions obtained in the 1980's by Ishihara [18].

In type II superstrings or M theory, it is known that the coefficient of the GB terms is zero and the first higher corrections start with R^4 terms [15]. We study this case and find interesting solutions of exponential and power-law expansions in Sec. IV.

Finally in Sec. V, we summarize our solutions and discuss inflationary solutions. We find that some solutions do not give inflations in the Einstein frame in four dimensions even though they appear to give inflations in the original frame, and that there are peculiar cases in which inflation appears to be realized in the Einstein frame though the external space is shrinking in the original frame. We discuss which solutions are suitable for interesting cosmologies.

II. VACUUM EINSTEIN EQUATIONS WITH HIGHER-ORDER TERMS

We consider the low-energy effective action for superstrings ($D = 10$) or M theory ($D = 11$) with higher-order corrections in D dimensions:

$$S = \sum_{n=1}^{4} S_n + S_S, \tag{2.1}$$

with

$$S_1 = S_{\rm EH} \equiv \frac{\alpha_1}{2\kappa_D^2} \int d^D x \sqrt{-g} R, \tag{2.2}$$

$$S_2 = S_{\rm GB} \equiv \frac{\alpha_2}{2\kappa_D^2} \int d^D x \sqrt{-g} [R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2], \tag{2.3}$$

$$S_3 = \frac{\alpha_3}{2\kappa_D^2} \int d^D x \sqrt{-g} \tilde{E}_6, \tag{2.4}$$

$$S_4 = \frac{\alpha_4}{2\kappa_D^2} \int d^D x \sqrt{-g} \tilde{E}_8, \tag{2.5}$$

$$S_S = \frac{\gamma}{2\kappa_D^2} \int d^D x \sqrt{-g} \tilde{J}_0, \tag{2.6}$$

where

$$\tilde{E}_{2n} = -\frac{1}{2^n (D-2n)!} \epsilon^{\alpha_1 \cdots \alpha_{D-2n} \mu_1 \nu_1 \ldots \mu_n \nu_n} \times \epsilon_{\alpha_1 \cdots \alpha_{D-2n} \rho_1 \sigma_1 \ldots \rho_n \sigma_n} R^{\rho_1 \sigma_1}{}_{\mu_1 \nu_1} \cdots R^{\rho_n \sigma_n}{}_{\mu_n \nu_n}, \tag{2.7}$$

$$\tilde{J}_0 = R^{\lambda\mu\nu\kappa} R_{\alpha\mu\nu\beta} R_\lambda{}^{\rho\sigma\alpha} R^\beta{}_{\rho\sigma\kappa} + \frac{1}{2} R^{\lambda\kappa\mu\nu} R_{\alpha\beta\mu\nu} R_\lambda{}^{\rho\sigma\alpha} R^\beta{}_{\rho\sigma\kappa}. \tag{2.8}$$

Here we have dropped contributions from forms and dilatons (if they exist), κ_D^2 is a D-dimensional gravitational constant, and we leave the coefficients $\alpha_1, \ldots, \alpha_4$ and γ free. The coefficient α_1 of the Einstein-Hilbert (EH) term is 1 by definition, and though α_3 is zero for all superstrings and M theory, we have included it since it will be useful for examining other cases. For the heterotic strings, the leading correction is given by the GB terms with the coefficient [13,14]:

$$\alpha_2 = \frac{1}{8}\alpha', \tag{2.9}$$

multiplied by an exponential factor of the dilaton, where α' is the Regge slope parameter. So it is not immediately obvious if our solutions are valid within the full string theories. Nevertheless they would give solutions for constant dilaton, and our results for these cases should be understood with this restriction. For the M theory in 11 dimensions, the coefficient for the GB terms α_2 vanishes, so we should consider fourth-order terms with the coefficients [15]:

$$\alpha_2 = \alpha_3 = 0, \qquad \alpha_4 = -\frac{\kappa_{11}^2 T_2}{3^2 \times 2^9 \times (2\pi)^4}, \qquad \gamma = -\frac{\kappa_{11}^2 T_2}{3 \times 2^4 \times (2\pi)^4}, \tag{2.10}$$

where $T_2 = (2\pi^2/\kappa_{11}^2)^{1/3}$ is the membrane tension. Type II superstring has the same couplings as M theory in 10 dimensions, so we can discuss this case if we keep the dilaton field constant and ignore the contributions from other fields. We should also note that contributions of the Ricci tensor $R_{\mu\nu}$ and scalar curvature R are not included in the fourth-order corrections (2.6) because these terms are not uniquely fixed.

A. Basic equations for cosmology

Since we are interested in cosmological solutions, we take the metric of our D-dimensional space as

$$ds_D^2 = -N^2(t)dt^2 + a^2(t)ds_p^2 + b^2(t)ds_q^2, \tag{2.11}$$

with

$$N(t) = e^{u_0(t)}, \qquad a(t) = e^{u_1(t)}, \qquad b(t) = e^{u_2(t)}, \tag{2.12}$$

where $D = 1 + p + q$. The external p- and internal q-dimensional spaces (ds_p^2 and ds_q^2) are chosen to be maximally symmetric. The curvature constants of those spaces are defined by σ_p and σ_q. The sign of σ_p (σ_q) determines the type of maximally symmetric spaces, i.e., σ_p (or σ_q) $= -1$, 0, and 1 denote a hyperbolic space, a flat Euclidean space, and a sphere, respectively. The volumes of the hyperbolic and flat spaces are made finite by dividing by discrete isometry.

From the variation of the total action (2.1) with respect to u_0, u_1 and u_2, we find three basic field equations:

$$F \equiv \sum_{n=1}^{4} F_n + F_S = 0, \tag{2.13}$$

$$F^{(p)} \equiv \sum_{n=1}^{4} f_n^{(p)} + X \sum_{n=1}^{4} g_n^{(p)} + Y \sum_{n=1}^{4} h_n^{(p)} + F_S^{(p)} = 0, \tag{2.14}$$

$$F^{(q)} \equiv \sum_{n=1}^{4} f_n^{(q)} + Y \sum_{n=1}^{4} g_n^{(q)} + X \sum_{n=1}^{4} h_n^{(q)} + F_S^{(q)} = 0, \tag{2.15}$$

where $X = \ddot{u}_1 - \dot{u}_0\dot{u}_1 + \dot{u}_1^2$, $Y = \ddot{u}_2 - \dot{u}_0\dot{u}_2 + \dot{u}_2^2$, and

$$\begin{gathered} F_n = F_n(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \qquad F_S = (u_0, u_1, u_2, \dot{u}_0, \dot{u}_1, \dot{u}_2, \ddot{u}_1, \ddot{u}_2, \dddot{u}_1, \dddot{u}_2, X, Y, \dot{X}, \dot{Y}), \\ f_n^{(p)} = f_n^{(p)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \qquad g_n^{(p)} = g_n^{(p)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \qquad h_n^{(p)} = h_n^{(p)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \\ F_S^{(p)} = F_S^{(p)}(u_0, u_1, u_2, \dot{u}_0, \dot{u}_1, \dot{u}_2, \ddot{u}_1, \ddot{u}_2, \dddot{u}_1, \dddot{u}_2, X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}), \qquad f_n^{(q)} = f_n^{(q)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \\ g_n^{(q)} = g_n^{(q)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \qquad h_n^{(q)} = h_n^{(q)}(u_0, \dot{u}_1, \dot{u}_2, A_p, A_q), \\ F_S^{(q)} = F_S^{(q)}(u_0, u_1, u_2, \dot{u}_0, \dot{u}_1, \dot{u}_2, \ddot{u}_1, \ddot{u}_2, \dddot{u}_1, \dddot{u}_2, X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}), \end{gathered} \tag{2.16}$$

are explicitly given in Appendix A. Here A_p and A_q are defined by

$$\begin{aligned} A_p &= \dot{u}_1^2 + \sigma_p \exp[2(u_0 - u_1)], \\ A_q &= \dot{u}_2^2 + \sigma_q \exp[2(u_0 - u_2)]. \end{aligned} \tag{2.17}$$

Since u_0 is a gauge freedom of time coordinate, we have three equations for two variables u_1 and u_2. It looks like an over-determinant system. However, these three equations are not independent. In fact, we can derive the following equation after bothersome calculation:

$$\dot{F} + (p\dot{u}_1 + q\dot{u}_2 - \dot{u}_0)F = p\dot{u}_1 F^{(p)} + q\dot{u}_2 F^{(q)}. \tag{2.18}$$

If $F = 0$ and $F^{(p)} = 0$ (or $F^{(q)} = 0$), we obtain $\dot{u}_2 F^{(q)} = 0$ (or $\dot{u}_1 F^{(p)} = 0$), since $F = 0$ is a constraint equation and its time derivative also vanishes. The third equation $F^{(q)} = 0$ (or $F^{(p)} = 0$) is then automatically satisfied unless $\dot{u}_2 = 0$ (or $\dot{u}_1 = 0$). On the other hand, suppose we have only $F^{(p)} = 0$ and $F^{(q)} = 0$. Then Eq. (2.18) gives

$$F = Ce^{u_0 - (pu_1 + qu_2)}, \tag{2.19}$$

where C is an integration constant. Upon imposing the initial condition $F = 0$ in (2.19), we get $C = 0$ and hence $F = 0$. This means that the constraint equation is satisfied if other dynamical equations are solved *and* it is initially satisfied. Consequently, it is in general not enough to solve the dynamical equations $F^{(p)} = F^{(q)} = 0$ only, but enough to solve the two equations $F = 0$ and $F^{(p)} = 0$ (or $F^{(q)} = 0$) instead of trying to solve all three equations.

B. Ansatz for solutions

We now analyze our basic equations (2.13), (2.14), and (2.15) for several models and look for inflationary solutions. Since we are interested in analytic solutions, we study the following two cases:

(i) *Generalized de Sitter solutions*: Using a cosmic time, i.e., $u_0 = 0$, an exponential expansion of each scale factor is given by $u_1 = \mu t + \ln a_0$, and $u_2 = \nu t + \ln b_0$, where μ, ν, a_0 and b_0 are constants.

(ii) *Power-law solutions*: To find a power-law solution, although we can discuss it with the above cosmic time, we use a different time gauge, which is defined by $u_0 = t$. Using this time coordinate, a power-law solution is given by $u_1 = \mu t + \ln a_0$, and $u_2 = \nu t + \ln b_0$, where μ and ν are constants.

The choice of time coordinate in (ii) is more convenient than the cosmic time in (i) because we can discuss both solutions in the same set up. Namely, we can write

$$u_0 = \epsilon t, \qquad u_1 = \mu t + \ln a_0, \quad \text{and} \quad u_2 = \nu t + \ln b_0, \tag{2.20}$$

where $\epsilon = 0$ for case (i), while $\epsilon = 1$ for case (ii). In the latter case, in terms of a new cosmic time $\tau = e^t$, we see that the solution gives the power-law behavior:

$$a = e^{u_1} = a_0 \tau^{\mu}, \quad \text{and} \quad b = e^{u_2} = b_0 \tau^{\nu}. \tag{2.21}$$

Note that when the curvature constant σ_p (or σ_q) vanishes, a_0 and b_0 are arbitrary but we can set $a_0 = 1$ (or $b_0 = 1$) because such a numerical constant can be absorbed by rescaling of the spatial coordinates.

C. Description in the Einstein frame

After the internal space is compactified, we observe physical variables in the $(1 + p)$-dimensional Einstein frame, which is defined by

$$ds_D^2 = e^{-2[q/(p-1)]\phi}(-dt_E^2 + a_E^2 ds_p^2) + e^{2\phi} ds_q^2, \tag{2.22}$$

where t_E, a_E, and $\phi(= u_2 = \ln b)$ are a cosmic time, a scale factor, and a scalar field parametrizing the size of the internal space in the Einstein frame, respectively. Comparing Eqs. (2.11) and (2.22), we find the relations

$$e^{u_0} dt = \pm e^{-[q/(p-1)]\phi} dt_E, \tag{2.23}$$

$$e^{u_1} = e^{-[q/(p-1)]\phi} a_E, \tag{2.24}$$

$$u_2 = \phi. \tag{2.25}$$

The sign $\pm$ in Eq. (2.23) is chosen so that two time coordinates proceed in the same (future) direction. The solutions in the form (2.20) can be rewritten in the Einstein frame as follows:

(i) $\epsilon = 0$ *and* $\nu > 0$

$$t_E = t_E^{(0)} e^{[q/(p-1)]\nu t}, \tag{2.26}$$

$$a_E = a_E^{(0)} \left| \frac{t_E}{t_E^{(0)}} \right|^{\lambda}, \tag{2.27}$$

$$\phi = \phi^{(0)} + \frac{(p-1)\nu}{q} \ln \left| \frac{t_E}{t_E^{(0)}} \right|, \tag{2.28}$$

where $t_E^{(0)}(>0)$, $a_E^{(0)}$, and $\phi^{(0)}$ are integration constants, and

$$\lambda = 1 + \frac{(p-1)\mu}{q\nu}. \tag{2.29}$$

If $\mu/\nu > 0$, this solution gives a power-law inflation in the Einstein frame. $t = -\infty$ and $t = \infty$ correspond to $t_E = 0$ and $t_E = \infty$, respectively.

(ii) $\epsilon = 0$ *and* $\nu < 0$

$$t_E = t_E^{(0)} e^{[q/(p-1)]\nu t}, \tag{2.30}$$

where $t_E^{(0)}(<0)$ is an integration constant, and a_E and ϕ are the same as Eqs. (2.27) and (2.28). $t \in (-\infty, \infty)$ is transformed into $t_E \in (-\infty, 0)$. $t = -\infty$ and $t = \infty$ correspond to $t_E = -\infty$ and $t_E = 0$, respectively. In this case the inflationary solutions in the Einstein frame are obtained for $\lambda < 0$, i.e., when $t_E \to 0_-$, a_E diverges as $|t_E|^{-|\lambda|}$. This is called a superinflation in Kaluza-Klein cosmology [16,25]. Since the asymptotic behavior as $t_E \to 0$ does not explain the present universe, we have to avoid a singularity at $t_E = 0$. Then, we have to clarify a mechanism to avoid the singularity at $t_E = 0$. The same problem was found in the Kaluza-Klein inflation in the 1980's [16]. In a pre–big bang scenario, we also find a similar inflation in the string frame [26].

Note that even for $\mu > 0$, this class of solutions in general do not give inflationary solutions in the Einstein frame.

(iii) $\epsilon = 0$ *and* $\nu = 0$

$$t_E = e^{[q/(p-1)]\phi^{(0)}} t, \tag{2.31}$$

$$a_E = a_E^{(0)} \exp[\mu e^{-[q/(p-1)]\phi^{(0)}} t_E], \tag{2.32}$$

$$\phi = \phi^{(0)}, \tag{2.33}$$

where $a_E^{(0)}$ and $\phi^{(0)}$ are constants. Rescaling the time coordinate, we can set $\phi^{(0)} = 0$, i.e., $t_E = t$ and $a_E \propto \exp(\mu t_E)$. This solution gives an exponential expansion even in the Einstein frame for $\mu > 0$.

(iv) $\epsilon = 1$ *and* $\nu > -[(p-1)/q]$

$$t_E = t_E^{(0)} e^{\{1+[q/(p-1)]\nu\}t}, \quad (2.34)$$

$$a_E = a_E^{(0)} \left| \frac{t_E}{t_E^{(0)}} \right|^{\lambda}, \quad (2.35)$$

$$\phi = \phi^{(0)} + \frac{(p-1)\nu}{(p-1)+q\nu} \ln \left| \frac{t_E}{t_E^{(0)}} \right|, \quad (2.36)$$

where $t_E^{(0)}(>0)$, $a_E^{(0)}$, and $\phi^{(0)}$ are integration constants, and

$$\lambda = \frac{(p-1)\mu + q\nu}{(p-1)+q\nu}. \quad (2.37)$$

This solution gives a power-law inflation in the Einstein frame if $\mu > 1$. Note that $\mu > 1$ gives a power-law inflation in the original frame. $t = -\infty$ and $t = \infty$ correspond to $t_E = 0$ and $t_E = \infty$, respectively.

(v) $\epsilon = 1$ *and* $\nu < -[(p-1)/q]$

$$t_E = t_E^{(0)} e^{\{1+[q/(p-1)]\nu\}t}, \quad (2.38)$$

where $t_E^{(0)}(<0)$ is an integration constant, and a_E and ϕ are the same as Eqs. (2.35) and (2.36). $t \in (-\infty, \infty)$ is transformed into $t_E \in (-\infty, 0)$. $t = -\infty$ and $t = \infty$ correspond to $t_E = -\infty$ and $t_E = 0$, respectively. Here the inflationary solutions in the Einstein frame are obtained for $\lambda < 0$ (a superinflation).

(vi) $\epsilon = 1$ *and* $\nu = -[(p-1)/q]$

$$t_E = e^{\{q/[(p-1)]\}\phi^{(0)}} t, \quad (2.39)$$

$$a_E = a_E^{(0)} \exp[(\mu - 1) e^{-\{q/[(p-1)]\}\phi^{(0)}} t_E], \quad (2.40)$$

$$\phi = \phi^{(0)} - \frac{(p-1)}{q} e^{-\{q/[(p-1)]\}\phi^{(0)}} t_E, \quad (2.41)$$

where $\phi^{(0)}$ can be set zero by rescaling time coordinate, i.e., $t_E = t$. This solution gives an exponential expansion in the Einstein frame if $\mu > 1$. Rescaling the time coordinate, we find $t_E = t$, $a_E \propto \exp[(\mu - 1)t_E]$, and $\phi = -[(p-1)/q]t_E$.

These will be useful in discussing the results in the Einstein frame.

III. SOLUTIONS IN HETEROTIC STRINGS

The higher-order corrections for heterotic strings start with the GB terms. So in this section, we first study various solutions of the field equations only with EH and GB terms, which are given by

$$F_1 + F_2 = 0, \quad (3.1)$$

$$F_1^{(p)} + F_2^{(p)} = 0, \quad (3.2)$$

$$F_1^{(q)} + F_2^{(q)} = 0, \quad (3.3)$$

where

$$\begin{aligned} F_1 &= F_1(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_2 &= F_2(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_1^{(p)} &= f_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) + X g_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + Y h_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_2^{(p)} &= f_2^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) + X g_2^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + Y h_2^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_1^{(q)} &= f_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) + Y g_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + X h_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_2^{(q)} &= f_2^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) + Y g_2^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + X h_2^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q), \end{aligned} \quad (3.4)$$

whose explicit expressions are given in Appendix B. Here we have three equations for two unknown parameters μ and ν. However, two of them are independent because we have one constraint Eq. (2.18).

From Eq. (B2), we expect there may exist no exact solution except for the case of $\sigma_p = \sigma_q = 0$. However, even for the case of $\sigma_p \neq 0$ or $\sigma_q \neq 0$, we may have some asymptotic analytic solutions either in the future direction ($t \to \infty$) or in the past direction ($t \to -\infty$), which describe cosmologies in these time regions. We classify solutions to Eqs. (3.1), (3.2), and (3.3) by the signatures of σ_p and σ_q.

A. $\sigma_p = \sigma_q = 0$

In this case, $A_p = \mu^2$ and $A_q = \nu^2$ are constants. We have two classes of solutions:

(i) *exact solutions* for $\epsilon = 0$ (generalized de Sitter solutions),

(ii) *asymptotic solutions* for $\epsilon = 1$ (power-law solutions),

which are summarized below.

1. Generalized de Sitter solutions ($\epsilon = 0$)

We have three basic equations one of which is a constraint equation. In this case, however, as discussed in Appendix C, if the solution is not the Minkowski space ($\mu = \nu = 0$), we can find two independent algebraic equations without any constraint equation:

$$F_1(\mu, \nu) + F_2(\mu, \nu) = 0, \quad (3.5)$$

$$H_1(\mu, \nu) + H_2(\mu, \nu) = 0, \quad (3.6)$$

which are given in Appendix C, i.e.,

$$\alpha_1[p_1\mu^2 + q_1\nu^2 + 2pq\mu\nu] + \alpha_2[p_3\mu^4 + 6p_1q_1\mu^2\nu^2 + q_3\nu^4 + 4\mu\nu(p_2q\mu^2 + pq_2\nu^2)] = 0, \quad (3.7)$$

$$(\mu - \nu)\{\alpha_1 + 2\alpha_2[(p-1)_2\mu^2 + 2(p-1)(q-1)\mu\nu + (q-1)_2\nu^2]\} = 0. \quad (3.8)$$

Now we have two branches of solutions: one is $\mu = \nu$, and the other is

$$\alpha_1 + 2\alpha_2[(p-1)_2\mu^2 + 2(p-1)(q-1)\mu\nu + (q-1)_2\nu^2] = 0. \quad (3.9)$$

(i) Inserting $\mu = \nu$ into Eq. (3.7), we find either Minkowski space $\mu = \nu = 0$, or another solution, i.e.,

$$\mu = \nu = \pm\sqrt{-\frac{\alpha_1(p_1 + q_1 + 2pq)}{\alpha_2[p_3 + q_3 + 6p_1q_1 + 4(p_2q + pq_2)]}}, \quad (3.10)$$

if $\alpha_2 < 0$. For $\alpha_2 > 0$, there is no real solution.

(ii) When we assume Eq. (3.9), eliminating α_2 from Eq. (3.7), we obtain the fourth-order equation:

$$(p+1)_2\mu^4 + 4p_1(p-1)(q-1)\mu^3\nu + 2(p-1)(q-1)(3pq - 2p - 2q)\mu^2\nu^2 + 4q_1(p-1)(q-1)\mu\nu^3 + (q+1)_2\nu^4 = 0. \quad (3.11)$$

If $\nu = 0$, we have $\mu = 0$, which gives Minkowski space. Except for this trivial solution, Eq. (3.11) is reduced to the fourth-order equation for $h = \mu/\nu$:

$$(p+1)_2h^4 + 4p_1(p-1)(q-1)h^3 + 2(p-1)(q-1)(3pq - 2p - 2q)h^2 + 4q_1(p-1)(q-1)h + (q+1)_2 = 0. \quad (3.12)$$

We then have four solutions for h. For a solution h of this equation, we get ν and then μ from Eq. (3.9), which is rewritten as

$$\alpha_1 + 2\alpha_2[(p-1)_2h^2 + 2(p-1)(q-1)h + (q-1)_2]\nu^2 = 0. \quad (3.13)$$

We thus find

$$\nu = \pm\sqrt{-\frac{\alpha_1}{2\alpha_2[(p-1)_2h^2 + 2(p-1)(q-1)h + (q-1)_2]}}, \quad (3.14)$$

$$\mu = h\nu. \quad (3.15)$$

For ν to take a real value, we have a constraint

$$\alpha_2[(p-1)_2h^2 + 2(p-1)(q-1)h + (q-1)_2] < 0. \quad (3.16)$$

For the heterotic strings with $\alpha_1 = 1$, $\alpha_2 = \alpha'/8$, $p = 3$, and $q = 6$, we have two real solutions for h in Eq. (3.12). Using these two solutions, we have the following four solutions in the unit $\alpha' = 1$:

$$(\mu, \nu) = (1.36601, -0.965665), (2.50608, -0.391608), \quad (3.17)$$

and the time-reversed ones $(-\mu, -\nu)$. In both solutions in (3.17), the external space inflates, while the internal space shrinks exponentially. It was shown that one solution is stable and the other is unstable [18]. Since the present universe is not in the phase of de Sitter expansion with this energy scale, we cannot use the stable solution for a realistic universe. If we adopt the unstable solution, on the other hand, we may not find sufficient inflation unless we fine-tune the initial values. We shall also discuss if these solutions give an inflation in the four-dimensional Einstein frame in Sec. V.

Though it is known that there is no GB terms for M theory, it may be instructive to find solutions for $p = 3$ and $q = 7$:

$$(\mu, \nu) = (1.45839, -0.838657), (2.53838, -0.331212), \quad (3.18)$$

and the time-reversed ones $(-\mu, -\nu)$. Thus we find that the result does not change qualitatively.

2. Power-law solutions ($\epsilon = 1$)

Setting $\epsilon = 1$ in Eqs. (B4) and (B5), we find that the EH action is dominant as $t \to \infty$, while the GB action becomes dominant as $t \to -\infty$. Here we present asymptotic power-law solutions for each case.

(i) *Future asymptotic solutions ($t \to \infty$)*: Our basic equations reduce to

$$p_1\mu^2 + q_1\nu^2 + 2pq\mu\nu = 0, \quad (3.19)$$

$$q\nu(\nu - \mu - 1) - (p-1)\mu = 0, \quad (3.20)$$

$$p\mu(\mu - \nu - 1) - (q-1)\nu = 0. \quad (3.21)$$

We can easily show that these three equations are equivalent to the following two equations, if it is not Minkowski space ($\mu = \nu = 0$):

$$p\mu^2 + q\nu^2 = 1, \qquad p\mu + q\nu = 1, \quad (3.22)$$

which is a special case of Kasner solutions. We have a solution

$$\mu = \frac{p \pm \sqrt{pq(p+q-1)}}{p(p+q)}, \quad \nu = \frac{q \mp \sqrt{pq(p+q-1)}}{q(p+q)}. \tag{3.23}$$

For $p = 3$ and $q = 6$, we find $(\mu, \nu) = (5/9, -1/9), (-1/3, 1/3)$. They are also future asymptotic solutions for type II superstrings. Note that a general Kasner solution is given by

$$\sum_{i=1}^{p+q} \mu_i^2 = 1, \qquad \sum_{i=1}^{p+q} \mu_i = 1, \tag{3.24}$$

where each scale factor is assumed as $e^{u_i} = \tau^{\mu_i}$ ($i = 1, \ldots, p+q$). Apparently $\mu < 1$ in this class of solutions and they do not give inflation according to the discussions in Sec. II C.

(ii) *Past asymptotic solutions ($t \to -\infty$)*: Our equations are

$$p_3\mu^4 + 6p_1q_1\mu^2\nu^2 + q_3\nu^4 + 4\mu\nu(p_2q\mu^2 + pq_2\nu^2) = 0, \tag{3.25}$$

$$q\nu[(q-1)_2\nu^3 + (q-1)(2p-q)\mu\nu^2 + (p-1)(p-2q)\mu^2\nu - (p-1)_2\mu^3] - [q_2\nu^3 + 3(p-1)q_1\mu\nu^2 + 3(p-1)_2q\mu^2\nu + (p-1)_3\mu^3] = 0, \tag{3.26}$$

$$p\mu[(p-1)_2\mu^3 - (p-1)(p-2q)\mu^2\nu - (q-1)(2p-q)\mu\nu^2 - (q-1)_2\nu^3] - [p_2\mu^3 + 3p_1(q-1)\mu^2\nu + 3p(q-1)_2\mu\nu^2 + (q-1)_3\nu^3] = 0. \tag{3.27}$$

We can show that Eq. (3.25) is derived from Eqs. (3.26) and (3.27), and these three equations are not independent. We can use any two of them to find the solutions.

We obtain the following equation from the difference between Eqs. (3.26) and (3.27):

$$(p\mu + q\nu - 3)[(p-1)_2\mu^3 - (p-1)(p-2q)\mu^2\nu - (q-1)(2p-q)\mu\nu^2 - (q-1)_2\nu^3] = 0. \tag{3.28}$$

Thus we have either

$$p\mu + q\nu - 3 = 0, \tag{3.29}$$

or

$$(p-1)_2\mu^3 - (p-1)(p-2q)\mu^2\nu - (q-1)(2p-q)\mu\nu^2 - (q-1)_2\nu^3 = 0. \tag{3.30}$$

(a) $p\mu + q\nu - 3 = 0$: Here $\nu = 0$ gives $\mu = 3/p$, which is incompatible with Eq. (3.25). Thus $\nu \neq 0$, and Eq. (3.25) is rewritten by $h = \mu/\nu$:

$$p_3h^4 + 4p_2qh^3 + 6p_1q_1h^2 + 4pq_2h + q_3 = 0. \tag{3.31}$$

Once we find the solution of this fourth-order equation, μ and ν are given as

$$\mu = \frac{3h}{ph+q}, \qquad \nu = \frac{3}{ph+q}. \tag{3.32}$$

If $p = 3$, Eq. (3.31) reduces to a third order equation. We can formally find three solutions for h as

$$h = -\frac{q-1}{2} + \sqrt{\frac{q^2-1}{3}}\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{1}{q}\sqrt{\frac{q^2-4}{3}}\right) + \frac{2\pi n}{3}\right]; \qquad n = 0, 1, 2. \tag{3.33}$$

For the heterotic strings with $p = 3$ and $q = 6$, we find three real solutions:

$$\begin{aligned} h &= -5.868\,61, & (\mu, \nu) &= (1.516\,98, -0.258\,49), \\ h &= -1.304\,95, & (\mu, \nu) &= (-1.877\,48, 1.438\,74), \\ h &= -0.326\,45, & (\mu, \nu) &= (-0.195\,06, 0.597\,53). \end{aligned} \tag{3.34}$$

The first solution gives an inflation and is interesting.

(b) *Equation (3.30)*: Using $h = \mu/\nu$, Eq. (3.30) reduces to a third order equation:

$$\begin{aligned} &(p-1)_2h^3 - (p-1)(p-2q)h^2 - (q-1)(2p-q)h - (q-1)_2 \\ &= (h-1)[(p-1)_2h^2 + 2(p-1)(q-1)h + (q-1)_2] = 0. \end{aligned} \tag{3.35}$$

We then have either $h = 1$, or

$$h = \frac{1}{(p-1)(p-2)}\left[-(p-1)(q-1) \pm \sqrt{(p-1)(q-1)(p+q-3)}\right]. \tag{3.36}$$

However, those are not consistent with Eq. (3.25). Hence we have no solution in this case.

B. $\sigma_p = 0, \sigma_q \neq 0$ (or $\sigma_p \neq 0, \sigma_q = 0$)

1. Generalized de Sitter solutions ($\epsilon = 0$)

Since $A_p = \mu^2$ and $A_q = \nu^2 + \tilde{\sigma}_q e^{-2\nu t}$, it is easy to see that ν must vanish for the existence of exact solutions, where $\tilde{\sigma}_q = \sigma_q/b_0^2$. (We also introduce $\tilde{\sigma}_p = \sigma_p/a_0^2$ for further calculations.) Setting $\nu = 0$, we have

$$\begin{aligned} F &= F_1 + F_2, \\ &= \alpha_1[p_1\mu^2 + q_1\tilde{\sigma}_q] + \alpha_2[p_3\mu^4 + 2p_1q_1\mu^2\tilde{\sigma}_q \\ &\quad + q_3\tilde{\sigma}_q^2] = 0, \\ F^{(p)} &= f_1^{(p)} + f_2^{(p)} + (g_1^{(p)} + g_2^{(p)})X + (h_1^{(p)} + h_2^{(p)})Y \\ &= \alpha_1[p_1\mu^2 + q_1\tilde{\sigma}_q] + \alpha_2[p_3\mu^4 + 2p_1q_1\mu^2\tilde{\sigma}_q \\ &\quad + q_3\tilde{\sigma}_q^2] = 0, \\ F^{(q)} &= f_1^{(q)} + f_2^{(q)} + (g_1^{(q)} + g_2^{(q)})X + (h_1^{(q)} + h_2^{(q)})Y \\ &= \alpha_1[(p+1)_0\mu^2 + (q-1)_2\tilde{\sigma}_q] \\ &\quad + \alpha_2[(p+1)_2\mu^4 + 2(p+1)_0(q-1)_2\mu^2\tilde{\sigma}_q \\ &\quad + (q-1)_4\tilde{\sigma}_q^2] = 0. \end{aligned} \tag{3.37}$$

Although the first and second equations are identical, the third one is different. Since we have two undetermined variables μ and $\tilde{\sigma}_q$ for two independent equations, we may have some solutions. However, we find that there is no real solution at least for $p = 3$ and $q = 6$.

If $\nu \neq 0$, we have only asymptotic solutions. If $\nu > 0 (< 0)$,

$$A_q \to \nu^2 \quad \text{as } t \to +\infty\ (-\infty), \tag{3.38}$$

$$A_q \to \tilde{\sigma}_q e^{-2\nu t} \quad \text{as } t \to -\infty\ (+\infty). \tag{3.39}$$

Then, for the case (3.38), as $t \to +\infty(-\infty)$, we recover the previous generalized de Sitter solutions (3.17). For the heterotic strings, we find that

$$\begin{aligned} (\mu, \nu) &\sim (-1.366\,01, 0.965\,665), (-2.506\,08, 0.391\,608), \\ &\quad \text{as } t \to +\infty, \\ (\mu, \nu) &\sim (1.366\,01, -0.965\,665), (2.506\,08, -0.391\,608), \\ &\quad \text{as } t \to -\infty. \end{aligned} \tag{3.40}$$

On the other hand, for the case (3.39), as $t \to -\infty(+\infty)$, we do not find any asymptotic solutions within our ansatz for solutions. This does not mean that there is no time-dependent solution to this system but simply implies that there is no solution within our ansatz. We can study the evolution of the system by a numerical analysis.

In the case of $\sigma_p \neq 0$ and $\sigma_q = 0$, we can obtain our result by exchanging p, μ and q, ν. We have only asymptotic solutions. For the heterotic strings, we find that

$$\begin{aligned} (\mu, \nu) &\sim (1.366\,01, -0.965\,665), (2.506\,08, -0.391\,608), \\ &\quad \text{as } t \to +\infty, \\ (\mu, \nu) &\sim (-1.366\,01, 0.965\,665), (-2.506\,08, 0.391\,608), \\ &\quad \text{as } t \to -\infty. \end{aligned} \tag{3.41}$$

There are solutions in which our space inflates and internal space shrinks at late times, but no such solutions at early era.

2. Power-law solutions ($\epsilon = 1$)

Next we turn to the power-law solutions. Let us classify the solutions into three cases depending on ν:

(i) $\nu > 1$: In this case, as $t \to \infty$, the EH term becomes dominant and we obtain the asymptotic solution in the previous Sec. III A 2. However, no solutions satisfy the condition $\nu > 1$ [see Eq. (3.22)]. Thus there is no asymptotic solution of our form. As $t \to -\infty$, the GB curvature terms become dominant, but we find no consistent solution since A_q diverges without any balancing term.

(ii) $\nu < 1$: As $t \to \infty$ with EH dominance, we again find no consistent solution. As $t \to -\infty$ with GB dominance, we obtain the asymptotic solutions (3.34) in the previous section. For the heterotic strings, imposing the condition of $\nu < 1$, we find two solutions, which are $(\mu, \nu) = (1.516\,98, -0.258\,49)$, and $(-0.195\,06, 0.597\,53)$.

(iii) $\nu = 1$: This case is a little bit special because both $A_p = \mu^2$ and $A_q = 1 + \tilde{\sigma}_q$ are constants. Then the time dependence in the basic Eqs. (B4) and (B5) is only e^{-t} from the EH action and e^{-3t} from the GB action. In the future asymptotic solutions, as $t \to \infty$, the EH term becomes dominant, and we are left with

$$p_1\mu^2 + q_1(1 + \tilde{\sigma}_q) + 2pq\mu = 0, \tag{3.42}$$

$$(p + q - 1)\mu = 0, \tag{3.43}$$

$$2p\mu(\mu - 2) - 2(q - 1)(1 + \tilde{\sigma}_q) = 0. \tag{3.44}$$

We then have an asymptotic solution $\mu = 0$, $\nu = 1$, and $\sigma_q = -1$ ($b_0 = 1$). This is just Minkowski spacetime with Milne-type time slicing. We find that this solution is also consistent with the GB term because $A_p = 0, A_q = 0$, and $\mu = 0$. Hence this Minkowski solution is an exact one to the whole system.

Though this appears a rather trivial solution in the frame we are discussing, it gives power-law solutions in the Einstein frame in $(p + 1)$ dimensions and a nontrivial solution, as discussed in Sec. II C. Unfortunately, the scale factor behaves like $a_E \sim t_E^{q/(p-1+q)}$ and it is not an inflationary solution.

In the past asymptotic region $t \to -\infty$, the GB term becomes dominant. We have

$$p_3\mu^4 + 4p_2q\mu^3 + 4p_1q_1\mu^2 + 2p_1q_1\mu^2(1+\tilde{\sigma}_q) + 4pq_2\mu(1+\tilde{\sigma}_q) + q_3(1+\tilde{\sigma}_q)^2 = 0, \quad (3.45)$$

$$(p+q-3)\mu[(p-1)_2\mu^2 + 2(p-1)q\mu + q_1(1+\tilde{\sigma}_q)] = 0, \quad (3.46)$$

$$p(p+q-3)\mu^2[(p-1)_2\mu^2 + 2(p-1)q\mu + q_1(1+\tilde{\sigma}_q)] = 0. \quad (3.47)$$

Then we have either $\mu = 0$ or

$$(p-1)_2\mu^2 + 2(p-1)q\mu + q_1(1+\tilde{\sigma}_q) = 0. \quad (3.48)$$

For $\mu = 0$, inserting it into Eq. (3.45) gives $\sigma_q = -1$ and $b_0 = 1$. It is just the Minkowski spacetime with Milne-type time slicing found above. When Eq. (3.48) is satisfied, eliminating $(1+\tilde{\sigma}_q)$ in Eq. (3.45) yields the equation for μ:

$$(p+q-2)\mu^2[(p-2)(2pq-3p-q+3)\mu^2 + 4q(pq-2p-q+3)\mu + 2q^2(q-3)] = 0. \quad (3.49)$$

We find the solution

$$\mu = \frac{q}{(p-2)(2pq-3p-q+3)} \times [-2(pq-2p-q+3) \pm \sqrt{2p(q-1)(p+q-3)}], \quad (3.50)$$

$$1+\tilde{\sigma}_q = \frac{(p-1)q\{6+2p(q-2)-2q \mp \sqrt{2p(q-1)(p+q-3)}\}}{(p-2)(q-1)(2pq-3p-q+3)^2} \times \{2p(q-1) \pm \sqrt{2p(q-1)(p+q-3)}\}. \quad (3.51)$$

For $p = 3$ and $q = 6$, we obtain $\mu = \frac{3}{2}(-3 \pm \sqrt{5})$ and $\tilde{\sigma}_q = \frac{1}{10}(5 \mp 3\sqrt{5})$, i.e., $(\mu, \nu) \approx (-1.1459, 1)$, $\sigma_q = -1$, $b_0 \approx 2.419\,53$, or $(\mu, \nu) \approx (-7.8541, 1)$, $\sigma_q = +1$, $b_0 \approx 0.924\,176$. In both cases, the external space is contracting.

For the case of $\sigma_p \neq 0$ and $\sigma_q = 0$, exchanging μ, p and ν, q, we obtain the solutions. For $p = 3$ and $q = 6$, we find $\mu = 1$, $\nu = 0, -1$ and $\tilde{\sigma}_p = -1, \frac{2}{3}$, i.e., $(\mu, \nu) = (1, 0)$, $\sigma_p = -1$, $a_0 = 1$, or $(\mu, \nu) = (1, -1)$, $\sigma_p = +1$, $a_0 = \frac{2}{3}$. The first is an exact solution similar to that found for $\sigma_p = 0$, $\sigma_q = -1$. Here the external space is expanding while the internal space is static or contracting and these are interesting solutions.

C. $\sigma_p\sigma_q \neq 0$

1. Generalized de Sitter solutions ($\epsilon = 0$)

If $\mu = \nu = 0$, our basic equations reduce to

$$\alpha_1[p_1\tilde{\sigma}_p + q_1\tilde{\sigma}_q] + \alpha_2[p_3\tilde{\sigma}_p^2 + 2p_1q_1\tilde{\sigma}_p\tilde{\sigma}_q + q_3\tilde{\sigma}_q^2] = 0,$$
$$\alpha_1[(p-1)_2\tilde{\sigma}_p + q_1\tilde{\sigma}_q] + \alpha_2[(p-1)_4\tilde{\sigma}_p^2 + 2(p-1)_2q_1\tilde{\sigma}_p\tilde{\sigma}_q + q_3\tilde{\sigma}_q^2] = 0,$$
$$\alpha_1[p_1\tilde{\sigma}_p + (q-1)_2\tilde{\sigma}_q] + \alpha_2[p_3\tilde{\sigma}_p^2 + 2p_1(q-1)_2\tilde{\sigma}_p\tilde{\sigma}_q + (q-1)_4\tilde{\sigma}_q^2] = 0. \quad (3.52)$$

It is easy to see that there is no consistent solution.

If either $\mu = 0$ or $\nu = 0$ and the other is nonzero, it is clear that there is no exact solution. For asymptotic solutions, we can search for them by setting $A_p = \tilde{\sigma}_p$, $A_q = \nu^2$ and $X = 0$, $Y = \nu^2$ for the first case. We find that there is no real asymptotic solution. The second case is similar. For $\mu\nu \neq 0$, if our ansatz for solutions is imposed, it is easy to see that there is no asymptotic solution if μ and ν are of the opposite signs. If they are of the same sign, either $t \to +\infty$ or $t \to -\infty$ gives $A_p \to \mu^2$, $A_q \to \nu^2$ and there may be solutions. However, we find that there is no solution for Eqs. (3.7) and (3.8). To study time evolution of the system, we need again a numerical analysis.

2. Power-law solutions ($\epsilon = 1$)

In this case, we first consider the cases when both μ and ν are not equal to 1.

(i) $\mu > 1$ *and* $\nu > 1$: As $t \to \infty$ with EH dominance, we obtain the asymptotic solutions in Sec. III A 2. However, no solutions satisfy the condition of $\mu > 1$ and $\nu > 1$ [see Eq. (3.22)]. This implies that there is no asymptotic solution of our form. As $t \to -\infty$, the GB terms become dominant, and we find no consistent solution.

(ii) $\mu < 1$ *and* $\nu < 1$: As $t \to \infty$ with EH dominance, we again find no consistent solution. As $t \to -\infty$ with GB dominance, we obtain the asymptotic solution in Sec. III A 2. For the heterotic strings, we find only one consistent solution, which is $(\mu, \nu) = (-0.195\,06, 0.597\,53)$.

(iii) $\mu > 1$ *and* $\nu < 1$: As $t \to \infty$, $A_p \to \mu^2$ and $A_q \to \tilde{\sigma}_q e^{2(1-\nu)t}$. This is similar to the case (ii) in the previous section. Then there is no asymptotic solution of our form. As $t \to -\infty$, $A_p \to \tilde{\sigma}_p e^{2(1-\mu)t}$ and $A_q \to \nu^2$. We find no solution.

(iv) $\mu < 1$ *and* $\nu > 1$: Here we reach the same result by exchanging p, μ and q, ν. No asymptotic solution of our form is obtained.

Next, we discuss the cases in which one of μ or ν is equal to 1 and the other is not:

(v) $\mu > 1$ *and* $\nu = 1$: As $t \to \infty$ with EH dominance, $A_p \to \mu^2$ and $A_q = 1 + \tilde{\sigma}_q$, and we recover the case of $\sigma_p = 0, \sigma_q \neq 0$. However, there is no solution with $\mu > 1$. We do not have any asymptotic solution of our form. As $t \to -\infty$, $A_p \to \tilde{\sigma}_p e^{2(1-\mu)t}$ and $A_q = 1 + \tilde{\sigma}_q$. We again do not have any asymptotic solution of our form.

(vi) $\mu < 1$ *and* $\nu = 1$: As $t \to \infty$, A_p diverges as $\tilde{\sigma}_p e^{2(1-\mu)t}$. There is no asymptotic solution of our form. As $t \to -\infty$, we again recover the case of $\sigma_p = 0, \sigma_q \neq 0$ with the GB-term dominance, namely $\mu = 0, \nu = 1, \sigma_q = -1$, and (3.50) and (3.51). Since these asymptotic solutions are consistent with $\mu < 1$, we have asymptotic power-law solutions. (Note that the first one was an exact solution for $\sigma_p = 0$, but here we are considering $\sigma_p \neq 0$.)

(vii) $\mu = 1$ *and* $\nu > 1$: The analysis is almost the same as the case (v). There are no asymptotic solutions.

(viii) $\mu = 1$ *and* $\nu < 1$: The analysis is almost the same as the case (vi), then we find the asymptotic solutions as $t \to -\infty$, which are the same as the case of $\sigma_p \neq 0, \sigma_q = 0$. We have $\mu = 1, \nu = 0, \sigma_p = -1$ and $\mu = 1, \nu = -1, \sigma_p = +1, a_0 = \frac{2}{3}$.

Finally, we consider the remaining case.

(ix) $\mu = 1$ *and* $\nu = 1$: Here we have constant $A_p = 1 + \tilde{\sigma}_p$ and $A_q = 1 + \tilde{\sigma}_q$. As $t \to +\infty$, the EH term is dominant, and we have

$$\begin{aligned} &p_1 A_p + q_1 A_q + 2pq = 0, \\ &(p-1)_2 A_p + q_1 A_q + 2(p-1)q = 0, \\ &p_1 A_p + (q-1)_2 A_q + 2p(q-1) = 0. \end{aligned} \tag{3.53}$$

The solution is given by

$$A_p = -\frac{q}{p-1}, \qquad A_q = -\frac{p}{q-1}. \tag{3.54}$$

This is the solution found in Ref. [11] which exhibits eternal accelerating expansion when higher-order effects are taken into account.

For $p = 3, q = 6$, we have $\tilde{\sigma}_p = -4, \tilde{\sigma}_q = -\frac{8}{5}$. For $t \to -\infty$, GB terms are dominant and we get two independent equations

$$p_3 A_p^2 + p_1 q_1 A_p A_q + 3p_2 q A_p + pq_2 A_q + 2p_1 q_1 = 0, \tag{3.55}$$

$$q_3 A_q^2 + p_1 q_1 A_p A_q + p_2 q A_p + 3pq_2 A_q + 2p_1 q_1 = 0. \tag{3.56}$$

For $p = 3$ and $q = 6$, we have only one real solution $A_p = -1.361\,56, A_q = -1.853\,05$, i.e., $\sigma_p = \sigma_q = -1$ and $a_0 = 0.650\,73, b_0 = 0.592\,032$.

IV. SOLUTIONS IN M AND TYPE II THEORIES

The higher-order corrections to M and type II theories do not involve GB terms, so we have to take the fourth-order corrections into account. From our ansatz for solutions, we have

$$F_1 + F_4 + F_S = 0, \tag{4.1}$$

$$F_1^{(p)} + F_4^{(p)} + F_S^{(p)} = 0, \tag{4.2}$$

$$F_1^{(q)} + F_4^{(q)} + F_S^{(q)} = 0,, \tag{4.3}$$

where

$$\begin{aligned} F_1 &= F_1(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_4 &= F_4(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_S &= F_S(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_1^{(p)} &= f_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) + X g_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + Y h_1^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_4^{(p)} &= f_4^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) + X g_4^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + Y h_4^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_S^{(p)} &= F_S^{(p)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_1^{(q)} &= f_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) + Y g_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + X h_1^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_4^{(q)} &= f_4^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) + Y g_4^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q) \\ &\quad + X h_4^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q), \\ F_S^{(q)} &= F_S^{(q)}(t, \epsilon, \mu, \nu, A_p, A_q), \end{aligned} \tag{4.4}$$

whose explicit expressions are given in Appendix B.

A. $\sigma_p = \sigma_q = 0$

In this case, $A_p = \mu^2, A_q = \nu^2$ are constants. We shall discuss the cases of $\epsilon = 0$ and $\epsilon = 1$ in order.

1. Generalized de Sitter solutions ($\epsilon = 0$)

From Appendix C, we have two algebraic equations:

$$F_1 + F_4 + F_S = 0, \tag{4.5}$$

$$H_1 + H_4 + H_S = 0, \tag{4.6}$$

where F_1, F_4, F_S, H_1, H_4, and H_S are functions with respect to μ and ν given in Appendix C. In what follows, we set $p = 3$. The explicit forms of equations are

$$\alpha_1[6\mu^2 + 6q\mu\nu + q_1\nu^2] + \alpha_4 q_4 \nu^5[336\mu^3 + 168(q-5)\mu^2\nu + 24(q-5)_6\mu\nu^2 + (q-5)_7\nu^3] - 21\gamma[24\mu^8 + 2q(\mu^2+\nu^2+\mu\nu)^2\mu^2\nu^2 + (q+1)_1\nu^8 + 2q(2\mu+\nu)^2\mu^4\nu^2 + q_1(\mu+2\nu)^2\mu^2\nu^4] + 24\gamma(3\mu+q\nu)[6\mu^7 + q_1\nu^7 + q(\mu+\nu)\mu^2\nu^2(\mu^2+\nu^2+\mu\nu)] = 0, \quad (4.7)$$

$$(\mu-\nu)[\alpha_1 + 4\alpha_4\{30(q-1)_4\mu^2\nu^4 + 12(q-1)_5\mu\nu^5 + (q-1)_6\nu^6\} + 2\gamma\{12\mu^6 - 6(2q-1)\mu^5\nu + 3(q-1)\mu^4\nu^2 - (q^2-15q+6)\mu^3\nu^3 - (2q^2-7q-3)\mu^2\nu^4 - (q-1)(q+12)\mu\nu^5 + 6(q-1)\nu^6\}] = 0. \quad (4.8)$$

Setting $\alpha_1 = 1$, we have solved these equations numerically. Before giving the solution, we note on the unit used in our solutions when the coupling constants α_4 and γ are free. If γ does not vanish, rescaling α_4, γ, μ, and ν as

$$\tilde{\alpha}_4 = \alpha_4/|\gamma|, \qquad \tilde{\gamma} = \gamma/|\gamma|(=1 \text{ or } -1), \qquad \tilde{\mu} = \mu|\gamma|^{1/6}, \quad \text{and} \quad \tilde{\nu} = \nu|\gamma|^{1/6}, \quad (4.9)$$

we can always set γ to -1 if it is negative (or 1 if positive). We also have to rescale time coordinate as $\tilde{t} = |\gamma|^{-1/6}t$. The typical dynamical time scale is then given by $|\gamma|^{1/6} \sim O(m_D^{-1})$, where $m_D = \kappa_D^{-2/(D-2)}$ is the fundamental Planck scale. In particular, for M theory, we find $|\gamma|^{1/6} = 6^{-1/6}(4\pi)^{-5/9}m_{11}^{-1} \sim 0.181\,8176 m_{11}^{-1}$ from Eq. (2.10). After this scaling, we have only one free parameter $\tilde{\alpha}_4$.

If $\gamma = 0$ and $\alpha_4 \neq 0$, we can always set α_4 to -1 if it is negative (or 1 if positive), by rescaling α_4, μ, and ν as

$$\bar{\alpha}_4 = \alpha_4/|\alpha_4|(=1 \text{ or } -1), \qquad \bar{\mu} = \mu|\alpha_4|^{1/6}, \quad \text{and} \quad \bar{\nu} = \nu|\alpha_4|^{1/6}. \quad (4.10)$$

Let us now present our results for M theory and type II superstrings, in which α_4 and γ are given by Eq. (2.10). In this paper, we use the above unit as in our previous paper [23]. (We have slightly changed our convention so the numerical results also a little change from those in [23].) For brevity, we omit a tilde for variables except for $\tilde{\alpha}_4$ and $\tilde{\gamma}$.

(i) *M theory*: For the M theory, we have

$$\tilde{\alpha}_4 = -\frac{1}{3 \times 2^5}, \quad \tilde{\gamma} = -1. \quad (4.11)$$

We then find three solutions

$$(\mu, \nu) = (0.40731, 0.40731), (0.79683, 0.10793), (0.55570, 0.34253), \quad (4.12)$$

and the time-reversed ones $(-\mu, -\nu)$.

(ii) *Type II superstrings*: In type II superstrings, the coefficients $\tilde{\alpha}_4$ and $\tilde{\gamma}$ are same as the M theory, but there are additional terms in the curvature as well as dilaton [15]. However, we examine what happens if we simply consider the above theory for ten dimensions ($q = 6$). Since the basic features of the obtained results are the same, we simply give the solutions. The same remark applies to the following discussions on type II superstrings.

With the couplings (4.11), we find three solutions

$$(\mu, \nu) = (0.50754, 0.50754), (0.79988, 0.12991), (0.49618, 0.51313), \quad (4.13)$$

and the time-reversed ones. We thus find that the solutions are qualitatively similar to those in M theory.

2. Power-law solutions ($\epsilon = 1$)

As $t \to \infty$ with EH dominance, we get the same results (3.23) in Sec. III A 2. For $p = 3$, $q = 7$, we get $(\mu, \nu) = ((1 \pm \sqrt{21})/10, (7 \mp 3\sqrt{21})/70)$.

As $t \to -\infty$, the fourth-order terms dominate. So let us briefly discuss asymptotic power-law solutions only with quartic terms. Assuming the metric (2.20) with $\epsilon = 1$, we obtain three algebraic equations:

$$\alpha_4 q_4 \nu^5[336\mu^3 + 168(q-5)\mu^2\nu + 24(q-5)_6\mu\nu^2 + (q-5)_7\nu^3] - 7\gamma[6\mu^4(\mu-1)^2(3\mu-1)^2 + q_1\nu^4(\nu-1)^2(3\nu-1)^2 + 18\mu^8 + 3q_2\nu^8 + 6q(2\mu+\nu)^2\mu^4\nu^2 + 3q_1(\mu+2\nu)^2\mu^2\nu^4 + 6q\mu^2\nu^2((\mu+\nu-1)^2 - \mu\nu)^2] + \gamma[24\mu^4(\mu-1)(2\mu-1)(3\mu-1) + 4q_1\nu^4(\nu-1)(2\nu-1)(3\nu-1) + 24q\mu^2\nu^2(\mu+\nu-1)((\mu+\nu-1)^2 - \mu\nu)](3\mu + q\nu - 7) = 0, \quad (4.14)$$

$$4\alpha_4 q_4 \nu^5\{14(q-5)\mu^2\nu + 4(q-5)_6\mu\nu^2 + 4\mu(\mu-1)[6\mu + (q-5)\nu] + 2(\nu-1)[30\mu^2 + 12(q-5)\mu\nu + (q-5)_6\nu^2]\} + \gamma\{6\mu^4(\mu-1)^2(3\mu-1)^2 + q_1\nu^4(\nu-1)^2(3\nu-1)^2 + 18\mu^8 + 3q_2\nu^8 + 6q(2\mu+\nu)^2\mu^4\nu^2 + 3q_1(\mu+2\nu)^2\mu^2\nu^4 + 6q\mu^2\nu^2[(\mu+\nu-1)^2 - \mu\nu]^2\} - \gamma\{4\mu^3(\mu-1)(2\mu-1)(3\mu-1)^2 + 4\mu^3(\mu-1)^2(3\mu-1)(6\mu-1) + 48\mu^7 + 8q\mu^3\nu^2(2\mu+\nu)(3\mu+\nu) + 4q_1\mu\nu^4(\mu+\nu)(\mu+2\nu) + 4q\mu\nu^2[(\mu+\nu-1)(3\mu+\nu-1) - 2\mu\nu][(\mu+\nu-1)^2 - \mu\nu]\}(3\mu + q\nu - 7) + \gamma\{8\mu^3(\mu-1)(2\mu-1)(3\mu-1) + 4q\mu\nu^2(\mu+\nu-1)[(\mu+\nu-1)^2 - \mu\nu]\} \times (3\mu + q\nu - 7)^2 = 0, \tag{4.15}$$

$$\alpha_4(q-1)_4\nu^4[(q-5)_8\nu^4 + 24(q-5)_7\mu\nu^3 + 168(q-5)_6\mu^2\nu^2 + 336(q-5)\mu^3\nu + 24\mu(\mu-1)(30\mu^2 + 12(q-5)\mu\nu + (q-5)_6\nu^2) + 8(\nu-1)(120\mu^3 + 90(q-5)\mu^2\nu + 18(q-5)_6\mu\nu^2 + (q-5)_7\nu^3)] + \gamma[6\mu^4(\mu-1)^2(3\mu-1)^2 + q_1\nu^4(\nu-1)^2(3\nu-1)^2 + 18\mu^8 + 3q_2\nu^8 + 6q(2\mu+\nu)^2\mu^4\nu^2 + 3q_1(\mu+2\nu)^2\mu^2\nu^4 + 6q\mu^2\nu^2((\mu+\nu-1)^2 - \mu\nu)^2] - 2\gamma\nu[(q-1)\nu^2(\nu-1)(2\nu-1)(3\nu-1)^2 + (q-1)\nu^2(\nu-1)^2(3\nu-1)(6\nu-1) + 12(q-1)_2\nu^6 + 12\mu^4(\mu+\nu)(2\mu+\nu) + 6(q-1)\mu^2\nu^2(\mu+2\nu)(\mu+3\nu) + 6\mu^2((\mu+\nu-1)(\mu+3\nu-1) - 2\mu\nu)((\mu+\nu-1)^2 - \mu\nu)](3\mu + q\nu - 7) + 4\gamma\nu[(q-1)\nu^2(\nu-1)(2\nu-1)(3\nu-1) + 3\mu^2(\mu+\nu-1)((\mu+\nu-1)^2 - \mu\nu)] \times (3\mu + q\nu - 7)^2 = 0. \tag{4.16}$$

Using the values for $\tilde{\alpha}_4$ and $\tilde{\gamma}$ in Eq. (4.11), we have solved these equations numerically and found the following four solutions:

$q = 7$ (M theory)

$$(\mu, \nu) = (0.87610, 0.62453), (0.53167, 0.77214), (0.32052, 0.000168), (-0.000877, 0.28898), \tag{4.17}$$

$q = 6$ (Type II superstrings)

$$(\mu, \nu) = (5.74269, 5.74269), (0.32052, 0.000168), (0.28829, 0.28829), (0.00133, 0.295437). \tag{4.18}$$

B. $\sigma_p = 0, \sigma_q \neq 0$ (or $\sigma_p \neq 0, \sigma_q = 0$)

1. Generalized de Sitter solutions ($\epsilon = 0$)

Here we have $A_p = \mu^2, A_q = \nu^2 + \tilde{\sigma}_q e^{-2\nu t}, X = \mu^2$, and $Y = \nu^2$. It is easy to see that there is no exact solution unless $\nu = 0$, in which case we have constant $A_p = X = \mu^2, A_q = \tilde{\sigma}_q$, and $Y = 0$. Our basic Eqs. (4.1) and (4.3) now give

$$\alpha_1[p_1\mu^2 + q_1\tilde{\sigma}_q] + \alpha_4[p_7\mu^8 + 4p_5q_1\mu^6\tilde{\sigma}_q + 6p_3q_3\mu^4\tilde{\sigma}_q^2 + 4p_1q_5\mu^2\tilde{\sigma}_q^3 + q_7\tilde{\sigma}_q^4] + 3\gamma[(p-7)p_1\mu^8 + q_2\tilde{\sigma}_q^4] = 0, \tag{4.19}$$

$$\alpha_1[(p+1)_0\mu^2 + (q-1)_2\tilde{\sigma}_q] + \alpha_4[(p+1)_6\mu^8 + 4(p+1)_4(q-1)_2\mu^6\tilde{\sigma}_q + 6(p+1)_2(q-1)_4\mu^4\tilde{\sigma}_q^2 + 4(p+1)_0(q-1)_6\mu^2\tilde{\sigma}_q^3 + (q-1)_8\tilde{\sigma}_q^4] + 3\gamma[(p+1)_1\mu^8 + (q-8)(q-1)_2\tilde{\sigma}_q^4] = 0. \tag{4.20}$$

We note that Eq. (4.2) gives the same equation as (4.19) for $\nu = 0$ and need not be taken into account.

For $p = 3$, we find the following solutions:

$q = 7$ (M theory)

$$(\mu, \tilde{\sigma}_q) = (\pm 0.65615, 0.28708), (\pm 0.61935, -0.61904), (\pm 0.60255, -0.08823), \tag{4.21}$$

$q = 6$ (Type II superstrings)

$$(\mu, \tilde{\sigma}_q) = (\pm 0.76553, 0.45670), (\pm 0.62004, -0.13097). \tag{4.22}$$

For the case of $\sigma_p \neq 0$ and $\sigma_q = 0$, exchanging μ, p and ν, q, we obtain the solutions with $\mu = 0$ and

$$\text{M theory:} \quad (\nu, \tilde{\sigma}_p) = (\pm 0.49021, 0.63074),$$
$$\text{Type II superstrings:} \quad (\nu, \tilde{\sigma}_p) = (\pm 0.62007, 0.86033). \tag{4.23}$$

2. Power-law solutions ($\epsilon = 1$)

Here we have $A_p = \mu^2, A_q = \nu^2 + \tilde{\sigma}_q e^{2(1-\nu)t}, X = \mu(\mu - 1)$, and $Y = \nu(\nu - 1)$. We have asymptotic solutions in most cases.

(i) $\nu > 1$: For $t \to \infty$, the EH term dominates and $A_q \to \nu^2$. The solutions are the same as $\sigma_p = \sigma_q = 0$ case in Sec. III A 2. However, there is no solution with $\nu > 1$.
For $t \to -\infty$, $A_q \to \tilde{\sigma}_q e^{2(1-\nu)t}$ and there is no solution.

(ii) $\nu < 1$: For $t \to \infty$, $A_q \to \tilde{\sigma}_q e^{2(1-\nu)t}$ and there is no solution.
For $t \to -\infty$, $A_q \to \nu^2$ and the solutions are the same as $\sigma_p = \sigma_q = 0$ case.

(iii) $\nu = 1$: We have $A_p = \mu^2, A_q = 1 + \tilde{\sigma}_q, X = \mu(\mu - 1)$, and $Y = 0$.

For $t \to \infty$, the EH term dominates and the solutions are the same as GB case. We have

$$\mu = 0, \nu = 1, \sigma_q = -1, b_0 = 1. \tag{4.24}$$

Actually this is an exact solution.

For $t \to -\infty$, fourth-order terms dominate. Our basic independent Eqs. (4.1) and (4.3) give

$$\alpha_4[p_7\mu^8 + 4p_5q_1\mu^6A_q + 6p_3q_3\mu^4A_q^2 + 4p_1q_5\mu^2A_q^3 + q_7A_q^4 + 8\mu\{p_6q\mu^6 + 3p_4q_2\mu^4A_q + 3p_2q_4\mu^2A_q^2 + pq_6A_q^3\} + 24\mu^2\{p_5q_1\mu^4 + 2p_3q_3\mu^2A_q + p_1q_5A_q^2\} + 32\mu^3\{p_4q_2\mu^2 + p_2q_4A_q\} + 16p_3q_3\mu^4] + \gamma[-7\tilde{L}_4 + (p\mu + q - 7)(p\mu M_X + qM_Y) + 2(A_q - 1)qN_q] = 0, \tag{4.25}$$

$$\tilde{f}_4^{(q)} + \tilde{h}_4^{(q)}\mu(\mu - 1) + \tilde{F}_S^{(q)} = 0, \tag{4.26}$$

where

$$\tilde{L}_4 = p_1\mu^4(\mu - 1)^2(3\mu - 1)^2 + 2pq\mu^4(\mu - 1)^2 + 3p_2\mu^8 + 3q_2A_q^4 + p_1q\mu^4(2\mu + 1)^2 + pq_1\mu^2(\mu + 2A_q)^2, \tag{4.27}$$

$$M_X = 4[(p - 1)\mu^3(\mu - 1)(2\mu - 1)(3\mu - 1) + q\mu^3(\mu - 1)], \tag{4.28}$$

$$M_Y = 4p\mu^4(\mu - 1), \tag{4.29}$$

$$N_q = 4(q - 1)[3(q - 2)A_q^3 + p\mu^2(\mu + 2A_q)], \tag{4.30}$$

$$\tilde{f}_4^{(q)} = \alpha_4[p_7\mu^8 + 4p_5(q - 1)_2\mu^6A_q + 6p_3(q - 1)_4\mu^4A_q^2 + 4p_1(q - 1)_6\mu^2A_q^3 + (q - 1)_8A_q^4 + 8\mu\{p_6(q - 1)\mu^6 + 3p_4(q - 1)_3\mu^4A_q + 3p_2(q - 1)_5\mu^2A_q^2 + p(q - 1)_7A_q^3\} + 24\mu^2\{p_5(q - 1)_2\mu^4 + 2p_3(q - 1)_4\mu^2A_q + p_1(q - 1)_6A_q^2\} + 32\mu^3\{p_4(q - 1)_3\mu^2 + p_2(q - 1)_5A_q\} + 16p_3(q - 1)_4\mu^4], \tag{4.31}$$

$$\tilde{h}_4^{(q)} = 8p\alpha_4[(p - 1)_6\mu^6 + 3(p - 1)_4(q - 1)_2\mu^4A_q + 3(p - 1)_2(q - 1)_4\mu^2A_q^2 + (q - 1)_6A_q^3 + 6\mu\{(p - 1)_5(q - 1)\mu^4 + 2(p - 1)_3(q - 1)_3\mu^2A_q + (p - 1)(q - 1)_5A_q^2\} + 12\mu^2\{(p - 1)_4(q - 1)_2\mu^2 + (p - 1)_2(q - 1)_4A_q\} + 8(p - 1)_3(q - 1)_3\mu^3], \tag{4.32}$$

$$\tilde{F}_S^{(q)} = \gamma[\tilde{L}_4 - (p\mu + q - 7)\{M_Y + 2N_q + p\mu U\} + (p\mu + q - 7)^2M_Y - 2(A_q - 1)N_q], \tag{4.33}$$

$$U = 4[\mu^3(\mu - 1)^2 + (p - 1)\mu^3(\mu + 1)(2\mu + 1) + (q - 1)\mu(\mu + A_q)(\mu + 2A_q)]. \tag{4.34}$$

For $p = 3$, we find the solution (4.24) and

$q = 7$ (M theory)

$$(\mu, \tilde{\sigma}_q) = (14.8319, -413.5411), (0.7335, -0.3062), \tag{4.35}$$

$q = 6$ (Type II string)

$$(\mu, \tilde{\sigma}_q) = (4.0305, 8.7771), (0.4484, -1.2490), (-9.7439, -94.7146). \tag{4.36}$$

[We also find a solution $(\mu, \tilde{\sigma}_q) = (0, 7)$ to Eqs. (4.25) and (4.26), but this is the special case of $\dot{u}_1 = 0$, and then Eq. (4.2) must be checked, as discussed in Sec. II A. We find that it is not satisfied asymptotically and this is not a solution.] The first of these gives an interesting inflationary solution.

For the case of $\sigma_p \neq 0$ and $\sigma_q = 0$, exchanging μ, p and ν, q, we obtain the solutions:

$$\begin{aligned} &\text{exact solution:} && (\mu, \nu, \sigma_p) = (1, 0, -1), \\ &\text{past asymptotic solutions:} && (\mu, \nu, \tilde{\sigma}_p) = (1, 0.7181, -1.5485), (1, 0.0417, -1.0204), \\ &&& (1, -14.1607, -138.1063), \end{aligned} \tag{4.37}$$

for M theory and

$$\begin{aligned} &\text{exact solution:} && (\mu, \nu, \sigma_p) = (1, 0, -1), \\ &\text{past asymptotic solutions:} && (\mu, \nu, \tilde{\sigma}_p) = (1, 6.1725, 75.9086), (1, 0.0358, -1.0173), \\ &&& (1, -26.8744, -961.1752), \end{aligned} \tag{4.38}$$

for type II superstrings.

C. $\sigma_p \sigma_q \neq 0$

1. Generalized de Sitter solutions ($\epsilon = 0$)

If $\mu = \nu = 0$, our basic equations reduce to

$$\begin{aligned} &\alpha_1[p_1\tilde{\sigma}_p + q_1\tilde{\sigma}_q] + \alpha_4[p_7\tilde{\sigma}_p^4 + 4p_5q_1\tilde{\sigma}_p^3\tilde{\sigma}_q + 6p_3q_3\tilde{\sigma}_p^2\tilde{\sigma}_q^2 + 4p_1q_5\tilde{\sigma}_p\tilde{\sigma}_q^3 + q_7\tilde{\sigma}_q^4] + 3\gamma[p_2\tilde{\sigma}_p^4 + q_2\tilde{\sigma}_q^4] = 0, \\ &\alpha_1[(p-1)_2\tilde{\sigma}_p + q_1\tilde{\sigma}_q] + \alpha_4[(p-1)_8\tilde{\sigma}_p^4 + 4(p-1)_6q_1\tilde{\sigma}_p^3\tilde{\sigma}_q + 6(p-1)_4q_3\tilde{\sigma}_p^2\tilde{\sigma}_q^2 + \\ &\qquad 4(p-1)_2q_5\tilde{\sigma}_p\tilde{\sigma}_q^3 + q_7\tilde{\sigma}_q^4] + 3\gamma[(p-8)(p-1)_2\tilde{\sigma}_p^4 + q_2\tilde{\sigma}_q^4] = 0, \end{aligned} \tag{4.39}$$

$$\begin{aligned} &\alpha_1[p_1\tilde{\sigma}_p + (q-1)_2\tilde{\sigma}_q] + \alpha_4[p_7\tilde{\sigma}_p^4 + 4p_5(q-1)_2\tilde{\sigma}_p^3\tilde{\sigma}_q + 6p_3(q-1)_4\tilde{\sigma}_p^2\tilde{\sigma}_q^2 + 4p_1(q-1)_6\tilde{\sigma}_p\tilde{\sigma}_q^3 + \\ &\qquad (q-1)_8\tilde{\sigma}_q^4] + 3\gamma[p_2\tilde{\sigma}_p^4 + (q-8)(q-1)_2\tilde{\sigma}_q^4] = 0. \end{aligned}$$

For both M theory with $p = 3, q = 7$ and type II theory with $p = 3, q = 6$, we find that there is no solution.

If either $\mu = 0$ or $\nu = 0$ and the other is nonzero, it is clear that there is no exact solution. For asymptotic solutions, we can search for them by setting $A_p = \tilde{\sigma}_p, A_q = \nu^2, X = 0$ and $Y = \nu^2$ for the first case. The solution is for $t \to +\infty(-\infty)$ for μ or $\nu > 0(<0)$. The basic equations are

$$\begin{aligned} &\alpha_1[p_1\tilde{\sigma}_p + q_1\nu^2] + \alpha_4[p_7\tilde{\sigma}_p^4 + 4p_5q_1\tilde{\sigma}_p^3\nu^2 + 6p_3q_3\tilde{\sigma}_p^2\nu^4 + 4p_1q_5\tilde{\sigma}_p\nu^6 + q_7\nu^8] + 3\gamma[p_2\tilde{\sigma}_p^4 + (q-7)q_1\nu^8] = 0, \\ &\alpha_1[(p-1)_2\tilde{\sigma}_p + (q+1)_0\nu^2] + \alpha_4[(p-1)_8\tilde{\sigma}_p^4 + 4(p-1)_6(q+1)_0\tilde{\sigma}_p^3\nu^2 + 6(p-1)_4(q+1)_2\tilde{\sigma}_p^2\nu^4 + \\ &\qquad 4(p-1)_2(q+1)_4\tilde{\sigma}_p\nu^6 + (q+1)_6\nu^8] + 3\gamma[(p-8)(p-1)_2\tilde{\sigma}_p^4 + (q+1)_1\nu^8] = 0. \end{aligned} \tag{4.40}$$

We find for M theory that there are solutions with $\mu = 0$ and

$$(\tilde{\sigma}_p, \nu) = (0.630\,74, \pm 0.490\,21), \tag{4.41}$$

and for type II superstrings

$$(\tilde{\sigma}_p, \nu) = (0.860\,33, \pm 0.620\,07), \tag{4.42}$$

for $t \to +\infty(-\infty)$ for $\nu > 0(<0)$.

The second case is obtained by exchanging p, μ and q, ν. The solutions are

$$\begin{aligned} (\tilde{\sigma}_q, \mu) = &(0.287\,08, \pm 0.656\,15), (-0.619\,04, \pm 0.619\,35), \\ &(-0.088\,23, \pm 0.602\,55), \end{aligned} \tag{4.43}$$

for M theory and

$$(\tilde{\sigma}_q, \mu) = (0.456\,70, \pm 0.765\,53), (-0.130\,97, \pm 0.620\,04), \tag{4.44}$$

for type II superstrings. They are qualitatively the same.

For $\mu\nu \neq 0$, if our ansatz for solutions is imposed, it is easy to see that there is no asymptotic solution if μ and ν are of the opposite signs. If they are of the same sign, either $t \to +\infty$ or $t \to -\infty$ gives $A_p \to \mu^2, A_q \to \nu^2$ and there may be solutions. This implies that inflationary solutions with positive eigenvalues are obtained for asymptotic infinite future, so that these are not interesting from the cosmological point of view. However, it may be useful to check if there are any solutions of this type. In fact we find that there are asymptotic solutions for M theory

$$\begin{aligned} (\mu, \nu) = &\pm(0.796\,83, 0.107\,92), \pm(0.555\,70, 0.342\,53), \\ &\pm(0.407\,31, 0.407\,31), \end{aligned} \tag{4.45}$$

where negative (positive) one is for $t \to -\infty(\infty)$. For type II superstrings, we have

$$(\mu, \nu) = \pm(0.79988, 0.12991), \pm(0.50754, 0.50754), \pm(0.49618, 0.51313). \tag{4.46}$$

2. Power-law solutions ($\epsilon = 1$)

In this case, we first consider the cases when both μ and ν are not equal to 1.

(i) $\mu > 1$ *and* $\nu > 1$: For $t \to \infty$, the EH term dominates and we obtain the asymptotic solutions in Sec. III A 2. Again no solutions satisfy the condition of $\mu > 1$ and $\nu > 1$ [see Eq. (3.22)] and hence there is no asymptotic solution of our form.
As $t \to -\infty$, the fourth-order terms become dominant and we find no consistent solution from the fourth-order terms.

(ii) $\mu < 1$ *and* $\nu < 1$: As $t \to \infty$ with EH dominance, we again find no consistent solution. As $t \to -\infty$ with fourth-order-term dominance, we obtain the asymptotic solutions in Eqs. (4.17) and (4.18) in Sec. IV A 2.

(iii) $\mu > 1$ *and* $\nu < 1$: As $t \to \infty$, $A_p \to \mu^2$ and $A_q \to \tilde{\sigma}_q e^{2(1-\nu)t}$. This is similar to the case (ii) in Sec. IV B 2. There is no asymptotic solution of our form. As $t \to -\infty$, $A_p \to \tilde{\sigma}_p e^{2(1-\mu)t}$, and $A_q \to \nu^2$. We find no solution.

(iv) $\mu < 1$ *and* $\nu > 1$: Here we reach the same result by exchanging p, μ and q, ν. No asymptotic solution of our form is obtained.

Next, we discuss the cases in which one of μ or ν is equal to 1 and the other is not:

(v) $\mu > 1$ *and* $\nu = 1$: As $t \to \infty$ with EH dominance, $A_p \to \mu^2$, and we recover the case of $\sigma_p = 0, \sigma_q \neq 0$. However, there is no solution with $\mu > 1$. We do not have any asymptotic solution of our form. As $t \to -\infty$ with fourth-order-term dominance, $A_p \to \tilde{\sigma}_p e^{2(1-\mu)t}$. We again do not have any asymptotic solution of our form.

(vi) $\mu < 1$ *and* $\nu = 1$: As $t \to \infty$ with EH dominance, A_p diverges as $\tilde{\sigma}_p e^{2(1-\mu)t}$. There is no asymptotic solution of our form. As $t \to -\infty$, we again recover the case of $\sigma_p = 0, \sigma_q \neq 0$ with the fourth-order-term dominance (4.24). Since this asymptotic solution is consistent with $\mu < 1$, we have an asymptotic power-law solution. (Note that this was an exact solution for $\sigma_p = 0$.)

(vii) $\mu = 1$ *and* $\nu > 1$: The analysis is almost the same as the case (v). There is no asymptotic solution.

(viii) $\mu = 1$ *and* $\nu < 1$: The analysis is almost the same as the case (vi), then we find the asymptotic solution as $t \to -\infty$, which is the same as the case of $\sigma_p \neq 0, \sigma_q = 0$.

Finally, we consider the remaining case.

(ix) $\mu = 1$ *and* $\nu = 1$: Here we have constant $A_p = 1 + \tilde{\sigma}_p$ and $A_q = 1 + \tilde{\sigma}_q$. As $t \to +\infty$ with EH dominance, and we recover the solution (3.54) of Sec. III C 2. For $p = 3, q = 7$, we get $(\tilde{\sigma}_p, \tilde{\sigma}_q) = (-\frac{9}{2}, -\frac{3}{2})$.
For $t \to -\infty$ with fourth-order-term dominance, we get two independent equations for $A_p = 1 + \tilde{\sigma}_p, A_q = 1 + \tilde{\sigma}_q$:

$$\alpha_4[p_7A_p^4 + 4p_5q_1A_p^3A_q + 6p_3q_3A_p^2A_q^2 + 4p_1q_5A_pA_q^3 + q_7A_q^4 + 8(p_6qA_p^3 + 3p_4q_2A_p^2A_q + 3p_2q_4A_pA_q^2 + pq_6A_q^3) + 24(p_5q_1A_p^2 + 2p_3q_3A_pA_q + p_1q_5A_q^2) + 32(p_4q_2A_p + p_2q_4A_q) + 16p_3q_3] + \gamma[-7\tilde{L}_4 + 2p(A_p - 1)N_p + 2q(A_q - 1)N_q] = 0, \tag{4.47}$$

$$\alpha_4[(p-1)_8A_p^4 + 4(p-1)_6q_1A_p^3A_q + 6(p-1)_4q_3A_p^2A_q^2 + 4(p-1)_2q_5A_pA_q^3 + q_7A_q^4 + 8\{(p-1)_7qA_p^3 + 3(p-1)_5q_2A_p^2A_q + 3(p-1)_3q_4A_pA_q^2 + (p-1)q_6A_q^3\} + 24\{(p-1)_6q_1A_p^2 + 2(p-1)_4q_3A_pA_q + (p-1)_2q_5A_q^2\} + 32\{(p-1)_5q_2A_p + (p-1)_3q_4A_q\} + 16(p-1)_4q_3] + \gamma[\tilde{L}_4 - (p+q-7)(2N_p + qU) - 2(A_p - 1)N_p] = 0, \tag{4.48}$$

where

$$\begin{aligned} \tilde{L}_4 &= 3p_2A_p^4 + 3q_2A_q^4 + p_1q(2A_p + 1)^2 + pq_1(2A_q + 1)^2, \\ N_p &= 4(p-1)[3(p-2)A_p^3 + q(2A_p + 1)], \\ N_q &= 4(q-1)[3(q-2)A_q^3 + p(2A_q + 1)], \\ U &= 4[(p-1)(A_p + 1)(2A_p + 1) + (q-1)(A_q + 1)(2A_q + 1)]. \end{aligned} \tag{4.49}$$

For $p = 3$, we have four real solutions

$$q = 7 \qquad \text{(M theory)}$$
$$(A_p, A_q) = (12.2143, -10.4313), (0.43403, -0.60288), (0.19127, 0.73878), (-2.10241, -0.19306) \tag{4.50}$$

$$q = 6 \qquad \text{(Type II string)}$$
$$(A_p, A_q) = (5.5316, 3.1354), (0.212\,14, -0.662\,02), (-1.3472, -0.311\,16), (-33.5609, 19.4154). \tag{4.51}$$

V. SUMMARY AND DISCUSSIONS

We have found generalized de Sitter solutions

$$a \propto e^{\mu t}, \qquad b \propto e^{\nu t}, \quad \text{for } \epsilon = 0, \tag{5.1}$$

and power-law solutions

$$a \propto \tau^{\mu}, b \propto \tau^{\nu}, \quad \text{for } \epsilon = 1. \tag{5.2}$$

In the Einstein frame,

$$a_E \propto t_E^{\lambda}, \qquad \phi \sim \phi^{(0)} + \phi_1 \ln[t_E/t_E^{(0)}], \tag{5.3}$$

where

$$\lambda = 1 + \frac{(p-1)\mu}{q\nu}, \qquad \phi_1 = \frac{(p-1)\nu}{q} \quad \text{for } \epsilon = 0,$$
$$\lambda = \frac{(p-1)\mu + q\nu}{(p-1) + q\nu}, \qquad \phi_1 = \frac{(p-1)\nu}{(p-1) + q\nu} \quad \text{for } \epsilon = 1. \tag{5.4}$$

Note that the values of μ and ν in generalized de Sitter solutions (5.1) depend on the choice of the unit. In the heterotic string theories, we adopt $\alpha' = 1$, while in the M theory and the type II string theory, we use the unit of $|\gamma| = 1$, i.e., $m_{11} = 6^{-1/2}(4\pi)^{-5/9} \sim 0.181\,8176$. If we set $m_{11} = 1$, the values of μ and ν in the following tables should be multiplied by the factor $6^{1/2}(4\pi)^{5/9} \sim 5.5$. On the other hand, the power exponent μ and ν in the power-law solutions (5.2) or λ in (5.3) are dimensionless and they do not depend on the choice of the unit.

We summarize our results in the following tables for the cases of the heterotic string theories and M theory in order. For asymptotic solutions, the time regions for t_E where the solutions are valid in the Einstein frame are also included. In the last lines of the tables, we include the type of two spaces (ds_p^2, ds_q^2). K means the kinetic dominant space, in which the curvature term (σ_p, or σ_q) is either zero or can be asymptotically ignored. M denote the Milne-type space, which is described by $ds^2 = -dt^2 + t^2 ds_p^2 + \cdots$ with $\sigma_p = -1$, or $ds^2 = -dt^2 + \cdots + t^2 ds_q^2$ with $\sigma_q = -1$. Similarly, we define a constant curvature space C by $\sigma_p = 1$ or $\sigma_q = 1$, and S_0 and $S_\pm$ are static spaces with zero curvature and positive (or negative) curvature, respectively. The result for the type II string model, which is similar to the case of M theory, is given in Appendix D.

A. Heterotic strings

Exact solutions are given in Table I, future asymptotic solutions in Table II and past asymptotic solutions in Table III, where $\cdots$ means that the radius can be arbitrary.

Since we are interested in inflation in string theories, we pick up such solutions and give comments on those. In the original frame, we find $HE1_+$ (HF3, HP3) give an exponential expansion whereas $HE2_+$ (HF4, HP4) and HP5 give a power-law inflation. In the former solutions the extra space expands exponentially, while the internal space shrinks exponentially. However, in the Einstein frame, they correspond to a noninflationary power-law expansion and $HE2_+$ (HF4, HP4) and HP5 give a power-law inflation. Another interesting observation is that we could obtain an expansion of the universe in the Einstein frame from an external space shrinking in the original frame [$HE1_-$ (HF1, HP1), HF6, HP6, HP7, HP8].

B. M theory

Exact solutions are summarized in Table IV, future asymptotic solutions in Table V and past asymptotic solutions in Table VI.

TABLE I. Heterotic string: exact solutions. K, M, and S_0 mean a kinetic dominance, a Milne-type space, and a flat static space, respectively. $\cdots$ means that the radius can be arbitrary.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	Type
$HE1_\pm$	0	0	0	± 1.366	∓ 0.9657	$\cdots$	$\cdots$	0.5285	∓ 0.3219	K K
$HE2_\pm$	0	0	0	± 2.506	∓ 0.3916	$\cdots$	$\cdots$	-1.132	∓ 0.1305	K K
HE3	1	0	-1	0	1	$\cdots$	1	0.75	0.25	S_0 M
HE4	1	-1	0	1	0	1	$\cdots$	1	0	M S_0

TABLE II. Heterotic string: future asymptotic solutions ($t \to \infty$). M means a Milne-type space. The time regions for t_E where the solutions are valid in the Einstein frame are also included.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
HF1	0	0	±1	-1.366	0.9657	···	···	0.5285	0.3219	$\to\infty$	HE1$_-$
HF2	0	0	±1	-2.506	0.3916	···	···	-1.132	0.1305	$\to\infty$	HE2$_-$
HF3	0	±1	0	1.366	-0.9657	···	···	0.5285	-0.3219	~0	HE1$_+$
HF4	0	±1	0	2.506	-0.3916	···	···	-1.132	-0.1305	~0	HE2$_+$
HF5	1	0	0	0.5556	-0.1111	···	···	0.3333	-0.1667	$\to\infty$	Kasner
HF6	1	0	0	-0.3333	0.3333	···	···	0.3333	0.1667	$\to\infty$	Kasner
HF7	1	-1	-1	1	1	0.5	0.7906	1	0.25	$\to\infty$	M M

TABLE III. Heterotic string: past asymptotic solutions ($t \to -\infty$). K, M, and C mean a kinetic dominance, a Milne-type space, and a constant curvature space, respectively.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
HP1	0	±1	0	-1.366	0.9657	···	···	0.5285	0.3219	~0	HE1$_-$
HP2	0	±1	0	-2.506	0.3916	···	···	-1.132	0.1305	~0	HE2$_-$
HP3	0	0	±1	1.366	-0.9657	···	···	0.5285	-0.3219	$\to-\infty$	HE1$_+$
HP4	0	0	±1	2.506	-0.3916	···	···	-1.132	-0.1305	$\to-\infty$	HE2$_+$
HP5	1	0	$0, \pm1$	1.517	-0.2585	···	···	3.3029	-1.15145	~0	K K
HP6	1	$0, \pm1$	0	-1.877	1.439	···	···	0.4589	0.2706	~0	K K
HP7	1	$0, \pm1$	$0, \pm1$	-0.1951	0.5975	···	···	0.5720	0.2140	~0	K K
HP8	1	$0, \pm1$	-1	-1.146	1	···	2.420	0.4635	0.25	~0	K M
HP9	1	$0, \pm1$	1	-7.854	1	···	0.9242	-1.214	0.25	~0	K C
HP10	1	±1	-1	0	1	···	1	0.75	0.25	~0	K M
HP11	1	-1	±1	1	0	1	···	1	0	~0	M K
HP12	1	1	$0, \pm1$	1	-1	1.2247	···	1	0.5	$\to-\infty$	C K
HP13	1	-1	-1	1	1	0.6507	0.5920	1	0.25	~0	M M

TABLE IV. M theory: exact solutions. K, S$_\pm$, S$_0$, and M mean a kinetic dominance, a static space with positive (or negative) curvature, a flat static space, and a Milne-type space.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	Type
ME1$_\pm$	0	0	0	±0.7968	±0.1079	···	···	3.1099	$\pm0.030\,83$	K K
ME2$_\pm$	0	0	0	±0.5557	±0.3425	···	···	1.4636	$\pm0.097\,86$	K K
ME3$_\pm$	0	0	0	±0.4073	±0.4073	···	···	1.2857	±0.1164	K K
ME4$_\pm$	0	0	1	±0.6562	0	···	1.866	$e^{\mu t_E}$	0	K S$_+$
ME5$_\pm$	0	0	-1	±0.6194	0	···	1.271	$e^{\mu t_E}$	0	K S$_-$
ME6$_\pm$	0	0	-1	±0.6026	0	···	3.367	$e^{\mu t_E}$	0	K S$_-$
ME7$_\pm$	0	1	0	0	±0.4902	1.259	···	1	±0.1401	S$_+$ K
ME8	1	0	-1	0	1	···	1	0.7778	0.7778	S$_0$ M
ME9	1	-1	0	1	0	1	···	1	0	M S$_0$

Here we also focus on inflationary solutions. In the original frame, ME1$_+$(MF5), ME2$_+$(MF6), ME3$_+$(MF7), ME4$_+$(MF2), ME5$_+$(MF3), ME6$_+$(MF4) give an exponential expansion for the external space. In the Einstein frame, we find either a power-law inflation [ME1$_+$(MF5), ME2$_+$(MF6), ME3$_+$(MF7)] or an exponential expansion [ME4$_+$(MF2), ME5$_+$(MF3), ME6$_+$(MF4)]. Just as the case of the heterotic strings, we obtain strange solutions MP5 $\sim$ 7 and MP11, in which the external space shrinks exponentially in the original frame, but it expands by a

TABLE V. M theory: future asymptotic solutions ($t \to \infty$). M means a Milne-type space.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
MF1	0	1	±1	0	0.4902	1.259	$\cdots$	1	0.14006	$\to \infty$	$ME7_+$
MF2	0	±1	1	0.6562	0	$\cdots$	1.866	$e^{\mu t_E}$	0	$\to \infty$	$ME4_+$
MF3	0	±1	-1	0.6194	0	$\cdots$	1.271	$e^{\mu t_E}$	0	$\to \infty$	$ME5_+$
MF4	0	±1	-1	0.6026	0	$\cdots$	3.367	$e^{\mu t_E}$	0	$\to \infty$	$ME6_+$
MF5	0	±1	±1	0.7968	0.1079	$\cdots$	$\cdots$	3.1099	0.030 83	$\to \infty$	$ME1_+$
MF6	0	±1	±1	0.5557	0.3425	$\cdots$	$\cdots$	1.4636	0.097 86	$\to \infty$	$ME2_+$
MF7	0	±1	±1	0.4073	0.4073	$\cdots$	$\cdots$	1.2857	0.1164	$\to \infty$	$ME3_+$
MF8	1	0	0	0.5583	-0.0964	$\cdots$	$\cdots$	0.3333	-0.1455	$\to \infty$	Kasner
MF9	1	0	0	-0.3583	0.2964	$\cdots$	$\cdots$	0.3333	0.1455	$\to \infty$	Kasner
MF10	1	-1	-1	1	1	0.4714	0.8165	1	0.2222	$\to \infty$	M M

TABLE VI. M theory: past asymptotic solutions ($t \to -\infty$). K, $S_\pm$, S_0, M and C mean a kinetic dominance, a static space with positive (or negative) curvature, a flat static space, a Milne-type space, and a constant curvature space, respectively.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
MP1	0	1	±1	0	-0.4902	1.259	$\cdots$	1	$-0.140 06$	$\to -\infty$	$ME7_-$
MP2	0	±1	1	-0.6562	0	$\cdots$	1.866	$e^{\mu t_E}$	0	$\to -\infty$	$ME4_-$
MP3	0	±1	-1	-0.6194	0	$\cdots$	1.271	$e^{\mu t_E}$	0	$\to -\infty$	$ME5_-$
MP4	0	±1	-1	-0.6026	0	$\cdots$	3.367	$e^{\mu t_E}$	0	$\to -\infty$	$ME6_-$
MP5	0	±1	±1	-0.7968	-0.1079	$\cdots$	$\cdots$	3.1099	$-0.030 83$	$\to -\infty$	$ME1_-$
MP6	0	±1	±1	-0.5557	-0.3425	$\cdots$	$\cdots$	1.4636	$-0.097 86$	$\to -\infty$	$ME2_-$
MP7	0	±1	±1	-0.4073	-0.4073	$\cdots$	$\cdots$	1.2857	-0.1164	$\to -\infty$	$ME3_-$
MP8	1	$0, \pm1$	$0, \pm1$	0.876 10	0.624 53	$\cdots$	$\cdots$	0.9611	0.1960	~0	K K
MP9	1	$0, \pm1$	$0, \pm1$	0.531 67	0.772 14	$\cdots$	$\cdots$	0.8735	0.2085	~0	K K
MP10	1	$0, \pm1$	$0, \pm1$	0.320 52	0.000 168	$\cdots$	$\cdots$	0.3209	0.000 1679	~0	K K
MP11	1	$0, \pm1$	$0, \pm1$	$-0.000 88$	0.288 98	$\cdots$	$\cdots$	0.5024	0.1437	~0	K K
MP12	1	0	-1	14.8319	1	$\cdots$	0.0492	4.0738	0.2222	~0	K M
MP13	1	0	-1	0.7335	1	$\cdots$	1.807	0.9408	0.2222	~0	K M
MP14	1	-1	0	1	0.7181	0.8036	$\cdots$	1	0.2044	~0	M K
MP15	1	-1	0	1	0.0417	0.9900	$\cdots$	1	0.0364	~0	M K
MP16	1	-1	0	1	-14.1607	0.0851	$\cdots$	1	0.2916	~0	M K
MP17	1	-1	±1	1	0	1	$\cdots$	1	0	~0	M K
MP18	1	±1	-1	0	1	$\cdots$	1	0.75	0.25	~0	$S_\pm$ M
MP19	1	1	-1	1	1	0.2986	0.2958	1	0.25	~0	C M
MP20	1	-1	-1	1	1	1.329	0.7899	1	0.25	~0	M M
MP21	1	-1	-1	1	1	1.112	1.957	1	0.25	~0	M M
MP22	1	-1	-1	1	1	0.5677	0.9155	1	0.25	~0	M M

power-law in the Einstein frame. In the past asymptotic solution MP12, we also find a power-law inflation both in the original and Einstein frames.

C. Concluding remarks

Before we apply our solutions to cosmology, we have to specify what kind of universe we are looking for. We wish to have an inflation in the early stage of the universe. We also hope to find an accelerated expansion in the present stage, if possible. Note that our cosmological model is higher dimensional, so that there are two kinds of frames that we can take to discuss cosmologies, the original frame and the Einstein frame in four dimensions. We must first determine in which frame we should consider the problem. Notice that the flatness and horizon problems should be explained in our four-dimensional spacetime for a successful inflationary scenario. If the radius of the internal space does not change, there is no difference between these frames. On the other hand, the four-dimensional gravitational constant depends on time in general unless we take the Einstein frame when the radius of the internal space changes as in the present case, and this is not preferable for a model of our universe. It thus appears more reasonable to consider the problem in the Einstein frame. Also the condition for the inflation in the Einstein frame is sufficient for that in the original frame. For these reasons, we require a successful inflation in the Einstein frame. This may be regarded as a criterion for the successful inflation independent of the mechanism of fixing the size of the internal space.

Next, we need at least 60 e foldings of inflationary expansion. This may give some constraint on the power exponent for a power-law inflation, that is, the power exponent should be significantly larger than unity. As we mentioned above, some solutions give an inflation in the Einstein frame but the external space shrinks in the original higher dimensions. Are such solutions suitable for a good cosmological model? The answer is no. Our four-dimensional universe makes sense only if it is much larger than the internal space, so the external space should expand faster than the internal space. Its expansion may not necessary to be inflationary, but at least the external space must be expanding in the whole space.

From these considerations and the above list of solutions, we conclude that the M theory (and type II string theories) can provide successful inflationary solutions. In the heterotic string theories with Gauss-Bonnet term, although we find exponential expansions of the external space in the original frame, those give noninflationary power-law expansions in the Einstein frame. There is a power-law inflation HP5 in the past asymptotic regime. However, the power exponent is 3.3, which may be too small to solve flatness and horizon problems, because we do not expect the expansion in these solutions continues so long. We also have a super inflation HE2$_+$ (HF4, HP4) in the Einstein frame [25]. In this case, we have to clarify a mechanism to avoid the singularity at $t_E = 0$.

In the M theory, we find seven candidates (ME1$_+$, ME2$_+$, ME3$_+$, ME4$_+$, ME5$_+$, ME6$_+$, and MP12). Among these, we can first exclude ME2$_+$ and ME3$_+$ because the internal space expands almost at the same rate as the external space. As for the solutions ME1$_+$ and MP12, we could also reject them because the power exponents in the Einstein frame are 3.1 and 4.1, which may be too small. However, this does not completely exclude the solutions because they may give large e foldings if the inflation lasts for a long time. To check this, we have to analyze how long such an inflationary period can continue. In our previous paper [23], we showed that although the period may be too short for the present value of $\tilde{\alpha}_4$ in the solution ME1$_+$, if we change the coupling constant slightly, we find a successful inflationary scenario with large extra dimensions. Such change of coupling constant is possible because there is intrinsic ambiguity in the terms of effective action involving Ricci tensors and scalar curvature.

For the solutions ME4$_+$, ME5$_+$, ME6$_+$, we find an exponential expansion of the external space both in the original and the Einstein frames, and the internal space is static. Hence those solutions may provide a successful inflationary scenario.

Which solution is preferable? In order to answer this question, we have to analyze the dynamics of our system. Then we should study the stability of those solutions both perturbatively and nonperturbatively and find how large e folding we can get. This study is in progress and the results will be reported in the forthcoming paper [24].

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APPENDIX A: EQUATIONS OF MOTION

Taking variation of the actions, we find the basic equations (2.13), (2.14), and (2.15), where each term is summarized here according to which action it originates from. We use the following notation throughout this paper.

$$\begin{aligned} &(p-m)_n \equiv (p-m)(p-m-1)(p-m-2)\cdots(p-n),\\ &(q-m)_n \equiv (q-m)(q-m-1)(q-m-2)\cdots(q-n),\\ &A_p \equiv \dot{u}_1^2 + \sigma_p e^{2(u_0-u_1)}, \qquad A_q \equiv \dot{u}_2^2 + \sigma_q e^{2(u_0-u_2)},\\ &X \equiv \ddot{u}_1 - \dot{u}_0\dot{u}_1 + \dot{u}_1^2, \qquad Y \equiv \ddot{u}_2 - \dot{u}_0\dot{u}_2 + \dot{u}_2^2. \end{aligned} \tag{A1}$$

The Einstein equations are given by the following three equations:

$$F = 0, \tag{A2}$$

$$F^{(p)} = 0, \tag{A3}$$

$$F^{(q)} = 0, \tag{A4}$$

where

$$F = \frac{\partial S}{\partial u_0} = \sum_{n=1}^{4} F_n + F_S, \tag{A5}$$

$$F^{(p)} = \frac{1}{p}\frac{\partial S}{\partial u_1} = \sum_{n=1}^{4} F_n^{(p)} + F_S^{(p)}, \tag{A6}$$

$$F^{(q)} = \frac{1}{q}\frac{\partial S}{\partial u_2} = \sum_{n=1}^{4} F_n^{(q)} + F_S^{(q)}, \tag{A7}$$

with

$$F_n^{(p)} = f_n^{(p)} + g_n^{(p)}X + h_n^{(p)}Y, \tag{A8}$$

$$F_n^{(q)} = f_n^{(q)} + g_n^{(q)}Y + h_n^{(q)}X. \tag{A9}$$

$f_n^{(p)}, g_n^{(p)}, h_n^{(p)}, f_n^{(q)}, g_n^{(q)}$, and $h_n^{(q)}$ are functionals of $u_0, \dot u_1, \dot u_2, A_p$, and A_q, while $F_S, F_S^{(p)}$, and $F_S^{(q)}$ are functionals of $u_0, u_1, u_2, \dot u_0, \dot u_1, \dot u_2, \ddot u_1, \ddot u_2, \dddot u_1, \dddot u_2, X, Y, \dot X, \dot Y, \ddot X$, and $\ddot Y$. The explicit forms of each term are listed here:

(i) *EH action* ($n = 1$)

$$F_1 = \alpha_1 e^{-u_0}[p_1 A_p + q_1 A_q + 2pq\dot u_1 \dot u_2], \quad \text{(A10)}$$

$$f_1^{(p)} = \alpha_1 e^{-u_0}[(p-1)_2 A_p + q_1 A_q + 2(p-1)q\dot u_1 \dot u_2], \quad \text{(A11)}$$

$$f_1^{(q)} = \alpha_1 e^{-u_0}[p_1 A_p + (q-1)_2 A_q + 2p(q-1)\dot u_1 \dot u_2], \quad \text{(A12)}$$

$$g_1^{(p)} = 2(p-1)\alpha_1 e^{-u_0}, \quad \text{(A13)}$$

$$g_1^{(q)} = 2(q-1)\alpha_1 e^{-u_0}, \quad \text{(A14)}$$

$$h_1^{(p)} = 2q\alpha_1 e^{-u_0}, \quad \text{(A15)}$$

$$h_1^{(q)} = 2p\alpha_1 e^{-u_0}. \quad \text{(A16)}$$

(ii) *GB action* ($n = 2$)

$$F_2 = \alpha_2 e^{-3u_0}[p_3 A_p^2 + 2p_1 q_1 A_p A_q + q_3 A_q^2 + 4\dot u_1 \dot u_2(p_2 q A_p + p q_2 A_q) + 4p_1 q_1 \dot u_1^2 \dot u_2^2], \quad \text{(A17)}$$

$$f_2^{(p)} = \alpha_2 e^{-3u_0}[(p-1)_4 A_p^2 + 2(p-1)_2 q_1 A_p A_q + q_3 A_q^2 + 4\dot u_1 \dot u_2((p-1)_3 q A_p + (p-1)q_2 A_q) + 4(p-1)_2 q_1 \dot u_1^2 \dot u_2^2], \quad \text{(A18)}$$

$$f_2^{(q)} = \alpha_2 e^{-3u_0}[(q-1)_4 A_q^2 + 2(q-1)_2 p_1 A_p A_q + p_3 A_p^2 + 4\dot u_1 \dot u_2((q-1)_3 p A_q + (q-1)p_2 A_p) + 4(q-1)_2 p_1 \dot u_1^2 \dot u_2^2], \quad \text{(A19)}$$

$$g_2^{(p)} = 4(p-1)\alpha_2 e^{-3u_0}[(p-2)_3 A_p + q_1 A_q + 2(p-2)q\dot u_1 \dot u_2], \quad \text{(A20)}$$

$$g_2^{(q)} = 4(q-1)\alpha_2 e^{-3u_0}[(q-2)_3 A_q + p_1 A_p + 2(q-2)p\dot u_1 \dot u_2], \quad \text{(A21)}$$

$$h_2^{(p)} = 4q\alpha_2 e^{-3u_0}[(p-1)_2 A_p + (q-1)_2 A_q + 2(p-1)(q-1)\dot u_1 \dot u_2], \quad \text{(A22)}$$

$$h_2^{(q)} = 4p\alpha_2 e^{-3u_0}[(q-1)_2 A_q + (p-1)_2 A_p + 2(p-1)(q-1)\dot u_1 \dot u_2]. \quad \text{(A23)}$$

(iii) *Lovelock action* ($n = 3, 4$)

$$F_3 = \alpha_3 e^{-5u_0}[p_5 A_p^3 + 3p_3 q_1 A_p^2 A_q + 3p_1 q_3 A_p A_q^2 + q_5 A_q^3 + 6\dot u_1 \dot u_2(p_4 q A_p^2 + 2p_2 q_2 A_p A_q + p q_4 A_q^2) + 12\dot u_1^2 \dot u_2^2(p_3 q_1 A_p + p_1 q_3 A_q) + 8p_2 q_2 \dot u_1^3 \dot u_2^3], \quad \text{(A24)}$$

$$f_3^{(p)} = \alpha_3 e^{-5u_0}[(p-1)_6 A_p^3 + 3(p-1)_4 q_1 A_p^2 A_q + 3(p-1)_2 q_3 A_p A_q^2 + q_5 A_q^3 + 6\dot u_1 \dot u_2((p-1)_5 q A_p^2 + 2(p-1)_3 q_2 A_p A_q + (p-1)q_4 A_q^2) + 12\dot u_1^2 \dot u_2^2((p-1)_4 q_1 A_p + (p-1)_2 q_3 A_q) + 8(p-1)_3 q_2 \dot u_1^3 \dot u_2^3], \quad \text{(A25)}$$

$$f_3^{(q)} = \alpha_3 e^{-5u_0}[(q-1)_6 A_q^3 + 3(q-1)_4 p_1 A_q^2 A_p + 3(q-1)_2 p_3 A_q A_p^2 + p_5 A_p^3 + 6\dot u_1 \dot u_2((q-1)_5 p A_q^2 + 2(q-1)_3 p_2 A_q A_p + (q-1)p_4 A_p^2) + 12\dot u_1^2 \dot u_2^2((q-1)_4 p_1 A_q + (q-1)_2 p_3 A_p) + 8(q-1)_3 p_2 \dot u_1^3 \dot u_2^3], \quad \text{(A26)}$$

$$g_3^{(p)} = 6(p-1)\alpha_3 e^{-5u_0}[(p-2)_5 A_p^2 + 2(p-2)_3 q_1 A_p A_q + q_3 A_q^2 + 4\dot u_1 \dot u_2((p-2)_4 q A_p + (p-2)q_2 A_q) + 4(p-2)_3 q_1 \dot u_1^2 \dot u_2^2], \quad \text{(A27)}$$

$$g_3^{(q)} = 6(q-1)\alpha_3 e^{-5u_0}[(q-2)_5 A_q^2 + 2(q-2)_3 p_1 A_p A_q + p_3 A_p^2 + 4\dot u_1 \dot u_2((q-2)_4 p A_q + (q-2)p_2 A_p) + 4(q-2)_3 p_1 \dot u_1^2 \dot u_2^2], \quad \text{(A28)}$$

$$h_3^{(p)} = 6q\alpha_3 e^{-5u_0}[(p-1)_4 A_p^2 + 2(p-1)_2 (q-1)_2 A_p A_q + (q-1)_4 A_q^2 + 4\dot u_1 \dot u_2((p-1)_3 (q-1) A_p + (p-1)(q-1)_3 A_q) + 4(p-1)_2 (q-1)_2 \dot u_1^2 \dot u_2^2], \quad \text{(A29)}$$

$$h_3^{(q)} = 6p\alpha_3 e^{-5u_0}[(q-1)_4 A_q^2 + 2(q-1)_2(p-1)_2 A_p A_q + (p-1)_4 A_p^2 + 4\dot{u}_1\dot{u}_2((q-1)_3(p-1)A_q + (q-1)(p-1)_3 A_p) + 4(p-1)_2(q-1)_2\dot{u}_1^2\dot{u}_2^2], \tag{A30}$$

$$F_4 = \alpha_4 e^{-7u_0}[p_7 A_p^4 + 4p_5 q_1 A_p^3 A_q + 6p_3 q_3 A_p^2 A_q^2 + 4p_1 q_5 A_p A_q^3 + q_7 A_q^4 + 8\dot{u}_1\dot{u}_2(p_6 q A_p^3 + 3p_4 q_2 A_p^2 A_q + 3p_2 q_4 A_p A_q^2 + p q_6 A_q^3) + 24\dot{u}_1^2\dot{u}_2^2(p_5 q_1 A_p^2 + 2p_3 q_3 A_p A_q + p_1 q_5 A_q^2) + 32\dot{u}_1^3\dot{u}_2^3(p_4 q_2 A_p + p_2 q_4 A_q) + 16p_3 q_3 \dot{u}_1^4\dot{u}_2^4], \tag{A31}$$

$$f_4^{(p)} = \alpha_4 e^{-7u_0}[(p-1)_8 A_p^4 + 4(p-1)_6 q_1 A_p^3 A_q + 6(p-1)_4 q_3 A_p^2 A_q^2 + 4(p-1)_2 q_5 A_p A_q^3 + q_7 A_q^4 + 8\dot{u}_1\dot{u}_2((p-1)_7 q A_p^3 + 3(p-1)_5 q_2 A_p^2 A_q + 3(p-1)_3 q_4 A_p A_q^2 + (p-1) q_6 A_q^3) + 24\dot{u}_1^2\dot{u}_2^2((p-1)_6 q_1 A_p^2 + 2(p-1)_4 q_3 A_p A_q + (p-1)_2 q_5 A_q^2) + 32\dot{u}_1^3\dot{u}_2^3((p-1)_5 q_2 A_p + (p-1)_3 q_4 A_q) + 16(p-1)_4 q_3 \dot{u}_1^4\dot{u}_2^4], \tag{A32}$$

$$f_4^{(q)} = \alpha_4 e^{-7u_0}[(q-1)_8 A_q^4 + 4(q-1)_6 p_1 A_q^3 A_p + 6(q-1)_4 p_3 A_p^2 A_q^2 + 4(q-1)_2 p_5 A_q A_p^3 + p_7 A_p^4 + 8\dot{u}_1\dot{u}_2((q-1)_7 p A_q^3 + 3(q-1)_5 p_2 A_q^2 A_p + 3(q-1)_3 p_4 A_q A_p^2 + (q-1) p_6 A_p^3) + 24\dot{u}_1^2\dot{u}_2^2((q-1)_6 p_1 A_q^2 + 2(q-1)_4 p_3 A_p A_q + (q-1)_2 p_5 A_p^2) + 32\dot{u}_1^3\dot{u}_2^3((q-1)_5 p_2 A_q + (q-1)_3 p_4 A_p) + 16(q-1)_4 p_3 \dot{u}_1^4\dot{u}_2^4], \tag{A33}$$

$$g_4^{(p)} = 8(p-1)\alpha_4 e^{-7u_0}[(p-2)_7 A_p^3 + 3(p-2)_5 q_1 A_p^2 A_q + 3(p-2)_3 q_3 A_p A_q^2 + q_5 A_q^3 + 6\dot{u}_1\dot{u}_2((p-2)_6 q A_p^2 + 2(p-2)_4 q_2 A_p A_q + (p-2) q_4 A_q^2) + 12\dot{u}_1^2\dot{u}_2^2((p-2)_5 q_1 A_p + (p-2)_3 q_3 A_q) + 8(p-2)_4 q_2 \dot{u}_1^3\dot{u}_2^3], \tag{A34}$$

$$g_4^{(q)} = 8(q-1)\alpha_4 e^{-7u_0}[(q-2)_7 A_q^3 + 3(q-2)_5 p_1 A_q^2 A_p + 3(q-2)_3 p_3 A_q A_p^2 + p_5 A_p^3 + 6\dot{u}_1\dot{u}_2((q-2)_6 p A_q^2 + 2(q-2)_4 p_2 A_p A_q + (q-2) p_4 A_p^2) + 12\dot{u}_1^2\dot{u}_2^2((q-2)_5 p_1 A_q + (q-2)_3 p_3 A_p) + 8(q-2)_4 p_2 \dot{u}_1^3\dot{u}_2^3], \tag{A35}$$

$$h_4^{(p)} = 8q\alpha_4 e^{-7u_0}[(p-1)_6 A_p^3 + 3(p-1)_4(q-1)_2 A_p^2 A_q + 3(p-1)_2(q-1)_4 A_p A_q^2 + (q-1)_6 A_q^3 + 6\dot{u}_1\dot{u}_2((p-1)_5(q-1)A_p^2 + 2(p-1)_3(q-1)_3 A_p A_q + (p-1)(q-1)_5 A_q^2) + 12\dot{u}_1^2\dot{u}_2^2((p-1)_4(q-1)_2 A_p + (p-1)_2(q-1)_4 A_q) + 8(p-1)_3(q-1)_3\dot{u}_1^3\dot{u}_2^3], \tag{A36}$$

$$h_4^{(q)} = 8p\alpha_4 e^{-7u_0}[(q-1)_6 A_q^3 + 3(q-1)_4(p-1)_2 A_q^2 A_p + 3(q-1)_2(p-1)_4 A_q A_p^2 + (p-1)_6 A_p^3 + 6\dot{u}_1\dot{u}_2((q-1)_5(p-1)A_q^2 + 2(p-1)_3(q-1)_3 A_p A_q + (q-1)(p-1)_5 A_p^2) + 12\dot{u}_1^2\dot{u}_2^2((q-1)_4(p-1)_2 A_q + (q-1)_2(p-1)_4 A_p) + 8(p-1)_3(q-1)_3\dot{u}_1^3\dot{u}_2^3]. \tag{A37}$$

(iv) S_S *action*

$$F_S = \gamma e^{-pu_1 - qu_2}\left[-7L_4 + 2\sigma_p e^{2(u_0-u_1)}\frac{\partial L_4}{\partial A_p} + 2\sigma_q e^{2(u_0-u_2)}\frac{\partial L_4}{\partial A_q} + \frac{d}{dt}\left(\dot{u}_1\frac{\partial L_4}{\partial X} + \dot{u}_2\frac{\partial L_4}{\partial Y}\right)\right], \tag{A38}$$

$$pF_S^{(p)} = \gamma e^{-pu_1 - qu_2}\left\{pL_4 - 2\sigma_p e^{2(u_0-u_1)}\frac{\partial L_4}{\partial A_p} + \frac{d}{dt}\left[(\dot{u}_0 - 2\dot{u}_1)\frac{\partial L_4}{\partial X} - 2\dot{u}_1\frac{\partial L_4}{\partial A_p} - \frac{\partial L_4}{\partial \dot{u}_1}\right] + \frac{d^2}{dt^2}\left(\frac{\partial L_4}{\partial X}\right)\right\}, \tag{A39}$$

$$qF_S^{(q)} = \gamma e^{-pu_1 - qu_2}\left\{qL_4 - 2\sigma_q e^{2(u_0-u_2)}\frac{\partial L_4}{\partial A_q} + \frac{d}{dt}\left[(\dot{u}_0 - 2\dot{u}_2)\frac{\partial L_4}{\partial Y} - 2\dot{u}_2\frac{\partial L_4}{\partial A_q} - \frac{\partial L_4}{\partial \dot{u}_2}\right] + \frac{d^2}{dt^2}\left(\frac{\partial L_4}{\partial Y}\right)\right\}, \tag{A40}$$

where

$$L_4 = e^{-7u_0+pu_1+qu_2}[p_1X^2(X+2A_p)^2 + q_1Y^2(Y+2A_q)^2 + 2pq(XY+(X+Y)\dot{u}_1\dot{u}_2)^2 + 3p_2A_p^4 + 3q_2A_q^4 + p_1q\dot{u}_1^2\dot{u}_2^2(\dot{u}_1\dot{u}_2+2A_p)^2 + pq_1\dot{u}_1^2\dot{u}_2^2(\dot{u}_1\dot{u}_2+2A_q)^2], \tag{A41}$$

$$\frac{\partial L_4}{\partial X} = 4pe^{-7u_0+pu_1+qu_2}[(p-1)X(X+A_p)(X+2A_p) + q(Y+\dot{u}_1\dot{u}_2)(XY+(X+Y)\dot{u}_1\dot{u}_2)], \tag{A42}$$

$$\frac{\partial L_4}{\partial Y} = 4qe^{-7u_0+pu_1+qu_2}[(q-1)Y(Y+A_q)(Y+2A_q) + p(X+\dot{u}_1\dot{u}_2)(XY+(X+Y)\dot{u}_1\dot{u}_2)], \tag{A43}$$

$$\frac{\partial L_4}{\partial A_p} = 4p_1e^{-7u_0+pu_1+qu_2}[X^2(X+2A_p) + 3(p-2)A_p^3 + q\dot{u}_1^2\dot{u}_2^2(\dot{u}_1\dot{u}_2+2A_p)], \tag{A44}$$

$$\frac{\partial L_4}{\partial A_q} = 4q_1e^{-7u_0+pu_1+qu_2}[Y^2(Y+2A_q) + 3(q-2)A_q^3 + p\dot{u}_1^2\dot{u}_2^2(\dot{u}_1\dot{u}_2+2A_q)], \tag{A45}$$

$$\frac{\partial L_4}{\partial \dot{u}_1} = 4pqe^{-7u_0+pu_1+qu_2}\dot{u}_2[(X+Y)(XY+(X+Y)\dot{u}_1\dot{u}_2) + (p-1)\dot{u}_1\dot{u}_2(\dot{u}_1\dot{u}_2+A_p)(\dot{u}_1\dot{u}_2+2A_p) + (q-1)\dot{u}_1\dot{u}_2(\dot{u}_1\dot{u}_2+A_q)(\dot{u}_1\dot{u}_2+2A_q)], \tag{A46}$$

$$\frac{\partial L_4}{\partial \dot{u}_2} = 4pqe^{-7u_0+pu_1+qu_2}\dot{u}_1[(X+Y)(XY+(X+Y)\dot{u}_1\dot{u}_2) + (p-1)\dot{u}_1\dot{u}_2(\dot{u}_1\dot{u}_2+A_p)(\dot{u}_1\dot{u}_2+2A_p) + (q-1)\dot{u}_1\dot{u}_2(\dot{u}_1\dot{u}_2+A_q)(\dot{u}_1\dot{u}_2+2A_q)]. \tag{A47}$$

APPENDIX B: INPUTTING OUR ANSATZ INTO SOLUTIONS

In order to find solutions, we assume

$$u_0 = \epsilon t, \qquad u_1 = \mu t + \ln a_0, \qquad u_2 = \nu t + \ln b_0. \tag{B1}$$

Inserting this form into the above equations [Eqs. (A10)–(A23)] and setting

$$A_p = \mu^2 + \tilde{\sigma}_p e^{2(\epsilon-\mu)t}, \qquad A_q = \nu^2 + \tilde{\sigma}_q e^{2(\epsilon-\nu)t}, \qquad \tilde{\sigma}_p = \frac{\sigma_p}{a_0^2}, \qquad \tilde{\sigma}_q = \frac{\sigma_q}{b_0^2}, \tag{B2}$$

$$X = \mu(\mu-\epsilon), \qquad Y = \nu(\nu-\epsilon), \tag{B3}$$

we obtain

$$F_1 = \alpha_1 e^{-\epsilon t}[p_1A_p + q_1A_q + 2pq\mu\nu], \qquad f_1^{(p)} = \alpha_1 e^{-\epsilon t}[(p-1)_2A_p + q_1A_q + 2(p-1)q\mu\nu],$$
$$f_1^{(q)} = \alpha_1 e^{-\epsilon t}[p_1A_p + (q-1)_2A_q + 2p(q-1)\mu\nu], \qquad g_1^{(p)} = 2(p-1)\alpha_1 e^{-\epsilon t}, \qquad g_1^{(q)} = 2(q-1)\alpha_1 e^{-\epsilon t}, \tag{B4}$$
$$h_1^{(p)} = 2q\alpha_1 e^{-\epsilon t}, \qquad h_1^{(q)} = 2p\alpha_1 e^{-\epsilon t},$$

$$\begin{aligned}
F_2 &= \alpha_2 e^{-3\epsilon t}[p_3 A_p^2 + 2p_1 q_1 A_p A_q + q_3 A_q^2 + 4\mu\nu(p_2 q A_p + p q_2 A_q) + 4p_1 q_1 \mu^2\nu^2],\\
f_2^{(p)} &= \alpha_2 e^{-3\epsilon t}[(p-1)_4 A_p^2 + 2(p-1)_2 q_1 A_p A_q + q_3 A_q^2 + 4\mu\nu((p-1)_3 q A_p + (p-1) q_2 A_q) + 4(p-1)_2 q_1 \mu^2\nu^2],\\
f_2^{(q)} &= \alpha_2 e^{-3\epsilon t}[(q-1)_4 A_q^2 + 2(q-1)_2 p_1 A_p A_q + p_3 A_p^2 + 4\mu\nu((q-1)_3 p A_q + (q-1) p_2 A_p) + 4(q-1)_2 p_1 \mu^2\nu^2],\\
g_2^{(p)} &= 4(p-1)\alpha_2 e^{-3\epsilon t}[(p-2)_3 A_p + q_1 A_q + 2(p-2) q\mu\nu],\\
g_2^{(q)} &= 4(q-1)\alpha_2 e^{-3\epsilon t}[(q-2)_3 A_q + p_1 A_p + 2(q-2) p\mu\nu],\\
h_2^{(p)} &= 4q\alpha_2 e^{-3\epsilon t}[(p-1)_2 A_p + (q-1)_2 A_q + 2(p-1)(q-1)\mu\nu],\\
h_2^{(q)} &= 4p\alpha_2 e^{-3\epsilon t}[(p-1)_2 A_p + (q-1)_2 A_q + 2(p-1)(q-1)\mu\nu],
\end{aligned} \tag{B5}$$

$$\begin{aligned}
F_4 = \alpha_4 e^{-7\epsilon t}[&p_7 A_p^4 + 4p_5 q_1 A_p^3 A_q + 6p_3 q_3 A_p^2 A_q^2 + 4p_1 q_5 A_p A_q^3 + q_7 A_q^4 + 8\mu\nu(p_6 q A_p^3 + 3p_4 q_2 A_p^2 A_q\\
&+ 3p_2 q_4 A_p A_q^2 + p q_6 A_q^3) + 24\mu^2\nu^2(p_5 q_1 A_p^2 + 2p_3 q_3 A_p A_q + p_1 q_5 A_q^2) + 32\mu^3\nu^3(p_4 q_2 A_p + p_2 q_4 A_q)\\
&+ 16 p_3 q_3 \mu^4\nu^4],
\end{aligned} \tag{B6}$$

$$\begin{aligned}
f_4^{(p)} = \alpha_4 e^{-7\epsilon t}[&(p-1)_8 A_p^4 + 4(p-1)_6 q_1 A_p^3 A_q + 6(p-1)_4 q_3 A_p^2 A_q^2 + 4(p-1)_2 q_5 A_p A_q^3 + q_7 A_q^4\\
&+ 8\mu\nu\{(p-1)_7 q A_p^3 + 3(p-1)_5 q_2 A_p^2 A_q + 3(p-1)_3 q_4 A_p A_q^2 + (p-1) q_6 A_q^3\} + 24\mu^2\nu^2\{(p-1)_6 q_1 A_p^2\\
&+ 2(p-1)_4 q_3 A_p A_q + (p-1)_2 q_5 A_q^2\} + 32\mu^3\nu^3\{(p-1)_5 q_2 A_p + (p-1)_3 q_4 A_q\} + 16(p-1)_4 q_3 \mu^4\nu^4],
\end{aligned} \tag{B7}$$

$$\begin{aligned}
g_4^{(p)} = 8(p-1)\alpha_4 e^{-7\epsilon t}[&(p-2)_7 A_p^3 + 3(p-2)_5 q_1 A_p^2 A_q + 3(p-2)_3 q_3 A_p A_q^2 + q_5 A_q^3 + 6\mu\nu\{(p-2)_6 q A_p^2\\
&+ 2(p-2)_4 q_2 A_p A_q + (p-2) q_4 A_q^2\} + 12\mu^2\nu^2\{(p-2)_5 q_1 A_p + (p-2)_3 q_3 A_q\} + 8(p-2)_4 q_2 \mu^3\nu^3],
\end{aligned} \tag{B8}$$

$$\begin{aligned}
h_4^{(p)} = 8q\alpha_4 e^{-7\epsilon t}[&(p-1)_6 A_p^3 + 3(p-1)_4 (q-1)_2 A_p^2 A_q + 3(p-1)_2 (q-1)_4 A_p A_q^2 + (q-1)_6 A_q^3\\
&+ 6\mu\nu\{(p-1)_5 (q-1) A_p^2 + 2(p-1)_3 (q-1)_3 A_p A_q + (p-1)(q-1)_5 A_q^2\}\\
&+ 12\mu^2\nu^2\{(p-1)_4 (q-1)_2 A_p + (p-1)_2 (q-1)_4 A_q\} + 8(p-1)_3 (q-1)_3 \mu^3\nu^3],
\end{aligned} \tag{B9}$$

$$\begin{aligned}
f_4^{(q)} = \alpha_4 e^{-7\epsilon t}[&(q-1)_8 A_q^4 + 4(q-1)_6 p_1 A_q^3 A_p + 6(q-1)_4 p_3 A_p^2 A_q^2 + 4(q-1)_2 p_5 A_q A_p^3 + p_7 A_p^4\\
&+ 8\mu\nu\{(q-1)_7 p A_q^3 + 3(q-1)_5 p_2 A_q^2 A_p + 3(q-1)_3 p_4 A_q A_p^2 + (q-1) p_6 A_p^3\} + 24\mu^2\nu^2\{(q-1)_6 p_1 A_q^2\\
&+ 2(q-1)_4 p_3 A_p A_q + (q-1)_2 p_5 A_p^2\} + 32\mu^3\nu^3\{(q-1)_5 p_2 A_q + (q-1)_3 p_4 A_p\} + 16(q-1)_4 p_3 \mu^4\nu^4],
\end{aligned} \tag{B10}$$

$$\begin{aligned}
g_4^{(q)} = 8(q-1)\alpha_4 e^{-7\epsilon t}[&(q-2)_7 A_q^3 + 3(q-2)_5 p_1 A_q^2 A_p + 3(q-2)_3 p_3 A_q A_p^2 + p_5 A_p^3 + 6\mu\nu\{(q-2)_6 p A_q^2\\
&+ 2(q-2)_4 p_2 A_p A_q + (q-2) p_4 A_p^2\} + 12\mu^2\nu^2\{(q-2)_5 p_1 A_q + (q-2)_3 p_3 A_p\} + 8(q-2)_4 p_2 \mu^3\nu^3],
\end{aligned} \tag{B11}$$

$$\begin{aligned}
h_4^{(q)} = 8p\alpha_4 e^{-7\epsilon t}[&(q-1)_6 A_q^3 + 3(q-1)_4 (p-1)_2 A_q^2 A_p + 3(q-1)_2 (p-1)_4 A_q A_p^2 + (p-1)_6 A_p^3\\
&+ 6\mu\nu\{(q-1)_5 (p-1) A_q^2 + 2(p-1)_3 (q-1)_3 A_p A_q + (q-1)(p-1)_5 A_p^2\}\\
&+ 12\mu^2\nu^2\{(q-1)_4 (p-1)_2 A_q + (q-1)_2 (p-1)_4 A_p\} + 8(p-1)_3 (q-1)_3 \mu^3\nu^3],
\end{aligned} \tag{B12}$$

$$\begin{aligned}
F_S = \gamma e^{-7\epsilon t}[&-7\tilde{L}_4 + (-7\epsilon + p\mu + q\nu)(p\mu M_X + q\nu M_Y) + 2\tilde{\sigma}_p e^{2(\epsilon-\mu)t} p\{N_p + \mu(\epsilon-\mu) P_{pX}\}\\
&+ 2\tilde{\sigma}_q e^{2(\epsilon-\nu)t} q\{N_q + \nu(\epsilon-\nu) P_{qY}\}],
\end{aligned} \tag{B13}$$

$$\begin{aligned}
F_S^{(p)} = \gamma e^{-7\epsilon t}[&\tilde{L}_4 + (-7\epsilon + p\mu + q\nu)\{(\epsilon - 2\mu) M_X - 2\mu N_p - q\nu U + 4(\epsilon-\mu)\tilde{\sigma}_p e^{2(\epsilon-\mu)t} P_{pX}\}\\
&+ (-7\epsilon + p\mu + q\nu)^2 M_X + 2\tilde{\sigma}_p e^{2(\epsilon-\mu)t}\{-N_p + (\epsilon-\mu)[(\epsilon-2\mu)P_{pX} - 2\mu Q_{pp} - (p-1)q\nu V_p]\}\\
&- 2(\epsilon-\nu)\tilde{\sigma}_q e^{2(\epsilon-\nu)t} q_1 \nu V_q + 4(\epsilon-\mu)^2\tilde{\sigma}_p e^{2(\epsilon-\mu)t} P_{pX} + 4(\epsilon-\mu)^2\tilde{\sigma}_p^2 e^{4(\epsilon-\mu)t} R_{pX}],
\end{aligned} \tag{B14}$$

$$F_S^{(q)} = \gamma e^{-7\epsilon t}[\tilde{L}_4 + (-7\epsilon + p\mu + q\nu)\{(\epsilon - 2\nu)M_Y - 2\nu N_q - p\mu U + 4(\epsilon - \nu)\tilde{\sigma}_q e^{2(\epsilon-\nu)t}P_{qY}\} + (-7\epsilon + p\mu + q\nu)^2 M_Y + 2\tilde{\sigma}_q e^{2(\epsilon-\nu)t}\{-N_q + (\epsilon - \nu)[(\epsilon - 2\nu)P_{qY} - 2\nu Q_{qq} - p(q-1)\mu V_q]\} - 2(\epsilon - \mu)\tilde{\sigma}_p e^{2(\epsilon-\mu)t} p_1 \mu V_p + 4(\epsilon - \nu)^2 \tilde{\sigma}_q e^{2(\epsilon-\nu)t} P_{qY} + 4(\epsilon - \nu)^2 \tilde{\sigma}_q^2 e^{4(\epsilon-\nu)t} R_{qY}], \quad \text{(B15)}$$

where

$$\tilde{L}_4 = p_1 X^2 (X + 2A_p)^2 + q_1 Y^2 (Y + 2A_q)^2 + 2pq(XY + (X+Y)\mu\nu)^2 + 3p_2 A_p^4 + 3q_2 A_q^4 + p_1 q \mu^2\nu^2(\mu\nu + 2A_p)^2 + pq_1\mu^2\nu^2(\mu\nu + 2A_q)^2, \quad \text{(B16)}$$

$$M_X = 4[(p-1)X(X + A_p)(X + 2A_p) + q(Y + \mu\nu)(XY + (X+Y)\mu\nu)], \quad \text{(B17)}$$

$$M_Y = 4[(q-1)Y(Y + A_q)(Y + 2A_q) + p(X + \mu\nu)(XY + (X+Y)\mu\nu)], \quad \text{(B18)}$$

$$N_p = 4(p-1)[X^2(X + 2A_p) + 3(p-2)A_p^3 + q\mu^2\nu^2(\mu\nu + 2A_p)], \quad \text{(B19)}$$

$$N_q = 4(q-1)[Y^2(Y + 2A_q) + 3(q-2)A_q^3 + p\mu^2\nu^2(\mu\nu + 2A_q)], \quad \text{(B20)}$$

$$P_{pX} = 4(p-1)X(3X + 4A_p), \quad \text{(B21)}$$

$$P_{qY} = 4(q-1)Y(3Y + 4A_q), \quad \text{(B22)}$$

$$Q_{pp} = 4(p-1)[2X^2 + 9(p-2)A_p^2 + 2q\mu^2\nu^2], \quad \text{(B23)}$$

$$Q_{qq} = 4(q-1)[2Y^2 + 9(q-2)A_q^2 + 2p\mu^2\nu^2], \quad \text{(B24)}$$

$$R_{pX} = 16(p-1)X, \quad \text{(B25)}$$

$$R_{qY} = 16(q-1)Y, \quad \text{(B26)}$$

$$U = 4[(X+Y)(XY + (X+Y)\mu\nu) + (p-1)\mu\nu(\mu\nu + A_p)(\mu\nu + 2A_p) + (q-1)\mu\nu(\mu\nu + A_q)(\mu\nu + 2A_q)], \quad \text{(B27)}$$

$$V_p = 4\mu\nu(3\mu\nu + 4A_p), \quad \text{(B28)}$$

$$V_q = 4\mu\nu(3\mu\nu + 4A_q). \quad \text{(B29)}$$

APPENDIX C: GENERALIZED DE SITTER SOLUTIONS WITH FLAT SPACES

If we assume $\epsilon = 0$, $\sigma_p = 0$, and $\sigma_q = 0$, we find two independent basic equations without constraint. Setting $X = \mu^2 = A_p$ and $Y = \nu^2 = A_q$, we obtain the following terms:

$$F_1 = \alpha_1[p_1\mu^2 + q_1\nu^2 + 2pq\mu\nu], \quad \text{(C1)}$$

$$F_1^{(p)} = F_1 - q\nu H_1, \quad \text{(C2)}$$

$$F_1^{(q)} = F_1 + p\mu H_1, \quad \text{(C3)}$$

$$F_2 = \alpha_2[p_3\mu^4 + 4\mu\nu(p_2 q\mu^2 + pq_2\nu^2) + 6p_1q_1\mu^2\nu^2 + q_3\nu^4], \quad \text{(C4)}$$

$$F_2^{(p)} = F_2 - q\nu H_2, \quad \text{(C5)}$$

$$F_2^{(q)} = F_2 + p\mu H_2, \quad \text{(C6)}$$

$$F_4 = \alpha_4[p_7\mu^8 + 8p_6 q\mu^7\nu + 28p_5q_1\mu^6\nu^2 + 56p_4q_2\mu^5\nu^3 + 70p_3q_3\mu^4\nu^4 + 56p_2q_4\mu^3\nu^5 + 28p_1q_5\mu^2\nu^6 + 8pq_6\mu\nu^7 + q_7\nu^8], \quad \text{(C7)}$$

$$F_4^{(p)} = F_4 - q\nu H_4, \quad \text{(C8)}$$

$$F_4^{(q)} = F_4 + p\mu H_4, \quad \text{(C9)}$$

$$F_S = \gamma[-7\tilde{L}_4 + (p\mu + q\nu)(p\mu M_X + q\nu M_Y)], \quad \text{(C10)}$$

$$F_S^{(p)} = F_S - \gamma q\nu H_S \quad \text{(C11)}$$

$$F_S^{(q)} = F_S + \gamma p\mu H_S, \quad \text{(C12)}$$

where

$$H_1 = 2\alpha_1(\mu - \nu), \quad \text{(C13)}$$

$$H_2 = 4\alpha_2(\mu - \nu)[(p-1)_2\mu^2 + 2(p-1)(q-1)\mu\nu + (q-1)_2\nu^2], \quad \text{(C14)}$$

$$H_4 = 8\alpha_4(\mu - \nu)[(p-1)_6\mu^6 + 6(p-1)_5(q-1)\mu^5\nu + 15(p-1)_4(q-1)_2\mu^4\nu^2 + 20(p-1)_3(q-1)_3\mu^3\nu^3 + 15(p-1)_2(q-1)_4\mu^2\nu^4 + 6(p-1)(q-1)_5\mu\nu^5 + (q-1)_6\nu^6], \tag{C15}$$

$$\tilde{L}_4 = 3(p+1)_1\mu^8 + 3(q+1)_1\nu^8 + 2pq(\mu^2+\nu^2+\mu\nu)^2\mu^2\nu^2 + p_1 q\mu^4\nu^2(2\mu+\nu)^2 + pq_1\mu^2\nu^4(\mu+2\nu)^2, \tag{C16}$$

$$M_X = 4[6(p-1)\mu^6 + q\mu\nu^2(\mu+\nu)(\mu^2+\nu^2+\mu\nu)], \tag{C17}$$

$$M_Y = 4[6(q-1)\nu^6 + p\mu^2\nu(\mu+\nu)(\mu^2+\nu^2+\mu\nu)], \tag{C18}$$

$$H_S = 4\gamma(\mu-\nu)[6(p-1)\mu^6 - (p-1)(p+6q-6)\mu^5\nu - (2p^2-7p-3q+6)\mu^4\nu^2 - \{p^2-(4q+3)p + q^2-3q+6\}\mu^3\nu^3 - (2q^2-7q-3p+6)\mu^2\nu^4 - (q-1)(q+6p-6)\mu\nu^5 + 6(q-1)\nu^6]. \tag{C19}$$

Since the second and third equations are given in the form

$$F^{(p)} = F - q\nu H, \quad \text{and} \quad F^{(p)} = F + p\mu H \tag{C20}$$

those are equivalent if $\mu \neq 0$ and $\nu \neq 0$. Then we can take the following two algebraic equations as our basic equations.

$$\sum_{n=1}^{4} F_n + F_S = 0, \tag{C21}$$

$$\sum_{n=1}^{4} H_n + H_S = 0. \tag{C22}$$

D. SOLUTIONS IN TYPE II STRING

Here we summarize the case of type II string. Exact solutions are in Table VII, future asymptotic solutions in Table VIII, and past asymptotic solutions in Table IX.

Here we have the similar results to the case of the M theory. Let us focus on inflationary solutions. In the original frame, $\mathrm{IIE1}_+$(IIF4), $\mathrm{IIE2}_+$(IIF5), $\mathrm{IIE3}_+$(IIF6), $\mathrm{IIE4}_+$(IIF2), $\mathrm{IIE5}_+$(IIF3) give an exponential expansion for the external space. In the Einstein frame, we find either a power-law inflation [$\mathrm{IIE1}_+$(IIF4), $\mathrm{IIE2}_+$(IIF5),

TABLE VII. Type II superstring: exact solutions. K, $S_\pm$, S_0, and M mean a kinetic dominance, a static space with positive (or negative) curvature, a flat static space, and a Milne-type space, respectively.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	Type
$\mathrm{IIE1}_\pm$	0	0	0	±0.7999	±0.1299	$\cdots$	$\cdots$	3.053	±0.0433	K K
$\mathrm{IIE2}_\pm$	0	0	0	±0.5075	±0.5075	$\cdots$	$\cdots$	1.333	±0.1692	K K
$\mathrm{IIE3}_\pm$	0	0	0	±0.4962	±0.5131	$\cdots$	$\cdots$	1.322	±0.1710	K K
$\mathrm{IIE4}_\pm$	0	0	1	±0.7655	0	$\cdots$	1.480	$e^{\mu t_E}$	0	K S_+
$\mathrm{IIE5}_\pm$	0	0	-1	±0.6200	0	$\cdots$	2.763	$e^{\mu t_E}$	0	K S_-
$\mathrm{IIE6}_\pm$	0	1	0	0	±0.6201	1.078	$\cdots$	1	±0.2067	S_+ K
IIE7	1	0	-1	0	1	$\cdots$	1	0.75	0.25	S_0 M
IIE8	1	-1	0	1	0	1	$\cdots$	1	0	M S_0

TABLE VIII. Type II superstrings: future asymptotic solutions ($t \to \infty$). M means a Milne-type space.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
IIF1	0	1	±1	0	0.6201	1.078	$\cdots$	1	0.2067	$\to\infty$	$\mathrm{IIE6}_+$
IIF2	0	±1	1	0.7655	0	$\cdots$	1.480	$e^{\mu t_E}$	0	$\to\infty$	$\mathrm{IIE4}_+$
IIF3	0	±1	-1	0.6200	0	$\cdots$	2.763	$e^{\mu t_E}$	0	$\to\infty$	$\mathrm{IIE5}_+$
IIF4	0	±1	±1	0.7999	0.1299	$\cdots$	$\cdots$	3.053	0.0433	$\to\infty$	$\mathrm{IIE1}_+$
IIF5	0	±1	±1	0.5075	0.5075	$\cdots$	$\cdots$	1.333	0.1692	$\to\infty$	$\mathrm{IIE2}_+$
IIF6	0	±1	±1	0.4962	0.5131	$\cdots$	$\cdots$	1.322	0.1710	$\to\infty$	$\mathrm{IIE3}_+$
IIF7	1	0	0	0.5556	-0.1111	$\cdots$	$\cdots$	0.3333	-0.1667	$\to\infty$	Kasner
IIF8	1	0	0	-0.3333	0.3333	$\cdots$	$\cdots$	0.3333	0.1667	$\to\infty$	Kasner
IIF9	1	-1	-1	1	1	0.5	0.7906	1	0.25	$\to\infty$	M M

TABLE IX. Type II superstrings: past asymptotic solutions ($t \to -\infty$). K, $S_\pm$, S_0, M, and C mean a kinetic dominance, a static space with positive (or negative) curvature, a flat static space, a Milne-type space, and a constant curvature space, respectively.

	ϵ	σ_p	σ_q	μ	ν	a_0	b_0	λ	ϕ_1	t_E	Type
IIP1	0	1	±1	0	-0.6201	1.078	$\cdots$	1	-0.2067	$\to -\infty$	$IIE6_-$
IIP2	0	±1	-1	-0.6200	0	$\cdots$	2.763	$e^{\mu t_E}$	0	$\to -\infty$	$IIE4_-$
IIP3	0	±1	1	-0.7655	0	$\cdots$	1.480	$e^{\mu t_E}$	0	$\to -\infty$	$IIE5_-$
IIP4	0	±1	±1	-0.7999	-0.1299	$\cdots$	$\cdots$	3.053	-0.0433	$\to -\infty$	$IIE1_-$
IIP5	0	±1	±1	-0.5075	-0.5075	$\cdots$	$\cdots$	1.333	-0.1692	$\to -\infty$	$IIE2_-$
IIP6	0	±1	±1	-0.4962	-0.5131	$\cdots$	$\cdots$	1.322	-0.1710	$\to -\infty$	$IIE3_-$
IIP7	1	$0, \pm1$	$0, \pm1$	5.7427	5.7427	$\cdots$	$\cdots$	1.2602	0.3150	~ 0	K K
IIP8	1	$0, \pm1$	$0, \pm1$	0.3205	0.000 17	$\cdots$	$\cdots$	0.3208	0.000 1699	~ 0	K K
IIP9	1	$0, \pm1$	$0, \pm1$	0.2883	0.2883	$\cdots$	$\cdots$	0.6184	0.1546	~ 0	K K
IIP10	1	$0, \pm1$	$0, \pm1$	0.0013	0.2954	$\cdots$	$\cdots$	0.4705	0.1566	~ 0	K K
IIP11	1	0	1	4.0305	1	$\cdots$	0.3375	1.7576	0.25	~ 0	K C
IIP12	1	0	-1	0.4484	1	$\cdots$	0.8948	0.8621	0.25	~ 0	K M
IIP13	1	0	-1	-9.7439	1	$\cdots$	0.1028	-1.6860	0.25	~ 0	K M
IIP14	1	1	0	1	6.1725	0.1148	$\cdots$	1	0.3163	~ 0	C K
IIP15	1	-1	0	1	0.0358	0.9915	$\cdots$	1	0.0323	~ 0	M K
IIP16	1	-1	0	1	-26.8744	0.0323	$\cdots$	1	0.3375	~ 0	M K
IIP17	1	-1	±1	1	0	1	$\cdots$	1	0	~ 0	M K
IIP18	1	±1	-1	0	1	$\cdots$	1	0.75	0.25	~ 0	K M
IIP19	1	1	1	1	1	0.4698	0.6843	1	0.25	~ 0	C C
IIP20	1	-1	-1	1	1	1.127	0.7757	1	0.25	~ 0	M M
IIP21	1	-1	-1	1	1	0.6527	0.8733	1	0.25	~ 0	M M
IIP22	1	-1	1	1	1	0.1701	0.2330	1	0.25	~ 0	M C

$IIE3_+$(IIF6)] or an exponential expansion [$IIE4_+$(IIF2), $IIE5_+$(IIF3)]. Just as the case of the heterotic strings, we obtain strange solutions IIP4 $\sim$ 6, in which the external space shrinks exponentially in the original frame, but it expands by a power-law in the Einstein frame. In some past asymptotic solutions [IIP7 and IIP11], we also find a power-law inflation both in the original and Einstein frames.

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