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Behavior of curvature and matter in the Penrose limit

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In the Penrose limit construction a constant parameter Ω is introduced by which coordinates as well as the metric are rescaled in a suitable way. It results in a metric which apart from a possible coordinate dependence has an additional dependence on this constant parameter Ω. Here the scaling with Ω of the components of the Weyl tensor and the energy-momentum tensor is determined in the Penrose limit, i.e. $\Omega \to 0$. In both cases, a type of peeling-off behavior in Ω is found. Examples of different types of matter are provided. The expansion and shear of the congruence of null geodesics along which the Penrose limit is taken are determined. Finally, the approach to the singularity in the Penrose limit of cosmological space-times is discussed.

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I. INTRODUCTION

Recent developments in string-/M theory have led to renewed interest in an argument by Penrose [1] that in a neighborhood of a null geodesic, which contains no conjugate points, any space-time has a plane wave as a limit ([2] and references therein). However, as this plane wave limit depends on the choice of null geodesic a space-time can have more than one Penrose limit.

The Penrose limit construction involves changing first to a suitable system of coordinates and then rescaling the coordinates and the metric in a nonuniform way by a constant parameter Ω. This leads to a metric which apart from a possible dependence on the coordinates has an additional dependence on the constant parameter Ω. The Penrose limit then consists in taking the limit $\Omega \to 0$ of this Ω-dependent metric. It has been shown that this always leads to a plane wave metric [1]. Here, we will be interested in the behavior of the curvature and the energy-momentum tensor of the explicitly Ω-dependent space-time. Thus, the problem to be addressed is, how the components of the Weyl tensor and the energy-momentum tensor, calculated with the rescaled, explicitly Ω-dependent metric, scale with Ω in the Penrose limit $\Omega \to 0$.

The asymptotic behavior of the components of the Weyl tensor has been studied in space-times of a radiating isolated source. It is described in terms of a suitably defined radial coordinate r, considering the limit $r \to \infty$ [3]. This situation is different from the behavior of curvature and matter under consideration here, where the scaling with the parameter Ω in the Penrose limit $\Omega \to 0$ is the main issue. Nevertheless, as it turns out, the behavior of the components of the Weyl tensor and the energy-momentum tensor is very similar in the two different settings.

In the case of the limiting space-time far away from an isolated radiating source the components of the Weyl tensor show a typical peeling-off behavior [3]. In the Newman-Penrose formalism the components of the Weyl tensor are described by five complex scalars Ψ_i. In this particular case, as the far-field limit is approached these five components become negligible one after the other until only Ψ_4 remains which is associated with a plane wave space-time. A peeling-off behavior was also found in space-times admitting two spacelike Killing vectors [4]. Furthermore, space-times describing lightlike signals propagating through a general Bondi-Sachs space-time have peeling properties as well. In certain cases these are different from those of a Bondi-Sachs space-time [5].

In the following, it will be demonstrated that there is a kind of peeling-off behavior in the parameter Ω of the components of the Weyl tensor as well as the energy-momentum tensor. This is analogous to the peeling-off behavior in the radial coordinate r found in space-times of an isolated radiating source.

The first step of the Penrose limit construction leads to a rescaled form of the metric which explicitly depends on the constant parameter Ω, whose zero limit results in the limiting plane wave space-time. It is interesting to investigate whether there are properties of the initial space-time which remain unchanged in the Penrose limit $\Omega \to 0$ and thus are the same in the resulting plane wave space-time. As it will be shown, the expansion and the shear of the congruence of null geodesics along which the Penrose limit is taken are quantities which are independent of the parameter Ω and thus remain unchanged in the Penrose limit.

In Sec. II, the Penrose limit construction in four dimensions is formulated using the Newman-Penrose formalism which is especially adapted to the treatment of null geodesics. Then the peeling-off property of the Weyl tensor and of the energy-momentum tensor is found. In Sec. III the kinematics of the null geodesics is determined and the approach to null power law singularities is discussed. Finally, in Sec. IV conclusions are presented.

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II. THE WEYL TENSOR AND ENERGY-MOMENTUM TENSOR IN THE PENROSE LIMIT

The Penrose limit is taken in the neighborhood of a segment of a null geodesic γ not containing any conjugate points. It is possible to set up a system of coordinates u, v, x, y by embedding γ into a twist-free congruence of null geodesics, given by $v = \text{const.}$, $x = \text{const.}$, $y = \text{const.}$, and u, an affine parameter on each of them. $\gamma(u)$ corresponds to the geodesic given by $v = 0$, $x = 0$, $y = 0$. This null coordinate system is the analogue of the synchronous coordinate system [6].

Explicitly, using the null coordinate system the metric in the neighborhood of a segment of a null geodesic containing no conjugate points is of the form [1,6]

$$ds^2 = dudv + \alpha dv^2 + \beta_1 dxdv + \beta_2 dydv - \frac{2e^{-\mu_1}}{Z+\bar{Z}}(dx + iZdy)(dx - i\bar{Z}dy), \tag{2.1}$$

where α, β_i, Z, and μ_1 are functions of all coordinates u, v, x, y. Z is a complex function. Complex conjugation is denoted by a bar. The parametrization of the components of the metric over the two-dimensional $x - y$ subspace is of the form used in colliding plane wave space-times [7,8]. Whereas in the case of colliding plane waves Z is just a function of u and v, here Z is a function of all variables. This form of the metric seems to be convenient, since plane waves, and thus the resulting metrics in the Penrose limit, are a particular solution of colliding plane wave space-times.

Recently, the Penrose limit has been formulated in a covariant form by making use of a parallel-transported frame along the chosen null geodesic [9–11]. Such a frame is obtained by introducing a null tetrad with two real null vectors l^μ, n^μ and two complex null vectors which are complex conjugates of each other m^μ, $\bar{m}^\mu$. They satisfy the relations [7,12]

$$l_\mu n^\mu = 1, \qquad m_\mu \bar{m}^\mu = -1,$$
$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \tag{2.2}$$

Explicit expressions for the null tetrad and the spin coefficients in the Newman-Penrose formalism are given in the Appendix.

The Penrose limit uses the freedom of rescaling coordinates by a constant parameter. Namely, new coordinates U, V, X, Y are introduced as follows [1],

$$u = U, \qquad v = \Omega^2 V, \qquad x = \Omega X, \qquad y = \Omega Y, \tag{2.3}$$

where $\Omega > 0$ is a constant parameter. Furthermore, a conformal transformation is applied to the metric (2.1) such that

$$ds^2_\Omega = \Omega^{-2} ds^2, \tag{2.4}$$

resulting in

$$ds^2_\Omega = dUdV + \Omega^2 A dV^2 + \Omega B_1 dXdV + \Omega B_2 dYdV - \frac{2e^{-M}}{\zeta+\bar{\zeta}}(dX + i\zeta dY)(dX - i\bar{\zeta}dY), \tag{2.5}$$

where A, B_i, M, and ζ are functions of U, V, X, Y and replace α, β_i, μ_1, and Z, respectively, of the metric (2.1).

The Penrose limit is defined as

$$\hat{ds}^2 = \lim_{\Omega\to 0} ds^2_\Omega. \tag{2.6}$$

Applying this to the metric (2.5) a plane wave metric is obtained. The approach to this plane wave, in powers of Ω, will be studied by means of the components of the Weyl tensor. Furthermore, for nonvacuum space-times the scaling with Ω of the components of the energy-momentum tensor will be discussed.

In the Newman-Penrose formalism the ten components of the Weyl tensor are encoded in five complex scalars Ψ_i (cf. Appendix). These involve the spin coefficients as well as their directional derivatives. The Ricci identities [12] together with the dependence on Ω of the spin coefficients determine the scaling with the Penrose parameter Ω of the components of the Weyl tensor.

Using the expressions for the spin coefficients as given in the Appendix [cf. Eq. (A4)] for the rescaled metric (2.5) leads to the following overall scaling with the Penrose parameter Ω, which determines the factor Ω^n in front of the U-dependent part,

$$\kappa(\Omega) = \mathcal{O}(\Omega^3), \qquad \sigma(\Omega) = \mathcal{O}(\Omega^2),$$
$$\lambda(\Omega) = \mathcal{O}(\Omega^0), \qquad \nu = 0, \qquad \rho(\Omega) = \mathcal{O}(\Omega^2),$$
$$\mu(\Omega) = \mathcal{O}(\Omega^0), \qquad \tau(\Omega) = \mathcal{O}(\Omega), \qquad \pi(\Omega) = \mathcal{O}(\Omega),$$
$$\epsilon(\Omega) = \mathcal{O}(\Omega^2), \qquad \gamma(\Omega) = \mathcal{O}(\Omega^0),$$
$$\alpha_{NP}(\Omega) = \mathcal{O}(\Omega), \qquad \beta(\Omega) = \mathcal{O}(\Omega). \tag{2.7}$$

The scaling with Ω of the directional derivatives [cf. Appendix, Eq. (A5)] is found by going back to the original coordinates u, v, x, y since all metric functions are functions of these coordinates and taking partial derivatives with respect to the new coordinates U, V, X, Y leads to derivatives with respect to the old coordinates with an additional factor of Ω. For example,

$$D(\Omega) = -\Omega^2 \frac{\alpha}{2}\frac{\partial}{\partial u} + \Omega^2 \frac{\partial}{\partial v},$$

where α is, as before, a function of u, v, x, y. Therefore, the directional derivatives scale as follows,

$$D(\Omega) = \mathcal{O}(\Omega^2), \qquad \Delta(\Omega) = \mathcal{O}(\Omega^0),$$
$$\delta(\Omega) = \mathcal{O}(\Omega), \qquad \bar{\delta}(\Omega) = \mathcal{O}(\Omega). \tag{2.8}$$

Using the following Ricci identities [12] and the expression for Ψ_2 derived from Ricci identities

$$
\begin{aligned}
D\sigma - \delta\kappa &= \sigma(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}) + \kappa(\bar{\pi} - \tau - 3\beta - \bar{\alpha}_{NP}) + \Psi_0,\\
D\beta - \delta\epsilon &= \sigma(\alpha_{NP} + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha}_{NP} - \bar{\pi}) + \Psi_1,\\
\Psi_2 &= \frac{1}{3}[D\gamma - \Delta\epsilon - \delta\alpha_{NP} + \bar{\delta}\beta + D\mu - \delta\pi - (\alpha_{NP} + \pi)(\tau + \bar{\pi} - \bar{\alpha} + \beta) - \beta(\bar{\tau} + \pi + \alpha - \bar{\beta})\\
&\quad + (\mu + \gamma)(\epsilon + \bar{\epsilon} + \rho - \bar{\rho}) + \epsilon(\gamma + \bar{\gamma} + \mu - \bar{\mu}) + 2\nu\kappa - 2\sigma\lambda],\\
\Delta\alpha_{NP} - \bar{\delta}\gamma &= \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha_{NP}(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3,\\
\Delta\lambda - \bar{\delta}\nu &= -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha_{NP} + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4
\end{aligned} \tag{2.9}
$$

shows that the components of the Weyl tensor scale as

$$\Psi_i(\Omega) = \mathcal{O}(\Omega^{4-i}), \tag{2.10}$$

where $i = 0, \ldots, 4$. This scaling behavior with the constant parameter Ω can be interpreted as a peeling-off behavior. However, contrary to the standard peeling-off property it is determined by an expansion in the constant parameter Ω and not one of the coordinates. As an example of a standard peeling-off property consider the behavior of the components of the Weyl tensor of the far field of an isolated source. In this case, the components of the Weyl tensor behave as [3]

$$\Psi_k = \mathcal{O}(r^{k-5}), \tag{2.11}$$

in the limit $r \rightarrow \infty$ and r is a radial coordinate. Thus the peeling-off property manifests itself as an expansion in the coordinate r. Furthermore, there is a crucial difference in the behavior of Ψ_4 in the two types of peeling-off properties. The standard peeling-off property leads to Ψ_4 scaling as r^{-1}, thus the dependence on the coordinate r remains. However, in the case at hand Ψ_4 is independent of the parameter Ω. Thus taking the Penrose limit $\Omega \rightarrow 0$ does not change Ψ_4. As was already observed in the covariant approach to the Penrose limit as formulated in [9,10] here the component of the Weyl tensor Ψ_4 is exactly the same as in the original space-time (2.1).

Using Einstein's equations in the tetrad basis

$$R_{(a)(b)} - \frac{1}{2}\eta_{(a)(b)}R = -T_{(a)(b)} \tag{2.12}$$

allows one to determine the scaling behavior with the constant parameter Ω of the components of the energy-momentum tensor. In the Newman-Penrose formalism the components of the Ricci tensor are denoted by Φ_{ab} [cf. Appendix, Eq. (A7)]. The Ricci identities as given in [12], e.g., can be used to find the scaling with Ω of Φ_{ab} and thus the behavior of the components of the energy-momentum tensor.

The useful Ricci identities to determine the scaling of the Φ_{ab} are given by [12]

$$
\begin{aligned}
D\rho - \bar{\delta}\kappa &= (\rho^2 + \sigma\bar{\sigma}) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha_{NP} + \bar{\beta} - \pi) + \Phi_{00},\\
D\alpha_{NP} - \bar{\delta}\epsilon &= \alpha_{NP}(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10},\\
D\lambda - \bar{\delta}\pi &= \rho\lambda + \bar{\sigma}\mu + \pi(\pi + \alpha_{NP} - \bar{\beta}) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20},\\
\delta\nu - \Delta\mu &= \mu^2 + \lambda\bar{\lambda} + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}_{NP}) + \Phi_{22},\\
\delta\gamma - \Delta\beta &= \gamma(\tau - \bar{\alpha}_{NP} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12},\\
\Delta\rho - \bar{\delta}\tau &= -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha_{NP} - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda,\\
\delta\alpha_{NP} - \bar{\delta}\beta &= \mu\rho - \lambda\sigma + \alpha_{NP}\bar{\alpha}_{NP} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda.
\end{aligned} \tag{2.13}
$$

For the case of the rescaled metric (2.5) this gives the following scaling behavior with Ω,

$$
\begin{aligned}
&\Phi_{00} = \mathcal{O}(\Omega^4), \qquad \Phi_{10} = \mathcal{O}(\Omega^3), \qquad \Phi_{20} = \mathcal{O}(\Omega^2),\\
&\Phi_{11} = \mathcal{O}(\Omega^2), \qquad \Phi_{12} = \mathcal{O}(\Omega), \qquad \Phi_{22} = \mathcal{O}(\Omega^0),\\
&\Lambda = \mathcal{O}(\Omega^2),
\end{aligned} \tag{2.14}
$$

which determines the remaining components. The component Φ_{22} is independent of the parameter Ω. Therefore, it remains unchanged once the Penrose limit $\Omega \rightarrow 0$ is taken. It is exactly the component Φ_{22} of the original space-time (2.1).

Furthermore, Einstein's equations imply that the components of the energy-momentum tensor scale as follows,

$$
\begin{aligned}
&T_{(1)(1)} = \mathcal{O}(\Omega^4), \qquad T_{(1)(3)} = \mathcal{O}(\Omega^3),\\
&T_{(1)(2)} = \mathcal{O}(\Omega^2), \qquad T_{(3)(3)} = \mathcal{O}(\Omega^2),\\
&T_{(3)(4)} = \mathcal{O}(\Omega^2), \qquad T_{(2)(3)} = \mathcal{O}(\Omega),\\
&T_{(2)(2)} = \mathcal{O}(\Omega^0).
\end{aligned} \tag{2.15}
$$

As a first example, consider a space-time containing a Maxwell field, $F_{\mu\nu}$. Its components are given in the Newman-Penrose formalism by [12]

$$\phi_0 = F_{(1)(3)} = F_{\mu\nu} l^\mu m^\nu,$$
$$\phi_1 = \frac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \frac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu),$$
$$\phi_2 = F_{(4)(2)} = F_{\mu\nu}\bar{m}^\mu n^\nu. \tag{2.16}$$

The nonvanishing components of the energy-momentum tensor are given by [12]

$$T_{(1)(1)} = -2\phi_0\bar{\phi}_0, \qquad T_{(1)(2)} + T_{(3)(4)} = -4\phi_1\bar{\phi}_1,$$
$$T_{(1)(3)} = -2\phi_0\bar{\phi}_1, \qquad T_{(2)(2)} = -2\phi_2\bar{\phi}_2, \tag{2.17}$$
$$T_{(2)(3)} = -2\phi_1\bar{\phi}_2, \qquad T_{(3)(3)} = -2\phi_0\bar{\phi}_2.$$

Thus the components of the Maxwell field scale in the Penrose limit as follows,

$$\phi_0 = \mathcal{O}(\Omega^2), \qquad \phi_1 = \mathcal{O}(\Omega), \qquad \phi_2 = \mathcal{O}(\Omega^0), \tag{2.18}$$

or equivalently

$$\phi_i = \mathcal{O}(\Omega^{2-i}), \tag{2.19}$$

where $i = 0, 1, 2$. Thus one can interpret this as a peeling-off behavior, where the scaling is determined by the constant parameter Ω. Furthermore, we find that the component ϕ_2 is independent of Ω. Contrary to the other components ϕ_0 and ϕ_1, it remains unchanged when the Penrose limit $\Omega \to 0$ is taken. Thus ϕ_2 is the component of the original space-time (2.1). In comparison, the standard peeling-off behavior of the components of the Maxwell tensor in the background of the far field of an isolated, radiating source is given by (see, for example, [13]),

$$\phi_i = \mathcal{O}(r^{i-3}), \tag{2.20}$$

in the limit $r \to \infty$ and where r is a radial coordinate. As in the case of the standard peeling-off property of the components of the Weyl tensor all components are changed once the limit $r \to \infty$ is taken. In the case of the peeling-off property in the Penrose limit one of the components, namely, ϕ_2 remains unchanged once the Penrose limit is taken.

As a second example, the scaling behavior with the constant parameter Ω of the energy-momentum tensor describing a perfect fluid will be discussed. The energy-momentum tensor is given in this case by

$$T^{\mu\nu} = (\rho_f + p_f)u^\mu u^\nu - p_f g^{\mu\nu}, \tag{2.21}$$

where ρ_f and p_f are the energy and pressure densities, respectively. Following [7] the velocity can be written in terms of the tetrad vectors as

$$u^\mu = \frac{1}{\sqrt{2}}(a l^\mu + b n^\mu), \tag{2.22}$$

where a and b are constants. For $ab = 1$, u^μ is a timelike vector field. This describes the matter field in the original space-time, for example, a perfect fluid cosmology. In order to determine the scaling behavior of the energy-momentum tensor, the constants a and b are chosen such that $a \equiv 1$ and $b \equiv \Omega^2$, resulting in $u_\mu u^\mu = \Omega^2 \neq 1$. Furthermore, the last term in Eq. (2.21) is rescaled by a factor Ω^2. The nonvanishing components of the energy-momentum tensor are found to be

$$T_{(1)(1)} = \frac{\Omega^4}{2}(\rho_f + p_f), \qquad T_{(1)(2)} = \frac{\Omega^2}{2}(\rho_f - p_f),$$
$$T_{(2)(2)} = \frac{1}{2}(\rho_f + p_f), \qquad T_{(3)(4)} = \Omega^2 p_f. \tag{2.23}$$

Thus in the limit $\Omega \to 0$ the fluid becomes null. $T_{(2)(2)}$ is independent of the constant parameter Ω and remains the only nonvanishing component once the Penrose limit $\Omega \to 0$ is taken. The scaling with Ω of Eq. (2.23) follows Eq. (2.15). Furthermore, as already found in [10] the Penrose limit of a perfect fluid space-time is flat only for $p_f + \rho_f = 0$, that is, for a cosmological constant.

Einstein's equations yield the nonvanishing components of the Ricci tensor and the Ricci scalar as

$$\Phi_{00} = \frac{\Omega^4}{4}(\rho_f + p_f), \qquad \Phi_{11} = \frac{\Omega^2}{8}(\rho_f + p_f),$$
$$\Phi_{22} = \frac{1}{4}(\rho_f + p_f), \qquad \Lambda = \frac{\Omega^2}{24}(\rho_f - 3p_f). \tag{2.24}$$

Thus the scaling behavior follows Eq. (2.14).

III. KINEMATICS OF THE NULL CONGRUENCE

The focusing behavior of a congruence of geodesics is described by the Raychaudhuri equation,

$$\frac{d\theta}{du} = -\frac{1}{2}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta, \tag{3.1}$$

where k^α in this case is the tangent vector along a congruence of null geodesics (cf. e.g. [14]). This involves the expansion θ, the shear tensor $\sigma_{\alpha\beta}$, and the vorticity tensor $\omega_{\alpha\beta}$ of the geodesics. The Penrose limit of a space-time is calculated along a null geodesic. Choosing this null geodesic to calculate the expansion, the shear and the vorticity allow one to determine further characteristics of the plane wave space-time obtained in the Penrose limit, such as the occurrence of caustics or singularities.

The expansion, the shear, and the vorticity are defined in terms of the null vector k^α as follows, e.g., [12]

$$\theta = \frac{1}{2}k^\alpha_{;\alpha}, \qquad \sigma^2 = \frac{1}{2}k^{\alpha;\beta}k_{(\alpha;\beta)} - \frac{1}{4}(k^\alpha_{;\alpha})^2,$$
$$\omega_{\alpha\beta}\omega^{\alpha\beta} = \frac{1}{2}k^{\alpha;\beta}k_{[\alpha;\beta]},$$

where $\sigma^2 \equiv \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$.

In the Penrose adapted coordinates [cf. metric (2.1)] the tangent vector along the congruence of null geodesics is given by the vector n^μ of the null tetrad introduced in the previous section. The expansion, shear, and vorticity in the original space-time with metric (2.1) are given by

$$\theta = -\frac{1}{2}\mu_{1,u}, \tag{3.2}$$

$$\sigma^2 = \frac{Z_{,u}\bar{Z}_{,u}}{(Z+\bar{Z})^2}, \tag{3.3}$$

$$\omega_{\alpha\beta}\omega^{\alpha\beta} = 0. \tag{3.4}$$

There is no vorticity by construction of the coordinate system [cf. Eq. (2.1)] [1,6].

Introducing the new coordinates and rescaling the metric accordingly [cf. Eqs. (2.3), (2.4), and (2.5)] shows that the quantities (3.2), (3.3), and (3.4) are independent of the parameter Ω. Thus taking the Penrose limit $\Omega \to 0$ does not change the expansion, shear, and vorticity of the congruence of null geodesics. They are the same as in the original space-time (2.1). This is consistent with the fact that only the transverse part of the metric enters into these expressions. The Raychaudhuri equation (3.1) determines the evolution of the expansion θ. However, in the case of the plane wave space-time resulting from taking the Penrose limit, the kinematic quantities expansion, shear, and vorticity are already known since they are directly derived from the original space-time. Thus it can be immediately determined if any of these quantities becomes unbounded at some point and thus a singularity develops.

As an example a Kasner type metric will be considered. The metric is given by

$$ds^2 = dt^2 - t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2, \tag{3.5}$$

where the Kasner exponents p_i satisfy the relations $\sum_{i=1}^{3} p_i = 1$. In vacuum there is an additional relation, that is, $\sum_{i=1}^{3} p_i^2 = 1$. For stiff perfect fluid cosmologies the right-hand side of this relation is equal to a constant unequal to 1.

Kasner-type metrics provide a good approximation for the description of space-times close to a singularity. Furthermore, for spatially homogeneous models it has been shown that there is a class of models for which the approach to the singularity is oscillatory and chaotic. The dynamics can be described by a succession of epochs in which the Kasner indices are interchanged [15].

Introducing conformal time T, defining null coordinates $u = T - z$, $v = T + z$ taking the Penrose limit according to (2.3), (2.4), (2.5), and (2.6) and redefining u and rescaling the remaining coordinates accordingly, the resulting plane wave space-time is given by

$$ds^2 = dudv - u^{\alpha_1}dx^2 - u^{\alpha_2}dy^2, \tag{3.6}$$

where $\alpha_1 \equiv \frac{2p_1}{2-(p_1+p_2)}$ and $\alpha_2 \equiv \frac{2p_2}{2-(p_1+p_2)}$. Determining μ_1 and Z results in

$$\mu_1 = -\frac{\alpha_1+\alpha_2}{2}\ln u, \qquad Z = u^{[(\alpha_2-\alpha_1)/2]}. \tag{3.7}$$

Thus the expansion and the shear are given by

$$\theta = \frac{\alpha_1+\alpha_2}{4}\frac{1}{u}, \tag{3.8}$$

$$\sigma^2 = \frac{1}{4}\left(\frac{\alpha_2-\alpha_1}{2}\right)^2\frac{1}{u^2}. \tag{3.9}$$

In spatially homogeneous space-times the ratio of shear over the expansion is a measure of anisotropy. Calculating this ratio in the case of the null geodesics it is found that

$$\frac{\sigma^2}{\theta^2} = \left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right)^2 = \left(\frac{p_2-p_1}{p_1+p_2}\right)^2. \tag{3.10}$$

The expansion θ and the shear σ^2 both diverge as $u \to 0$. However, their ratio is constant. Thus the amount of anisotropy measured by the ratio σ^2/θ^2 equals a constant, determined by the Kasner exponents in the x and y directions. The amount of anisotropy is not changed by taking the Penrose limit, that is, it is independent of the parameter Ω.

In comparison, the anisotropy along a congruence of timelike geodesics is determined by the constant ratio

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3}\left(\sum_i p_i^2 - p_1p_2 - p_1p_3 - p_2p_3\right). \tag{3.11}$$

Spatially homogeneous space-times with form fields [16] as well as vacuum Bianchi IX space-times show chaotic behavior in the approach to the initial singularity. This behavior is well described by a succession of Kasner epochs, each determined by a different set of Kasner exponents (p_1, p_2, p_3). Einstein's equations determine the effective change in the Kasner exponents. However, in the case of the Penrose limit the approach to the null singularity at $u = 0$ is simpler than in the case of the approach to the spacelike singularity of the original space-time at $t = 0$. The initial set of Kasner exponents (p_1, p_2, p_3) remains unchanged, since the only Einstein equation of a plane wave is given by

$$2\mu_{1,uu} - \mu_{1,u}^2 - 4\frac{Z_{,u}\bar{Z}_{,u}}{(Z+\bar{Z})^2} = 4\Phi_{22}, \tag{3.12}$$

which in vacuum is satisfied by metrics of type (3.6) with $\sum_i p_i = 1 = \sum_i p_i^2$. The disappearance of the oscillating behavior in the plane wave space-time obtained in the Penrose limit might also be understood from the point of view of functional genericity of the metric. General vacuum Bianchi IX space-times, for example, are described by eight free functions. In the case of metrics of the type of Eq. (2.1) it has been argued that the most general solution of this type has eight arbitrary functions [17]. However, taking the Penrose limit of this metric results in the vanishing of the VV, the XV, and YV components. Thus the resulting plane wave space-time has less degrees of freedom than the general type of the original space-time. Since chaotic behavior in the approach to the singularity is related to the functional genericity of the space-time any chaotic behavior of the original space-time is not expected to persist in the Penrose limit.

IV. CONCLUSIONS

The Penrose limit of any space-time is a plane wave. The Penrose limit construction involves choosing a null geodesic around which the Penrose limit is taken. Thus a space-time can have more than one Penrose limit. However, once a null geodesic and appropriate coordinates are chosen the components of the Weyl tensor and the energy-momentum tensor are fixed. Here, the behavior of the components of the Weyl tensor and the energy-momentum tensor has been investigated with respect to its scaling in the parameter Ω appearing in the Penrose limit construction. This parameter is introduced to rescale the metric as well as the coordinates and the Penrose limit is the limit $\Omega \to 0$ of the rescaled metric. Here, it has been found that the components of the Weyl tensor and the energy-momentum tensor show a scaling with Ω which can be interpreted as a peeling-off behavior.

The components of the curvature tensor are known to show a peeling-off behavior in the case of the space-time of an isolated radiating source or in the case of space-times admitting two Abelian Killing vectors [3–5]. In these cases the peeling-off behavior manifests itself in the expansion, in terms of one of the coordinates, of the components of the Weyl tensor. This is different from the case of the approach to the Penrose limit where the analogue of this behavior in the parameter Ω, which is not a coordinate, has been found.

The approach taken here is in the line of the covariant approach to the Penrose limit [9,10]. Using the Newman-Penrose formalism it has been shown that the component Ψ_4 of the Weyl tensor of the original metric in coordinates adapted to the neighborhood around a null geodesic is independent of the parameter Ω. Thus it is not changed once the Penrose limit $\Omega \to 0$ is taken. It remains the only nonvanishing component of the Weyl tensor in the resulting plane wave space-time. Similarly, Φ_{22} is the only nonvanishing component of the Ricci tensor. This is different from the behavior found in the standard peeling-off behavior in, for example, the space-time of an isolated radiating source. Here all components of the Weyl tensor scale with some negative power of the radial coordinate r. Thus all components are affected by taking the limit $r \to \infty$.

The peeling-off behavior in the parameter Ω might be used to find corrections to the resulting plane wave space-time also with respect to the discussion of singularities in these types of space-times.

The expansion, shear, and vorticity of the congruence of null geodesic around which the Penrose limit is taken has been determined. These expressions do not depend on the constant parameter Ω. Their values are determined by functions characterizing the original metric. By construction of the coordinate system the congruence of null geodesics has vanishing vorticity.

A large class of cosmological solutions close to the initial singularity is well approximated by Kasner-like space-times. For this type of metrics it has been found that the anisotropy as measured by the ratio of shear over the expansion is a constant. This is the case for congruences of timelike as well as null geodesics.

In the case of Bianchi IX vacuum and of spatially homogeneous models with form fields it is known that the approach to the initial (spacelike) singularity is oscillatory and chaotic. The dynamics is described by the succession of epochs characterized by different Kasner exponents. Taking the Penrose limit in one of these epochs the corresponding Kasner indices remain unchanged. Therefore, the approach to the appearing null singularity in the plane wave space-time is not chaotic. This could also be interpreted in terms of the functional genericity of the plane wave space-time resulting in the Penrose limit.

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APPENDIX: QUANTITIES IN THE NEWMAN-PENROSE FORMALISM

Several quantities in the Newman-Penrose formalism are given (cf., e.g., [12]). The tetrad metric is given by

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{A1}$$

Tetrad indices are enclosed in brackets ( ). They run from 1 to 4.

For a space-time with line element (2.1) a null tetrad is provided by

$$\begin{aligned} e_{(1)} &= e^{(2)} = l_\mu = \begin{pmatrix} 1 & \frac{\alpha}{2} & \beta_1 & \beta_2 \end{pmatrix}, \\ e_{(2)} &= e^{(1)} = n_\mu = (0 \quad 1 \quad 0 \quad 0), \\ e_{(3)} &= -e^{(4)} = m_\mu = \begin{pmatrix} 0 & 0 & -\frac{e^{-(\mu_1/2)}}{(Z+\bar{Z})^{1/2}} & \frac{i\bar{Z}e^{-(\mu_1/2)}}{(Z+\bar{Z})^{1/2}} \end{pmatrix}, \\ e_{(4)} &= -e^{(3)} = \bar{m}_\mu = \begin{pmatrix} 0 & 0 & -\frac{e^{-(\mu_1/2)}}{(Z+\bar{Z})^{1/2}} & -\frac{iZe^{-(\mu_1/2)}}{(Z+\bar{Z})^{1/2}} \end{pmatrix}. \end{aligned} \tag{A2}$$

The corresponding contravariant components are given by

$$l^\mu = \begin{pmatrix} -\frac{\alpha}{2} & 1 & 0 & 0 \end{pmatrix}, \qquad n^\mu = (1 \quad 0 \quad 0 \quad 0), \qquad m^\mu = \begin{pmatrix} -\frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}}(\beta_1 \bar Z - i\beta_2) & 0 & \frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}}\bar Z & -i\frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}} \end{pmatrix},$$

$$\bar m^\mu = \begin{pmatrix} -\frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}}(\beta_1 Z + i\beta_2) & 0 & \frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}} Z & i\frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}} \end{pmatrix}. \tag{A3}$$

The Newman-Penrose spin coefficients derived from the Ricci rotation coefficients, $\gamma_{(a)(b)(c)} = e^\mu_{(a)} e_{(b)\mu;\nu} e^\nu_{(c)}$, are given by

$$\begin{aligned}
\kappa &= \frac{e^{\mu_1/2}}{(Z+\bar Z)^{1/2}}\left[\bar Z\left(\frac{1}{2}\beta_1\alpha_{,u} - \frac{1}{2}\alpha\beta_{1,u} + \beta_{1,v} - \frac{1}{2}\alpha_{,x}\right) - i\left(\frac{1}{2}\alpha_{,u}\beta_2 - \frac{1}{2}\alpha\beta_{2,u} + \beta_{2,v} - \frac{1}{2}\alpha_{,y}\right)\right], \quad \sigma = \frac{-\frac{\alpha}{2}\bar Z_{,u} + \bar Z_{,v}}{Z+\bar Z},\\
\lambda &= -\frac{Z_{,u}}{Z+\bar Z}, \quad \nu = 0, \quad \rho = \frac{1}{2}\left[\mu_{1,v} - \frac{\alpha}{2}\mu_{1,u} + ie^{\mu_1}(\beta_1\beta_{2,u} - \beta_2\beta_{1,u} + \beta_{1,y} - \beta_{2,x})\right], \quad \mu = -\frac{1}{2}\mu_{1,u},\\
\tau &= \frac{1}{2}\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_{1,u}\bar Z - i\beta_{2,u}), \quad \pi = -\frac{1}{2}\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_{1,u}Z + i\beta_{2,u}),\\
\epsilon &= \frac{1}{4}\left[-\frac{Z_{,v} - \bar Z_{,v}}{Z+\bar Z} + \frac{\alpha}{2}\frac{Z_{,u} - \bar Z_{,u}}{Z+\bar Z} + ie^{\mu_1}(\beta_1\beta_{2,u} - \beta_2\beta_{1,u} + \beta_{1,y} - \beta_{2,x}) - \alpha_{,u}\right], \quad \gamma = \frac{1}{4}\frac{\bar Z_{,u} - Z_{,u}}{Z+\bar Z},\\
\alpha_{NP} &= -\frac{1}{2}\left[\frac{1}{2}\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_{1,u}Z + i\beta_{2,u}) + ie^{\mu_1}\left(-\left(\frac{e^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,u}\beta_2 + i\left(\frac{Ze^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,u}\beta_1 + \left(\frac{e^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,y} - i\left(\frac{Ze^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,x}\right)\right],\\
\beta &= -\frac{1}{2}\left[\frac{1}{2}\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_{1,u}\bar Z - i\beta_{2,u}) + ie^{\mu_1}\left(-\left(\frac{e^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,u}\beta_2 - i\left(\frac{\bar Ze^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,u}\beta_1 + \left(\frac{e^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,y} + i\left(\frac{\bar Ze^{-(\mu_1/2)}}{\sqrt{Z+\bar Z}}\right)_{,x}\right)\right],
\end{aligned} \tag{A4}$$

where , u denotes $\frac{\partial}{\partial u}$ etc.

The directional derivatives for the metric (2.1) are given by

$$\begin{aligned}
D &= e_{(1)} = e^{(2)} = -\frac{\alpha}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \qquad \Delta = e_{(2)} = e^{(1)} = \frac{\partial}{\partial u},\\
\delta &= e_{(3)} = -e^{(4)} = -\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_1\bar Z - i\beta_2)\frac{\partial}{\partial u} + \frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}\bar Z\frac{\partial}{\partial x} - i\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}\frac{\partial}{\partial y},\\
\bar\delta &= e_{(4)} = -e^{(3)} = -\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}(\beta_1 Z + i\beta_2)\frac{\partial}{\partial u} + \frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}Z\frac{\partial}{\partial x} + i\frac{e^{\mu_1/2}}{\sqrt{Z+\bar Z}}\frac{\partial}{\partial y}.
\end{aligned} \tag{A5}$$

The components of the Weyl tensor are given by

$$\begin{aligned}
\Psi_0 &= -C_{(1)(3)(1)(3)} = -C_{\mu\nu\lambda\kappa}l^\mu m^\nu l^\lambda m^\kappa,\\
\Psi_1 &= -C_{(1)(2)(1)(3)} = -C_{\mu\nu\lambda\kappa}l^\mu n^\nu l^\lambda m^\kappa,\\
\Psi_2 &= -C_{(1)(3)(4)(2)} = -C_{\mu\nu\lambda\kappa}l^\mu m^\nu \bar m^\lambda n^\kappa,\\
\Psi_3 &= -C_{(1)(2)(4)(2)} = -C_{\mu\nu\lambda\kappa}l^\mu n^\nu \bar m^\lambda n^\kappa,\\
\Psi_4 &= -C_{(2)(4)(2)(4)} = -C_{\mu\nu\lambda\kappa}n^\mu \bar m^\nu n^\lambda \bar m^\kappa.
\end{aligned} \tag{A6}$$

The components of the Ricci tensor are denoted as

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{(1)(1)}, \qquad \Phi_{22} = -\frac{1}{2}R_{(2)(2)},\\
\Phi_{02} &= -\frac{1}{2}R_{(3)(3)}, \qquad \Phi_{20} = -\frac{1}{2}R_{(4)(4)},\\
\Phi_{11} &= -\frac{1}{4}(R_{(1)(2)} + R_{(3)(4)}), \qquad \Phi_{01} = -\frac{1}{2}R_{(1)(3)},\\
\Lambda &= \frac{1}{24}R = \frac{1}{12}(R_{(1)(2)} - R_{(3)(4)}), \qquad \Phi_{10} = -\frac{1}{2}R_{(1)(4)},\\
\Phi_{12} &= -\frac{1}{2}R_{(2)(3)}, \qquad \Phi_{21} = -\frac{1}{2}R_{(2)(4)}.
\end{aligned} \tag{A7}$$

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